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LONG-TIME BEHAVIOR OF LÉVY-DRIVEN ORNSTEIN–UHLENBECK PROCESSES WITH REGIME SWITCHING

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Abstract

We investigate the long-time behavior of the Ornstein–Uhlenbeck process driven by Lévy noise with regime switching. We provide explicit criteria on the transience and recurrence of this process. Contrasted with the Ornstein–Uhlenbeck process driven simply by Brownian motion, whose stationary distribution must be light-tailed, both the jumps caused by the Lévy noise and the regime switching described by a Markov chain can derive the heavy-tailed property of the stationary distribution. The different role played by the Lévy measure and the regime-switching process is clearly characterized.

Keywords: Ornstein–Uhlenbeck process; Lévy process; regime switching; recurrence

2010 Mathematics Subject Classification: Primary 60J75; Secondary 60K37; 60J60

1. Introduction

In this work we are concerned with the existence of stationary distributions of regime-switching processes driven by Lévy noise, and further analyze the tail behavior of these distributions. We are motivated by the works of de Saporta and Yao [6] and Bardet et al. [1]. It has been shown in different ways that in contrast with the classical Ornstein–Uhlenbeck process whose stationary distribution must be light-tailed, the stationary distribution of the Ornstein–Uhlenbeck process with regime switching may have a heavy-tailed stationary distribution. Moreover, in a recent work, Hou and Shao [7] showed that the Cox–Ingersoll–Ross (CIR) process with regime switching may have a heavy-tailed stationary distribution, which provides a suitable modification of the classical CIR model due to its poor empirical performance as shown by Brown and Dybvig [4]. Hence, from the application viewpoint, it is important to study stochastic processes with regime switching.

The Lévy-driven Ornstein–Uhlenbeck process has a lot of applications in the study of mathematical finance (see, e.g., [2, 3, 5, 8, 9], amongst others). In this work we shall extend the study of [6] and [1] to Lévy-driven Ornstein–Uhlenbeck processes with regime switching. We shall provide explicit conditions to justify the recurrence and transience of such a process, and determine the role played by the random change of the environment. Furthermore, given the existence of a stationary distribution, we provide explicit conditions to justify whether it is light- or heavy-tailed. In this part we are mainly interested in determining the different role

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played by Lévy jumps and regime switching in determining the heavy-tailed property of the stationary distribution.

Precisely, we consider the following stochastic processes (X_t, Λ_t):

$$\mathrm{d}X_t = \alpha_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t} \mathrm{d}Z_t, \qquad X_0 = x_0 \in \mathbb{R}, \tag{1.1}$$

where (Λ_t) is a continuous-time Markov chain on a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$, $2 \leq N < \infty$, with the transition rate matrix $Q = (q_{ij})_{i,j\in\mathcal{S}}$, which is assumed to be conservative and irreducible. Here, $\alpha : \mathcal{S} \to \mathbb{R}$ and $\sigma : \mathcal{S} \to \mathbb{R}$ are measurable functions, and $Z = (Z_t)_{t\geq 0}$ is a Lévy process on $\mathbb{R}$ given by

$$Z_t = bt + \sqrt{a}B_t + \int_0^t \int_{0<|z|<1} z\,\widetilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_0^t \int_{|z|\geq 1} zN(\mathrm{d}t, \mathrm{d}z), \tag{1.2}$$

where $b \in \mathbb{R}$, $a > 0$, $B = (B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion, $N(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure on $[0, \infty) \times \mathbb{R}\backslash\{0\}$, and $\widetilde{N}(\mathrm{d}t, \mathrm{d}z)$ is its corresponding compensated Poisson random measure relative to the Lévy measure ν, i.e. $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$. Throughout, it is assumed that (Λ_t) is independent of the Lévy process (Z_t). As a Lévy measure, ν is a σ-finite measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and

$$\int_{z\neq 0} (1 \wedge |z|^2)\, \nu(\mathrm{d}z) < \infty. \tag{1.3}$$

Since the Markov chain (Λ_t) is in a finite state space with a conservative and irreducible Q-matrix, there exists a unique stationary distribution, denoted by $\mu = (\mu_i)_{i\in\mathcal{S}}$. Our first result is on the recurrent property of (X_t, Λ_t).

Theorem 1.1.

(i) *Assume that*

$$\int_{|z|\neq 0} \log(1 + |z|)\nu(\mathrm{d}z) < \infty; \tag{1.4}$$

then (X_t, Λ_t) is positive recurrent if $\sum_{i\in\mathcal{S}} \mu_i\alpha_i < 0$.

(ii) *Assume that*

$$\int_{|z|\neq 0} |z| \vee |z|^2 \nu(\mathrm{d}z) < \infty; \tag{1.5}$$

then (X_t, Λ_t) is transient if $\sum_{i\in\mathcal{S}} \mu_i\alpha_i > 0$.

Restricted to the set $\{0 < |z| < 1\}$, each of (1.4) and (1.5) can yield $\int_{0<|z|<1} |z|\nu(\mathrm{d}z) < \infty$, which is stronger than the necessary condition (1.3) and seems not to be a necessary one. Nevertheless, restricted to the set $\{|z| \geq 1\}$, our condition (1.4) is just condition (1.3) in [10]. In [10], Sato and Yamazato showed that this condition is a sharp condition on the integrability of the Lévy measure ν so that the associated Ornstein–Uhlenbeck process without regime switching is positive recurrent. Later, Shiga [14] provided a recurrence criterion for this kind of process in one dimension and discussed several symmetric multidimensional cases. References [11] and [12] also investigated the recurrence criterion in multidimensional cases, and finally Watanabe [15] used the Fourier analytic method to provide a general recurrence criterion for Lévy-driven Ornstein–Uhlenbeck processes which solved Sato's conjecture in the affirmative.

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Next, when (X_t, Λ_t) is positive recurrent, it owns a unique stationary distribution π. Concerning the tail behavior of π, we establish the following result:

Theorem 1.2. *Assume that* (1.4) *and* $\sum_{i\in\mathcal{S}} \mu_i\alpha_i < 0$ *hold, and denote by* π *the stationary distribution of* (X_t, Λ_t).

