FUZZY SOLUTIONS FOR TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

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ABSTRACT

A new methodology via using an adaptive fuzzy algorithm to obtain solutions of "Two-dimensional Navier-Stokes equations" (2-D NSE) is presented in this investigation. The design objective is to find two fuzzy solutions to satisfy precisely the 2-D NSE frequently encountered in practical applications. In this study, a rough fuzzy solution is formulated with adjustable parameters firstly, and then, a set of adaptive laws for optimally tuning the free parameters in the consequent parts of the proposed fuzzy solutions are derived from minimizing an error cost function which is the square summation of approximation errors of boundary conditions, continuum equation and Navier-Stokes equations. In addition, elegant approximated error bounds between the exact solution and the proposed fuzzy solution with respect to the number of fuzzy rules and solution errors have also been proven. Furthermore, the error equations in mesh points can be proven to converge to zero for the 2-D NSE with two sufficient conditions.

Keywords: Navier-Stokes equations, Adaptive Fuzzy Solutions, Approximation error bounds.

1. INTRODUCTION

Recently, the solution finding problem of the NSE has become one of the main topics of modern mathematical analysis and attracted many attentions. Navier-Stokes equations (NSE) have been widely used in many applications, such as aeronautical engineering [1], ocean engineering, naval mechatronics engineering [2] and biomedical technology [3] hence the solution finding problem of NSE is a very important task and also a challenging issue due to NSE is too nonlinear and very complex. Many methods such as finite-element method [4], finite-difference method [5], finite-volume method [6], sine collocation method [7], boundary element method [8] and global weak solutions [9] are developed for the possibility of obtaining accurate solutions. However, these methods usually need a great amount of mathematical calculation and a lot of computational power. In addition, these methods have realization problems, which have singularities occur within or on the boundaries and also the difficulty to handle semi-infinite and infinite domains. According to these revealed reasons, it is obvious that some of the above-mentioned methods are not good enough for building up numerical solutions and maybe are only applicable to some particular situations for NSE. Furthermore, the accuracy between the computed solution and the exact solution can't easily check over in practice based on the above mentioned methods. In the past decades, fuzzy logic system has been widely used in control designs and it has also been proven to be a very good representation for unknown nonlinear dynamic systems by the common schemes. Theoretically and practically, any nonlinear unknown system can be approximated to any desired accuracy as possible by adopting the approach of the fuzzy approximator. This excellent system identification ability offers us a direction for regarding the solutions of NSE as unknown systems and can be identified by the fuzzy approximator. Based on reasons depicted in the above, this study tries to propose an advanced method which relies on the function approximation capabilities of the fuzzy approximator to find out acceptable and accurate solutions for the complicated NSE. From a series of systematic and mathematical analysis, there are three special and useful features can be summarized for this proposed method. First, the solutions of NSE expressed by the fuzzy approximator are continuously differentiable forms. In the existing literature most other techniques offer the discrete solution or a solution of limited differentiability. Second, a set of adjusting algorithm for searching the best solution parameters are elegantly derived and two sufficient conditions for guaranteeing the convergence of this proposed solution is proven and given. Third, solutions of NSE constructed via using fuzzy approximators with excellent superior interpolation properties.

2. DESCRIPTION OF THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

2.1 Two-Dimensional Navier-Stokes Equations

Considering the continuum, Newtonian and incompressible fluid with constant properties of density and

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viscosity at a constant temperature, the governing equation: Navier-Stokes equations, can be described as below [10]:

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{1a}$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{1b}$$

with boundary conditions (BCs):

$$u\Big|_{\substack{y=a\\x=0}}=0 \tag{1c}$$

$$u\Big|_{\substack{y=-a\\x=0}}=0 \tag{1d}$$

where μ is viscosity parameter of fluid, ρ is water density, $\partial p/\partial x$ and $\partial p/\partial y$ are pressure gradients, respectively.

The continuity equation is expressed as the following:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1e}$$

2.2 Problem Formulation

Suppose a steady flow is considered in this study, *i.e.*, $\lim_{t\to\infty}\frac{\partial u}{\partial t}=0$ and $\lim_{t\to\infty}\frac{\partial v}{\partial t}=0$ in Eq. (1), and denote variables e_1, e_2, e_3, e_4 and e_5 as error equations when rough fuzzy solutions $\tilde{u}$ and $\tilde{v}$ (descriptions of $\tilde{u}$ and $\tilde{v}$ will be introduced later) are given for Eq. (1a) to Eq. (1e) as:

$$e_1=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 \tilde{u}}{\partial x^2}+\frac{\partial^2 \tilde{u}}{\partial y^2}\right] \tag{2a}$$

$$e_2=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 \tilde{v}}{\partial x^2}+\frac{\partial^2 \tilde{v}}{\partial y^2}\right] \tag{2b}$$

$$e_3=\tilde{u}\Big|_{\substack{y=a\\x=0}} \tag{2c}$$

$$e_4=\tilde{u}\Big|_{\substack{y=-a\\x=0}} \tag{2d}$$

$$e_5=\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y} \tag{2e}$$

If exact solutions u and v for NSE in Eqs. (1a), (1b), the BCs in Eqs. (1c), (1d) and the continuity equation in Eq. (1e) can be found analytically, $e_1(\cdot)$, $e_2(\cdot)$, $e_3(\cdot)$, $e_4(\cdot)$ and $e_5(\cdot)$ will be zero automatically. However, it is almost impossible to find exact solutions from NSE with the BCs mentioned above by mathematical analysis or tools. For this reason, we will develop approximated fuzzy solutions $\tilde{u}$ and $\tilde{v}$ to satisfy the NSE in Eqs. (1a), (1b) and the BCs in Eqs. (1c), (1e) as precisely as possible.

Design objective: Consider the NSE in Eqs. (1a), (1b) and the BCs in Eqs. (1c), (1e) and the design objective is to find suitable and approximated fuzzy solutions $\tilde{u}$ and $\tilde{v}$ such that $e_1(\cdot)$, $e_2(\cdot)$, $e_3(\cdot)$, $e_4(\cdot)$ and $e_5(\cdot)$ are as small as possible.

