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# A Multidimensional Central Sets Theorem

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The theorems of Hindman and van der Waerden belong to the classical theorems of partition Ramsey Theory. The Central Sets Theorem is a strong simultaneous extension of both theorems that applies to general commutative semigroups. We give a common extension of the Central Sets Theorem and Ramsey's theorem.

## 1. Introduction

Van der Waerden's theorem [9] states that, for any partition of the positive integers $\mathbb{N}$, one of the cells of the partition contains arbitrarily long arithmetic progressions.

To formulate Hindman's theorem [5] and the Central Sets Theorem we set up some notation. By $\mathscr{P}_f(\omega)$ we denote the set of all finite nonempty subsets of $\omega = \mathbb{N} \cup \{0\}$. For a sequence $\langle x_n \rangle_{n=0}^{\infty}$ in $\mathbb{N}$ we put $\mathrm{FS}(\langle x_n \rangle_{n=0}^{\infty}) := \{\sum_{t\in\alpha} x_t : \alpha \in \mathscr{P}_f(\omega)\}$. A set $A \subseteq \mathbb{N}$ is called an *IP-set* if and only if there exists a sequence $\langle x_n \rangle_{n=0}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}(\langle x_n \rangle_{n=0}^{\infty}) \subseteq A$. (These definitions make perfect sense in any semigroup $(S, \cdot)$ and we indeed plan to use them in this context. FS is an abbreviation of *finite sums* and will be replaced by FP if we use multiplicative notation for the semigroup operation.) Now Hindman's theorem states that in any finite partition of $\mathbb{N}$ one of the cells is an IP-set.

K. Milliken and A. Taylor [7, 8] found a quite natural common extension of the theorems of Hindman and Ramsey. For a sequence $\langle x_n \rangle_{n=0}^{\infty}$ in $\mathbb{N}$ and $k \geqslant 1$ put $[\mathrm{FS}(\langle x_n \rangle_{n=0}^{\infty})]_<^k := \{\{\sum_{t\in\alpha_1} x_t, \ldots, \sum_{t\in\alpha_k} x_t\} : \alpha_1 < \cdots < \alpha_k \in \mathscr{P}_f(\omega)\}$, where we write $\alpha < \beta$ for $\alpha, \beta \in \mathscr{P}_f(\omega)$ if and only if $\max \alpha < \min \beta$. For an arbitrary set $S$ let $[S]^k$ be the set of all finite subsets of $S$ consisting of exactly $k$ elements. If $[\mathbb{N}]^k = \bigcup_{i=1}^r A_i$ then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=0}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}[\langle x_n \rangle_{n=0}^{\infty}]_<^k \subseteq A_i$.

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Let $\Phi$ be the set of all functions $f : \omega \to \omega$ such that $f(n) \leqslant n$ for all $n \in \omega$. Then our main theorem may be stated as follows.

**Theorem 1.1.** *Let $(S, \cdot)$ be a commutative semigroup and assume that there exists a nonprincipal minimal idempotent in $\beta S$. For each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=0}^{\infty}$ be a sequence in $S$. Let $k, r \geqslant 1$ and let $[S]^k = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \dots, r\}$, a sequence $\langle a_n\rangle_{n=0}^{\infty}$ in $S$ and a sequence $\alpha_0 < \alpha_1 < \cdots$ in $\mathscr{P}_f(\omega)$ such that, for each $g \in \Phi$, $[\mathrm{FP}(\langle a_n \prod_{t\in\alpha_n} y_{g(n),t}\rangle_{n=0}^{\infty})]_{<}^{k} \subseteq A_i$.*

We will review some properties of the Stone–Čech compactification as well as the definition of a minimal idempotent in the next section. In the case $k = 1$, the somewhat odd assumption that $\beta S$ should contain a nonprincipal minimal idempotent is not needed. In general this condition will be satisfied if $S$ is *weakly (left) cancellative*, *i.e.*, for all $u, v \in S$ the set $\{s \in S : us = v\}$ is finite and $S$ itself is infinite (see [6, Theorem 4.3.7]). In particular, the conclusion of Theorem 1.1 holds in the semigroups $(\mathbb{N}, +), (\mathbb{N}, \cdot), (\mathscr{P}_f(\omega), \cup)$.