(i) *If there exists some* $i_0 \in \mathcal{S}$ *such that, for any* $\lambda > 0$,

$$\int_{|z|\geq 1} (e^{\lambda\sigma_{i_0}z} - 1)\nu(dz) = \infty, \tag{1.6}$$

then π *is heavy-tailed, i.e.*

$$\sum_{j\in\mathcal{S}} \int_{\mathbb{R}} e^{\lambda|x|}\pi(dx, j) = \infty, \quad \text{for all } \lambda > 0.$$

(ii) *Suppose there exists a* $\lambda_0 > 0$ *such that, for each* $i \in \mathcal{S}$,

$$\int_{|z|\geq 1} (e^{\lambda_0|\sigma_i z|} - 1)\nu(dz) < \infty. \tag{1.7}$$

If $\max_{i\in\mathcal{S}} \alpha_i < 0$, *then* π *is light-tailed. If* $\max_{i\in\mathcal{S}} \alpha_i > 0$ *then, for any* $p > \kappa \vee 2$,

$$\sum_{j\in\mathcal{S}} \int_{\mathbb{R}} |x|^p\pi(dx, j) = \infty,$$

which means that π *is heavy-tailed, where* $\kappa > 0$ *is given in* (1.9) *below.*

For any $p > 0$, define $Q_p = Q + p\,\mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, where $Q = (q_{ij})$ is the transition rate matrix of (Λ_t), and $\mathrm{diag}(\alpha_1, \ldots, \alpha_N)$ is the diagonal matrix generated by the vector $(\alpha_1, \ldots, \alpha_N)$. Let

$$\eta_p = -\max_{\gamma\in\mathrm{Spec}(Q_p)} \mathrm{Re}\,\gamma, \tag{1.8}$$

where $\mathrm{Spec}(Q_p)$ stands for the spectrum of Q_p. Define

$$\kappa = \sup\{p > 0; \eta_p > 0\}. \tag{1.9}$$

According to [1, Propositions 4.1, 4.2], $\kappa = \infty$ if $\max_{i\in\mathcal{S}} \alpha_i \leq 0$; otherwise, $\kappa \in (0, \min\{\frac{q_i}{\alpha_i}; \alpha_i > 0\})$. Moreover, for $p > \kappa$, $\eta_p < 0$, and for $p \in (0, \kappa)$, $\eta_p > 0$.

Remark 1.1. According to the characterization of the tail behavior for the Lévy-driven Ornstein–Uhlenbeck process in Lemma 2.1 below, assertion (i) of Theorem 1.2 tells us that if in some fixed environment the Lévy measure ν makes the stationary distribution for the Ornstein–Uhlenbeck process without switching be heavy-tailed, then the stationary distribution of the Ornstein–Uhlenbeck process with switching must be heavy-tailed regardless of the random switching of the environment.

Assertion (ii) of Theorem 1.2 tells us that under a slightly stronger condition on ν which ensures that the Ornstein–Uhlenbeck process in every fixed environment must own a light-tailed stationary distribution, the random switching process (Λ_t) can change the tail behavior of the stationary distribution of (X_t, Λ_t) according to the signal of $\max_{i\in\mathcal{S}} \alpha_i$.

The proofs of Theorems 1.1 and 1.2 are presented in the next section.

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2. Proofs of the main results

In this section we shall first present the argument of Theorem 1.1 on the recurrent property of the process (X_t, Λ_t). To this end, we need to extend the Lyapunov-type criterion established by Shao in [13] for stochastic processes driven by Brownian motion to the current situation.

Let us first introduce some notation. For each $i \in \mathcal{S}$, define

$$\begin{aligned}
\mathcal{D}^{(i)}f(\cdot, i)(x) &= (\alpha_i x + b\sigma_i)\frac{\partial f(x, i)}{\partial x} + \frac{a\sigma_i^2}{2}\frac{\partial^2 f(x, i)}{\partial x^2},\\
\mathcal{J}^{(i)}f(\cdot, i)(x) &= \int_{z\neq 0}\left[f(x+\sigma_i z, i) - f(x, i) - \sigma_i z\frac{\partial f(x, i)}{\partial x}\mathbf{1}_{0<|z|<1}\right]\nu(\mathrm{d}z),\\
L^{(i)}f(\cdot, i)(x) &= \mathcal{D}^{(i)}f(\cdot, i)(x) + \mathcal{J}^{(i)}f(\cdot, i)(x)
\end{aligned}$$

for $f(\cdot, i) \in C^2(\mathbb{R})$. Then the infinitesimal generator of the process (X_t, Λ_t) is given by

$$\begin{aligned}
\mathcal{A}f(x, i) &= L^{(i)}f(\cdot, i)(x) + Qf(x, \cdot)(i)\\
&= L^{(i)}f(\cdot, i)(x) + \sum_{j\neq i} q_{ij}(f(x, j) - f(x, i)), \qquad f(\cdot, i) \in C^2(\mathbb{R}).
\end{aligned}$$

Following the same approach as [13, Theorems 2.1, 3.1], we can establish the following criterion to justify the recurrence of the Lévy-driven process (X_t, Λ_t). The details are omitted.

Proposition 2.1. *Let μ be the stationary distribution of (Λ_t).*

(i) *Suppose that there exist constants $r_0 > 0$, $\beta_i \in \mathbb{R}$, $i \in \mathcal{S}$, and a positive function $V(x)$ such that*

$$L^{(i)}V(x) \le \beta_i V(x), \qquad |x| \ge r_0, \quad i \in \mathcal{S},$$

satisfying $\sum_{i\in\mathcal{S}} \mu_i\beta_i < 0$. Then (X_t, Λ_t) is exponentially ergodic if $\lim_{|x|\to\infty} V(x) = \infty$, and is transient if $\lim_{|x|\to\infty} V(x) = 0$.

(ii) *Suppose that there exist constants $r_0 > 0$, $\beta_i \in \mathbb{R}$, $i \in \mathcal{S}$, and two positive functions h, $g \in C^2(\mathbb{R})$ such that, for each $i \in \mathcal{S}$,*

$$L^{(i)}h(x) \le \beta_i g(x), \qquad |x| > r_0, \tag{2.1}$$

and

$$\lim_{|x|\to\infty}\frac{g(x)}{h(x)} = 0, \qquad \lim_{|x|\to\infty}\frac{L^{(i)}g(x)}{g(x)} = 0.$$

Assume that

$$\sum_{i\in\mathcal{S}} \mu_i\beta_i < 0.$$

Then (X_t, Λ_t) given by (1.1) and (1.2) is positive recurrent if $\lim_{|x|\to\infty} h(x) = \infty$, and is transient if $\lim_{|x|\to\infty} h(x) = 0$.