3. FUZZY SOLUTION AND SOLUTION FINDING METHODOLOGY FOR 2-D NSE

3.1 Fuzzy Solution

Intuitively, the unknown solutions $\tilde{u}$ and $\tilde{v}$ of 2-D NSE in Eqs. (1a), (1b) with boundary conditions in Eqs. (1c), (1e) can be regarded as unknown systems and can be excellent approximated via the help of the fuzzy approximator, hence the following solutions will be introduced to approximate unknown solutions $\tilde{u}$ and $\tilde{v}$ of 2-D NSE.

The fuzzy solutions $\tilde{u}$ and $\tilde{v}$ of 2-D NSE with boundary conditions can be described by the following rules:

$$\begin{aligned}&R^l:\ \text{IF } x \text{ is } \mu_i^x \text{ and } y \text{ is } \mu_j^y\\&\text{Then } \tilde{u}(x,y)=\theta_{ij} \text{ and } \tilde{v}(x,y)=\lambda_{ij}\end{aligned} \tag{3}$$

where $R^l (l=1,\ldots,\alpha_x\times\alpha_y)$ are fuzzy rules, x and y are the input variables of the fuzzy proposed solutions $\tilde{u}(x,y)=\theta_{ij}$ and $\tilde{v}(x,y)=\lambda_{ij}$ with respect to the lth rule, and β_i^x $(i=1,\ldots,\alpha_x)$ and β_j^y $(j=1,\ldots,\alpha_y)$ are the membership functions which characterize the ith and jth fuzzy sets defined in the space of the variables of x and y coordinates, respectively. These membership functions in this study are chosen as follows:

$$\beta_i^x=\exp\left[-\frac{1}{2}\left(\frac{x-m_i^x}{\pi_i^x}\right)\right]^2 \tag{4a}$$

$$\beta_j^y=\exp\left[-\frac{1}{2}\left(\frac{y-m_j^y}{\pi_j^y}\right)\right]^2 \tag{4b}$$

where m_i^x, m_j^y, π_i^x and π_l^y are defined as mean values and standard deviations of the membership distributions, respectively. From the mathematical expressions of membership function given in Eq. (4), it is easy to check over that the used membership functions are nonlinear function in C^∞. In this paper, standard deviations of the membership distributions π_i^x and π_l^y in Eq. (4) are arranged as follows:

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$$\pi_i^x = \frac{a-o}{\alpha_x - 1} \tag{5a}$$

$$\pi_j^y = \frac{b-o}{\alpha_y - 1} \tag{5b}$$

where α_x and $\alpha_y \geq 2$. The parameters o, a, and b are defined as the bounds of the analyzed region $\Delta = [o, a] \times [o, b]$.

The proposed fuzzy solutions for velocity components $u(x, y)$ and $v(x, y)$ of 2-D NSE can be formulated via using Eqs. (4) and (5) as

$$\tilde{u}(x, y|\underline{\theta}) = \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\theta_{ij}\,\xi_{ij} = \underline{\theta}^T\underline{\xi} \tag{6a}$$

$$\tilde{v}(x, y|\underline{\lambda}) = \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\lambda_{ij}\,\xi_{ij} = \underline{\lambda}^T\underline{\xi} \tag{6b}$$

where

$$\underline{\theta} = [\theta_{11} \quad \ldots \quad \theta_{\alpha_x\alpha_y}]^T\,,\ \underline{\lambda} = [\lambda_{11} \quad \ldots \quad \lambda_{\alpha_x\alpha_y}]^T$$

and $\xi_{ij} = \dfrac{\beta_i^x\beta_j^y}{\sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\beta_i^x\beta_j^y}$

The physical meaning in Eq. (6) is that the exact solutions $u(x, y)$ and $v(x, y)$ of 2-D NSE are approximated by interpolating $\underline{\theta}$ and $\underline{\lambda}$ at $i, j = (1, 1), \ldots, (\alpha_x, \alpha_y)$ with the fuzzy smooth function ξ_{ij} for $i, j = (1, 1), \ldots, (\alpha_x, \alpha_y)$.

3.2 Solution Finding Methodology

In this section, the approximation error bounds between the above fuzzy solutions Eq. (6) and the exact solutions of NSE will be discussed.

Theorem 1: Let the fuzzy solutions $\tilde{u}(x, y)$ and $\tilde{v}(x, y)$ defined on $\Delta = [o, a] \times [o, b]$ for 2-D NSE be differentiable and continuous functions. Then, the approximation error bounds between the exact solutions $u(x, y)$ and $\tilde{u}(x, y|\underline{\theta})$ and $v(x, y)$ and $\tilde{v}(x, y|\underline{\lambda})$ in Eq. (6) can be described as follows:

$$\left|u(x, y) - \tilde{u}(x, y|\underline{\theta})\right| \leq 0.83452(S_{ux\infty}\Delta_x + S_{uy\infty}\Delta_y) + \max|e_u^{ij}| \tag{7a}$$

$$\left|v(x, y) - \tilde{v}(x, y|\underline{\lambda})\right| \leq 0.83452(S_{vx\infty}\Delta_x + S_{vy\infty}\Delta_y) + \max|e_v^{ij}| \tag{7b}$$

where $\Delta_x = a - o/\alpha_x$ and $\Delta_y = b - o/\alpha_y$ are defined as the distance factors. $e_u^{ij} = u(\alpha_i^x, \alpha_j^y) - \theta_{ij}$ and $e_v^{ij} = v(\alpha_i^x, \alpha_j^y) - \lambda_{ij}$ are called as the solution error factors. In addition, the four factors, $S_{ux\infty} = \sup_{(x,y)\in\Delta}\left|\frac{\partial\tilde{u}}{\partial x}\right|$, $S_{uy\infty} = \sup_{(x,y)\in\Delta}\left|\frac{\partial\tilde{u}}{\partial y}\right|$, $S_{vx\infty} = \sup_{(x,y)\in\Delta}\left|\frac{\partial\tilde{v}}{\partial x}\right|$ and $S_{vy\infty} = \sup_{(x,y)\in\Delta}\left|\frac{\partial\tilde{v}}{\partial y}\right|$ are defined as smoothness factors.

Proof of Theorem 1 is given in Appendix A.

