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Bivariate incomplete-Bessel kernels for the first nonlinear Vlasov–Maxwell response

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The weakly nonlinear response of a homogeneous magnetised plasma is usually written as a double cyclotron-harmonic expansion. This representation is explicit, but the repeated use of the Jacobi–Anger expansion produces long sums of Bessel functions and nested resonance denominators. In the linear problem we recently obtained an alternative formulation by keeping the Larmor phase unexpanded and by evaluating the characteristic integral in terms of the newly introduced incomplete-Bessel function $G_\mu(z, \psi)$. In this article we apply the same idea directly to the first nonlinear Vlasov–Maxwell equation. The nonlinear source contains the linear response of an inner mode. When this linear response is written in incomplete-Bessel form, the outer characteristic integral produces a bivariate orbit-resolvent $G^{(r)}_{\mu,\nu}(z, \psi; w, \chi)$. This function emerges therefore as the natural orbit integral generated by the first nonlinear characteristic problem. We derive the nonlinear distribution function in terms of these bivariate functions, collect the identities needed for the current projection, recover an extension of the classical double-harmonic Liu–Tripathi formula by expansion and indicate how the nonlinear susceptibility tensor is obtained from bivariate angular contractions.

Key words: plasma nonlinear phenomena, plasma dynamics

1. Introduction

The kinetic theory of nonlinear wave coupling in magnetised plasmas is a classical problem. The standard weakly nonlinear treatment expands the Vlasov–Maxwell system about a homogeneous magnetised equilibrium and expresses the response in cyclotron harmonics. This gives the familiar picture of resonant mode coupling, but the explicit formulas rapidly become cumbersome when electromagnetic waves propagate at arbitrary angle with respect to the background magnetic field. In the first nonlinear correction, one already encounters two coupled harmonic indices: one from the outer characteristic integration of the output mode and one from the inner linear response entering the nonlinear source. This is the algebraic origin of the complicated double sums appearing in Liu–Tripathi and similar calculations (Liu & Tripathi 1986; Sugaya 1989; Yoon 2024).

In the linear problem, the same cyclotron structure can be organised in a different way. Instead of expanding the Larmor phase at the beginning of the calculation, one keeps the phase along the unperturbed orbit and evaluates the characteristic integral directly. This produces the incomplete-Bessel function $G_\mu(z, \psi)$, which replaces the cyclotron-harmonic sum before the susceptibility tensor is projected (Ricci 2026). This idea is closely related to the sum-rule approach of Newberger and subsequent developments (Newberger 1982; Bakker & Temme 1984; Swanson 2003; Qin, Phillips & Davidson 2007; Lerche, Schlickeiser & Tautz 2008; Qin, Phillips & Davidson 2008).

The purpose of the present paper is to show that the first nonlinear problem admits an exactly analogous organisation. The first nonlinear equation has the same left-hand side as the linear equation; only the source term changes. More precisely, the nonlinear source is the Lorentz force of one mode acting on the linear response of another mode. If the inner linear response is written in its incomplete-Bessel form, then the outer characteristic integral acts on a shifted $G_{\mu_{\boldsymbol{q}}}$ response. The resulting object is a bivariate incomplete-Bessel function:

$$G^{(r)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}; z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}). \tag{1.1}$$

The integer r records the finite harmonic shifts generated by transverse velocity factors and gyrophase derivatives in the nonlinear force.

The terminology ‘bivariate incomplete-Bessel function’ is used in the structural sense: $G^{(r)}_{\mu,\nu}$ is the bivariate orbit-resolvent generated by applying a second incomplete-Bessel-type characteristic transform to the one-variable incomplete-Bessel kernel G_ν. It is not meant to imply that the function is a classical two-variable Bessel function in the sense of the standard theory of multivariable Bessel functions.

This changes the exposition of the nonlinear calculation. The bivariate function appears directly from the nonlinear characteristic solution, exactly as G_μ appears directly from the linear characteristic solution. The double-harmonic formula is then recovered by expanding the bivariate function. In this sense the traditional harmonic expression becomes a validation of the orbit formulation rather than its starting point.

The article is organised as follows. We first state the perturbative Vlasov–Maxwell problem and introduce the notation for the cyclotron orbit. We recall the linear solution in incomplete-Bessel form, using the integral definition of G_μ as primary. We then take the nonlinear characteristic formula from the standard calculation and rewrite its source with compact differential operators. Inserting the $G_{\mu_{\boldsymbol{q}}}$ form of the linear response produces the bivariate incomplete-Bessel function. We derive the explicit nonlinear distribution function in this notation, study the main properties of the bivariate functions, derive independently the traditional double-harmonic formula and recover the same formula by expansion. We finally describe the current projection leading to the nonlinear susceptibility tensor, by providing an explicit compact formula.

2. Perturbative Vlasov–Maxwell problem and orbit notation

We consider a homogeneous plasma in a uniform background magnetic field:

$$\boldsymbol{B}_0 = B_0 \boldsymbol{b}, \tag{2.1}$$

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where $\boldsymbol{b}$ is a fixed unit vector, assumed along the z direction of an otherwise arbitrarily oriented Cartesian frame. For one species, the Vlasov equation is

$$\frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} F + \frac{e}{m} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \partial_{\boldsymbol{v}} F = 0. \tag{2.2}$$

Here F is the species distribution function, e is the signed charge and m denotes the particle mass. We write

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \cdots, \qquad \boldsymbol{E} = \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)} + \cdots \tag{2.3}$$

and

$$F = f_0 + f^{(1)} + f^{(2)} + \cdots. \tag{2.4}$$

The equilibrium distribution $f_0 = f_0(v_\perp, v_\parallel)$ is assumed gyrotropic. Then the linear problem is described by

$$\left[\partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} + \frac{e}{m}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \partial_{\boldsymbol{v}}\right] f^{(1)}(\boldsymbol{x}, \boldsymbol{v}, t) = -\frac{e}{m} [\boldsymbol{v} \times \boldsymbol{B}_0] \cdot \partial_{\boldsymbol{v}} f_0(v_\perp, v_\parallel). \tag{2.5}$$

The first nonlinear equation considered in this work has the same structure, apart from the source term:

$$\left[\partial_t + \boldsymbol{v}\partial_{\boldsymbol{x}} + \frac{e}{m}(\boldsymbol{v} \times \boldsymbol{B}_0)\partial_{\boldsymbol{v}}\right] f^{(2)}(\boldsymbol{x}, \boldsymbol{v}, t) = -\frac{e}{m}\left[\boldsymbol{E}^{(1)}(\boldsymbol{x}, t) + \boldsymbol{v} \times \boldsymbol{B}^{(1)}(\boldsymbol{x}, t)\right] \partial_{\boldsymbol{v}} f^{(1)}(\boldsymbol{x}, \boldsymbol{v}, t). \tag{2.6}$$

At all perturbative levels, the left-hand side is the same unperturbed cyclotron operator. After Laplace–Fourier transforming, the equations can be solved by using the method of characteristics.

We adopt the following conventions. The unperturbed signed cyclotron frequency is

$$\omega := \frac{eB_0}{m}. \tag{2.7}$$

For a Laplace–Fourier mode $(\boldsymbol{k}, \Omega)$, the characteristic parameters are

$$z_{\boldsymbol{k}} := \frac{k_\perp v_\perp}{\omega}, \qquad \mu_{\boldsymbol{k},\Omega} := \frac{k_\parallel v_\parallel - \Omega}{\omega}, \qquad \psi_{\boldsymbol{k}} := \phi - \phi_{\boldsymbol{k}}. \tag{2.8}$$

In the following, we set $\Omega = \Omega_{\boldsymbol{k}}$, where $\Omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} + \mathrm{i}\,\gamma_{\boldsymbol{k}}$ denotes a generally complex solution of the linear dispersion equation, assumed one-branched for simplicity.

Along the unperturbed orbit we use

$$\lambda = \omega\tau, \qquad \psi_{\boldsymbol{k}}(\lambda) = \psi_{\boldsymbol{k}} + \lambda. \tag{2.9}$$

The linear and nonlinear responses at mode $\boldsymbol{k}$ have the same external Larmor dressing:

$$f_{\boldsymbol{k}}^{(s)}(\boldsymbol{v}) = \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{k}} \sin\psi_{\boldsymbol{k}}}\, \tilde{f}_{\boldsymbol{k}}^{(s)}(\boldsymbol{v}), \tag{2.10}$$

where $s = 1, 2, \ldots$. This factorisation is a consequence of the common cyclotron characteristic operator. It is the basic reason why the incomplete-Bessel construction can be iterated from the linear to the first nonlinear response.

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We drop for simplicity the superscript from $\boldsymbol{E}^{(1)}$ and use circular transverse components:

$$E_{\boldsymbol{k},1} := \frac{E_{\boldsymbol{k},x} - \mathrm{i}\, E_{\boldsymbol{k},y}}{\sqrt{2}}, \qquad E_{\boldsymbol{k},-1} := \frac{E_{\boldsymbol{k},x} + \mathrm{i}\, E_{\boldsymbol{k},y}}{\sqrt{2}}, \qquad E_{\boldsymbol{k},0} = E_{\boldsymbol{k},z}. \tag{2.11}$$

The two velocity-space differential operators used below are

$$\mathcal{G}_{\boldsymbol{k}} := \partial_{v_\perp} + \frac{k_\parallel}{\Omega_{\boldsymbol{k}}} \left(v_\perp \partial_{v_\parallel} - v_\parallel \partial_{v_\perp} \right) \tag{2.12}$$

and

$$\mathcal{G} := v_\parallel \partial_{v_\perp} - v_\perp \partial_{v_\parallel}. \tag{2.13}$$

For the first nonlinear response we impose the resonance condition

$$\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}, \qquad \Omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}}. \tag{2.14}$$

The prime on sums over $\boldsymbol{p}$ and $\boldsymbol{q}$ denotes the standard exclusion of zero modes.

3. The linear solution in incomplete-Bessel form

The one-variable incomplete-Bessel function introduced in Ricci (2026) is primarily defined as the orbit integral

$$G_\mu(z, \psi) := \mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}\, z \sin(\psi + \lambda)}\, \mathrm{e}^{-\mathrm{i}\, \mu \lambda}\, \mathrm{d}\, \lambda, \tag{3.1}$$

with the usual causal prescription understood in the oscillatory integral. This function is 2π-periodic in the variable ψ. Its harmonic representation is obtained by using the Jacobi–Anger formula (DLMF, § 10.12, equations (10.12.1)–(10.12.2)) to expand the Larmor phase:

$$G_\mu(z, \psi) = \sum_{n=-\infty}^{\infty} \frac{J_n(z)\, \mathrm{e}^{-\mathrm{i}\, n \psi}}{n + \mu}. \tag{3.2}$$

This is the linear prototype: one cyclotron characteristic integral acting on one Larmor phase produces one cyclotron denominator.