Remark 2.1. In Proposition 2.1 we state that (X_t, Λ_t) is positive recurrent when $\sum_{i\in\mathcal{S}} \mu_i\beta_i < 0$ and $\lim_{|x|\to\infty} h(x) = \infty$, which is a little stronger than the result presented in [13, Theorem 3.1]. However, the argument in [13] is sufficient for this statement according to the Foster–Lyapunov criterion.

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Proof of Theorem 1.1. First, let us prove the positive recurrence of (X_t, Λ_t) in the situation $\sum_{i\in\mathcal{S}} \mu_i\alpha_i < 0$ by part (ii) of Proposition 2.1. Consider the auxiliary functions

$$h(x) = \log(1+x^2), \qquad g(x) = \frac{2x^2}{1+x^2}.$$

It is easy to check that $\lim_{|x|\to\infty} h(x) = \infty$ and $\lim_{|x|\to\infty} {}^{g(x)}/_{h(x)} = 0$.

Now we verify that $\lim_{|x|\to\infty} \frac{L^{(i)}g(x)}{g(x)} = 0$. Note that

$$\frac{\mathcal{D}^{(i)}g(x)}{g(x)} = \frac{2\alpha_i}{1+x^2} + \frac{2b\sigma_i}{x(1+x^2)} + \frac{a\sigma_i^2(1-3x^2)}{x^2(1+x^2)^2} \longrightarrow 0 \qquad \text{as } |x|\to\infty. \tag{2.2}$$

For the jumping component, we have

$$\begin{aligned}\frac{\mathcal{J}^{(i)}g(x)}{g(x)} &= \frac{1}{g(x)}\int_{z\neq 0}(g(x+\sigma_i z) - g(x) - \sigma_i z g'(x)\mathbf{1}_{\{0<|z|<1\}}(z))\nu(\mathrm{d}z)\\ &= \left(1+\frac{1}{x^2}\right)\int_{z\neq 0}\frac{\sigma_i^2z^2+2\sigma_i xz}{(1+x^2)(1+(x+\sigma_i z)^2)}\nu(\mathrm{d}z)\\ &\quad - \frac{2\sigma_i}{x(1+x^2)}\int_{z\neq 0} z\mathbf{1}_{|z|<1}\nu(\mathrm{d}z).\end{aligned}$$

Due to condition (1.4), it is obvious that

$$\lim_{|x|\to\infty}\frac{2\sigma_i}{x(1+x^2)}\int_{z\neq 0}(1\wedge|z|)\nu(\mathrm{d}z) = 0.$$

Because

$$\left|\frac{\sigma_i^2z^2+2\sigma_i xz}{(1+x^2)(1+(x+\sigma_i z)^2)}\right| = \left|\frac{(\sigma_i z+x)^2-x^2}{(1+x^2)(1+(x+\sigma_i z)^2)}\right| \le 1$$

and

$$\left|\frac{\sigma_i^2z^2+2\sigma_i xz}{(1+x^2)(1+(x+\sigma_i z)^2)}\right| \le \left|\frac{\sigma_i^2z^2+2\sigma_i xz}{(1+x^2)}\right| \le \sigma_i^2z^2+|\sigma_i z|,$$

we obtain that

$$\left|\frac{\sigma_i^2z^2+2\sigma_i xz}{(1+x^2)(1+(x+\sigma_i z)^2)}\right| \le \min\{1, \sigma_i^2z^2+|\sigma_i z|\}.$$

Invoking condition (1.4), the dominated convergence theorem yields that

$$\lim_{|x|\to\infty}\left(1+\frac{1}{x^2}\right)\int_{z\neq 0}\frac{\sigma_i^2+2\sigma_i xz}{(1+x^2)(1+(x+\sigma_i z)^2)}\nu(\mathrm{d}z) = 0.$$

Hence,

$$\lim_{|x|\to\infty}\frac{\mathcal{J}^{(i)}g(x)}{g(x)} = 0.$$

Combining with (2.2), this further implies that

$$\lim_{|x|\to\infty}\frac{L^{(i)}g(x)}{g(x)} = 0.$$

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Next, we shall show that for any ε satisfying $-\sum_{i\in\mathcal{S}}\mu_i\alpha_i > \varepsilon > 0$, there exists a constant $r_0 > 0$ such that, for every $i \in \mathcal{S}$,

$$L^{(i)}h(x) \le (\alpha_i + \varepsilon)g(x) \qquad \text{as } |x| \ge r_0. \tag{2.3}$$

So, by taking $\beta_i = \alpha_i + \varepsilon$ in (2.1), Proposition 2.1 yields that (X_t, Λ_t) is positive recurrent.

Indeed, for the diffusion part of $L^{(i)}h$ we have

$$\begin{aligned}\mathcal{D}^{(i)}h(x) &= (\alpha_i x + b\sigma_i)\frac{2x}{1+x^2} + a\sigma_i^2\frac{1-x^2}{(1+x^2)^2}\\ &\le \left(\alpha_i + \frac{b\sigma_i}{x} + \frac{a\sigma_i^2|1-x^2|}{2x^2(1+x^2)}\right)g(x).\end{aligned} \tag{2.4}$$

Due to the finiteness of the number of states in $\mathcal{S}$, it is clear that there exists $r_1 > 0$ such that

$$\frac{b\sigma_i}{x} + \frac{a\sigma_i^2|1-x^2|}{2x^2(1+x^2)} \le \frac{\varepsilon}{4} \qquad \text{for all } |x| \ge r_1,\ i \in \mathcal{S}. \tag{2.5}$$

For the jump part of $L^{(i)}h$, direct calculation leads to

$$\mathcal{J}^{(i)}h(x) = \int_{z\neq 0}[\log(1+(x+\sigma_i z)^2) - \log(1+x^2)]\nu(\mathrm{d}z) - \frac{2\sigma_i x}{1+x^2}\int_{0<|z|<1} z\,\nu(\mathrm{d}z).$$

Since

$$\begin{aligned}\log(1+(x+\sigma_i z)^2) - \log(1+x^2) &\le \log\left(1 + \frac{\sigma_i^2 z^2}{1+x^2} + \frac{2x}{1+x^2}|\sigma_i z|\right)\\ &\le \log(1 + |\sigma_i z| + \sigma_i^2 z^2) \le 2\log(1+|\sigma_i z|),\end{aligned}$$

we obtain from condition (1.4) and the dominated convergence theorem that

$$\lim_{|x|\to\infty}\mathcal{J}^{(i)}h(x) = 0.$$

Noting that $g(x) \ge 1$ for $|x| \ge 1$, there exists $r_2 > 1$ such that, for any $|x| \ge r_2$,

$$\mathcal{J}^{(i)}h(x) \le \frac{\varepsilon}{4}g(x). \tag{2.6}$$

Consequently, (2.3) follows from (2.4), (2.5), and (2.6) by taking $r_0 = \max\{r_1, r_2\}$.