3.3 Adaptive Law for Adjustable Parameters of the Proposed Fuzzy Solutions at Mesh Points

The following five approximated error equations with the fuzzy solutions $\tilde{u}(x, y|\underline{\theta})$ and $\tilde{v}(x, y|\underline{\lambda})$ for 2-D NSE are defined firstly,

$$\tilde{e}_1\left(x, y, \tilde{u}(x, y|\underline{\theta}), \tilde{v}(x, y|\underline{\lambda})\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2\tilde{u}}{\partial x^2} + \frac{\partial^2\tilde{u}}{\partial y^2}\right] \tag{9a}$$

$$\tilde{e}_2\left(x, y, \tilde{u}(x, y|\underline{\theta}), \tilde{v}(x, y|\underline{\lambda})\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left[\frac{\partial^2\tilde{v}}{\partial x^2} + \frac{\partial^2\tilde{v}}{\partial y^2}\right] \tag{9b}$$

$$\tilde{e}_3\left(x, y, \tilde{u}(x, y|\underline{\theta}), \tilde{v}(x, y|\underline{\lambda})\right) = \tilde{u}\Big|_{\substack{y=a\\x=0}} \tag{9c}$$

$$\tilde{e}_4\left(x, y, \tilde{u}(x, y|\underline{\theta}), \tilde{v}(x, y|\underline{\lambda})\right) = \tilde{u}\Big|_{\substack{y=-a\\x=0}} \tag{9d}$$

$$\tilde{e}_5\left(x, y, \tilde{u}(x, y|\underline{\theta}), \tilde{v}(x, y|\underline{\lambda})\right) = \frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{v}}{\partial y} \tag{9e}$$

However, analysis of solutions with all point (x, y) in domain Δ for 2-D NSE is impractical because engineers and analysts just need to know the tendency of 2-D NSE solutions with certain of mesh points in real simulation analysis hence only a finite number of mesh points $\left(x(i) = m_i^x, y(j) = m_j^y\right)$ are needed to be approached hence the collocation method [11] is adopted to discretize the domain Δ and its boundary $\partial\Delta$ into a set of points $\hat{\Delta}$ and $\partial\hat{\Delta}$ which are denoted by $(x(i), y(j))$, for $i = 1, \ldots, \alpha_x$ and $j = 1, \ldots, \alpha_y$, respectively and these points locate in $x(i) = o + i\Delta_x$ and $y(j) = o + i\Delta_y$, respectively. Approximation error equations in Eq. (9) can be converted into the following discretized error equations:

$$\tilde{e}_1\left(x(i), y(j), \underline{\theta}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2\tilde{u}}{\partial x^2} + \frac{\partial^2\tilde{u}}{\partial y^2}\right]\Big|_{\substack{x=x(i)\\y=y(j)}} \tag{10a}$$

$$\tilde{e}_2\left(x(i), y(j), \underline{\lambda}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left[\frac{\partial^2\tilde{v}}{\partial x^2} + \frac{\partial^2\tilde{v}}{\partial y^2}\right]\Big|_{\substack{x=x(i)\\y=y(j)}} \tag{10b}$$

$$\tilde{e}_3\left(x(i), y(j), \underline{\theta}\right) = \tilde{u}\Big|_{\substack{y=a\\x=0}} \tag{10c}$$

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$$\tilde{e}_4\left(x(i), y(j), \underline{\theta}\right) = \tilde{u}\Big|_{\substack{y=-a \\ x=0}} \tag{10d}$$

$$\tilde{e}_5\left(x(i), y(j), \underline{\theta}, \underline{\lambda}\right) = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\Big|_{\substack{x=x(i) \\ y=y(j)}} \tag{10e}$$

Let $\tilde{u}$ and $\tilde{v}$ denote approximated solutions to the 2-D NSE problem in Eq. (9) and Eq. (10) where $\underline{\theta}$ and $\underline{\lambda}$ stand for two sets of fuzzy parameters which are need to be adjusted optimally. The NSE solution finding problem in Eq. (10) can be expressed as an error cost function with constraints

$$\min_{\underline{\theta},\underline{\lambda}} \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\left\{\tilde{e}_1\left(x(i), y(j), \underline{\theta}\right)^2 + \tilde{e}_2\left(x(i), y(j), \underline{\lambda}\right)^2\right\} \tag{11}$$

subject to the constraints imposed by the BCs

$$\sum_{\zeta=3}^{5}\sum_{(x(i),y(j))\in\Delta} \tilde{e}_\zeta\left(x(i), y(j), \underline{\theta}, \underline{\lambda}\right)^2 = 0 \tag{12}$$

The above nonlinear constrained optimization problem in Eqs. (11) and (12) can be solved with a number of optimal methods as, Lagrange multipliers, penalty function approach, and Lyapunov method, *etc*. In this investigation, an energy perspective is adopted to handle the convergent properties of the NSE solution finding problem in mesh points, and the original constrained optimization problem in Eq. (12) has been then reduced as an unconstrained one as the following error energy function:

$$\begin{aligned} V = &\sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\left\{\frac{1}{2}\left[\tilde{e}_1\left(x(i), y(j), \underline{\theta}\right)^2\right]\right\}^2 \\ &+ \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\left\{\frac{1}{2}\left[\tilde{e}_2\left(x(i), y(j), \underline{\lambda}\right)^2\right]\right\}^2 \\ &+ \sum_{\zeta=3}^{5}\sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\frac{1}{2}\left[\tilde{e}_\zeta\left(x(i), y(j), \underline{\theta}, \underline{\lambda}\right)\right]^2 \end{aligned} \tag{13}$$

Then, our design objective is to specify the optimal parameters $\underline{\theta}$ and $\underline{\lambda}$ of fuzzy solutions in Eq. (6) to minimize the error energy function V in Eq. (13), and the following vector can be used to express Eq. (13):

$$V = \frac{1}{2}\tilde{E}^T\tilde{E} \tag{14}$$

where $\tilde{E}$ is the vector which contains the error equations in mesh points

$$\begin{aligned} \tilde{E} = \big[&\tilde{e}_1\left(x(1), y(1), \underline{\theta}\right) \quad \tilde{e}_1\left(x(1), y(2), \underline{\theta}\right) \\ &\cdots \ \tilde{e}_\zeta\left(x(l_x)y(l_y), \underline{\theta}, \underline{\lambda}\right)\big]_{(\alpha_x\alpha_y+5(\alpha_x+\alpha_y))\times 5\alpha_x\alpha_y} \end{aligned} \tag{15}$$

Assumption 1: Suppose V is at least C^2 with respect to parameters $\underline{\theta}$ and $\underline{\lambda}$ of fuzzy solutions $\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$.