The linear characteristic solution can be written as

$$f^{(1)}_{\boldsymbol{q}}(\boldsymbol{v}) = \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{q}} \sin \psi_{\boldsymbol{q}}}\, \tilde{f}^{(1)}_{\boldsymbol{q}}(\boldsymbol{v}), \tag{3.3}$$

where

$$\tilde{f}^{(1)}_{\boldsymbol{q}} = \frac{\mathrm{i}\, e}{m\omega} \sum_{a \in \mathcal{I}} \mathrm{e}^{\mathrm{i}\, a \psi_{\boldsymbol{q}}}\, G_{\mu_{\boldsymbol{q}} - a}(z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}) X_{\boldsymbol{q},a}(v_\perp, v_\parallel). \tag{3.4}$$

Here $\mathcal{I} := \{-1, 0, 1\}$ and

$$X_{\boldsymbol{q},1} := \left(\frac{\mathrm{e}^{\mathrm{i}\, \phi_{\boldsymbol{q}}}}{\sqrt{2}} E_{\boldsymbol{q},1} \mathcal{G}_{\boldsymbol{q}} + \frac{q_\perp E_{\boldsymbol{q},0}}{2\Omega_{\boldsymbol{q}}} \mathcal{G} \right) f_0, \tag{3.5}$$

$$X_{\boldsymbol{q},-1} := \left(\frac{\mathrm{e}^{-\mathrm{i}\, \phi_{\boldsymbol{q}}}}{\sqrt{2}} E_{\boldsymbol{q},-1} \mathcal{G}_{\boldsymbol{q}} + \frac{q_\perp E_{\boldsymbol{q},0}}{2\Omega_{\boldsymbol{q}}} \mathcal{G} \right) f_0 \tag{3.6}$$

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and

$$X_{\boldsymbol{q},0} := E_{\boldsymbol{q},0}\partial_{v_\parallel} f_0. \tag{3.7}$$

The shifted factors $G_{\mu_{\boldsymbol{q}}-1}$, $G_{\mu_{\boldsymbol{q}}+1}$ and $G_{\mu_{\boldsymbol{q}}}$ are the result of the linear orbit integral and multiply the corresponding velocity-space source terms.

The standard harmonic coefficients are recovered by expanding the shifted incomplete-Bessel factors. The familiar coefficient form is

$$f^{(1)}_{\boldsymbol{q}}(\boldsymbol{v}) = \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{q}} \sin\psi_{\boldsymbol{q}}} \sum_{N=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\, N\psi_{\boldsymbol{q}}}\, \tilde{f}^{(1)}_{\boldsymbol{q},N}(v_\perp, v_\parallel), \tag{3.8}$$

with

$$\tilde{f}^{(1)}_{\boldsymbol{q},N} = \mathrm{i}\,\frac{e}{m}\frac{1}{N\omega - (\Omega_{\boldsymbol{q}} - q_\parallel v_\parallel)}\left[\left(J_{N+1}(z_{\boldsymbol{q}})\frac{\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{q}}}}{\sqrt{2}}E_{\boldsymbol{q},1} + J_{N-1}(z_{\boldsymbol{q}})\frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{q}}}}{\sqrt{2}}E_{\boldsymbol{q},-1}\right)\mathcal{G}_{\boldsymbol{q}}\right.$$
$$\left. + J_N(z_{\boldsymbol{q}})E_{\boldsymbol{q},0}\left(\partial_{v_\parallel} + \frac{N\omega}{v_\perp\Omega_{\boldsymbol{q}}}\mathcal{G}\right)\right] f_0. \tag{3.9}$$

4. The nonlinear characteristic formula

The first nonlinear perturbation satisfies the same characteristic equation as the linear perturbation. For a triad $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$, the nonlinear source is the field of the mode $\boldsymbol{p}$ acting on the linear response of the mode $\boldsymbol{q}$. The characteristic solution obtained in the standard calculation is

$$f^{(2)}_{\boldsymbol{k}}(\boldsymbol{v}) = -\frac{e}{m\omega}\,\mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{k}} \sin\psi_{\boldsymbol{k}}}\,\frac{1}{V}{\sum_{\boldsymbol{p}}}'{\sum_{\boldsymbol{q}}}'\delta_{\boldsymbol{q},\,\boldsymbol{k}-\boldsymbol{p}}\frac{1}{\Omega_{\boldsymbol{p}}}\int_0^\infty \mathrm{e}^{-\mathrm{i}\, z_{\boldsymbol{k}}\sin\psi_{\boldsymbol{k}}(\lambda)}\,\mathrm{e}^{-\mathrm{i}\,\mu_{\boldsymbol{k}}\lambda}\left\{\mathrm{e}^{\mathrm{i}\,\psi_{\boldsymbol{k}}(\lambda)}\,\mathcal{P}_{\boldsymbol{p},1}\right.$$
$$\left. + \mathrm{e}^{-\mathrm{i}\,\psi_{\boldsymbol{k}}(\lambda)}\,\mathcal{P}_{\boldsymbol{p},-1} + \mathcal{P}_{\boldsymbol{p},0}\right\} f^{(1)}_{\boldsymbol{q}}\,|_{\boldsymbol{v}(\lambda)}\,\mathrm{d}\,\lambda. \tag{4.1}$$

The differential blocks are independent of λ. Their explicit form is

$$\mathcal{P}_{\boldsymbol{p},1} := \mathrm{e}^{\mathrm{i}(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}\left[\frac{\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}}E_{\boldsymbol{p},1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}} - \mathrm{i}\,\frac{\mu_{\boldsymbol{p}}\omega}{v_\perp}\partial_\phi\right) + E_{\boldsymbol{p},0}\frac{p_\perp}{2}\left(\mathcal{G} + \mathrm{i}\,\frac{v_\parallel}{v_\perp}\partial_\phi\right)\right]; \tag{4.2}$$

$$\mathcal{P}_{\boldsymbol{p},-1} := \mathrm{e}^{-\mathrm{i}(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}\left[\frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}}E_{\boldsymbol{p},-1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}} + \mathrm{i}\,\frac{\mu_{\boldsymbol{p}}\omega}{v_\perp}\partial_\phi\right) + E_{\boldsymbol{p},0}\frac{p_\perp}{2}\left(\mathcal{G} - \mathrm{i}\,\frac{v_\parallel}{v_\perp}\partial_\phi\right)\right]; \tag{4.3}$$

$$\mathcal{P}_{\boldsymbol{p},0} := \mathrm{i}\,p_\perp\frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}\,E_{\boldsymbol{p},-1} - \mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}\,E_{\boldsymbol{p},1}}{\sqrt{2}}\partial_\phi + E_{\boldsymbol{p},0}\Omega_{\boldsymbol{p}}\partial_{v_\parallel}. \tag{4.4}$$

These differential blocks act on the full linear response $f^{(1)}_{\boldsymbol{q}}$ and the resulting expression is evaluated at $\boldsymbol{v}(\lambda)$. In particular, $v_\perp$, $v_\parallel$, $z_{\boldsymbol{q}}$ and $\mu_{\boldsymbol{q}}$ do not change; only the inner gyrophase acquires a dependence on λ through

$$\psi_{\boldsymbol{q}} \mapsto \psi_{\boldsymbol{q}}(\lambda) = \psi_{\boldsymbol{q}} + \lambda. \tag{4.5}$$

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5. Bivariate incomplete-Bessel functions

For later convenience, we anticipate the definition of the bivariate incomplete-Bessel function. A more detailed analysis of this function and the investigation of its main properties is deferred to § 7.

DEFINITION 1 (Bivariate incomplete-Bessel function). *For* $r \in \mathbb{Z}$, *define*

$$G^{(r)}_{\mu,\nu}(z, \psi; w, \chi) := \mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}\, z \sin(\psi+\lambda)}\, \mathrm{e}^{\mathrm{i}\, w \sin(\chi+\lambda)}\, \mathrm{e}^{-\mathrm{i}\,\mu\lambda}\, \mathrm{e}^{\mathrm{i}\, r\lambda}\, G_\nu(w, \chi+\lambda)\, \mathrm{d}\,\lambda. \quad (5.1)$$

This function emerges as the natural orbit-resolvent kernel associated with the first nonlinear cyclotron response. The first triplet of variables (z, ψ, μ) belongs to the outer output mode. The second triplet (w, χ, ν) belongs to the inner linear response. The exponential factor with positive sign is the inner Larmor factor already present in the full linear distribution. Therefore $G^{(r)}_{\mu,\nu}$ is the exact orbit integral produced when the outer characteristic acts on a full inner incomplete-Bessel response.

The elementary replacement repeatedly used in the following is

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\, z_{\boldsymbol{k}} \sin(\psi_{\boldsymbol{k}}+\lambda)}\, \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{q}} \sin(\psi_{\boldsymbol{q}}+\lambda)}\, \mathrm{e}^{-\mathrm{i}\,\mu_{\boldsymbol{k}}\lambda}\, \mathrm{e}^{\mathrm{i}\, r\lambda}\, G_{\nu_{\boldsymbol{q}}}(z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}+\lambda)\, \mathrm{d}\,\lambda$$

$$= -\mathrm{i}\, G^{(r)}_{\mu_{\boldsymbol{k}},\nu_{\boldsymbol{q}}}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}; z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}). \quad (5.2)$$

This identity is the nonlinear analogue of the linear characteristic evaluation. It contains the full inner Larmor factor and therefore also accounts for the phase-gradient terms produced when the nonlinear velocity-space derivatives act on the inner linear response.

6. Nonlinear distribution in bivariate form

We write the full inner linear response as

$$f^{(1)}_{\boldsymbol{q}} = \frac{\mathrm{i}\, e}{m\omega} \sum_{a\in\mathcal{I}} K_{\boldsymbol{q},a} X_{\boldsymbol{q},a}, \quad (6.1)$$

where

$$K_{\boldsymbol{q},a}(z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}) := \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{q}} \sin\psi_{\boldsymbol{q}} + \mathrm{i}\, a\psi_{\boldsymbol{q}}}\, G_{\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}). \quad (6.2)$$

Then the characteristic formula for the nonlinear response can be written in the compact form

$$f^{(2)}_{\boldsymbol{k}}(\boldsymbol{v}) = -\frac{e^2}{m^2\omega^2}\, \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{k}} \sin\psi_{\boldsymbol{k}}}\, \frac{1}{V} {\sum_{\boldsymbol{p}}}' {\sum_{\boldsymbol{q}}}' \delta_{\boldsymbol{q},\,\boldsymbol{k}-\boldsymbol{p}} \frac{1}{\Omega_{\boldsymbol{p}}}$$

$$\mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}\, z_{\boldsymbol{k}} \sin(\psi_{\boldsymbol{k}}+\lambda)}\, \mathrm{e}^{-\mathrm{i}\,\mu_{\boldsymbol{k}}\lambda} \sum_{a\in\mathcal{I}} \sum_{b\in\mathcal{I}} \mathrm{e}^{\mathrm{i}\, b\psi_{\boldsymbol{k}}}\, \mathrm{e}^{\mathrm{i}\, b\lambda}\, \mathcal{P}_{\boldsymbol{p},b}(K_{\boldsymbol{q},a} X_{\boldsymbol{q},a})|_{\boldsymbol{v}(\lambda)}\, \mathrm{d}\,\lambda, \quad (6.3)$$

where

$$\mathcal{I} := \{-1, 0, 1\}. \quad (6.4)$$

The index $a \in \mathcal{I}$ labels the inner linear block, while the index $b \in \mathcal{I}$ labels the outer gyrophase block generated by the nonlinear force.