Second, we go on to prove the transience of (X_t, Λ_t) in the case $\sum_{i\in\mathcal{S}}\mu_i\alpha_i > 0$. To this end, we consider the function

$$V(x) = \frac{1}{\delta + x^2}$$

for some fixed $0 < \delta < 1$, and prove the desired result according to part (i) of Proposition 2.1. We have,

$$\mathcal{D}^{(i)}V(x) = \left(-2\alpha_i\frac{x^2}{\delta+x^2} - 2b\sigma_i\frac{x}{\delta+x^2} + a\sigma_i^2\frac{3x^2-\delta}{(\delta+x^2)^2}\right)V(x), \tag{2.7}$$

and clearly

$$\lim_{|x|\to\infty} -2\alpha_i\frac{x^2}{\delta+x^2} - 2b\sigma_i\frac{x}{\delta+x^2} + a\sigma_i^2\frac{3x^2-\delta}{(\delta+x^2)^2} = -2\alpha_i.$$

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For the jump part of $L^{(i)}V$ we have

$$\mathcal{J}^{(i)}V(x) = \left(\int_{z\neq 0} \frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\nu(\mathrm{d}z) + \frac{2\sigma_i x}{\delta+x^2}\int_{0<|z|<1} z\nu(\mathrm{d}z)\right)V(x). \tag{2.8}$$

Moreover,

$$\begin{aligned}\int_{z\neq 0} \frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\nu(\mathrm{d}z) &= \int_{|z|\geq 1} \frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\nu(\mathrm{d}z) \\ &\quad + \int_{0<|z|<1} \frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\nu(\mathrm{d}z) \\ &=: I_1 + I_2.\end{aligned}$$

To estimate the term I_1, since $0<\delta<1$ we have

$$\begin{aligned}\left|\frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\right| &\leq \frac{2|x\sigma_i z|}{\delta+(x+\sigma_i z)^2} + \frac{\sigma_i^2 z^2}{\delta} \\ &\leq \frac{(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2} + \frac{\sigma_i^2 z^2}{\delta} \\ &\leq \frac{1}{\delta} + \frac{\sigma_i^2}{\delta} z^2,\end{aligned}$$

and $\frac{1}{\delta}+\frac{\sigma_i^2 z^2}{\delta}$ is an integrable function with respect to $\mathbf{1}_{|z|\geq 1}\nu(\mathrm{d}z)$ by condition (1.5). Then, the dominated convergence theorem leads to

$$\lim_{|x|\to\infty} \int_{|z|\geq 1} \frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\nu(\mathrm{d}z) = 0. \tag{2.9}$$

To estimate the term I_2, let us consider

$$u(x) := \frac{2x\sigma_i z}{\delta+(x+\sigma_i z)^2} \qquad \text{for } 0<|z|<1,$$

and direct calculation leads to

$$u'(x) = \frac{2\sigma_i z\delta - 2\sigma_i z x^2 + 2(\sigma_i z)^3}{(\delta+(x+\sigma_i z)^2)^2}.$$

Hence, there exists $r_1>0$ such that, for $|x|\geq r_1$, $u'(x)>0$ if $\sigma_i z<0$ and $u'(x)<0$ if $\sigma_i z>0$, which means that $|u(x)|$ is monotone when $|x|\geq r_1$. Noting that $\lim_{|x|\to\infty}|u(x)|=0$, we have

$$\sup_{|x|\geq r_1}|u(x)| = \sup_{|x|=r_1}|u(x)| = \max\left\{\frac{2r_1\sigma_i z}{\delta+(r_1+\sigma_i z)^2}, \frac{-2r_1\sigma_i z}{\delta+(\sigma_i z - r_1)^2}\right\}. \tag{2.10}$$

Since

$$\left|\frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\right| \leq |u(x)| + \frac{\sigma_i^2 z^2}{\delta},$$

it follows from (2.10), condition (1.5), and the dominated convergence theorem that

$$\lim_{|x|\to\infty} \int_{0<|z|<1} \frac{x^2-(x+\sigma_i z)^2}{\delta+(x+\sigma_i z)^2}\nu(\mathrm{d}z) = 0. \tag{2.11}$$

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Invoking (2.7), (2.9), (2.11), and condition (1.4), by (2.8), for any ε in $(0, \sum_{i\in\mathcal{S}} \mu_i\alpha_i)$ there exists $r_0 > r_1 > 0$ such that

$$L^{(i)}V(x) = \mathcal{D}^{(i)}V(x) + \mathcal{J}^{(i)}V(x) \le (-2\alpha_i + \varepsilon)V(x), \qquad |x| \ge r_0.$$

According to Proposition 2.1 (X_t, Λ_t) is transient, which completes the proof. □

Next, we proceed to investigate the tail property of the stationary distribution $\pi(\mathrm{d}x, i)$ of (X_t, Λ_t). Before this, we present a lemma on the tail property of the stationary distribution of a Lévy-driven Ornstein–Uhlenbeck process without regime switching, which tells us how the jumping component impacts the tail behavior of its stationary distribution.

Consider the process

$$\mathrm{d}Y_t = \alpha Y_t \mathrm{d}t + \sigma \mathrm{d}Z_t, \qquad Y_0 = y_0 \in \mathbb{R}, \tag{2.12}$$

where $\alpha < 0$, $\sigma \in \mathbb{R}$, and $(Z_t)_{t\ge 0}$ is still given by (1.2) satisfying condition (1.4). Then the process (Y_t) is positive recurrent (cf. Theorem 1.1 or [10, Theorem 4.1]), and its stationary distribution is denoted by $\tilde{\pi}$.

Lemma 2.1. *Assume condition (1.4) holds. Consider the process (Y_t) given by (2.12) and its stationary distribution $\tilde{\pi}$.*

(i) *If, for some $\lambda_0 > 0$,*

$$\int_{|z|\ge 1} (\mathrm{e}^{\lambda_0\sigma z} - 1)\nu(\mathrm{d}z) < \infty, \tag{2.13}$$

then $\int_{\mathbb{R}} \mathrm{e}^{\lambda_0|x|}\tilde{\pi}(\mathrm{d}x) < \infty$, which means that $\tilde{\pi}$ is light-tailed.