For the solution finding problem of 2-D NSE, the following main results which contains a set of ordinary differential equations for finding the optimal parameters $\underline{\theta}$ and $\underline{\lambda}$ of the fuzzy solutions ($\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$), and two sufficient conditions which guarantee the convergence of approximation errors in mesh points are derived.

Theorem 2: The optimal parameters $\underline{\theta}$ and $\underline{\lambda}$ of fuzzy solutions for the NSE with BCs in Eq. (1) can be found optimally via the following adaptive laws.

$$\frac{d\underline{\theta}}{dt} = -\rho_\theta \frac{\partial V}{\partial \underline{\theta}} \tag{16a}$$

$$\frac{d\underline{\lambda}}{dt} = -\rho_\lambda \frac{\partial V}{\partial \underline{\lambda}} \tag{16b}$$

where the learning rates $\rho_\lambda > 0$ and $\rho_\theta > 0$.

Error equations Eq. (10) in mesh points can be proven to converge to zero finally if the following two sufficient conditions are satisfied.

$$\frac{\partial \underline{E}}{\partial \underline{\theta}}\frac{\partial \underline{E}^T}{\partial \underline{\theta}} > 0, \quad \text{and} \quad \frac{\partial \underline{E}}{\partial \underline{\lambda}}\frac{\partial \underline{E}^T}{\partial \underline{\lambda}} > 0 \tag{17}$$

Proof of Theorem 2 is given in Appendix 2.

The convergent properties of $\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$ for $l_x \to \infty$ and $l_y \to \infty$, *i.e.*, all points (x, y) in the domain Δ are considered, will be discussed based on the result derived in Theorem 1 in the following theorem.

Assumption 2: Suppose there exists a set of parameters for the consequent parts of the proposed fuzzy solutions $\theta_{ij} = u(a^x_i, b^y_j)$ and $\lambda_{ij} = v(a^x_i, b^y_j)$ which can be obtained via adaptive laws in (16) as $\alpha_x \to \infty$ and $\alpha_y \to \infty$.

The following theorem can be further proven with Assumption 2 and Theorems 1 and 2.

Theorem 3: Let $\alpha_x \to \infty$ and $\alpha_y \to \infty$, then the proposed fuzzy solutions will converge to the exact solutions finally as follows:

$$\lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\tilde{u}(x, y \mid \underline{\theta}) = u(x, y) \tag{18a}$$

$$\lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\tilde{v}(x, y \mid \underline{\lambda}) = v(x, y) \tag{18b}$$

Proof of Theorem 3 is given in Appendix C.

4. SIMULATION RESULTS AND DISCUSSIONS

In the following, there are two cases are used to verify the approximation ability of this proposed method.

Case 1: A domain $\Delta = [-10, 10] \times [-10, 10]$, *i.e.*, $o = -10$, $a = 10$ and $b = 10$, is defined for realizing and verifying the solution finding performance to NSE of this proposed method, and 100 mesh points are selected by ten equidistant of $x = [-10, 10]$ and $y = [-10, 10]$, *i.e.* $\Delta x = 2.22$ and $\Delta y = 2.22$, *i.e.*, $x(i) = -10, -7.78, -5.56,$

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…, 10 and $y(j) = -10, -7.78, -5.56, \ldots, 10$, for $i, j = 1, 2, \ldots, 10$, for illustrating the profile of the calculated fuzzy solutions. In this simulation, the fluid viscosity $\mu = 0.29(\text{N}\cdot\text{s})/\text{m}^2$, fluid density $\rho = 0.85\text{kg}/\text{m}^3$ and the Reynolds number Re = 0.0027 are set up, *i.e.*, the laminar flow is considered, for 2-D NSE in Eq. (1). The learning rates of adaptive laws are $\rho_\lambda = 25$ and $\rho_\theta = 25$. In this simulation, profiles of input pressure gradients are generated by $P_x = x_1 e^{(-x_1^2 - x_2^2)}$, where ($x_1 \in [-0.1, 0.1]$, $x_2 \in [-27, 27]$) and $P_y = y_1 e^{(-2y_1^2 - y_2^2)}$, where ($y_1 \in [0, 18]$, $y_2 \in [-60, 67]$) and are shown as Figs. 1 ~ 2.

Figure 3 shows the convergence history of the total error energy function V, which includes messages of the approximation errors $(\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_5)$ in mesh points, and obviously, the error energy function V converges quickly and closely to zero. This implies each of approximation errors: 2-D NSE, continuity equation, and boundary conditions, converge asymptotically to zero, hence we can conclude that the proposed fuzzy solutions closely approach to the exact solution of 2-D NSE from the observation of Fig. 3.

In the following, Figs. 4 and 5 show us the profile of the fuzzy solutions $\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$. Based on the result as shown in Fig. 3, these two convergent solutions can precisely exhibit the velocity component characteristics of a fluid with respect to pressure gradients P_x and P_y indicated in Figs. 1 and 2 with simulation conditions: Fluid viscosity $\mu = 0.29(\text{N}\cdot\text{s})/\text{m}^2$, fluid density $\rho = 0.85\text{kg}/\text{m}^3$ and the Reynolds number Re = 0.0027 within a defined domain $\Delta = [-10, 10] \times [-10, 10]$.

Figures 6 and 7 illustrate the histories of the adjustable parameters of θ_{ij} and λ_{ij} of the propsed fuzzy solutions $\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$.

The interpolation results (400 mesh points) of $\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$ which are constructed via learned optimal solution parameters θ_{ij} and λ_{ij} in steady-state situation (after 2 seconds as Figs. 6 ~ 7) are plotted as Figs. 8 ~ 9. From Figs. 8 ~ 9, these results show us more smooth and elegent profiles of $\tilde{u}(x, y \mid \underline{\theta})$ and $\tilde{v}(x, y \mid \underline{\lambda})$.