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The operator acting on the inner block is

$$\mathcal{P}_{\boldsymbol{p},b} := \mathrm{e}^{\mathrm{i}\, b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})} \left\{ E_{\boldsymbol{p},b} \left[b^2 \frac{\mathrm{e}^{\mathrm{i}\, b\phi_{\boldsymbol{p}}}}{\sqrt{2}} \left(\Omega_{\boldsymbol{p}} \mathcal{G}_{\boldsymbol{p}} - \mathrm{i}\, b \frac{\mu_{\boldsymbol{p}}\omega}{v_\perp} \partial_\phi \right) + (1-b^2)\Omega_{\boldsymbol{p}} \partial_{v_\parallel} \right] \right.$$
$$\left. + \sum_{c=\pm 1} E_{\boldsymbol{p},b+c} \frac{1-bc}{2} \left[b^2 \frac{p_\perp}{2} \left(\mathcal{G} - \mathrm{i}\, c \frac{v_\parallel}{v_\perp} \partial_\phi \right) - (1-b^2)\, \mathrm{i}\, p_\perp c\sqrt{2}\, \mathrm{e}^{\mathrm{i}\, c\phi_{\boldsymbol{p}}}\, \partial_\phi \right] \right\}. \tag{6.5}$$

Equivalently, by regrouping according to velocity derivatives,

$$\mathcal{P}_{\boldsymbol{p},b} = \mathrm{e}^{\mathrm{i}\, b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})} \left[C^{(\phi)}_{\boldsymbol{p},b} \partial_\phi + C^{(\perp)}_{\boldsymbol{p},b} \partial_{v_\perp} + C^{(\parallel)}_{\boldsymbol{p},b} \partial_{v_\parallel} \right], \tag{6.6}$$

where

$$C^{(\phi)}_{\boldsymbol{p},b} := -\mathrm{i}\, b^3 E_{\boldsymbol{p},b} \frac{\mathrm{e}^{\mathrm{i}\, b\phi_{\boldsymbol{p}}}}{\sqrt{2}} \frac{\mu_{\boldsymbol{p}}\omega}{v_\perp}$$
$$+ \sum_{c=\pm 1} E_{\boldsymbol{p},b+c} \frac{1-bc}{2} \left[-\mathrm{i}\, cb^2 \frac{p_\perp v_\parallel}{2v_\perp} - (1-b^2)\, \mathrm{i}\, p_\perp c\sqrt{2}\, \mathrm{e}^{\mathrm{i}\, c\phi_{\boldsymbol{p}}} \right], \tag{6.7}$$

$$C^{(\perp)}_{\boldsymbol{p},b} := b^2 E_{\boldsymbol{p},b} \frac{\mathrm{e}^{\mathrm{i}\, b\phi_{\boldsymbol{p}}}}{\sqrt{2}} \left(\Omega_{\boldsymbol{p}} - p_\parallel v_\parallel \right) + \sum_{c=\pm 1} E_{\boldsymbol{p},b+c} \frac{1-bc}{2} b^2 \frac{p_\perp v_\parallel}{2} \tag{6.8}$$

and

$$C^{(\parallel)}_{\boldsymbol{p},b} := E_{\boldsymbol{p},b} \left[b^2 \frac{\mathrm{e}^{\mathrm{i}\, b\phi_{\boldsymbol{p}}}}{\sqrt{2}} p_\parallel v_\perp + (1-b^2)\Omega_{\boldsymbol{p}} \right] - \sum_{c=\pm 1} E_{\boldsymbol{p},b+c} \frac{1-bc}{2} b^2 \frac{p_\perp v_\perp}{2}. \tag{6.9}$$

Values of $E_{\boldsymbol{p},j}$ with $j \notin \mathcal{I}$ are understood to vanish.

The product rule gives

$$\mathcal{P}_{\boldsymbol{p},b}(K_{\boldsymbol{q},a} X_{\boldsymbol{q},a}) = (\mathcal{P}_{\boldsymbol{p},b} K_{\boldsymbol{q},a}) X_{\boldsymbol{q},a} + K_{\boldsymbol{q},a} (\mathcal{P}_{\boldsymbol{p},b} X_{\boldsymbol{q},a}). \tag{6.10}$$

Since

$$z_{\boldsymbol{q}} = \frac{q_\perp v_\perp}{\omega}, \qquad \psi_{\boldsymbol{q}} = \phi - \phi_{\boldsymbol{q}}, \qquad \mu_{\boldsymbol{q}} = \frac{q_\parallel v_\parallel - \Omega_{\boldsymbol{q}}}{\omega}, \tag{6.11}$$

one has

$$\partial_\phi K_{\boldsymbol{q},a} = \partial_{\psi_{\boldsymbol{q}}} K_{\boldsymbol{q},a}, \tag{6.12}$$

$$\partial_{v_\perp} K_{\boldsymbol{q},a} = \frac{q_\perp}{\omega} \partial_{z_{\boldsymbol{q}}} K_{\boldsymbol{q},a} \tag{6.13}$$

and

$$\partial_{v_\parallel} K_{\boldsymbol{q},a} = \frac{q_\parallel}{\omega}\, \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{q}} \sin\psi_{\boldsymbol{q}} + \mathrm{i}\, a\psi_{\boldsymbol{q}}}\, \partial_{\mu_{\boldsymbol{q}}} G_{\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}). \tag{6.14}$$

Along the orbit

$$\psi_{\boldsymbol{q}} \mapsto \psi_{\boldsymbol{q}} + \lambda, \tag{6.15}$$

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whereas $z_{\boldsymbol{q}}$, $\mu_{\boldsymbol{q}}$ and $X_{\boldsymbol{q},a}$ are constant. Hence, grouping together the λ-dependent terms and using the definition of $G^{(r)}_{\mu,\nu}$ in (5.1), we obtain the replacement rules

$$K_{\boldsymbol{q},a} \mapsto \mathrm{e}^{\mathrm{i}\,a\psi_{\boldsymbol{q}}}\, G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}), \tag{6.16}$$

$$\partial_{\psi_{\boldsymbol{q}}} K_{\boldsymbol{q},a} \mapsto \mathrm{e}^{\mathrm{i}\,a\psi_{\boldsymbol{q}}}(\partial_{\psi_{\boldsymbol{q}}} + \mathrm{i}\,a) G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}), \tag{6.17}$$

$$\partial_{z_{\boldsymbol{q}}} K_{\boldsymbol{q},a} \mapsto \mathrm{e}^{\mathrm{i}\,a\psi_{\boldsymbol{q}}}\, \partial_{z_{\boldsymbol{q}}} G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}) \tag{6.18}$$

and

$$\mathrm{e}^{\mathrm{i}\,z_{\boldsymbol{q}}\sin\psi_{\boldsymbol{q}}+\mathrm{i}\,a\psi_{\boldsymbol{q}}}\, \partial_{\mu_{\boldsymbol{q}}} G_{\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}) \mapsto \mathrm{e}^{\mathrm{i}\,a\psi_{\boldsymbol{q}}}\, \partial_{\nu} G^{(a+b)}_{\mu_{\boldsymbol{k}},\nu}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}})|_{\nu=\mu_{\boldsymbol{q}}-a}. \tag{6.19}$$

Therefore, by introducing the shortcut notations

$$\mathbb{G}_{\boldsymbol{k},\boldsymbol{q},a,b} := G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}), \tag{6.20}$$

$$\mathbb{G}^{(\phi)}_{\boldsymbol{k},\boldsymbol{q},a,b} := (\partial_{\psi_{\boldsymbol{q}}} + \mathrm{i}\,a) G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}), \tag{6.21}$$

$$\mathbb{G}^{(\perp)}_{\boldsymbol{k},\boldsymbol{q},a,b} := \partial_{z_{\boldsymbol{q}}} G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}}) \tag{6.22}$$

and

$$\mathbb{G}^{(\parallel)}_{\boldsymbol{k},\boldsymbol{q},a,b} := \partial_{\nu} G^{(a+b)}_{\mu_{\boldsymbol{k}},\nu}(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}};\, z_{\boldsymbol{q}},\, \psi_{\boldsymbol{q}})|_{\nu=\mu_{\boldsymbol{q}}-a}, \tag{6.23}$$

the reduced nonlinear distribution can be eventually written as

$$\begin{aligned} \tilde{f}^{(2)}_{\boldsymbol{k}} = -\frac{e^2}{m^2\omega^2}\frac{1}{V}\sum_{\boldsymbol{p}}{}' \sum_{\boldsymbol{q}}{}' \delta_{\boldsymbol{q},\,\boldsymbol{k}-\boldsymbol{p}} \frac{1}{\Omega_{\boldsymbol{p}}} \sum_{a\in\mathcal{I}}\sum_{b\in\mathcal{I}} \mathrm{e}^{\mathrm{i}\,b\psi_{\boldsymbol{k}}}\, \mathrm{e}^{\mathrm{i}\,b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}\, \mathrm{e}^{\mathrm{i}\,a\psi_{\boldsymbol{q}}} \\ \left[\left(C^{(\phi)}_{\boldsymbol{p},b}\mathbb{G}^{(\phi)}_{\boldsymbol{k},\boldsymbol{q},a,b} + C^{(\perp)}_{\boldsymbol{p},b}\frac{q_\perp}{\omega}\mathbb{G}^{(\perp)}_{\boldsymbol{k},\boldsymbol{q},a,b} + C^{(\parallel)}_{\boldsymbol{p},b}\frac{q_\parallel}{\omega}\mathbb{G}^{(\parallel)}_{\boldsymbol{k},\boldsymbol{q},a,b}\right) X_{\boldsymbol{q},a} \right. \\ \left. + \mathbb{G}_{\boldsymbol{k},\boldsymbol{q},a,b}\left[C^{(\perp)}_{\boldsymbol{p},b}\partial_{v_\perp} + C^{(\parallel)}_{\boldsymbol{p},b}\partial_{v_\parallel}\right] X_{\boldsymbol{q},a}\right]. \end{aligned} \tag{6.24}$$

Only the bivariate orders $a+b$ occur. Since $a, b \in \{-1, 0, 1\}$, these orders range from -2 to 2.

7. Properties of the bivariate functions

We now collect the identities needed to manipulate the nonlinear response and to compare the bivariate characteristic representation with the classical double-harmonic formula. The bivariate incomplete-Bessel function is defined by

$$G^{(r)}_{\mu,\nu}(z,\, \psi;\, w,\, \chi) := \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}\,z\sin(\psi+\lambda)}\, \mathrm{e}^{\mathrm{i}\,w\sin(\chi+\lambda)}\, \mathrm{e}^{-\mathrm{i}\,\mu\lambda}\, \mathrm{e}^{\mathrm{i}\,r\lambda}\, G_\nu(w,\, \chi+\lambda)\, \mathrm{d}\,\lambda. \tag{7.1}$$

The same causal prescription used for the one-variable incomplete-Bessel function is understood.

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It is useful to introduce the dressed inner factor

$$B_\nu(w,\chi) := \mathrm{e}^{\mathrm{i}\, w \sin\chi}\, G_\nu(w,\chi). \tag{7.2}$$

Its Fourier expansion is

$$B_\nu(w,\chi) = \sum_{\ell=-\infty}^{\infty} T_\ell(w,\nu)\, \mathrm{e}^{-\mathrm{i}\,\ell\chi}, \tag{7.3}$$

where

$$T_\ell(w,\nu) := \sum_{n=-\infty}^{\infty} \frac{J_n(w) J_{n-\ell}(w)}{n+\nu} \tag{7.4}$$

is called Turkin's function in Newberger (1982).