The case $k = 1$ of Theorem 1.1 is exactly the Central Sets Theorem. (More precisely, this is the version stated in [6, Corollary 14.12]. A discussion on the origin of the Central Sets Theorem can also be found there.) By further specifying $(S, \cdot) = (\mathbb{N}, +)$ and $\langle y_{l,n}\rangle_{n=0}^{\infty} = \langle l, l, \dots\rangle$ we get that all finite sums of elements of the arithmetic progressions $a_n, a_n + |\alpha_n|, \dots, a_n + n|\alpha_n|, n \geqslant 0$ are guaranteed to be monochrome.

Theorem 1.1 may be seen as a generalization of the Central Sets Theorem in the same sense as the Milliken–Taylor theorem is a multidimensional version of Hindman's theorem.

## 2. Preliminaries on ultrafilters

For a set $S$ let $\beta S$ be the set of all ultrafilters on $S$. For $s \in S$ we will identify $s$ with the principal ultrafilter of all subsets of $S$ that contain $s$. If $(S, \cdot)$ is a semigroup, the operation $\cdot\,\cdot$ may be extended to a semigroup operation on $\beta S$ by defining

$$A \in p \cdot q \;:\Leftrightarrow\; \{s \in S : s^{-1}A \in q\} \in p.^{1} \tag{2.1}$$

If $\beta S$ is properly topologized it turns out to be the Stone–Čech compactification of $S$ (where we regard $S$ to be a discrete space). It can be shown that the operation $\cdot\,\cdot : \beta S \times \beta S \to \beta S$ defined in (2.1) is the unique extension of $\cdot\,\cdot : S \times S \to S$, such that, for each $s \in S$ and each $q \in \beta S$, the functions $\lambda_s, \rho_q : \beta S \to \beta S$ defined by $\lambda_s(r) := sr, \rho_q(r) := rq$ are continuous.

Applications of the algebraic structure of $\beta S$ in partition Ramsey Theory are abundant. Examples are simple proofs of the theorems of Hindman and van der Waerden. Idempotent ultrafilters (*i.e.*, ultrafilters $e \in \beta S$ satisfying $ee = e$) turn out to be tightly connected with IP-sets in $S$: a subset $A$ of $S$ is an IP-set if and only if there is an idempotent $e \in \beta S$ such that $A \in e$. By a theorem of Ellis $\beta S$ always contains an idempotent ultrafilter $e$ and by the ultrafilter properties of $e$ for any partition $A_1, A_2, \dots, A_r$ of $S$ there exists an $i \in \{1, 2, \dots, r\}$ such that $A_i \in e$. Thus $A_i$ is an IP-set.

[1] $S$ is a semigroup, so $s$ might not have an inverse. We may avoid this obstacle by defining $s^{-1}A := \{t \in S : st \in A\}$.

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$\beta S$ always has a smallest (two-sided) ideal which will be denoted by $K(\beta S)$. It turns out that for $(S,\cdot)=(\mathbb{N},+)$ the elements of $K(\beta\mathbb{N})$ are well suited for van der Waerden's theorem.

Idempotents in $K(\beta S)$ (which are always present) are called *minimal idempotents*. Not at all surprisingly, minimal idempotents are particularly interesting for combinatorial applications. Subsets of $S$ which are contained in some minimal idempotent are called *central sets* and that these sets satisfy the conclusion of the Central Sets Theorem reveals the source of the theorem's name.

See [6] for an elementary introduction to the semigroup $\beta S$ as well as for the combinatorial applications mentioned in this section.

If $S$ is an infinite set an arbitrary nonprincipal ultrafilter $p\in\beta S$ may be used to give a proof of Ramsey's theorem. (This proof is by now classical. See [2, p. 39] for a discussion of its origins.) It is an idea of V. Bergelson and N. Hindman that, in the case $S=\mathbb{N}$, something might be gained by using an ultrafilter with special algebraic properties. Via this approach in [1] a short proof of the Milliken–Taylor theorem is given and a very strong simultaneous generalization of Ramsey's theorem and numerous single-dimensional Ramsey-type theorems (including van der Waerden's theorem) is obtained. Our proof is a variation on this idea.

## 3. The proof of the main theorem

The following lemma is the basic tool in the ultrafilter proof of Ramsey's theorem.