Case 2: A driven cavity problem [14] which carries out with a set of parameters: $\mu = 1(\text{N}\cdot\text{s})/\text{m}^2$, density $\rho = 1000\text{kg}/\text{m}^3$ and the Reynolds number Re = 25.0 is adopted to verify the approximation ability of this proposed method via comparing with the famous continuous method: SINC collocation method [14]. Figures 10 ~ 11 are the approximation solutions for u and v within the driven cavity. Figures 12 ~ 13 are the approximation solutions based on SINC collocation method, and from the comparisons of these two methods, it is obvious that similar profiles for u and v are obtained. However, the convergence property can be guaranteed from the convergent history of error energy function V in Fig. 14 by our proposed method.

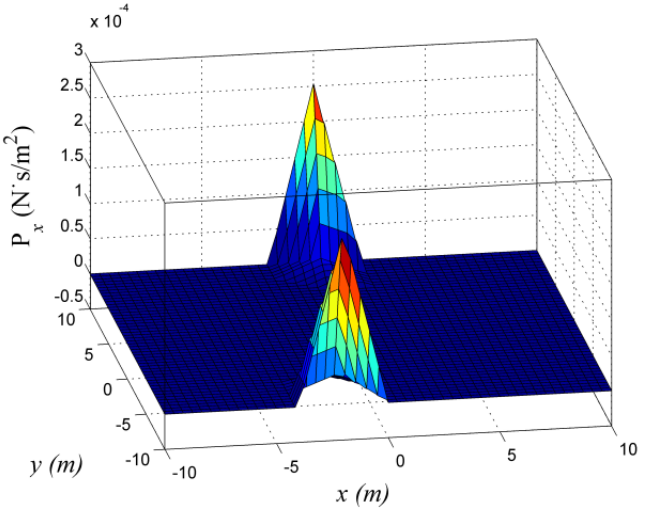


Fig. 1 Profile of the pressure gradient P_x.

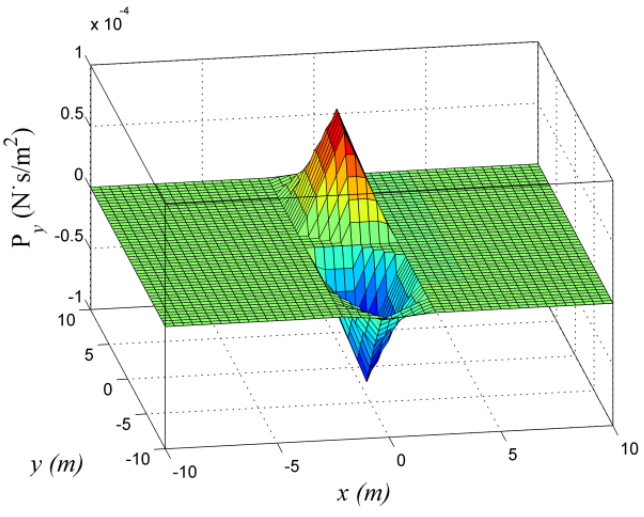


Fig. 2 Profile of pressure gradient P_y.

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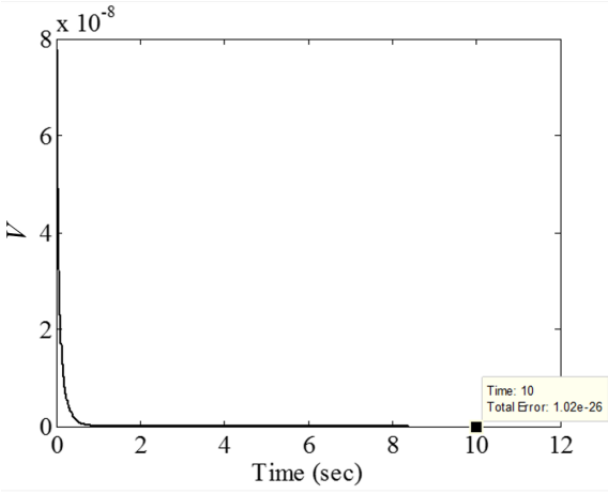


Fig. 3 Convergent history of error energy function V.

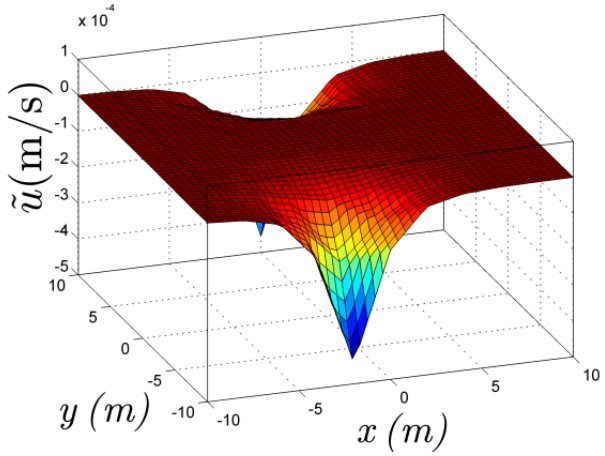


Fig. 4 Profile of the proposed fuzzy solution $\tilde{u}(x, y \mid \underline{\theta})$.

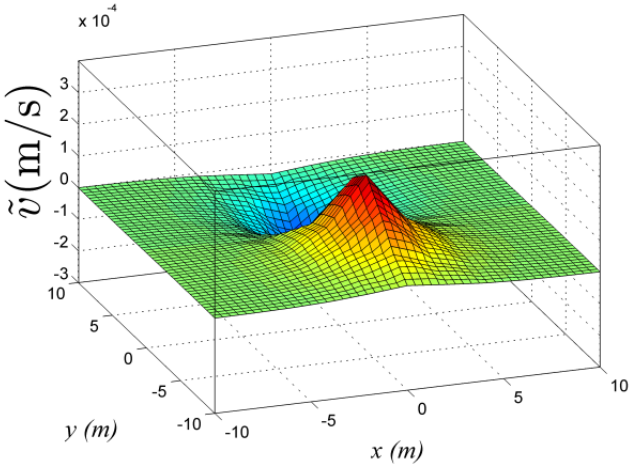


Fig. 5 Profile of the proposed fuzzy solution $\tilde{v}(x, y \mid \underline{\lambda})$.