PROPOSITION 1 (Superposition of one-variable incomplete-Bessel functions). *The bivariate function admits the representation*

$$G^{(r)}_{\mu,\nu}(z,\psi;w,\chi) = \sum_{\ell=-\infty}^{\infty} T_\ell(w,\nu)\, \mathrm{e}^{-\mathrm{i}\,\ell\chi}\, G_{\mu+\ell-r}(z,\psi)\,. \tag{7.5}$$

Proof. Use the Fourier expansion of B_ν at $\chi+\lambda$ in the defining integral. The remaining λ-integral is exactly $G_{\mu+\ell-r}(z,\psi)$. □

PROPOSITION 2 (Harmonic expansion). *The bivariate function has the double expansion*

$$G^{(r)}_{\mu,\nu}(z,\psi;w,\chi) = \sum_{N=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \frac{J_N(z) T_\ell(w,\nu)\, \mathrm{e}^{-\mathrm{i}\,N\psi}\, \mathrm{e}^{-\mathrm{i}\,\ell\chi}}{N+\ell+\mu-r}. \tag{7.6}$$

Proof. This follows from the superposition formula and from

$$G_\eta(z,\psi) = \sum_{N=-\infty}^{\infty} \frac{J_N(z)\, \mathrm{e}^{-\mathrm{i}\,N\psi}}{N+\eta}. \tag{7.7}$$

□

PROPOSITION 3 (Outer resolvent identity). *The bivariate function satisfies*

$$(\mathrm{i}\,\partial_\psi + \mathrm{i}\,\partial_\chi + \mu - r) G^{(r)}_{\mu,\nu}(z,\psi;w,\chi) = \mathrm{e}^{-\mathrm{i}\, z\sin\psi}\, B_\nu(w,\chi). \tag{7.8}$$

Proof. In the harmonic expansion the operator multiplies the term (N,ℓ) by $N+\ell+\mu-r$, cancelling the denominator. □

PROPOSITION 4 (Outer Bessel equation). *The bivariate function satisfies the homogeneous Bessel equation in the outer variables:*

$$\left[z^2\partial_z^2 + z\partial_z + z^2 + \partial_\psi^2\right] G^{(r)}_{\mu,\nu}(z,\psi;w,\chi) = 0. \tag{7.9}$$

Proof. Each outer harmonic is proportional to $J_N(z)\,\mathrm{e}^{-\mathrm{i}\,N\psi}$, which satisfies the corresponding Bessel equation. □

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PROPOSITION 5 (Outer differential recurrence). *The derivative with respect to the outer Larmor radius satisfies*

$$\partial_z G^{(r)}_{\mu,\nu}(z,\psi;w,\chi) = \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}\psi}\, G^{(r)}_{\mu+1,\nu}(z,\psi;w,\chi) - \mathrm{e}^{\mathrm{i}\psi}\, G^{(r)}_{\mu-1,\nu}(z,\psi;w,\chi)\right]. \tag{7.10}$$

Proof. Use the superposition formula and the corresponding one-variable recurrence. □

PROPOSITION 6 (Inner Turkin recurrence). *The contracted inner coefficients satisfy*

$$(\nu+\ell)T_\ell(w,\nu) + \frac{w}{2}\left[T_{\ell-1}(w,\nu) + T_{\ell+1}(w,\nu)\right] = \delta_{\ell,0}. \tag{7.11}$$

Consequently,

$$\begin{aligned}(\mathrm{i}\,\partial_\chi + \nu)G^{(r)}_{\mu,\nu}(z,\psi;w,\chi) + \frac{w}{2}\left[\mathrm{e}^{\mathrm{i}\chi}\, G^{(r+1)}_{\mu,\nu}(z,\psi;w,\chi) + \mathrm{e}^{-\mathrm{i}\chi}\, G^{(r-1)}_{\mu,\nu}(z,\psi;w,\chi)\right]\\ = G_{\mu-r}(z,\psi).\end{aligned} \tag{7.12}$$

Proof. The dressed factor satisfies

$$(\mathrm{i}\,\partial_\chi + \nu + w\cos\chi)B_\nu(w,\chi) = 1. \tag{7.13}$$

Comparing Fourier coefficients gives the recurrence for T_ℓ. The bivariate identity follows by multiplying by $\mathrm{e}^{-\mathrm{i}\ell\chi}\, G_{\mu+\ell-r}(z,\psi)$ and summing over ℓ. □

PROPOSITION 7 (Phase insertions). *For every $s \in \mathbb{Z}$,*

$$\begin{aligned}&\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}z\sin(\psi+\lambda)}\,\mathrm{e}^{\mathrm{i}w\sin(\chi+\lambda)}\,\mathrm{e}^{-\mathrm{i}\mu\lambda}\,\mathrm{e}^{\mathrm{i}r\lambda}\,\mathrm{e}^{\mathrm{i}s(\chi+\lambda)}\, G_\nu(w,\chi+\lambda)\,\mathrm{d}\lambda\\ &\quad = \mathrm{e}^{\mathrm{i}s\chi}\, G^{(r+s)}_{\mu,\nu}(z,\psi;w,\chi).\end{aligned} \tag{7.14}$$

Proof. The factor $\mathrm{e}^{\mathrm{i}s(\chi+\lambda)}$ gives $\mathrm{e}^{\mathrm{i}s\chi}$ and shifts r to $r+s$. □

PROPOSITION 8 (Inner derivative rules). *The derivatives with respect to the inner variables act on the full dressed inner orbit factor:*

$$\partial_w G^{(r)}_{\mu,\nu} = \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}z\sin(\psi+\lambda)}\,\mathrm{e}^{-\mathrm{i}\mu\lambda}\,\mathrm{e}^{\mathrm{i}r\lambda}\,\partial_w\left[\mathrm{e}^{\mathrm{i}w\sin(\chi+\lambda)}\, G_\nu(w,\chi+\lambda)\right]\mathrm{d}\lambda, \tag{7.15}$$

$$\partial_\chi G^{(r)}_{\mu,\nu} = \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}z\sin(\psi+\lambda)}\,\mathrm{e}^{-\mathrm{i}\mu\lambda}\,\mathrm{e}^{\mathrm{i}r\lambda}\,\partial_\chi\left[\mathrm{e}^{\mathrm{i}w\sin(\chi+\lambda)}\, G_\nu(w,\chi+\lambda)\right]\mathrm{d}\lambda \tag{7.16}$$

and

$$\partial_\nu G^{(r)}_{\mu,\nu} = \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}z\sin(\psi+\lambda)}\,\mathrm{e}^{\mathrm{i}w\sin(\chi+\lambda)}\,\mathrm{e}^{-\mathrm{i}\mu\lambda}\,\mathrm{e}^{\mathrm{i}r\lambda}\,\partial_\nu G_\nu(w,\chi+\lambda)\,\mathrm{d}\lambda. \tag{7.17}$$

PROPOSITION 9 (Double-characteristic representation). *The bivariate function can also be written as*

$$\begin{aligned}&G^{(r)}_{\mu,\nu}(z,\psi;w,\chi)\\ &\quad = -\int_0^\infty\int_0^\infty \mathrm{e}^{-\mathrm{i}z\sin(\psi+\lambda)}\,\mathrm{e}^{\mathrm{i}w\sin(\chi+\lambda)}\,\mathrm{e}^{-\mathrm{i}w\sin(\chi+\lambda+\sigma)}\,\mathrm{e}^{-\mathrm{i}(\mu-r)\lambda}\,\mathrm{e}^{-\mathrm{i}\nu\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\lambda.\end{aligned} \tag{7.18}$$

Proof. Insert the one-variable characteristic representation of G_ν into the definition of $G^{(r)}_{\mu,\nu}$. □

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PROPOSITION 10 (Degenerate one-orbit limit). *At zero inner Larmor radius one has*

$$G^{(r)}_{\mu,\nu}(z,\psi;0,\chi)=\frac{1}{\nu}G_{\mu-r}(z,\psi). \tag{7.19}$$

Proof. For $w=0$, the inner dressed factor reduces to $1/\nu$, and the defining integral becomes the one-variable incomplete-Bessel function with detuning $\mu-r$. □

8. Derivation of the double-harmonic formula

We provide in the following a classical derivation of the double-harmonic formula for the nonlinear response, obtained by Fourier expanding the cyclic gyrophase-depending components before performing the orbit integral. The result is a generalisation of the classical Liu–Tripathi formula (Liu & Tripathi 1986) and will be used in the following for a consistency check of the correctness of the bivariate representation.

We start from the harmonic form of the inner linear response (Ricci 2026):

$$f^{(1)}_{\boldsymbol{q}}(\boldsymbol{v})=\mathrm{e}^{\mathrm{i}\,z_{\boldsymbol{q}}\sin\psi_{\boldsymbol{q}}}\sum_{M=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,M\psi_{\boldsymbol{q}}}\,\tilde{f}^{(1)}_{\boldsymbol{q},M}. \tag{8.1}$$

Equivalently, for each elementary block $K_{\boldsymbol{q},a}X_{\boldsymbol{q},a}$, one may write

$$K_{\boldsymbol{q},a}X_{\boldsymbol{q},a}=\sum_{M=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,M\psi_{\boldsymbol{q}}}\,K_{\boldsymbol{q},a,M}X_{\boldsymbol{q},a}. \tag{8.2}$$

Since $\partial_\phi=\partial_{\psi_{\boldsymbol{q}}}$ on the inner response, the action of the operator $\mathcal{P}_{\boldsymbol{p},b}$ in the form of (6.6) on the Mth inner harmonic is

$$\begin{aligned}\mathcal{P}_{\boldsymbol{p},b}\left(\mathrm{e}^{-\mathrm{i}\,M\psi_{\boldsymbol{q}}}\,K_{\boldsymbol{q},a,M}X_{\boldsymbol{q},a}\right)=\mathrm{e}^{\mathrm{i}\,b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}\,\mathrm{e}^{-\mathrm{i}\,M\psi_{\boldsymbol{q}}}\big[&-\mathrm{i}\,MC^{(\phi)}_{\boldsymbol{p},c}+C^{(\perp)}_{\boldsymbol{p},c}\partial_{v_\perp}\\&+C^{(\parallel)}_{\boldsymbol{p},c}\partial_{v_\parallel}\big](K_{\boldsymbol{q},a,M}X_{\boldsymbol{q},a}).\end{aligned} \tag{8.3}$$

This is the essential reduction. The gyrophase derivative in the bivariate formula becomes multiplication by $-\mathrm{i}\,M$ in the harmonic formula.

Along the outer orbit,

$$\psi_{\boldsymbol{q}}\mapsto\psi_{\boldsymbol{q}}+\lambda, \tag{8.4}$$

and the outer block labelled by b contributes the phase

$$\mathrm{e}^{\mathrm{i}\,b\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\,b\lambda}\,\mathrm{e}^{-\mathrm{i}\,M(\psi_{\boldsymbol{q}}+\lambda)}=\mathrm{e}^{\mathrm{i}\,b\psi_{\boldsymbol{k}}}\,\mathrm{e}^{-\mathrm{i}\,M\psi_{\boldsymbol{q}}}\,\mathrm{e}^{-\mathrm{i}(M-b)\lambda}. \tag{8.5}$$

The remaining outer characteristic integral therefore selects the harmonic denominator with index $N=M-b$, or equivalently the Bessel factor $J_{N-M+b}(z_{\boldsymbol{p}})$ in the standard notation. Thus the three values of b give the three blocks:

$$\begin{aligned}b=1&\Longrightarrow J_{N-M+1}(z_{\boldsymbol{p}}),\\b=-1&\Longrightarrow J_{N-M-1}(z_{\boldsymbol{p}}),\\b=0&\Longrightarrow J_{N-M}(z_{\boldsymbol{p}}).\end{aligned} \tag{8.6}$$

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It remains only to evaluate the three coefficient combinations. For $b=1$ one has

$$-\mathrm{i}\,MC^{(\phi)}_{\boldsymbol{p},1}+C^{(\perp)}_{\boldsymbol{p},1}\partial_{v_\perp}+C^{(\parallel)}_{\boldsymbol{p},1}\partial_{v_\parallel}=\frac{\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}}E_{\boldsymbol{p},1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}}-M\frac{\omega\mu_{\boldsymbol{p}}}{v_\perp}\right)+\frac{p_\perp E_{\boldsymbol{p},0}}{2}\left(\mathcal{G}+M\frac{v_\parallel}{v_\perp}\right). \tag{8.7}$$