**Lemma 3.1.** *Let $S$ be a set, let $e\in\beta S\setminus S$, let $k,r\geqslant 1$, and let $[S]^k=\bigcup_{i=1}^r A_i$. For each $i\in\{1,2,\dots,r\}$, each $t\in\{1,2,\dots,k\}$ and each $E\in[S]^{t-1}$, define $B_t(E,i)$ by downward induction on $t$:*

(1) *for $E\in[S]^{k-1}$, $B_k(E,i):=\{y\in S\setminus E : E\cup\{y\}\in A_i\}$,*
(2) *for $1\leqslant t<k$ and $E\in[S]^{t-1}$, $B_t(E,i):=\{y\in S\setminus E : B_{t+1}(E\cup\{y\},i)\in e\}$.*

*Then there exists some $i\in\{1,2,\dots,r\}$ such that $B_1(\emptyset,i)\in e$.*

**Proof.** For each $E\in[S]^{k-1}$ we have $S=E\cup\bigcup_{i=1}^r B_k(E,i)$, so there exists $i\in\{1,2,\dots,r\}$ such that $B_k(E,i)\in e$. Next let $E\in[S]^{k-2}$ and $y\in S\setminus E$. Then there exists $i\in\{1,2,\dots,r\}$ such that $B_k(E\cup\{y\},i)\in e$. Thus $S=E\cup\bigcup_{i=1}^r B_{k-1}(E,i)$. After iterating this argument $k-1$ times we achieve $S=\emptyset\cup\bigcup_{i=1}^r B_1(\emptyset,i)$, which clearly proves the statement. □

To formulate our key lemma we need to introduce some notation. Let $S$ be a set and put $S^{<\omega}=\bigcup_{n=0}^{\infty}S^{\{0,\dots,n-1\}}$. A non-empty set $T\subseteq S^{<\omega}$ is a *tree* in $S$ if and only if, for all $f\in S^{<\omega}$, $g\in T$ such that $\operatorname{dom} f\subseteq\operatorname{dom} g$, $g_{\restriction\operatorname{dom} f}=f$, we have $f\in T$. We will identify a function $f\in S^{\{0,1,\dots,n-1\}}$ with the sequence $\langle f(0),f(1),\dots,f(n-1)\rangle$. If $s\in S$ then $f^\frown s:=\langle f(0),f(1),\dots,f(n-1),s\rangle$. For $f\in S^{<\omega}$ we put $T(f):=\{s\in S : f^\frown s\in T\}$.

**Lemma 3.2.** *Let $(S,\cdot)$ be a semigroup such that there exists an idempotent $e\in\beta S\setminus S$, let $k,r\geqslant 1$ and let $[S]^k=\bigcup_{i=1}^r A_i$. Then there exist $i\in\{1,2,\dots,r\}$ and a tree $T\subseteq S^{<\omega}$ such*

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*that, for all* $f \in T$ *and* $\alpha_1 < \alpha_2 < \cdots < \alpha_k \subseteq \operatorname{dom} f, \alpha_i \in \mathscr{P}_f(\omega)$, *we have:*

(1) $T(f) \in e$,
(2) $\{\prod_{t\in\alpha_1} f(t), \prod_{t\in\alpha_2} f(t), \ldots, \prod_{t\in\alpha_k} f(t)\} \in A_i$.

In the proof we will employ some basic properties of idempotent ultrafilters. For a set $A \subseteq S$ we have $A \in e = ee$ if and only if $\{s \in S : s^{-1}A \in e\} \in e$ by the definition of the multiplication in $\beta S$. For $A \in e$ let

$$A^\star := \{s \in A : s^{-1}A \in e\} = A \cap \{s \in S : s^{-1}A \in e\} \in e.$$

Then $t^{-1}A^\star \in e$ for all $t \in A^\star$, as we now show. It is clear that, for $t \in A^\star$, $t^{-1}A \in e$. Furthermore

$$t^{-1}\{s \in S : s^{-1}A \in e\} = \{s \in S : s^{-1}(t^{-1}A) \in e\} \in e.$$

Thus, in fact, $t^{-1}A^\star = t^{-1}A \cap t^{-1}\{s \in S : s^{-1}A \in e\} \in e$. (This is [6, Lemma 4.14].)