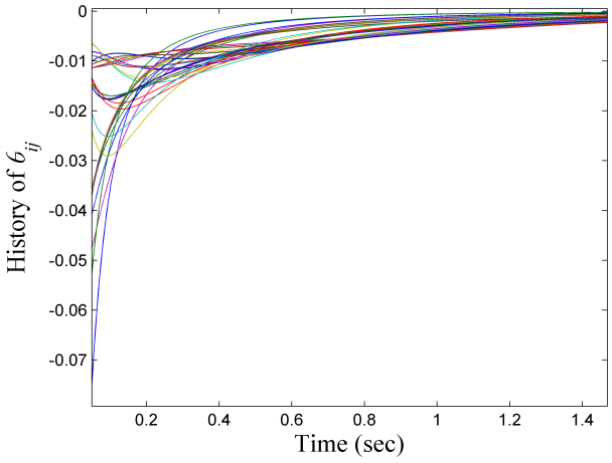


Fig. 6 Histories of the adjustable parameters θ_{ij}.

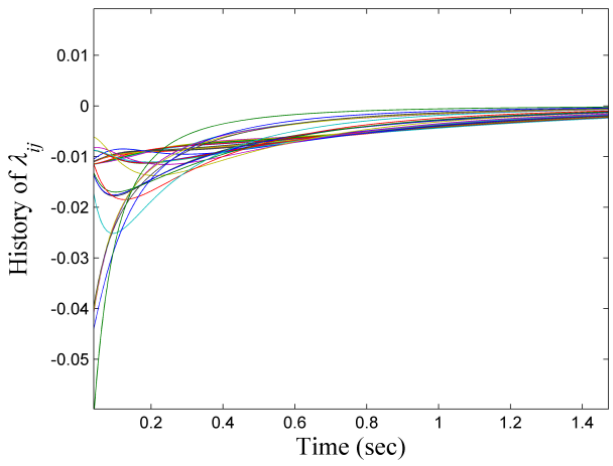


Fig. 7 Histories of the adjustable parameters λ_{ij}.

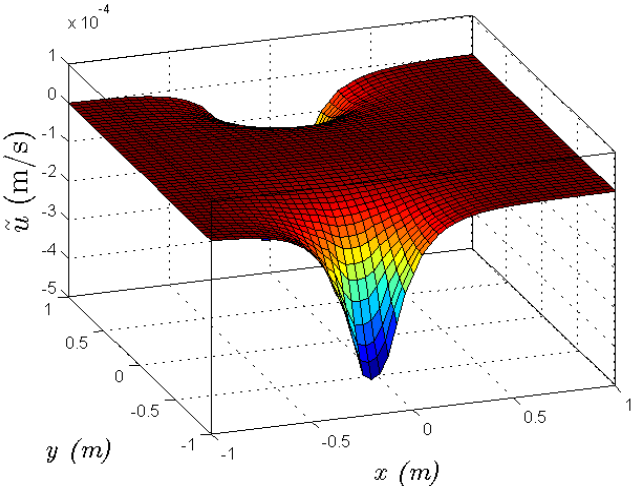


Fig. 8 Interpolation result (400 mesh points) of $\tilde{u}(x, y \mid \underline{\theta})$.

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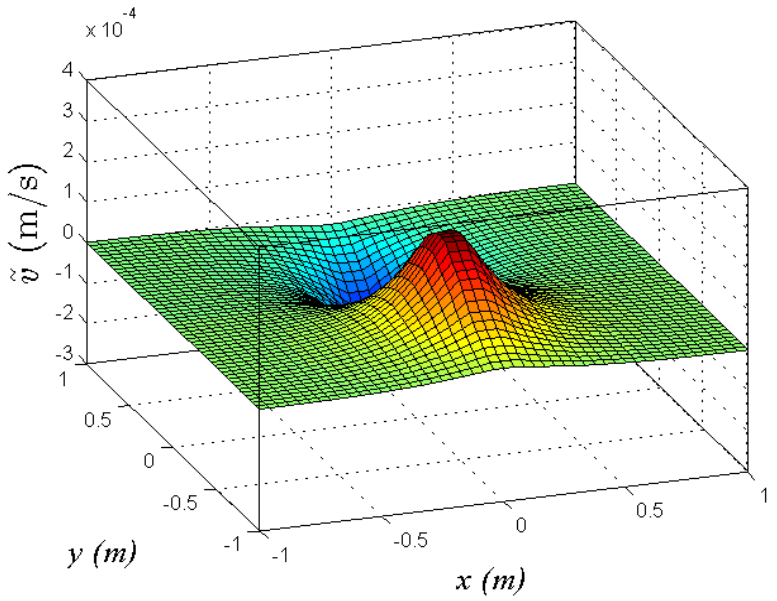


Fig. 9 Interpolation result (400 mesh points) of $\tilde{v}(x, y \mid \underline{\lambda})$.