For $b=-1$,

$$\begin{aligned}-\mathrm{i}\,MC^{(\phi)}_{\boldsymbol{p},-1}+C^{(\perp)}_{\boldsymbol{p},-1}\partial_{v_\perp}+C^{(\parallel)}_{\boldsymbol{p},-1}\partial_{v_\parallel}=&\frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}}E_{\boldsymbol{p},-1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}}+M\frac{\omega\mu_{\boldsymbol{p}}}{v_\perp}\right)\\&+\frac{p_\perp E_{\boldsymbol{p},0}}{2}\left(\mathcal{G}-M\frac{v_\parallel}{v_\perp}\right).\end{aligned} \tag{8.8}$$

Finally, for $b=0$,

$$-\mathrm{i}\,MC^{(\phi)}_{\boldsymbol{p},0}+C^{(\perp)}_{\boldsymbol{p},0}\partial_{v_\perp}+C^{(\parallel)}_{\boldsymbol{p},0}\partial_{v_\parallel}=M\frac{p_\perp}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}\,E_{\boldsymbol{p},-1}-\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}\,E_{\boldsymbol{p},1}\right)+\Omega_{\boldsymbol{p}}E_{\boldsymbol{p},0}\partial_{v_\parallel}. \tag{8.9}$$

Combining these three cases gives

$$f^{(2)}_{\boldsymbol{k}}(\boldsymbol{v})=\mathrm{e}^{\mathrm{i}\,z_{\boldsymbol{k}}\sin\psi_{\boldsymbol{k}}}\sum_{N=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,N\psi_{\boldsymbol{k}}}\,\tilde{f}^{(2)}_{\boldsymbol{k},N}, \tag{8.10}$$

with

$$\begin{aligned}\tilde{f}^{(2)}_{\boldsymbol{k},N}=&\,\mathrm{i}\,\frac{q}{m}\frac{1}{N\omega-(\Omega_{\boldsymbol{k}}-k_\parallel v_\parallel)}\frac{1}{V}{\sum_{\boldsymbol{p}}}'{\sum_{\boldsymbol{q}}}'\delta_{\boldsymbol{q},\,\boldsymbol{k}-\boldsymbol{p}}\frac{1}{\Omega_{\boldsymbol{p}}}\\&\sum_{M=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\left[N\phi_{\boldsymbol{k}}-(N-M)\phi_{\boldsymbol{p}}-M\phi_{\boldsymbol{q}}\right]}\,\mathcal{L}^{(\boldsymbol{p})}_{NM}\tilde{f}^{(1)}_{\boldsymbol{q},M},\end{aligned} \tag{8.11}$$

where

$$\mathcal{L}^{(\boldsymbol{p})}_{NM}=\mathcal{L}^{(\boldsymbol{p})}_{NM,1}+\mathcal{L}^{(\boldsymbol{p})}_{NM,-1}+\mathcal{L}^{(\boldsymbol{p})}_{NM,0} \tag{8.12}$$

and

$$\mathcal{L}^{(\boldsymbol{p})}_{NM,1}=J_{N-M+1}(z_{\boldsymbol{p}})\left[\frac{\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}}E_{\boldsymbol{p},1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}}-M\frac{\omega\mu_{\boldsymbol{p}}}{v_\perp}\right)+\frac{p_\perp E_{\boldsymbol{p},0}}{2}\left(\mathcal{G}+M\frac{v_\parallel}{v_\perp}\right)\right], \tag{8.13}$$

$$\mathcal{L}^{(\boldsymbol{p})}_{NM,-1}=J_{N-M-1}(z_{\boldsymbol{p}})\left[\frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}}E_{\boldsymbol{p},-1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}}+M\frac{\omega\mu_{\boldsymbol{p}}}{v_\perp}\right)+\frac{p_\perp E_{\boldsymbol{p},0}}{2}\left(\mathcal{G}-M\frac{v_\parallel}{v_\perp}\right)\right], \tag{8.14}$$

$$\mathcal{L}^{(\boldsymbol{p})}_{NM,0}=J_{N-M}(z_{\boldsymbol{p}})\left[\Omega_{\boldsymbol{p}}E_{\boldsymbol{p},0}\partial_{v_\parallel}+M\frac{p_\perp}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}\,E_{\boldsymbol{p},-1}-\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}\,E_{\boldsymbol{p},1}\right)\right]. \tag{8.15}$$

This is the classical double-harmonic formula. In the next section we show that this formula can be directly obtained from (6.24) using the properties of the bivariate incomplete-Bessel functions discussed in § 7.

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9. Recovery of the traditional double-harmonic formula

We now verify that the bivariate response reproduces the double-harmonic expression. The point of the verification is to start from (6.24), use only the identities of § 7, and recover the formula independently obtained in § 8.

We first use the superposition property of the bivariate function, equation (7.5). With $\nu = \mu_{\boldsymbol{q}} - a$, $r = a + b$ and $\ell = m + a$, it gives

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}\, G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}; z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\psi_{\boldsymbol{q}}}\, T_{m+a}\left(z_{\boldsymbol{q}}, \mu_{\boldsymbol{q}} - a\right) G_{\mu_{\boldsymbol{k}}+m-b}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}). \tag{9.1}$$

The differentiated kernels are expanded in the same way:

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}(\partial_{\psi_{\boldsymbol{q}}} + \mathrm{i}\,a) G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\psi_{\boldsymbol{q}}}(-\mathrm{i}\,m) T_{m+a}(z_{\boldsymbol{q}}, \mu_{\boldsymbol{q}} - a) G_{\mu_{\boldsymbol{k}}+m-b}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}), \tag{9.2}$$

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}\, \partial_{z_{\boldsymbol{q}}} G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\psi_{\boldsymbol{q}}}\, \partial_{z_{\boldsymbol{q}}} T_{m+a}(z_{\boldsymbol{q}}, \mu_{\boldsymbol{q}} - a) G_{\mu_{\boldsymbol{k}}+m-b}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}) \tag{9.3}$$

and

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}\, \partial_{\nu} G^{(a+b)}_{\mu_{\boldsymbol{k}},\nu}\Big|_{\nu=\mu_{\boldsymbol{q}}-a} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\psi_{\boldsymbol{q}}}\, \partial_{\mu_{\boldsymbol{q}}} T_{m+a}\left(z_{\boldsymbol{q}}, \mu_{\boldsymbol{q}} - a\right) G_{\mu_{\boldsymbol{k}}+m-b}\left(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}\right). \tag{9.4}$$

Thus the sum over a in (6.24) reconstructs the mth harmonic of the full inner linear response. More explicitly, if

$$Y_{\boldsymbol{q},m} := \sum_{a\in\mathcal{I}} T_{m+a}(z_{\boldsymbol{q}}, \mu_{\boldsymbol{q}} - a) X_{\boldsymbol{q},a}, \tag{9.5}$$

then

$$f^{(1)}_{\boldsymbol{q},m} = \frac{\mathrm{i}\,e}{m\omega} Y_{\boldsymbol{q},m}. \tag{9.6}$$

The differentiated bivariate blocks reconstruct the corresponding derivatives of $Y_{\boldsymbol{q},m}$. Consequently the bracket in (6.24) becomes

$$G_{\mu_{\boldsymbol{k}}+m-b}\,(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}})\, \mathcal{D}^{(m)}_{\boldsymbol{p},b} Y_{\boldsymbol{q},m}, \tag{9.7}$$

where

$$\mathcal{D}^{(m)}_{\boldsymbol{p},b} := -\mathrm{i}\,m C^{(\phi)}_{\boldsymbol{p},b} + C^{(\perp)}_{\boldsymbol{p},b}\partial_{v_\perp} + C^{(\parallel)}_{\boldsymbol{p},b}\partial_{v_\parallel}. \tag{9.8}$$

Using $Y_{\boldsymbol{q},m} = m\omega f^{(1)}_{\boldsymbol{q},m}/(\mathrm{i}\,e)$, the reduced bivariate response becomes

$$\tilde{f}^{(2)}_{\boldsymbol{k}} = \frac{\mathrm{i}\,e}{m\omega}\frac{1}{V}{\sum_{\boldsymbol{p}}}'{\sum_{\boldsymbol{q}}}'\delta_{\boldsymbol{q},\,\boldsymbol{k}-\boldsymbol{p}}\frac{1}{\Omega_{\boldsymbol{p}}}$$
$$\sum_{b\in\mathcal{I}}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}b\psi_{\boldsymbol{k}}}\, \mathrm{e}^{\mathrm{i}b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}\, \mathrm{e}^{-\mathrm{i}m\psi_{\boldsymbol{q}}}\, G_{\mu_{\boldsymbol{k}}+m-b}\,(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}})\, \mathcal{D}^{(m)}_{\boldsymbol{p},b} f^{(1)}_{\boldsymbol{q},m}. \tag{9.9}$$

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Since

$$\psi_{\boldsymbol{q}} = \psi_{\boldsymbol{k}} + \phi_{\boldsymbol{k}} - \phi_{\boldsymbol{q}}, \tag{9.10}$$

the phase factor satisfies

$$\mathrm{e}^{\mathrm{i}\, b\psi_{\boldsymbol{k}}}\, \mathrm{e}^{-\mathrm{i}\, m\psi_{\boldsymbol{q}}} = \mathrm{e}^{-\mathrm{i}(m-b)\psi_{\boldsymbol{k}}}\, \mathrm{e}^{-\mathrm{i}\, m(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{q}})}\,. \tag{9.11}$$

We now expand the remaining one-variable incomplete-Bessel function:

$$G_{\mu_{\boldsymbol{k}}+m-b}\,(z_{\boldsymbol{k}},\, \psi_{\boldsymbol{k}}) = \sum_{P=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\, P\psi_{\boldsymbol{k}}}\, \frac{J_P(z_{\boldsymbol{k}})}{P + \mu_{\boldsymbol{k}} + m - b}. \tag{9.12}$$

Setting

$$N := P + m - b, \tag{9.13}$$

or equivalently $P = N - m + b$, gives

$$P + \mu_{\boldsymbol{k}} + m - b = N + \mu_{\boldsymbol{k}}. \tag{9.14}$$

Therefore,

$$\frac{1}{P + \mu_{\boldsymbol{k}} + m - b} = \frac{\omega}{N\omega - \left(\Omega_{\boldsymbol{k}} - k_{\parallel} v_{\parallel}\right)}. \tag{9.15}$$

At this stage the bivariate formula gives the intermediate harmonic form

$$\tilde{f}^{(2)}_{\boldsymbol{k},N} = \mathrm{i}\, \frac{e}{m} \frac{1}{N\omega - \left(\Omega_{\boldsymbol{k}} - k_{\parallel} v_{\parallel}\right)} \frac{1}{V} {\sum_{\boldsymbol{p}}}' {\sum_{\boldsymbol{q}}}' \delta_{\boldsymbol{q},\, \boldsymbol{k}-\boldsymbol{p}} \frac{1}{\Omega_{\boldsymbol{p}}}$$
$$\sum_{b\in\mathcal{I}} \sum_{m=-\infty}^{\infty} J_{N-m+b}(z_{\boldsymbol{k}})\, \mathrm{e}^{-\mathrm{i}\, m(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{q}})}\, \mathrm{e}^{\mathrm{i}\, b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}\, \mathcal{D}^{(m)}_{\boldsymbol{p},b} f^{(1)}_{\boldsymbol{q},m}. \tag{9.16}$$

This formula is already the harmonic expansion of the bivariate response. It remains only to rewrite the full inner harmonic $f^{(1)}_{\boldsymbol{q},m}$ in terms of the reduced inner harmonic coefficients used in the traditional formula. By definition,

$$f^{(1)}_{\boldsymbol{q}} = \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{q}} \sin\psi_{\boldsymbol{q}}} \sum_{M=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\, M\psi_{\boldsymbol{q}}}\, \tilde{f}^{(1)}_{\boldsymbol{q},M}, \tag{9.17}$$

and therefore

$$f^{(1)}_{\boldsymbol{q},m} = \sum_{M=-\infty}^{\infty} J_{M-m}(z_{\boldsymbol{q}})\, \tilde{f}^{(1)}_{\boldsymbol{q},M}. \tag{9.18}$$