**Proof.** Let $i \in \{1, 2, \ldots, r\}$ be such that $B_1(\emptyset, i) \in e$. (We use the notation of Lemma 3.1. Since $i$ will be fixed in the rest of the proof, we will suppress it and write $B_r(E)$ instead of $B_r(E, i)$.) We will inductively construct an increasing sequence of trees $\langle T_n\rangle_{n=0}^\infty$, satisfying for each $n \geqslant 0$, $T_n = \{f_{\restriction\{1,2,\ldots,n-1\}} : f \in T_{n+1}\}$ such that, for each $f \in T_n$, the following holds:

(i) if $\operatorname{dom} f \subseteq \{0, 1, \ldots, n-2\}$ then $T_n(f) \in e$,
(ii) if $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathscr{P}_f(\omega), r \in \{1, 2, \ldots, k\}$ satisfy $\alpha_1 < \alpha_2 < \cdots < \alpha_r \subseteq \operatorname{dom} f$ and if $x_i = \prod_{t\in\alpha_i} f(t)$ then $x_r \in B_r(\{x_1, x_2, \ldots, x_{r-1}\})^\star$[2].

Trivially, we may put $T_0 = \{\emptyset\}$. Assume now that $T_0, T_1, \ldots, T_n$ have already been defined. Fix $f \in T_n$ with $\operatorname{dom} f = \{0, 1, \ldots, n-1\}$. For $\alpha_1 < \alpha_2 < \cdots < \alpha_r \subseteq \operatorname{dom} f$ let $x_i = \prod_{t\in\alpha_i} f(t)$. By assumption $x_r \in B_r(\{x_1, x_2, \ldots, x_{r-1}\})$ and thus $B_{r+1}(\{x_1, x_2, \ldots, x_r\}) \in e$ for $r \in \{1, 2, \ldots, k-1\}$. Since $x_r \in B_r(\{x_1, x_2, \ldots, x_{r-1}\})^\star$ we have $x_r^{-1}B_r(\{x_1, x_2, \ldots, x_{r-1}\})^\star \in e$ for $r \in \{0, 1, \ldots, k\}$. Define $T_n(f)$ to be the intersection of all sets $B_{r+1}(\{x_1, x_2, \ldots, x_r\})^\star$, $r \in \{0, 1, \ldots, k-1\}$ and $x_r^{-1}B_r(\{x_1, x_2, \ldots, x_{r-1}\})^\star, r \in \{0, 1, \ldots, k\}$ such that indeed $T_n(f) \in e$. Using this, put

$$T_{n+1} = T_n \cup \{f^\frown t : f \in T_n,\ \operatorname{dom} f = \{0, 1, \ldots, n-1\},\ t \in T_n(f)\}.$$

It is not hard to verify that this implies that the inductive construction can be continued. This is only interesting for $\operatorname{dom} f = \{0, 1, \ldots, n\}$ and $n \in \alpha_r$ (where $r \in \{1, 2, \ldots, k\}$). Fix $f' : \{0, 1, \ldots, n-1\} \to S$ such that $f'^\frown f(n) = f$. If $\alpha_r = \{n\}$, then $x_r = f(n) \in T_n(f') \subseteq B(\{x_1, x_2, \ldots, x_{r-1}\})^\star$, so we are done. If $\alpha_r = \alpha_r' \cup \{n\}$ for some non-empty $\alpha_r' \subseteq \{0, 1, \ldots, n-1\}$, we have

$$f(n) \in T_n(f') \subseteq \left(\prod_{t\in\alpha_r'} f'(t)\right)^{-1} B_r(\{x_1, x_2, \ldots, x_{r-1}\})^\star,$$

[2] For $r = 1$ this is meant to be $B_1(\emptyset)^\star$.

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and this implies

$$x_r = \prod_{t\in\alpha_r} f(t) \in B_r(\{x_1, x_2, \dots, x_{r-1}\})^\star.$$

Finally put $T = \bigcup_{n=0}^{\infty} T_n$. Obviously $T(f) \in e$ for all $f \in T$. Since

$$\prod_{t\in\alpha_k} f(t) \in B_k\left(\left\{\prod_{t\in\alpha_1} f(t), \dots, \prod_{t\in\alpha_{k-1}} f(t)\right\}\right)$$

for all $f \in T$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_k \subseteq \operatorname{dom} f$, we see that (2) holds. □

From this lemma one may directly derive the following strong version of the Milliken–Taylor theorem.