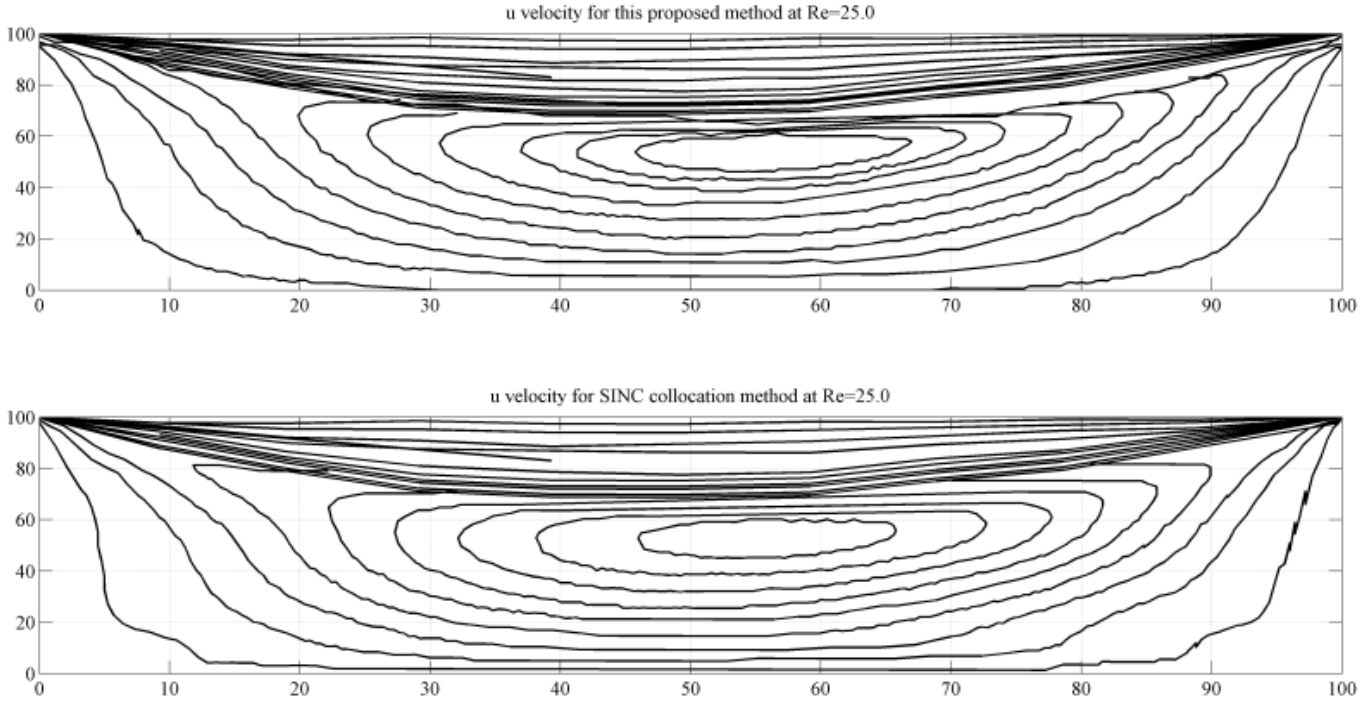


Fig. 10 *u* velocity for this proposed method and SINC collocation of the driven cavity problem.

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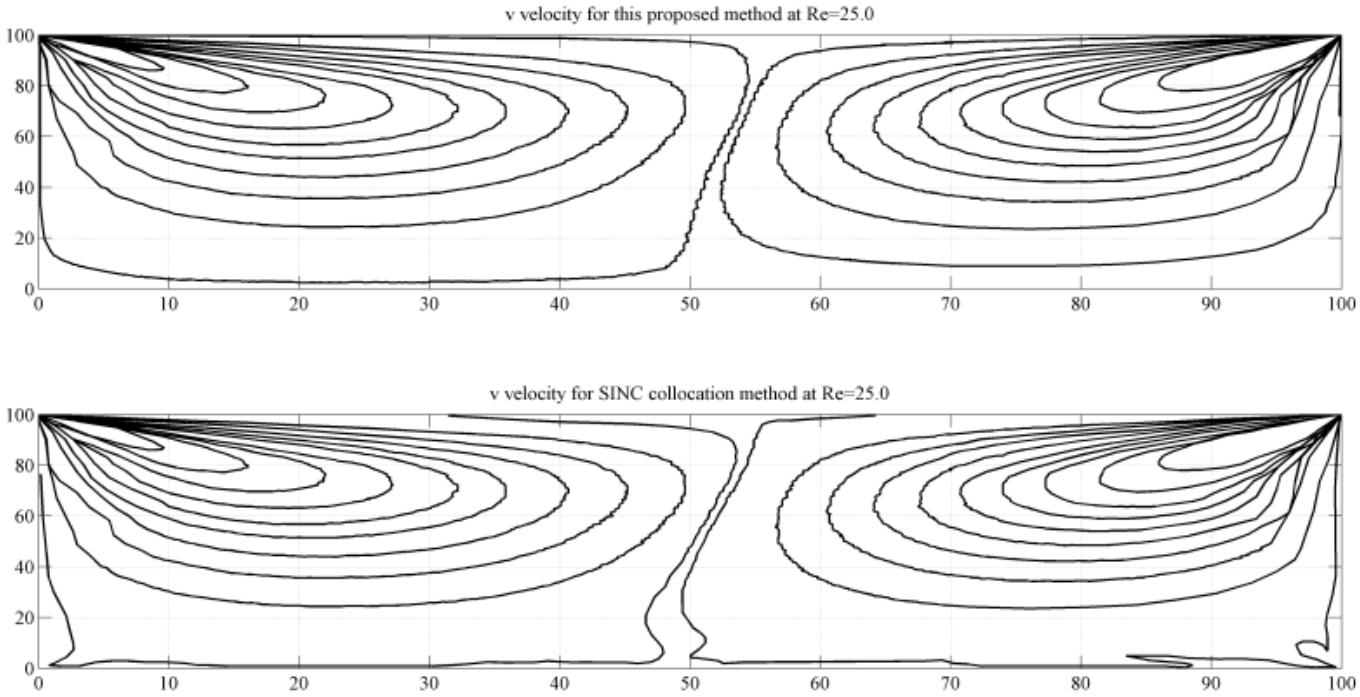


Fig. 11 v velocity for this proposed method and SINC collocation of the driven cavity problem.