(ii) *If, for any $\lambda > 0$,*

$$\int_{|z|\ge 1} (\mathrm{e}^{\lambda\sigma z} - 1)\nu(\mathrm{d}z) = \infty, \tag{2.14}$$

then, for any $\lambda > 0$, $\int_{\mathbb{R}} \mathrm{e}^{\lambda x}\tilde{\pi}(\mathrm{d}x) = \infty$, which means that $\tilde{\pi}$ is heavy-tailed.

Proof. (i) The characteristic function of the Lévy process (Z_t) given by (1.2) is

$$\mathrm{E}(\mathrm{e}^{\hat{\imath}uZ_t}) = \mathrm{e}^{-t\Phi(u)}, \qquad u \in \mathbb{R},\ t \ge 0,$$

where $\hat{\imath}$ is the imaginary unit, and Φ is given by the Lévy–Khintchine representation

$$\Phi(u) = \frac{1}{2}au^2 - \hat{\imath}bu + \int_{z\ne 0} (1 - \mathrm{e}^{\hat{\imath}uz} + \hat{\imath}uz\mathbf{1}_{0<|z|<1})\nu(\mathrm{d}z). \tag{2.15}$$

The stationary distribution $\tilde{\pi}$ of (Y_t) has the characteristic function

$$\widehat{\tilde{\pi}}(z) = \exp\left(\int_0^\infty -\Phi(\mathrm{e}^{\alpha t}\sigma z)\mathrm{d}t\right). \tag{2.16}$$

This yields

$$\int_{\mathbb{R}} \mathrm{e}^{\lambda_0 x}\tilde{\pi}(\mathrm{d}x) = \widehat{\tilde{\pi}}(-\hat{\imath}\lambda_0) = \exp\left(\int_0^\infty -\Phi(-\hat{\imath}\lambda_0\sigma\mathrm{e}^{\alpha t})\mathrm{d}t\right). \tag{2.17}$$

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Due to (2.15),

$$\begin{aligned}\int_0^\infty -\Phi(-\hat{\imath}\lambda_0\sigma e^{\alpha t})\mathrm{d}t &= \int_0^\infty \left(\frac{1}{2}a\lambda_0^2\sigma^2 e^{2\alpha t} + b\lambda_0\sigma e^{\alpha t}\right)\mathrm{d}t \\ &\quad + \int_0^\infty \int_{z\neq 0} (\exp(z\lambda_0\sigma e^{\alpha t}) - 1 - z\lambda_0\sigma e^{\alpha t}\mathbf{1}_{0<|z|<1})\nu(\mathrm{d}z)\mathrm{d}t \\ &=: I_1 + I_2. \end{aligned} \tag{2.18}$$

As $\alpha < 0$, it is clear that

$$I_1 = -\frac{a\lambda_0^2\sigma^2}{4\alpha} - \frac{b\lambda_0\sigma}{\alpha} < \infty.$$

Performing the variable substitution $u = e^{\alpha t}$, we get

$$\begin{aligned} I_2 &= -\frac{1}{\alpha}\int_0^1 \int_{z\neq 0} \frac{1}{u}(e^{z\lambda_0\sigma u} - 1)\nu(\mathrm{d}z)\mathrm{d}u + \frac{\lambda_0\sigma}{\alpha}\left(\int_{z\neq 0} z\mathbf{1}_{0<|z|<1}\nu(\mathrm{d}z)\right) \\ &\le -\frac{1}{\alpha}\int_{z\neq 0}(e^{\lambda_0\sigma z} - 1)\nu(\mathrm{d}z) + \frac{\lambda_0\sigma}{\alpha}\left(\int_{z\neq 0} z\mathbf{1}_{0<|z|<1}\nu(\mathrm{d}z)\right), \end{aligned}$$

where we used the fact that the function $f(x) := (e^{cx} - 1)/x$ is increasing for any $c \neq 0$. Combining these estimates on I_1, I_2 with (2.17), (2.18), by (1.4) and (2.13), we finally obtain that

$$\int_{\mathbb{R}} e^{\lambda_0 x}\tilde{\pi}(\mathrm{d}x) < \infty.$$

Similarly, we can show that $\int_{\mathbb{R}} e^{-\lambda_0 x}\tilde{\pi}(\mathrm{d}x) < \infty$. Noting that $e^{\lambda_0|x|} \le e^{\lambda_0 x} + e^{-\lambda_0 x}$, we obtain that

$$\int_{\mathbb{R}} e^{\lambda_0|x|}\tilde{\pi}(\mathrm{d}x) < \infty.$$

(ii) Replacing λ_0 with any positive constant λ in (2.17) and (2.18), we still have that

$$I_1 = -\frac{a\lambda^2\sigma^2}{4\alpha} - \frac{b\lambda\sigma}{\alpha} < \infty$$

since $\alpha < 0$. However, in the current case,

$$I_2 = -\frac{1}{\alpha}\int_0^1 \int_{z\neq 0} \frac{1}{u}(e^{z\lambda\sigma u} - 1)\nu(\mathrm{d}z)\mathrm{d}u + \frac{\lambda\sigma}{\alpha}\left(\int_{z\neq 0} z\mathbf{1}_{0<|z|<1}\nu(\mathrm{d}z)\right).$$

By conditions (1.4) and (2.14), there exists some $C > 0$ such that

$$\int_{0<|z|<1} \frac{1}{u}(e^{z\lambda\sigma u} - 1)\nu(\mathrm{d}z) \le C\int_{0<|z|<1} |z|\nu(\mathrm{d}z) < \infty,$$

and

$$\int_{|z|\ge 1} \frac{1}{u}(e^{z\lambda\sigma u} - 1)\nu(\mathrm{d}z) = \infty, \qquad u \neq 0.$$

As a consequence,

$$\int_{z\neq 0} \frac{1}{u}(e^{z\lambda\sigma u} - 1)\nu(\mathrm{d}z) = \infty,$$

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which implies $I_2 = \infty$ and

$$\int_{\mathbb{R}} \mathrm{e}^{\lambda x} \tilde{\pi}(\mathrm{d}x) = \infty.$$

The arbitrariness of the positive constant λ implies that $\tilde{\pi}$ is heavy-tailed, which completes the proof of this lemma. □

Proof of Theorem 1.2.