**Corollary 3.3.** *Let $k, r \geqslant 1$, let $(S, \cdot)$ be a semigroup, let $\langle x_n\rangle_{n=0}^{\infty}$ be a sequence in $S$ and let $[S]^k = \bigcup_{i=1}^{r} A_i$. Assume that for every idempotent $s \in S$ there exists some $m \in \mathbb{N}$ such that $s \notin \mathrm{FP}(\langle x_n\rangle_{n=m}^{\infty})$. Then there exist $i \in \{1, 2, \dots, r\}$ and a sequence $\alpha_0 < \alpha_1 < \cdots$ in $\mathscr{P}_f(\omega)$ such that*

$$\left[\mathrm{FP}\left(\left\langle \prod_{t\in\alpha_n} x_t \right\rangle_{n=0}^{\infty}\right)\right]_<^k \subseteq A_i.$$

**Proof.** By [6, Lemma 5.11] there exists an idempotent $e \in \beta S$, such that, for all $m \geqslant 0$, $\mathrm{FP}(\langle x_n\rangle_{n=m}^{\infty}) \in e$, and by our assumption we have $e \in \beta S \setminus S$. Let $i \in \{1, 2, \dots, r\}$ and $T \subseteq S^{<\omega}$ be as provided by Lemma 3.2. We have $T(\emptyset) \cap \mathrm{FP}(\langle x_n\rangle_{n=0}^{\infty}) \in e$. In particular, this set is not empty, so we may choose $\alpha_0 \in \mathscr{P}_f(\omega)$ such that $\prod_{t\in\alpha_0} x_t \in T(\emptyset)$. Let $m_0 := \max \alpha_0$. As before,

$$T\left(\left(\prod_{t\in\alpha_0} x_t\right)\right) \cap \mathrm{FP}\left(\langle x_n\rangle_{n=m_0}^{\infty}\right) \in e,$$

so we find $\alpha_1 > \alpha_0, \alpha_1 \in \mathscr{P}_f(\omega)$ such that

$$\prod_{t\in\alpha_1} x_t \in T\left(\left\langle \prod_{t\in\alpha_0} x_t \right\rangle\right).$$

By continuing in this fashion we achieve a sequence with the required properties. □

We remark that our restriction on the idempotents contained in $\mathrm{FP}(\langle x_n\rangle_{n=0}^{\infty})$ cannot be dropped. Consider for example $(S, \cdot) = (\mathbb{Z}, +)$ and $\langle x_n\rangle_{n=0}^{\infty} = \langle 0, 0, \dots\rangle$. In this case

$$\left[\mathrm{FP}\left(\langle x_n\rangle_{n=0}^{\infty}\right)\right]_<^k = \{\{0\}\}, \quad \text{for any } k \in \mathbb{N}.$$

Another possibility to avoid this difficulty is presented in [6, Corollary 18.9], where, instead of partitions of $[S]^k$, partitions of $\bigcup_{i=1}^{k} [S]^i$ are considered.

In the proof of Theorem 1.1 we will require the following.

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**Theorem 3.4.** *Let $(S,\cdot)$ be a commutative semigroup, let $A \in e \in K(\beta S)$, let $l \in \mathbb{N}$ and for each $j \in \{0,1,\dots,l-1\}$ let $\langle y_{j,n}\rangle_{n=0}^{\infty}$ be a sequence in $S$. Then there exist $a \in S$ and $\alpha \in \mathscr{P}_f(\omega)$ such that $a\prod_{t\in\alpha} y_{j,t} \in A$ for each $j \in \{0,1,\dots,l-1\}$.*

Theorem 3.4 is a special case of the Central Sets Theorem and may easily be derived from the Hales–Jewett theorem [4].

We are now able to prove our main theorem.

**Proof of Theorem 1.1.** Fix a minimal idempotent $e \in \beta S \setminus S$. Let $i \in \{1,2,\dots,r\}$ and $T \subseteq S^{<\omega}$ be as provided by Lemma 3.2. We will inductively construct sequences $\langle a_n\rangle_{n=0}^{\infty}$ in $S$ and $\alpha_0 < \alpha_1 < \cdots$ in $\mathscr{P}_f(\omega)$ such that, for all $n \in \mathbb{N}$ and all $g \in \Phi$,

$$\left\langle a_0 \prod_{t\in\alpha_0} y_{g(0),t}, a_1 \prod_{t\in\alpha_1} y_{g(1),t}, \dots, a_{n-1} \prod_{t\in\alpha_{n-1}} y_{g(n-1),t} \right\rangle \in T. \tag{3.1}$$

By the properties of $T$ this is sufficient to prove the theorem.