The differential operator $\mathcal{D}^{(m)}_{\boldsymbol{p},b}$ acts on the product $J_{M-m}(z_{\boldsymbol{q}})\, \tilde{f}^{(1)}_{\boldsymbol{q},M}$. The only identities needed to move the derivatives through the Bessel factor are

$$\Omega_{\boldsymbol{p}} \mathcal{G}_{\boldsymbol{p}} J_{M-m}(z_{\boldsymbol{q}}) = -\mu_{\boldsymbol{p}} q_{\perp} J'_{M-m}(z_{\boldsymbol{q}}), \tag{9.19}$$

$$\mathcal{G} J_{M-m}(z_{\boldsymbol{q}}) = \frac{v_{\parallel} q_{\perp}}{\omega} J'_{M-m}(z_{\boldsymbol{q}}) \tag{9.20}$$

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and

$$J_s'(z_{\boldsymbol{q}}) = \frac{1}{2}\left[J_{s-1}(z_{\boldsymbol{q}}) - J_{s+1}(z_{\boldsymbol{q}})\right]. \tag{9.21}$$

After these elementary reductions, all terms contain sums of the form

$$\sum_{m=-\infty}^{\infty} J_{N-m+b}(z_{\boldsymbol{k}})\, J_{M-m}(z_{\boldsymbol{q}})\, \mathrm{e}^{-\mathrm{i}\, m(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{q}})}\, \mathrm{e}^{\mathrm{i}\, b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})}. \tag{9.22}$$

The remaining sum over the intermediate harmonic m is evaluated by Graf's addition theorem. With the phase convention used here and $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$, this theorem may be written as

$$\mathrm{e}^{\mathrm{i}\, l(\phi_{\boldsymbol{p}}-\phi_{\boldsymbol{k}})}\, J_l(z_{\boldsymbol{p}}) = \sum_{h=-\infty}^{\infty} J_{l+h}(z_{\boldsymbol{k}})\, J_h(z_{\boldsymbol{q}})\, \mathrm{e}^{\mathrm{i}\, h(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{q}})}; \tag{9.23}$$

see, for instance, DLMF (equation (10.23.7)). Taking

$$l = N - M + b, \qquad h = M - m, \tag{9.24}$$

one obtains

$$\sum_{m=-\infty}^{\infty} J_{N-m+b}(z_{\boldsymbol{k}})\, J_{M-m}(z_{\boldsymbol{q}})\, \mathrm{e}^{-\mathrm{i}\, m(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{q}})}\, \mathrm{e}^{\mathrm{i}\, b(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})} = J_{N-M+b}(z_{\boldsymbol{p}})\, \mathrm{e}^{-\mathrm{i}\left[N\phi_{\boldsymbol{k}}-(N-M)\phi_{\boldsymbol{p}}-M\phi_{\boldsymbol{q}}\right]}. \tag{9.25}$$

This is the precise point at which the $z_{\boldsymbol{p}}$ Bessel factor of the classical formula appears. It is produced by the Graf contraction over the intermediate full harmonic m.

Evaluating the three possible values of b now gives the three blocks of the classical formula. For $b = 1$,

$$\mathcal{L}^{(\boldsymbol{p})}_{NM,1} = J_{N-M+1}(z_{\boldsymbol{p}})\left[\frac{\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}} E_{\boldsymbol{p},1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}} - M\frac{\omega\mu_{\boldsymbol{p}}}{v_\perp}\right) + \frac{p_\perp E_{\boldsymbol{p},0}}{2}\left(\mathcal{G} + M\frac{v_\parallel}{v_\perp}\right)\right]. \tag{9.26}$$

For $b = -1$,

$$\mathcal{L}^{(\boldsymbol{p})}_{NM,-1} = J_{N-M-1}(z_{\boldsymbol{p}})\left[\frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}}{\sqrt{2}} E_{\boldsymbol{p},-1}\left(\Omega_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}} + M\frac{\omega\mu_{\boldsymbol{p}}}{v_\perp}\right) + \frac{p_\perp E_{\boldsymbol{p},0}}{2}\left(\mathcal{G} - M\frac{v_\parallel}{v_\perp}\right)\right]. \tag{9.27}$$

For $b = 0$,

$$\mathcal{L}^{(\boldsymbol{p})}_{NM,0} = J_{N-M}(z_{\boldsymbol{p}})\left[\Omega_{\boldsymbol{p}} E_{\boldsymbol{p},0}\partial_{v_\parallel} + M\frac{p_\perp}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{p}}}\, E_{\boldsymbol{p},-1} - \mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{p}}}\, E_{\boldsymbol{p},1}\right)\right]. \tag{9.28}$$

Thus,

$$\mathcal{L}^{(\boldsymbol{p})}_{NM} = \mathcal{L}^{(\boldsymbol{p})}_{NM,1} + \mathcal{L}^{(\boldsymbol{p})}_{NM,-1} + \mathcal{L}^{(\boldsymbol{p})}_{NM,0}. \tag{9.29}$$

Substitution gives

$$f^{(2)}_{\boldsymbol{k}}(\boldsymbol{v}) = \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{k}} \sin\psi_{\boldsymbol{k}}} \sum_{N=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\, N\psi_{\boldsymbol{k}}}\, \tilde{f}^{(2)}_{\boldsymbol{k},N}, \tag{9.30}$$

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with

$$\tilde{f}^{(2)}_{k,N} = \mathrm{i}\,\frac{e}{m}\,\frac{1}{N\omega - \left(\Omega_{\boldsymbol{k}} - k_{\parallel} v_{\parallel}\right)}\,\frac{1}{V}\,{\sum_{\boldsymbol{p}}}'\,{\sum_{\boldsymbol{q}}}'\,\delta_{\boldsymbol{q},\,\boldsymbol{k}-\boldsymbol{p}}\,\frac{1}{\Omega_{\boldsymbol{p}}}$$
$$\sum_{M=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\left[N\phi_{\boldsymbol{k}} - (N-M)\phi_{\boldsymbol{p}} - M\phi_{\boldsymbol{q}}\right]}\,\mathcal{L}^{(\boldsymbol{p})}_{NM}\,\tilde{f}^{(1)}_{\boldsymbol{q},M}. \tag{9.31}$$

This is exactly the traditional double-harmonic formula derived independently in § 8. The proof shows explicitly that the bivariate formula is not a different approximation: it is the same harmonic response before the final expansion of the inner full harmonic and the Graf contraction over the intermediate index m.

10. Projected kernels and nonlinear susceptibility

The bivariate representation obtained above gives a compact expression for the second-order distribution function. The nonlinear susceptibility tensor is obtained by projecting this distribution on the velocity moments entering the current. In this section we perform this projection without returning to the double-harmonic expansion. The result is a susceptibility written directly in terms of projected bivariate kernels. This form keeps the orbit structure of the nonlinear response explicit and isolates the angular contractions into a small number of reusable special-function blocks.

The second-order nonlinear plasma susceptibility is the third-rank tensor implicitly defined by the relation (Sitenko 1982)

$$\frac{1}{V}\,{\sum_{\boldsymbol{p}}}'\,{\sum_{\boldsymbol{q}}}'\,\delta_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{p}}\,\chi^{(2)}_{ijk}(\boldsymbol{p},\,\Omega_{\boldsymbol{p}};\,\boldsymbol{q},\,\Omega_{\boldsymbol{q}})\,E_{\boldsymbol{p},j}\,E_{\boldsymbol{q},k} = \frac{\mathrm{i}}{\epsilon_0\Omega_{\boldsymbol{k}}}\,j^{(2)}_{\boldsymbol{k},i}, \tag{10.1}$$

where Einstein's convention is adopted for repeated indices. The determination of the nonlinear susceptibility requires the calculation of the second-order total nonlinear current density.

For a single species the nonlinear current is

$$\boldsymbol{j}^{(2)}_{\boldsymbol{k}} = e\int \boldsymbol{v}\,f^{(2)}_{\boldsymbol{k}}(\boldsymbol{v})\,\mathrm{d}^3\,v. \tag{10.2}$$

Since

$$\boldsymbol{v} = \frac{v_{\perp}}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\,\phi}\,\boldsymbol{e}_L + \mathrm{e}^{\mathrm{i}\,\phi}\,\boldsymbol{e}_R\right) + v_{\parallel}\boldsymbol{b}, \tag{10.3}$$

the angular integral selects only fixed gyrophase Fourier components of the full distribution. Thus the susceptibility calculation only requires angular projections of the bivariate kernels and of their derivatives.

Let

$$\Delta := \phi_{\boldsymbol{k}} - \phi_{\boldsymbol{q}}. \tag{10.4}$$

Along the output angular integration one has

$$\psi_{\boldsymbol{q}} = \psi_{\boldsymbol{k}} - \Delta. \tag{10.5}$$

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For the elementary block labelled by $a \in \mathcal{I}$ and $b \in \mathcal{I}$, we define the projected bivariate kernel

$$\mathcal{K}_N^{(r,a)}(z_{\boldsymbol{k}}, z_{\boldsymbol{q}}, \Delta; \mu_{\boldsymbol{k}}, \nu) := \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\, N\psi_{\boldsymbol{k}}}\, \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{k}} \sin\psi_{\boldsymbol{k}}}\, \mathrm{e}^{\mathrm{i}\, a(\psi_{\boldsymbol{k}}-\Delta)}\, G^{(r)}_{\mu_{\boldsymbol{k}},\nu}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}; z_{\boldsymbol{q}}, \psi_{\boldsymbol{k}} - \Delta)\, \mathrm{d}\,\psi_{\boldsymbol{k}}. \tag{10.6}$$

This is the angular Fourier projection of exactly the factor that appears in the current moment. In the calculation of the susceptibility we have specifically

$$r = a + b, \qquad \nu = \mu_{\boldsymbol{q}} - a. \tag{10.7}$$

The integer N is the output harmonic selected by the current projection. With the convention used in the definition of $\mathcal{K}_N^{(r,a)}$, the angular factors coming from the velocity projection are $N_1 = -1$, $N_{-1} = 1$ and $N_0 = 0$. Equivalently, the left and right current components select the -1 and $+1$ harmonics of the full distribution, respectively.

Using the superposition formula

$$G^{(r)}_{\mu,\nu}(z, \psi; w, \chi) = \sum_{\ell=-\infty}^{\infty} T_\ell(w, \nu)\, \mathrm{e}^{-\mathrm{i}\,\ell\chi}\, G_{\mu+\ell-r}(z, \psi)\,, \tag{10.8}$$

one obtains the coefficient representation

$$\mathcal{K}_N^{(r,a)}(z_{\boldsymbol{k}}, z_{\boldsymbol{q}}, \Delta; \mu_{\boldsymbol{k}}, \nu) = \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\ell-a)\Delta}\, T_\ell(z_{\boldsymbol{q}}, \nu) T_{N+a-\ell}(z_{\boldsymbol{k}}, \mu_{\boldsymbol{k}} + \ell - r)\,, \tag{10.9}$$

where the one-variable projected coefficient is

$$T_M(z, \eta) := \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\, M\psi}\, \mathrm{e}^{\mathrm{i}\, z \sin\psi}\, G_\eta(z, \psi)\, \mathrm{d}\,\psi. \tag{10.10}$$

Equivalently, after setting $n = \ell - a$,

$$\mathcal{K}_N^{(r,a)}(z_{\boldsymbol{k}}, z_{\boldsymbol{q}}, \Delta; \mu_{\boldsymbol{k}}, \nu) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\, n\Delta}\, T_{n+a}(z_{\boldsymbol{q}}, \nu) T_{N-n}(z_{\boldsymbol{k}}, \mu_{\boldsymbol{k}} + n + a - r)\,. \tag{10.11}$$

This is the projected analogue of the bivariate function. It is the object that replaces the full two-angle kernel in the current moment.