(i) We prove this assertion by contradiction. Assume that there exists a $\delta_0 > 0$ such that

$$\sum_{j\in\mathcal{S}} \int_{\mathbb{R}} \mathrm{e}^{2\delta_0|x|} \pi(\mathrm{d}x, j) < \infty. \tag{2.19}$$

Applying Itô's formula, we obtain that

$$\begin{aligned}
\mathrm{E}(\mathrm{e}^{\delta_0 X_t}) - \mathrm{E}(\mathrm{e}^{\delta_0 X_0}) &= \mathrm{E}\int_0^t \delta_0 \mathrm{e}^{\delta_0 X_{s-}} \left(\alpha_{\Lambda_{s-}} X_{s-} + b\sigma_{\Lambda_{s-}} + \frac{a\delta_0}{2}\sigma^2_{\Lambda_{s-}}\right)\mathrm{d}s \\
&\quad + 2\mathrm{E}\int_0^t \int_{|z|\geq 1} \mathrm{e}^{\delta_0 X_{s-}} (\mathrm{e}^{\delta_0 \sigma_{\Lambda_{s-}} z} - 1)\nu(\mathrm{d}z)\mathrm{d}s \\
&\quad + \mathrm{E}\int_0^t \int_{0<|z|<1} \mathrm{e}^{\delta_0 X_{s-}} (\mathrm{e}^{\delta_0 \sigma_{\Lambda_{s-}} z} - 1 - \delta_0 \sigma_{\Lambda_{s-}} z)\nu(\mathrm{d}z)\mathrm{d}s \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{2.20}$$

If we take the initial distribution of (X_0, Λ_0) to be the stationary distribution π, then for every $t > 0$ the distribution of (X_t, Λ_t) remains as π, which implies immediately that the left-hand side of (2.20) equals 0. From $\sum_{j\in\mathcal{S}} \int_{\mathbb{R}} \mathrm{e}^{2\delta_0|x|}\pi(\mathrm{d}x, j) < \infty$ and the finiteness of the number of states in $\mathcal{S}$, it is easy to see that

$$\mathrm{E}\left[\mathrm{e}^{\delta_0 X_{s-}} \left(|\alpha_{\Lambda_{s-}} X_{s-}| + |b\sigma_{\Lambda_{s-}}| + \frac{a\delta_0}{2}\sigma^2_{\Lambda_{s-}}\right)\right] < \infty,$$

and hence the term $I_1 < \infty$. Condition (1.4) ensures that there exists a constant $C > 0$ such that, for every $j \in \mathcal{S}$,

$$\int_{0<|z|<1} |\mathrm{e}^{\delta_0\sigma_j z} - 1 - \delta_0\sigma_j z|\nu(\mathrm{d}z) \leq C \int_{0<|z|<1} |z|\nu(\mathrm{d}z) < \infty.$$

Then, again using (2.19), we have $I_3 < \infty$. Nevertheless, the term I_2 is equal to ∞. Indeed, when $\Lambda_{s-} = i_0$, it follows from condition (1.6) that

$$\int_{|z|\geq 1} (\mathrm{e}^{\delta_0 \sigma_{\Lambda_{s-}} z} - 1)\nu(\mathrm{d}z) = \infty.$$

As an irreducible Markov chain, (Λ_t) arrives at the state i_0 with a positive probability, so $I_2 = \infty$. Therefore, we get a contradiction that the left-hand side of (2.20) equals 0, but the right-hand side of (2.20) equals ∞. Consequently, π cannot be light-tailed in the current situation.

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(ii) First, we show that π is light-tailed under the conditions (1.7) and $\max_{i\in\mathcal{S}}\alpha_i<0$. For $\delta\in(0,\lambda_0)$, similar to (2.20), Itô's formula yields that

$$\begin{aligned}
&\mathrm{E}(\mathrm{e}^{\delta|X_t|})-\mathrm{E}(\mathrm{e}^{\delta|X_0|})\\
&=\mathrm{E}\int_0^t\Big[\delta\mathrm{e}^{\delta|X_{s-}|}\mathrm{sgn}(X_{s-})(\alpha_{\Lambda_{s-}}X_{s-}+b\sigma_{\Lambda_{s-}})+\frac{a\delta^2}{2}\mathrm{e}^{\delta|X_{s-}|}\sigma^2_{\Lambda_{s-}}\Big]\mathrm{d}s\\
&\quad+2\mathrm{E}\int_0^t\int_{|z|\ge1}(\mathrm{e}^{\delta|X_{s-}+\sigma_{\Lambda_{s-}}z|}-\mathrm{e}^{\delta|X_{s-}|})\nu(\mathrm{d}z)\mathrm{d}s\\
&\quad+\mathrm{E}\int_0^t\int_{0<|z|<1}(\mathrm{e}^{\delta|X_{s-}+\sigma_{\Lambda_{s-}}z|}-\mathrm{e}^{\delta|X_{s-}|}-\delta\mathrm{e}^{\delta|X_{s-}|}\mathrm{sgn}(X_{s-})\sigma_{\Lambda_{s-}}z)\nu(\mathrm{d}z)\mathrm{d}s\\
&=:J_1+J_2+J_3,
\end{aligned}\tag{2.21}$$

where $\mathrm{sgn}(x)$ denotes the sign function of x.

Since $\max_{i\in\mathcal{S}}\alpha_i<0$, for any $c>0$ there exists $M>0$ such that

$$\alpha_{\Lambda_{s-}}|X_{s-}|+\mathrm{sgn}(X_{s-})b\sigma_{\Lambda_{s-}}\le-c+M\mathrm{e}^{-\delta|X_{s-}|}.$$

Therefore,

$$J_1\le\int_0^t\mathrm{E}(\delta M-c\delta\mathrm{e}^{\delta|X_{s-}|})\mathrm{d}s.\tag{2.22}$$

For J_2, we have

$$\begin{aligned}
J_2&\le\mathrm{E}\int_0^t\mathrm{e}^{\delta|X_{s-}|}\Big[\int_{|z|\ge1}(\mathrm{e}^{\delta|\sigma_{\Lambda_{s-}}z|}-1)\nu(\mathrm{d}z)\Big]\mathrm{d}s\\
&\le K_1(\delta)\int_0^t\mathrm{E}[\mathrm{e}^{\delta|X_{s-}|}]\mathrm{d}s,
\end{aligned}\tag{2.23}$$

where

$$K_1(\delta)=\max_{i\in\mathcal{S}}\int_{|z|\ge1}(\mathrm{e}^{\delta|\sigma_iz|}-1)\nu(\mathrm{d}z)<\infty$$

due to (1.7) and $\delta<\lambda_0$. Moreover,

$$\lim_{\delta\to0}\frac{K_1(\delta)}{\delta}\le\lim_{\delta\to0}\int_{|z|\ge1}\frac{1}{\delta}(\mathrm{e}^{\delta\sigma_*|z|}-1)\nu(\mathrm{d}z)=\int_{|z|\ge1}\sigma_*|z|\nu(\mathrm{d}z)<\infty,$$

where $\sigma_*:=\max_{i\in\mathcal{S}}|\sigma_i|$.