Assume that $a_0, a_1, \dots, a_{n-1} \in S$ and $\alpha_0 < \cdots < \alpha_{n-1} \in \mathscr{P}_f(\omega)$ have already been constructed such that (3.1) is true for all $g \in \Phi$. We have

$$G_n := \bigcap_{g\in\Phi} T\left(\left\langle a_0 \prod_{t\in\alpha_0} y_{g(0),t},\ a_1 \prod_{t\in\alpha_1} y_{g(1),t}, \dots, a_{n-1} \prod_{t\in\alpha_{n-1}} y_{g(n-1),t} \right\rangle\right) \in e.$$

Let $m := \max \alpha_{n-1}$. By applying Theorem 3.4 to the set $G_n$ and the sequences

$$\langle y_{0,k}\rangle_{k=m}^{\infty}, \langle y_{1,k}\rangle_{k=m}^{\infty}, \dots, \langle y_{n,k}\rangle_{k=m}^{\infty},$$

we find $a_n \in S$ and $\alpha_n \in \mathscr{P}_f(\mathbb{N})$, $\alpha_n > \alpha_{n-1}$ such that

$$a_n \prod_{t\in\alpha_n} y_{0,t}, a_n \prod_{t\in\alpha_n} y_{1,t}, \dots, a_n \prod_{t\in\alpha_n} y_{n,t} \in G_n.$$

Thus, for all $g \in \Phi$,

$$\left\langle a_0 \prod_{t\in\alpha_0} y_{g(0),k}, a_1 \prod_{t\in\alpha_1} y_{g(1),k}, \dots, a_n \prod_{t\in\alpha_n} y_{g(n),k} \right\rangle \in T,$$

as we wanted to show. □

We conclude this section by giving a strengthening of Theorem 1.1 that applies to partitions of the spaces $[S]^1, [S]^2, \dots, [S]^k$ simultaneously. It is not hard to verify that a similar extension of Corollary 3.3 is also valid. To avoid confusion about indices we use colourings instead of partitions.

**Corollary 3.5.** *Let $(S,\cdot)$ be a commutative semigroup and assume that there exists a non-principal minimal idempotent in $\beta S$. For each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=0}^{\infty}$ be a sequence in $S$. Let $k \geqslant 1$ and assume that for each $m \in \{1,2,\dots,k\}$, $[S]^m$ is finitely coloured. There exist a sequence $\langle a_n\rangle_{n=0}^{\infty}$ in $S$, a sequence $\alpha_0 < \alpha_1 < \cdots$ in $\mathscr{P}_f(\omega)$ and for each $m \in \{1,2,\dots,k\}$ a*

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*monochrome set* $A^{(m)}$ *such that for each* $g \in \Phi$ *and each* $m \in \{1, 2, \ldots, k\}$,

$$\left[ \mathrm{FP}\left( \left\langle a_n \prod_{t \in \alpha_n} y_{g(n),t} \right\rangle_{n=0}^{\infty} \right) \right]_{<}^{m} \subseteq A^{(m)}.$$

**Proof.** We describe two ways to prove Corollary 3.5. Firstly, fix a linear ordering $\prec$ on $S$. For $m \in \{1, 2, \ldots, k\}$ let $f^{(m)} : [S]^m \to \{1, 2, \ldots, r_m\}$ be the colouring at hand. Define a colouring $g^{(m)} : [S]^k \to \{1, 2, \ldots, r_m\}$ by letting

$$g^{(m)}(\{x_1, x_2, \ldots, x_k\}) = f(m)(\{x_1, x_2, \ldots, x_m\}),$$

where $\{x_1, x_2, \ldots, x_m\}$ are the $m$ smallest elements of $\{x_1, x_2, \ldots, x_k\}$ with respect to $\prec$. Then apply Theorem 1.1 to the colouring

$$\begin{aligned} f : [S]^k &\to \{1, 2, \ldots, r_1\} \times \{1, 2, \ldots, r_2\} \times \cdots \times \{1, 2, \ldots, r\}_k \\ E &\mapsto (g^{(1)}(E), g^{(2)}(E), \ldots, g^{(k)}(E)). \end{aligned}$$

It is clear that the resulting sequences $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle \alpha_n \rangle_{n=0}^{\infty}$ satisfy the conclusion of Corollary 3.5.