The derivative blocks appearing in the nonlinear distribution have equally simple projections. Since

$$\psi_{\boldsymbol{q}} = \psi_{\boldsymbol{k}} - \Delta, \tag{10.12}$$

differentiation with respect to the inner gyrophase at fixed $\psi_{\boldsymbol{k}}$ is equivalent to

$$\partial_{\psi_{\boldsymbol{q}}} = -\partial_\Delta. \tag{10.13}$$

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Moreover,

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}\left(\partial_{\psi_{\boldsymbol{q}}}+\mathrm{i}\,a\right)G^{(r)}_{\mu_{\boldsymbol{k}},\nu}=\partial_{\psi_{\boldsymbol{q}}}\left[\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}G^{(r)}_{\mu_{\boldsymbol{k}},\nu}\right]. \tag{10.14}$$

Therefore its projected form is

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}N\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}z_{\boldsymbol{k}}\sin\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}a(\psi_{\boldsymbol{k}}-\Delta)}\left(\partial_{\psi_{\boldsymbol{q}}}+\mathrm{i}\,a\right)G^{(r)}_{\mu_{\boldsymbol{k}},\nu}(z_{\boldsymbol{k}},\psi_{\boldsymbol{k}};z_{\boldsymbol{q}},\psi_{\boldsymbol{k}}-\Delta)\,\mathrm{d}\,\psi_{\boldsymbol{k}}=-\partial_\Delta\mathcal{K}^{(r,a)}_N. \tag{10.15}$$

Similarly,

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}N\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}z_{\boldsymbol{k}}\sin\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}a(\psi_{\boldsymbol{k}}-\Delta)}\,\partial_{z_{\boldsymbol{q}}}G^{(r)}_{\mu_{\boldsymbol{k}},\nu}(z_{\boldsymbol{k}},\psi_{\boldsymbol{k}};z_{\boldsymbol{q}},\psi_{\boldsymbol{k}}-\Delta)\,\mathrm{d}\,\psi_{\boldsymbol{k}}=\partial_{z_{\boldsymbol{q}}}\mathcal{K}^{(r,a)}_N \tag{10.16}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}N\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}z_{\boldsymbol{k}}\sin\psi_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}a(\psi_{\boldsymbol{k}}-\Delta)}\,\partial_\nu G^{(r)}_{\mu_{\boldsymbol{k}},\nu}(z_{\boldsymbol{k}},\psi_{\boldsymbol{k}};z_{\boldsymbol{q}},\psi_{\boldsymbol{k}}-\Delta)\,\mathrm{d}\,\psi_{\boldsymbol{k}}=\partial_\nu\mathcal{K}^{(r,a)}_N. \tag{10.17}$$

Thus all angular contractions required by the nonlinear current are finite combinations of

$$\mathcal{K}^{(r,a)}_N,\qquad \partial_\Delta\mathcal{K}^{(r,a)}_N,\qquad \partial_{z_{\boldsymbol{q}}}\mathcal{K}^{(r,a)}_N,\qquad \partial_\nu\mathcal{K}^{(r,a)}_N. \tag{10.18}$$

We note that the derivative with respect to the relative angle may also be written in terms of the multiplier operator

$$\mathfrak{D}_\Delta:=-\mathrm{i}\,\partial_\Delta, \tag{10.19}$$

so that

$$\mathfrak{D}_\Delta\mathcal{K}^{(r,a)}_N=\sum_{n=-\infty}^{\infty}n\,\mathrm{e}^{\mathrm{i}n\Delta}\,T_{n+a}(z_{\boldsymbol{q}},\nu)T_{N-n}\,(z_{\boldsymbol{k}},\mu_{\boldsymbol{k}}+n+a-r)\,. \tag{10.20}$$

The bivariate representation gives the nonlinear current by substituting the reduced distribution of § 6 into the current moment and applying the projected kernels above. More explicitly, each occurrence of

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}\,G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}},\psi_{\boldsymbol{k}};z_{\boldsymbol{q}},\psi_{\boldsymbol{q}}) \tag{10.21}$$

is replaced, in the Nth current projection, by

$$\mathcal{K}^{(a+b,a)}_N(z_{\boldsymbol{k}},z_{\boldsymbol{q}},\Delta;\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a). \tag{10.22}$$

Likewise,

$$\mathrm{e}^{\mathrm{i}a\psi_{\boldsymbol{q}}}(\partial_{\psi_{\boldsymbol{q}}}+\mathrm{i}\,a)G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a} \tag{10.23}$$

is replaced by

$$-\partial_\Delta\mathcal{K}^{(a+b,a)}_N(z_{\boldsymbol{k}},z_{\boldsymbol{q}},\Delta;\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a), \tag{10.24}$$

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while the $z_{\boldsymbol{q}}$- and ν-derivative terms are replaced by

$$\partial_{z_{\boldsymbol{q}}} \mathcal{K}_N^{(a+b,a)}(z_{\boldsymbol{k}}, z_{\boldsymbol{q}}, \Delta; \mu_{\boldsymbol{k}}, \mu_{\boldsymbol{q}} - a) \tag{10.25}$$

and

$$\partial_{\nu} \mathcal{K}_N^{(a+b,a)}(z_{\boldsymbol{k}}, z_{\boldsymbol{q}}, \Delta; \mu_{\boldsymbol{k}}, \nu)|_{\nu=\mu_{\boldsymbol{q}}-a}. \tag{10.26}$$

Consequently the nonlinear current is built from the projected family

$$\mathcal{K}_N^{(a+b,a)}(z_{\boldsymbol{k}}, z_{\boldsymbol{q}}, \Delta; \mu_{\boldsymbol{k}}, \mu_{\boldsymbol{q}} - a), \qquad a, b \in \mathcal{I}, \tag{10.27}$$

together with the three projected derivatives

$$\partial_{\Delta}, \qquad \partial_{z_{\boldsymbol{q}}}, \qquad \partial_{\nu}. \tag{10.28}$$

This is exactly parallel to the linear susceptibility calculation. In the linear case the angular contractions are generated by the one-variable incomplete-Bessel function G_{μ}. In the first nonlinear case they are generated by the projected bivariate kernels $\mathcal{K}_N^{(r,a)}$, which encode the outer characteristic propagation and the full inner linear orbit response in a single object.

We now make the current-response formula explicit. We use the circular basis

$$\sigma, \rho \in \{1, -1, 0\} \tag{10.29}$$

and decompose the coefficients of the $\boldsymbol{p}$-field as

$$C_{\boldsymbol{p},b}^{(\alpha)} = \sum_{\sigma \in \{1,-1,0\}} E_{\boldsymbol{p},\sigma} C_{\boldsymbol{p},b;\sigma}^{(\alpha)}, \qquad \alpha \in \{\phi, \perp, \|\}. \tag{10.30}$$

The indices b and σ take values in the same set but have different meanings: b labels the gyrophase block of the nonlinear force operator, whereas σ labels the actual polarisation component of the field $E_{\boldsymbol{p},\sigma}$.

Similarly, the inner amplitudes are written as

$$X_{\boldsymbol{q},a} = \sum_{\rho \in \{1,-1,0\}} E_{\boldsymbol{q},\rho} X_{\boldsymbol{q},a;\rho}. \tag{10.31}$$

The coefficients $C_{\boldsymbol{p},b;\sigma}^{(\alpha)}$ are read off from the definitions of $C_{\boldsymbol{p},c}^{(\alpha)}$, while $X_{\boldsymbol{q},a;\rho}$ are read off from the linear response amplitudes. In the following explicit formulas the species label is suppressed.

Explicitly,

$$C_{\boldsymbol{p},b;1}^{(\phi)} = -\,\mathrm{i}\, b \delta_{b,1} \frac{\mathrm{e}^{\mathrm{i}\, b\phi_{\boldsymbol{p}}}}{\sqrt{2}} \frac{\mu_{\boldsymbol{p}}\omega}{v_{\perp}} + \sum_{c=\pm 1} \delta_{b+c,1} \frac{1-bc}{2} \left[-(1-b^2)\,\mathrm{i}\, p_{\perp} c \sqrt{2}\, \mathrm{e}^{\mathrm{i}\, c\phi_{\boldsymbol{p}}} \right], \tag{10.32}$$

$$C_{\boldsymbol{p},b;-1}^{(\phi)} = -\,\mathrm{i}\, b \delta_{b,-1} \frac{\mathrm{e}^{\mathrm{i}\, b\phi_{\boldsymbol{p}}}}{\sqrt{2}} \frac{\mu_{\boldsymbol{p}}\omega}{v_{\perp}} + \sum_{c=\pm 1} \delta_{b+c,-1} \frac{1-bc}{2} \left[-(1-b^2)\,\mathrm{i}\, p_{\perp} c \sqrt{2}\, \mathrm{e}^{\mathrm{i}\, c\phi_{\boldsymbol{p}}} \right], \tag{10.33}$$

$$C_{\boldsymbol{p},b;0}^{(\phi)} = \sum_{c=\pm 1} \delta_{b+c,0} \frac{1-bc}{2} \left[-\,\mathrm{i}\, c b^2 \frac{p_{\perp} v_{\|}}{2 v_{\perp}} \right], \tag{10.34}$$

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$$C^{(\perp)}_{\boldsymbol{p},b;1} = b^2\delta_{b,1}\frac{\mathrm{e}^{\mathrm{i}\,b\phi_{\boldsymbol{p}}}}{\sqrt{2}}\left(\Omega_{\boldsymbol{p}} - p_\parallel v_\parallel\right), \tag{10.35}$$

$$C^{(\perp)}_{\boldsymbol{p},b;-1} = b^2\delta_{b,-1}\frac{\mathrm{e}^{\mathrm{i}\,b\phi_{\boldsymbol{p}}}}{\sqrt{2}}\left(\Omega_{\boldsymbol{p}} - p_\parallel v_\parallel\right), \tag{10.36}$$

$$C^{(\perp)}_{\boldsymbol{p},b;0} = \sum_{c=\pm1}\delta_{b+c,0}\frac{1-bc}{2}b^2\frac{p_\perp v_\parallel}{2}, \tag{10.37}$$

$$C^{(\parallel)}_{\boldsymbol{p},b;1} = \delta_{b,1}b^2\frac{\mathrm{e}^{\mathrm{i}\,b\phi_{\boldsymbol{p}}}}{\sqrt{2}}p_\parallel v_\perp, \tag{10.38}$$

$$C^{(\parallel)}_{\boldsymbol{p},b;-1} = \delta_{b,-1}b^2\frac{\mathrm{e}^{\mathrm{i}\,b\phi_{\boldsymbol{p}}}}{\sqrt{2}}p_\parallel v_\perp \tag{10.39}$$

and

$$C^{(\parallel)}_{\boldsymbol{p},b;0} = \delta_{b,0}\Omega_{\boldsymbol{p}} - \sum_{c=\pm1}\delta_{b+c,0}\frac{1-bc}{2}b^2\frac{p_\perp v_\perp}{2}. \tag{10.40}$$

