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Letter to the Editor

On the existence of resistive magnetohydrodynamic equilibria

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Abstract. A necessary condition for the existence of general dissipative magnetohydrodynamic equilibria is derived. The derivation comprises Ohm's law and the existence of magnetic surfaces, only in the sense of the Kolmogorov–Arnold–Moser (KAM) theorem. All other equations describing the system are only required for evaluating the condition for a specific case.

Determining the conditions for the existence of long-term equilibria of hot magnetically confined plasmas is an important area of fusion research. Such plasmas are weakly collisional and obey the kinetic Maxwell equations. The conditions for the existence of such equilibria are known neither within the framework of neoclassical theory nor in resistive magnetohydrodynamics (MHD). Since neoclassical theory should match classical theory in the highly collisional limit, it is of interest to derive long-term equilibrium conditions for the macroscopic equations. The purpose of this paper is to derive the necessary equilibrium conditions within the frame of resistive viscous MHD.

The basic equations for resistive viscous MHD equilibria are (see, e.g., [1])

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla p - \nu\nabla\times\nabla\times\mathbf{v} + \text{source (momentum)}, \tag{1}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \boldsymbol{\eta}\cdot\mathbf{j} + \frac{\lambda}{\rho}(\mathbf{j}\times\mathbf{B} - \nabla p_{\mathrm{e}}), \tag{2}$$

$$\nabla\cdot\rho\mathbf{v} = \text{source (mass)}, \tag{3}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{4}$$

$$\nabla\times\mathbf{E} = 0, \tag{5}$$

$$\nabla\times\mathbf{B} = \mathbf{j}, \tag{6}$$

where ρ is the mass density, ν is the viscosity, $\boldsymbol{\eta}$ is the anisotropic resistivity, p is the total pressure, p_{e} is the pressure of the electron fluid, $\mathbf{v}$ is the velocity, $\mathbf{B}$ is the

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magnetic field and $\mathbf{E}$ and $\mathbf{j}$ are the electric field and the electric current density, respectively.

Since only Ohm's law (2) and (4) and (5) will be used in the derivation, the other equations can be modified to accommodate for any additional physical sophistication. Another important factor is the existence of a volume bounded by magnetic surfaces. As a result of the Kolmogorov–Arnold–Moser (KAM) theorem (see, e.g., [2]) this is a mild assumption even for three-dimensional equilibria.

Let us take the scalar product of (2) with the magnetic field to obtain

$$\mathbf{B}\cdot\nabla\Phi = \mathbf{B}\cdot\boldsymbol{\eta}\cdot\mathbf{j} - \frac{\lambda}{\rho}\mathbf{B}\cdot\nabla p_{\mathrm{e}}, \tag{7}$$

where $\mathbf{E}$ has been replaced by $\nabla\Phi$ because of (5). An equation similar to (7) has been derived previously by Kruskal and Kulsrud [3] assuming $\mathbf{j}\times\mathbf{B} = \nabla p$ instead of (1). However, Kruskal and Kulsrud did not attempt to obtain an existence condition as demonstrated here.

We now integrate (7) throughout any volume bounded by two magnetic surfaces. By Gauss' theorem and (4) the gradient term on the right-hand side of (7) vanishes if the electron fluid is barotropic, $p_{\mathrm{e}}(\rho)$, or if the electron temperature is constant along the magnetic field lines. Let us note that it is likely that for short mean-free paths the electrons behave adiabatically, while for moderate mean-free paths the electron temperature becomes constant along magnetic field lines. The reason why Gauss' theorem applies is that $\nabla p_{\mathrm{e}}/\rho = \nabla F(\rho)$ as the magnetic field is divergence free. The integral of the first term on the left-hand side either vanishes if the potential Φ is single valued or is proportional to the loop voltage if the potential is multi valued.

- *Tokamak case*. Since the transformer produces a toroidal loop voltage, the above integration leads to

$$FV_{\mathrm{L}} = \int_{\mathrm{KAM}} d\tau\, \mathbf{B}\cdot\boldsymbol{\eta}\cdot\mathbf{j}, \tag{8}$$

  where F is the toroidal magnetic flux, V_{L} the toroidal loop voltage and $d\tau$ the volume element.

- *Stellarator case*. Since no electric field is induced the potential is single valued, so the above integration leads to

$$\int_{\mathrm{KAM}} d\tau\, \mathbf{B}\cdot\boldsymbol{\eta}\cdot\mathbf{j} = 0. \tag{9}$$

Conditions (8) and (9) are only necessary since they do not need the full system of (1)–(6) for their derivation. However, they can give a qualitative insight of how the current density behaves in order to fulfil the stationary condition as explained below. The evaluation of the non-local conditions (8) and (9) requires knowledge of the KAM surfaces in the three-dimensional MHD equilibrium as well as the resistivity tensor, the magnetic field and the current density.

(1) Condition (9), for instance, indicates that, for a scalar Spitzer resistivity, the volume average of the parallel current density multiplied by the magnetic field between nearby KAM surfaces should vanish. For low 'beta' zero flow plasmas, the current density can be found from $\mathbf{j}\times\mathbf{B} = \nabla p$, $\nabla\cdot\mathbf{j} = 0$ assuming $\mathbf{B}$ is a vacuum magnetic field whose KAM surfaces can be approximated numerically.

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For the resistive viscous finite beta case, a great amount of work has yet to be done. However, it is encouraging to know that in some of these cases existence theorems have been proved (see [4]), which gives hope to the idea of constructing reliable codes with condition (9) as a benchmark.

(2) For many axisymmetric equilibria the magnetic surfaces are known as well as the magnetic field and the current density. In the latter case one can find local conditions for the existence of resistive MHD incompressible equilibria and even solve the full system of (1)–(6) as demonstrated in [5]. However, the non-local conditions (8) and (9) apply to any toroidal MHD equilibrium without restricting the number of dimensions by symmetry.

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