For J_3, we have

$$\begin{aligned}
J_3&\le\mathrm{E}\int_0^t\int_{0<|z|<1}\mathrm{e}^{\delta|X_{s-}|}(\mathrm{e}^{\delta|\sigma_{\Lambda_{s-}}z|}-1+\delta|\sigma_{\Lambda_{s-}}z|)\nu(\mathrm{d}z)\mathrm{d}s\\
&\le K_2(\delta)\int_0^t\mathrm{E}(\mathrm{e}^{\delta|X_{s-}|})\mathrm{d}s,
\end{aligned}\tag{2.24}$$

where

$$K_2(\delta)=\max_{i\in\mathcal{S}}\int_{0<|z|<1}(\mathrm{e}^{\delta|\sigma_iz|}-1+\delta|\sigma_iz|)\nu(\mathrm{d}z)<\infty,$$

due to (1.4). Furthermore,

$$\lim_{\delta\to0}\frac{K_2(\delta)}{\delta}\le\int_{0<|z|<1}2\sigma_*|z|\nu(\mathrm{d}z)<\infty.$$

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Inserting (2.22), (2.23), and (2.24) into (2.21), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(\mathrm{e}^{\delta|X_t|}) \le M\delta + K_3(\delta)\mathrm{E}(\mathrm{e}^{\delta|X_t|}),$$

with $K_3(\delta) = K_1(\delta) + K_2(\delta) - c\delta$. By the arbitrariness of c, we can take $\delta > 0$ sufficiently small and $c > 0$ large enough so that $K_3(\delta) < 0$. Applying Gronwall's inequality, we obtain

$$\mathrm{E}(\mathrm{e}^{\delta|X_t|}) \le \mathrm{E}(\mathrm{e}^{\delta|X_0|}\mathrm{e}^{K_3(\delta)t}) + M\delta\int_0^t \mathrm{e}^{K_3(\delta)(t-s)}\mathrm{d}s.$$

The negativeness of $K_3(\delta)$ implies that

$$\sup_{t\ge 0}\mathrm{E}(\mathrm{e}^{\delta|X_t|}) < \infty \tag{2.25}$$

if $\mathrm{E}(\mathrm{e}^{\delta|X_0|}) < \infty$. In particular, we take (X_0, Λ_0) starting from some fixed point (x_0, i), and the positive recurrence of (X_t, Λ_t) yields that the distribution of (X_t, Λ_t) weakly converges to its stationary distribution π. Together with the estimate (2.25), we finally get

$$\sum_{j\in\mathcal{S}}\int_{\mathbb{R}} \mathrm{e}^{\delta|x|}\pi(\mathrm{d}x, j) \le \liminf_{t\to\infty}\mathrm{E}(\mathrm{e}^{\delta|X_t|}) < \sup_{t>0}\mathrm{E}(\mathrm{e}^{\delta|X_t|}) < \infty,$$

which is the desired conclusion.

Second, we proceed to show that π is heavy-tailed under conditions (1.7) and $\max_{i\in\mathcal{S}}\alpha_i > 0$. For $p > 2\vee\kappa$, thanks to Itô's formula,

$$\begin{aligned}
&|X_t|^p - |X_0|^p\\
&= \int_0^t \left(p|X_{s-}|^{p-1}\mathrm{sgn}(X_{s-})(\alpha_{\Lambda_{s-}}X_{s-} + b\sigma_{\Lambda_{s-}}) + \frac{ap(p-1)}{2}\sigma_{\Lambda_{s-}}^2|X_{s-}|^{p-2}\right)\mathrm{d}s\\
&\quad + \int_0^t\int_{z\neq 0}(|X_{s-} + \sigma_{\Lambda_{s-}}z|^p - |X_{s-}|^p - p\sigma_{\Lambda_{s-}}z\,\mathrm{sgn}(X_{s-})|X_{s-}|^{p-1}\mathbf{1}_{0<|z|<1})\nu(\mathrm{d}z)\mathrm{d}s\\
&\quad + \int_0^t p\sqrt{a}\sigma_{\Lambda_{s-}}\mathrm{sgn}(X_{s-})|X_{s-}|^{p-1}\mathrm{d}B_s + \int_0^t\int_{z\neq 0}(|X_{s-} + \sigma_{\Lambda_{s-}}z|^p - |X_{s-}|^p)\widetilde{N}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Given the initial distribution γ of (X_0, Λ_0), let $\mathscr{F}^\Lambda = \sigma\{\Lambda_s; 0 \le s < \infty\}$ and

$$G_p^\gamma(t) = \mathrm{E}[|X_{t-}|^p|\mathscr{F}^\Lambda], \qquad t \ge 0, \tag{2.26}$$

where the superscript γ in $G_p^\gamma(t)$ emphasizes the initial distribution γ. By virtue of the independence of (Λ_t) and (Z_t), taking conditional expectation with respect to $\mathscr{F}^\Lambda$ yields that

$$\frac{\mathrm{d}G_p^\gamma(t)}{\mathrm{d}t} = p\alpha_{\Lambda_{t-}}G_p^\gamma(t) + I(t),$$

where $I(t) = I_1(t) + I_2(t)$ with

$$\begin{aligned}
I_1(t) &:= pb\sigma_{\Lambda_{t-}}\mathrm{E}[\mathrm{sgn}(X_{t-})|X_{t-}|^{p-1}|\mathscr{F}^\Lambda] + \frac{ap(p-1)\sigma_{\Lambda_{t-}}^2}{2}G_{p-2}^\gamma(t),\\
I_2(t) &:= \mathrm{E}\left[\int_{z\neq 0}(|X_{t-} + \sigma_{\Lambda_{t-}}z|^p - |X_{t-}|^p - p\sigma_{\Lambda_{t-}}z\,\mathrm{sgn}(X_{t-})|X_{t-}|^{p-1}\mathbf{1}_{0<|z|<1})\nu(\mathrm{d}z)\,\middle|\,\mathscr{F}^\Lambda\right].
\end{aligned}$$