The more complicated way to prove Corollary 3.5 is to start by extending Lemma 3.2. Pick a minimal idempotent $e \in \beta S \backslash S$. Choose by Lemma 3.2 for each $m \in \{1, 2, \ldots, k\}$ a monochrome set $A^{(m)} \subseteq [S]^m$ and a tree $T^{(m)} \subseteq S^{<\omega}$ such that, for all $f \in T^{(m)}$ and all $\alpha_1 < \alpha_2 < \cdots < \alpha_m \subseteq \operatorname{dom} f, \alpha_i \in \mathscr{P}_f(\omega)$, we have $T^{(m)}(f) \in e$ and

$$\left\{ \prod_{t \in \alpha_1} f(t), \prod_{t \in \alpha_2} f(t), \ldots, \prod_{t \in \alpha_m} f(t) \right\} \in A^{(m)}.$$

But then $T := \bigcap_{m=1}^{k} T^{(m)}$ is a tree such that, for all $f \in T$, $T(f) \in e$, and for all $m \in \{1, 2, \ldots, k\}$ and all $\alpha_1 < \alpha_2 < \cdots < \alpha_m \subseteq \operatorname{dom} f, \alpha_i \in \mathscr{P}_f(\omega)$,

$$\left\{ \prod_{t \in \alpha_1} f(t), \prod_{t \in \alpha_2} f(t), \ldots, \prod_{t \in \alpha_m} f(t) \right\} \in A^{(m)}.$$

By performing the proof of Theorem 1.1 with this tree $T$ we again see that Corollary 3.5 is valid. □

## 4. Conclusion

When applying ultrafilters to Ramsey theory one typically establishes that a set is non-empty by showing that it is actually large, *i.e.*, contained in a certain ultrafilter $e$. The Milliken–Taylor theorem mentioned in the Introduction states that, for any partition $A_1, A_2, \ldots, A_r$ of $\mathbb{N}$, there exist $i$ and a sequence $\langle x_n \rangle_{n=0}^{\infty}$ such that $[\mathrm{FS}(\langle x_n \rangle_{n=0}^{\infty})]_{<}^{k} \subseteq A_i$. In the spirit of the principle stated above, one could expect that after constructing the first $n$ elements $x_0, x_1, \ldots, x_{n-1}$ of the sequence, the set of possible choices of the element $x_n$ is contained in an ultrafilter $e$. Lemma 3.2 gives this idea a rigorous meaning. The combinatorial gain is that the sequence $\langle x_n \rangle_{n=0}^{\infty}$ can be forced to satisfy additional properties. In our generalizations (Corollary 3.3) of the Milliken–Taylor theorem the

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sequence may be chosen from a predefined IP-set in a quite general semigroup. In appropriate commutative semigroups the variety of possible sequences is large enough to achieve the multidimensional extension (Theorem 1.1) of the Central Sets Theorem.

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## References

[1] Bergelson, V. and Hindman, N. (1989) Ultrafilters and multidimensional Ramsey theorems. *Combinatorica* **9** 1–7.
[2] Comfort, W. (1977) Some recent applications of ultrafilters to topology. In *Proc. Fourth Prague Topological Symposium, 1976* (J. Novak, ed.), Vol. 609 of *Lecture Notes in Mathematics*, pp. 34–42.
[3] Furstenberg, H. (1981) *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton.
[4] Hales, A. and Jewett, R. (1963) Regularity and positional games. *Trans. Amer. Math. Soc.* **106** 222–229.
[5] Hindman, N. (1974) Finite sums from a sequences within cells of a partition of $\mathbb{N}$. *J. Combin. Theory Ser. A* **17** 1–11.
[6] Hindman, N. and Strauss, D. (1998) *Algebra in the Stone–Čech Compactification: Theory and Applications*, de Gruyter, Berlin.
[7] Milliken, K. (1975) Ramsey's theorem with sums or unions. *J. Combin. Theory Ser. A* **18** 276–290.
[8] Taylor, A. (1976) A canonical partition relation for finite subsets of $\omega$. *J. Combin. Theory Ser. A* **21** 137–146.
[9] van der Waerden, B. (1927) Beweis einer Bedautschen Vermutung. *Nieuw Arch. Wiskunde* **19** 212–216.

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