The inner amplitudes are

$$X_{\boldsymbol{q},1;1} = \frac{\mathrm{e}^{\mathrm{i}\,\phi_{\boldsymbol{q}}}}{\sqrt{2}}\mathcal{G}_{\boldsymbol{q},\Omega_{\boldsymbol{q}}}f_0, \qquad X_{\boldsymbol{q},1;-1} = 0, \qquad X_{\boldsymbol{q},1;0} = \frac{q_\perp}{2\Omega_{\boldsymbol{q}}}\mathcal{H}f_0, \tag{10.41}$$

$$X_{\boldsymbol{q},-1;1} = 0, \qquad X_{\boldsymbol{q},-1;-1} = \frac{\mathrm{e}^{-\mathrm{i}\,\phi_{\boldsymbol{q}}}}{\sqrt{2}}\mathcal{G}_{\boldsymbol{q},\Omega_{\boldsymbol{q}}}f_0, \qquad X_{\boldsymbol{q},-1;0} = \frac{q_\perp}{2\Omega_{\boldsymbol{q}}}\mathcal{H}f_0 \tag{10.42}$$

and

$$X_{\boldsymbol{q},0;1} = 0, \qquad X_{\boldsymbol{q},0;-1} = 0, \qquad X_{\boldsymbol{q},0;0} = \partial_{v_\parallel}f_0. \tag{10.43}$$

For each projected current component, introduce the selector

$$S_1(v_\perp) := \frac{v_\perp}{\sqrt{2}}, \qquad S_{-1}(v_\perp) := \frac{v_\perp}{\sqrt{2}}, \qquad S_0(v_\parallel) := v_\parallel. \tag{10.44}$$

The corresponding output harmonic indices are

$$N_1 = -1, \qquad N_{-1} = 1, \qquad N_0 = 0. \tag{10.45}$$

Then, after performing the gyrophase projection, the single-species current-response coefficient in the circular basis becomes

$$\begin{aligned}\mathscr{J}^{(2,s)}_{\tau\sigma\rho}(\boldsymbol{k};\boldsymbol{p},\boldsymbol{q}) = -\frac{e_s^3}{m_s^2\omega_s^2}\frac{1}{\Omega_{\boldsymbol{p}}}\sum_{a\in\mathcal{I}}\sum_{b\in\mathcal{I}}\int_0^\infty\int_{-\infty}^\infty S_\tau\Bigg[\Bigg(&C^{(\phi,s)}_{\boldsymbol{p},b;\sigma}\mathbb{K}^{(\phi,s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau} \\ &+ C^{(\perp,s)}_{\boldsymbol{p},b;\sigma}\frac{q_\perp}{\omega_s}\mathbb{K}^{(\perp,s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau} + C^{(\parallel,s)}_{\boldsymbol{p},b;\sigma}\frac{q_\parallel}{\omega_s}\mathbb{K}^{(\parallel,s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau}\Bigg)X^{(s)}_{\boldsymbol{q},a;\rho} \\ &+ \mathbb{K}^{(s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau}\left[C^{(\perp,s)}_{\boldsymbol{p},b;\sigma}\partial_{v_\perp} + C^{(\parallel,s)}_{\boldsymbol{p},b;\sigma}\partial_{v_\parallel}\right]X^{(s)}_{\boldsymbol{q},a;\rho}\Bigg]v_\perp\,\mathrm{d}\,v_\perp\,\mathrm{d}\,v_\parallel,\end{aligned} \tag{10.46}$$

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where $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$ is understood, s identifies the species and

$$\mathbb{K}^{(s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau} := \mathcal{K}^{(a+b,a)}_{N_\tau+b}(z_{\boldsymbol{k},s}, z_{\boldsymbol{q},s}, \Delta; \mu_{\boldsymbol{k},s}, \mu_{\boldsymbol{q},s} - a), \tag{10.47}$$

$$\mathbb{K}^{(\phi,s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau} := -\partial_\Delta \mathcal{K}^{(a+b,a)}_{N_\tau+b}(z_{\boldsymbol{k},s}, z_{\boldsymbol{q},s}, \Delta; \mu_{\boldsymbol{k},s}, \mu_{\boldsymbol{q},s} - a), \tag{10.48}$$

$$\mathbb{K}^{(\perp,s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau} := \partial_{z_{\boldsymbol{q},s}} \mathcal{K}^{(a+b,a)}_{N_\tau+b}(z_{\boldsymbol{k},s}, z_{\boldsymbol{q},s}, \Delta; \mu_{\boldsymbol{k},s}, \mu_{\boldsymbol{q},s} - a), \tag{10.49}$$

$$\mathbb{K}^{(\parallel,s)}_{\boldsymbol{k},\boldsymbol{q},a,b;\tau} := \partial_\nu \mathcal{K}^{(a+b,a)}_{N_\tau+b}(z_{\boldsymbol{k},s}, z_{\boldsymbol{q},s}, \Delta; \mu_{\boldsymbol{k},s}, \nu)|_{\nu=\mu_{\boldsymbol{q},s}-a}. \tag{10.50}$$

The total nonlinear current is the sum of the single-species current coefficients. Hence, comparing with the defining relation for the nonlinear susceptibility, one obtains

$$\chi^{(2)}_{\tau\sigma\rho}(\boldsymbol{p}, \Omega_{\boldsymbol{p}}; \boldsymbol{q}, \Omega_{\boldsymbol{q}}) = \frac{\mathrm{i}}{\epsilon_0 \Omega_{\boldsymbol{k}}} \sum_s \mathscr{J}^{(2,s)}_{\tau\sigma\rho}(\boldsymbol{k}; \boldsymbol{p}, \boldsymbol{q}) \,. \tag{10.51}$$

Here $\tau \in \{1, -1, 0\}$ labels the output current component. This formula gives the nonlinear susceptibility explicitly as a finite sum over the nine elementary bivariate blocks (a, b), followed by a two-dimensional velocity integral. The Cartesian tensor is obtained from $\chi^{(2)}_{\tau\sigma\rho}$ by the fixed change of basis between the circular basis $\{\boldsymbol{e}_1, \boldsymbol{e}_{-1}, \boldsymbol{e}_0\}$ and the Cartesian basis.

11. Outlook towards higher orders

The factorisation

$$f^{(2)}_{\boldsymbol{k}} = \mathrm{e}^{\mathrm{i}\, z_{\boldsymbol{k}} \sin\psi_{\boldsymbol{k}}}\, \tilde{f}^{(2)}_{\boldsymbol{k}} \tag{11.1}$$

has the same form as the linear factorisation. This is not accidental. In the weakly nonlinear hierarchy considered here, the left-hand characteristic operator is the same at every perturbative order; only the source term changes. Consequently, once the source at a given order has been written in terms of orbit kernels, the next order is obtained by applying the same characteristic integral once more.

The linear response is governed by the one-variable incomplete-Bessel kernel

$$G_{\mu_{\boldsymbol{k}}}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}). \tag{11.2}$$

The first nonlinear response is governed by the bivariate kernel

$$G^{(a+b)}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}}-a}(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}; z_{\boldsymbol{q}}, \psi_{\boldsymbol{q}}), \tag{11.3}$$

which couples the outer characteristic of the output mode $\boldsymbol{k}$ to the inner linear orbit associated with the mode $\boldsymbol{q}$. At the next perturbative step, the nonlinear source contains the first nonlinear response itself. Applying the outer characteristic integral to this source therefore produces a trivariate kernel, schematically of the form

$$G^{(\boldsymbol{r})}_{\mu_{\boldsymbol{k}},\mu_{\boldsymbol{q}_1},\mu_{\boldsymbol{q}_2}}\big(z_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}; z_{\boldsymbol{q}_1}, \psi_{\boldsymbol{q}_1}; z_{\boldsymbol{q}_2}, \psi_{\boldsymbol{q}_2}\big), \tag{11.4}$$

with discrete shifts $\boldsymbol{r}$ determined by the circular and parallel polarisation blocks appearing in the nested nonlinear source.

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More generally, within the fixed-characteristic perturbative hierarchy, the nth order distribution is expected to be expressible in terms of n-variate orbit kernels. These kernels are obtained recursively by adding one characteristic time integral for each nonlinear interaction. Their harmonic expansions reproduce the usual multiple cyclotron-harmonic sums, while their projected versions provide the angular contractions needed for the corresponding higher-order susceptibility tensors.

This recursive statement has the standard limitation of weakly nonlinear response theory. It assumes that the nonlinear corrections remain on the right-hand side of the Vlasov equation, so that the unperturbed cyclotron characteristics are not modified. If nonlinear orbit corrections, trapping, ponderomotive drifts or self-consistent changes of the guiding-centre motion are promoted to the left-hand side, then the characteristic operator itself changes and the simple incomplete-Bessel recursion is deformed.

Thus the present construction should be understood as a compact orbit-kernel reformulation of the fixed-characteristic perturbative hierarchy. In this setting the first nonlinear susceptibility is generated by projected bivariate kernels, and higher-order susceptibilities should be generated by projected multivariate kernels. Whether these higher-rank projected kernels admit closed forms as compact as those obtained here depends on the corresponding angular-contraction calculus.

12. Conclusion

We have reorganised the first nonlinear Vlasov–Maxwell response around the characteristic structure of the problem. The linear response is obtained by applying one cyclotron orbit integral to one Larmor phase, producing the incomplete-Bessel function G_{μ}. The first nonlinear response is obtained by applying an outer cyclotron orbit integral to an inner $G_{\mu_{\boldsymbol{q}}}$-resolved linear response. This produces the bivariate incomplete-Bessel function

$$G^{(r)}_{\mu,\nu}(z, \psi; w, \chi). \tag{12.1}$$

The bivariate function is therefore not an auxiliary notation, but the natural orbit-resolvent kernel associated with the first nonlinear cyclotron response.

The resulting nonlinear distribution function is explicit and preserves all physical factors of the original calculation through compact differential blocks. Expanding the bivariate function recovers a generalisation of the traditional double-harmonic Liu–Tripathi expression, while keeping it unexpanded gives a direct route to the nonlinear susceptibility tensor through projected bivariate angular contractions. In this sense the bivariate formulation is equivalent to the harmonic representation, but reorganises it around the geometry of the characteristic flow.

The projected kernels introduced above are also relevant for numerical implementation. In the traditional formulation one has to handle double cyclotron-harmonic sums, together with several shifted Bessel factors and derivative-generated recurrence terms. In the present formulation these structures are absorbed into a small family of projected kernels:

$$\mathcal{K}^{(r,a)}_{N}, \qquad \partial_{\Delta}\mathcal{K}^{(r,a)}_{N}, \qquad \partial_{z_{\boldsymbol{q}}}\mathcal{K}^{(r,a)}_{N}, \qquad \partial_{v}\mathcal{K}^{(r,a)}_{N}. \tag{12.2}$$

These kernels may be evaluated either from their angular-integral definition, from their single-sum coefficient representation in terms of the contracted coefficients T_{ℓ}, or from the original characteristic integral when this is more stable. This flexibility

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is expected to be advantageous in regimes where direct double-harmonic summation is slowly convergent, strongly oscillatory or numerically delicate because of cancellations among shifted harmonics.

The numerical advantage should not be interpreted as a universal reduction of computational cost. Its usefulness depends on the values of the Larmor radii, detunings, damping prescription and truncation strategy. Nevertheless, the bivariate formulation offers a more structured computational target: instead of repeatedly summing and differentiating long harmonic expressions, one can tabulate projected kernels and their derivatives, reuse them across tensor components and exploit their recurrence, resolvent and projection identities. This suggests a practical route towards efficient evaluation of nonlinear susceptibilities in parameter regimes where the harmonic representation is algebraically unwieldy.

The main advantage of the formulation is therefore both structural and potentially computational. Structurally, it identifies the special-function object underlying the first nonlinear cyclotron response. Computationally, it replaces the unorganised double-harmonic expansion by a small set of orbit-projected kernels whose analytic properties can be studied independently and then used systematically in the susceptibility tensor.

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During the preparation of this work the author used ChatGPT in order to double-check the correctness of part of the calculations. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

Declaration of interests

The author reports no conflict of interest.

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