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For $\theta > 0$, $\varepsilon > 0$, and $0 < r < p$, by Hölder's inequality and the elementary inequality

$$\theta x^{\frac{r}{p}} \le \varepsilon x + c_0, \quad x > 0, \qquad \text{with } c_0 = (\varepsilon/\theta)^{\frac{-r}{p+r}},$$

we have

$$\theta G_r^\gamma(t) \le \theta G_p^\gamma(t)^{\frac{r}{p}} \le \varepsilon G_p^\gamma(t) + c_0. \tag{2.27}$$

Therefore, for any $\varepsilon > 0$ there exists a constant $C_1 > 0$ independent of t such that

$$|I_1(t)| \le p\sigma_*|b|G_{p-1}^\gamma(t) + \frac{ap(p-1)\sigma_*^2}{2} G_{p-2}^\gamma(t) \le \frac{\varepsilon}{2} G_p^\gamma(t) + C_1. \tag{2.28}$$

To estimate the term $I_2(t)$, we note that the mean value theorem implies that

$$|x+z|^p - |x|^p = p(\vartheta|x+z| + (1-\vartheta)|z|)^{p-1}(|x+z| - |x|), \qquad x, z \in \mathbb{R}$$

for some $\vartheta \in [0, 1]$ depending on x, z, and, further,

$$\begin{aligned} ||x+z|^p - |x|^p| &\le p \max\{|x+z|^{p-1}, |z|^{p-1}\}|z| \\ &\le p \max\{2^{p-2}(|x|^{p-1} + |z|^{p-1}), |z|^{p-1}\}|z|. \end{aligned} \tag{2.29}$$

Also, conditions (1.4) and (1.7) imply that, for every $n \ge 1$, $\int_{z\neq 0} |z|^n \nu(\mathrm{d}z) < \infty$. Finally, invoking (2.27) and (2.29), we obtain that there exists some $C_2 > 0$, independent of t, such that

$$|I_2(t)| \le \frac{\varepsilon}{2} G_p^\gamma(t) + C_2. \tag{2.30}$$

Inserting the estimates (2.28), (2.30) into (2.26), we get

$$\frac{\mathrm{d}G_p^\gamma(t)}{\mathrm{d}t} \ge (p\alpha_{\Lambda_{t-}} - \varepsilon)G_p^\gamma - M,$$

where $M = C_1 + C_2 > 0$. Then,

$$\begin{aligned} G_p^\gamma(t) &\ge G_p^\gamma(0) \exp\left\{\int_0^t (p\alpha_{\Lambda_{s-}} - \varepsilon)\mathrm{d}s\right\} - M\int_0^t \exp\left\{\int_s^t (p\alpha_{\Lambda_{r-}} - \varepsilon)\mathrm{d}r\right\}\mathrm{d}s \\ &\ge G_p^\gamma(0) \exp\left\{\int_0^t (p\alpha_{\Lambda_{s-}} - \varepsilon)\mathrm{d}s\right\} - M\int_0^t \exp\left\{(p\max_{i\in\mathcal{S}}\alpha_i - \varepsilon)(t-s)\right\}\mathrm{d}s \\ &= G_p^\gamma(0) \exp\left\{\int_0^t (p\alpha_{\Lambda_{s-}} - \varepsilon)\mathrm{d}s\right\} \\ &\quad + \frac{M}{p\max_{i\in\mathcal{S}}\alpha_i - \varepsilon}\left(\exp\left\{(p\max_{i\in\mathcal{S}}\alpha_i - \varepsilon)t\right\} - 1\right). \end{aligned} \tag{2.31}$$

We shall prove statement (ii) by contradiction. Assume that $\sum_{i\in\mathcal{S}} \int_{\mathbb{R}} |x|^p \pi(\mathrm{d}x, i) < \infty$. Now let us take γ in (2.26) to be the stationary distribution π of (X_t, Λ_t), so for every $t > 0$ the distribution of (X_t, Λ_t) is also π. Since (Λ_t) is an irreducible Markov chain in a finite state space, denote by μ its stationary distribution on $\mathcal{S}$. Then, for every $i \in \mathcal{S}$, $\mu_i > 0$. Moreover, it holds that $\mu_i = \pi(\mathbb{R}, i)$, which can be seen by the weak convergence of the semigroup of (X_t, Λ_t) to π and the semigroup of (Λ_t) to μ. Consequently, we have

$$\min_{i\in\mathcal{S}} \int_{\mathbb{R}} |x|^p \pi(\mathrm{d}x, i) > 0.$$

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Taking expectation on both sides of (2.31), we obtain

$$\begin{aligned}\sum_{i\in\mathcal{S}}\int_{\mathbb{R}}|x|^p\pi(\mathrm{d}x,i) &= \mathrm{E}|X_t|^p\\ &\geq \Big(\min_{i\in\mathcal{S}}\int_{\mathbb{R}}|x|^p\pi(\mathrm{d}x,i)\Big)\mathrm{E}\Big(\exp\Big\{\int_0^t(p\alpha_{\Lambda_{s-}}+\varepsilon)\mathrm{d}s\Big\}\Big)\\ &\quad+\frac{M}{p\max_{i\in\mathcal{S}}\alpha_i+\varepsilon}\big(\exp\big\{(p\max_{i\in\mathcal{S}}\alpha_i+\varepsilon)t\big\}-1\big)\\ &\geq c_p\Big(\min_{i\in\mathcal{S}}\int_{\mathbb{R}}|x|^p\pi(\mathrm{d}x,i)\Big)\mathrm{e}^{(-\eta_p+\varepsilon)t}\\ &\quad+\frac{M}{p\max_{i\in\mathcal{S}}\alpha_i+\varepsilon}\big(\exp\big\{(p\max_{i\in\mathcal{S}}\alpha_i+\varepsilon)t\big\}-1\big),\end{aligned}\tag{2.32}$$

where c_p is a positive constant, $\eta_p<0$ is defined by (1.8), and in the last step we have used [1, Proposition 4.1]. Choosing $\varepsilon>0$ such that $\eta_p+\varepsilon<0$, and letting $t\to\infty$ in (2.32), the left-hand side is finite but the right-hand side goes to ∞, which is a contradiction. Therefore, the pth moment of π is infinite, and hence π is heavy-tailed, completing the proof. □

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