5. CONCLUSIONS

Elegant solutions of 2-D Navier Stokes equations are successfully solved by an advanced solution finding methodology based on the fuzzy approximator and a set of adaptive laws. In this investigation, the approximation error bounds between the exact solutions ($u(x, y)$, $v(x, y)$) of 2-D Navier-Stokes equations and the proposed fuzzy solution ($\tilde{u}(x, y \mid \underline{\theta})$, $\tilde{v}(x, y \mid \underline{\lambda})$) are mathematically derived. After mathematical derivation, this investigation proves that the accuracy of solutions of 2-D Navier-Stokes equations depend on the number of membership functions $l = \alpha_x \times \alpha_y$ and the specification of parameters $(\underline{\theta}, \underline{\lambda})$ in the consequent parts. From the proof of Theorem 1, two important issues for effectively enhancing the accuracy of the proposed fuzzy solutions are given as the following: (1) a more complex fuzzy solution structure is adopted; and (2) a set of optimal adaptive laws for the adjustable parameters in the proposed fuzzy solutions should be developed. Based on these design suggestions, a set of optimal adaptive laws for the adjustable parameters of the proposed fuzzy solutions of 2-D Navier-Stokes equations are derived from minimizing a convex and bounded error energy function, and precise fuzzy solutions for the encountered solution finding problem of 2-D Navier-Stokes equations can be found. Finally, two simulations for verifying the solution finding performance of this proposed method based on two sets of predefined parameters: (μ = 0.29(N·s)/m^2, ρ = 0.85kg/m^3 and the Reynolds number Re = 0.0027) and (μ = 1(N·s)/m^2, density ρ = 1000kg/m^3 and the Reynolds number Re = 25.0) are given. From the simulation results, it is obvious that this proposed method can precisely offer acceptable and satisfactory solutions for 2-D Navier-Stokes equations, and details between mesh points can be observed easily by interpolation of fuzzy membership functions of the proposed solutions.

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APPENDICES

A. Proof of Theorem 1

In this section, the approximation error bound between the exact solution $u(x, y)$ and the proposed fuzzy solution $\tilde{u}(x, y \mid \underline{\theta})$ with Gaussian membership functions will be discussed firstly and the similar analysis will be applied to $v(x, y)$ and $\tilde{v}(x, y \mid \underline{\lambda})$ again. From the one dimension result in [12], the approximation error bound between $u(x, y)$ and $\tilde{u}(x, y \mid \underline{\theta})$ can be described as

$$\left|u(x, y) - \tilde{u}(x, y \left|\underline{\theta}\right.)\right| \le 0.83452(S_{ux\infty}\Delta_x + S_{uy\infty}\Delta_y) + \max_{i,j} \mid e_u^{ij} \mid \quad \text{(A1)}$$

Similarity, it is easy to obtain approximation error bound between $v(x, y)$ and $\tilde{v}(x, y \mid \underline{\lambda})$ as

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$$\left|v(x,y)-\tilde{v}(x,y\,|\,\underline{\lambda})\right| \le 0.83452(S_{vx\infty}\Delta_x + S_{vy\infty}\Delta_y) + \max|e_v^{ij}| \tag{A2}$$

Then, Theorem 1 is proven. QED

B. Proof of Theorem 2

It is straightforward to demonstrate the convergences of the proposed fuzzy solutions $\tilde{u}(x,y\,|\,\underline{\theta})$ and $\tilde{v}(x,y\,|\,\underline{\lambda})$ in mesh points via the following mathematical derivation.

Differentiating straightly Eq. (14) with the Assumption 1 of bounded conditions for V, the following differentiation result can be calculated:

$$\frac{dV}{dt} = \left(\frac{\partial \tilde{E}}{\partial \underline{\theta}}\frac{d\underline{\theta}}{dt}\right)^T \tilde{E} + \left(\frac{\partial \tilde{E}}{\partial \underline{\lambda}}\frac{d\underline{\lambda}}{dt}\right)^T \tilde{E} \tag{B1}$$

where the derivative of the vector $\tilde{E}$ with respect to a vector $\underline{\theta}$ is defined by

$$\frac{\partial \tilde{E}}{\partial \underline{\theta}} = \left[\frac{\partial e_q}{\partial \theta_{ij}}\right], \text{ for } q = 1,\cdots,\left(\alpha_x\alpha_y + 5(\alpha_x+\alpha_y)\right)\times 5\alpha_x\alpha_y,$$
$$i = 1,\cdots,\alpha_x \text{ and } j = 1,\cdots,\alpha_y.$$

and by the same mathematical operation,

$$\frac{\partial \tilde{E}}{\partial \underline{\lambda}} = \left[\frac{\partial e_q}{\partial \lambda_{ij}}\right]$$

with adaptive laws as shown in Theorem 3 and the fact ρ_θ and $\rho_\lambda > 0$, we have

$$\frac{dV}{dt} = -\rho_\theta \tilde{E}^T \frac{\partial \tilde{E}}{\partial \underline{\theta}}\frac{\partial \tilde{E}}{\partial \underline{\theta}}^T \tilde{E} - \rho_\lambda \tilde{E}^T \frac{\partial \tilde{E}}{\partial \underline{\lambda}}\frac{\partial \tilde{E}}{\partial \underline{\lambda}}^T \tilde{E} \le 0 \tag{B2}$$

Then, differentiating $\dot{V}$ in (B2) with respect to t, and from Assumption 1, the double differentiation of V is

$$\ddot{V} < \infty \tag{B3}$$

(B3) implies that $\dot{V}$ is uniformly continuous with respect to time hence by using Barbalat's lemma [13], the following result can be obtained

$$\lim_{t\to\infty}\dot{V} = -\rho_\theta \tilde{E}^T \frac{\partial \tilde{E}}{\partial \underline{\theta}}\frac{\partial \tilde{E}}{\partial \underline{\theta}}^T \tilde{E} - \rho_\lambda \tilde{E}^T \frac{\partial \tilde{E}}{\partial \underline{\lambda}}\frac{\partial \tilde{E}}{\partial \underline{\lambda}}^T \tilde{E} = 0 \tag{B4}$$

From (B4), two sufficient conditions for ensuring the convergences of error equations in mesh points for NSE can be obtained. They are

$$\frac{\partial \tilde{E}}{\partial \underline{\theta}}\frac{\partial \tilde{E}}{\partial \underline{\theta}}^T > 0, \text{ and } \frac{\partial \tilde{E}}{\partial \underline{\lambda}}\frac{\partial \tilde{E}}{\partial \underline{\lambda}}^T > 0 \tag{B5}$$

From the facts that V in Eq. (14) is positive definite and $\dot{V}$ in (B2) is negative semi-definite, it follows that V is upper bounded, *i.e.*, $0 \le V(t) \le V(0)$.

Suppose NSE with BC's in Eq. (1) satisfies the sufficient conditions in (B5), the convergences of the error equations for two solutions $\tilde{u}(x,y\,|\,\underline{\theta})$ and $\tilde{v}(x,y\,|\,\underline{\lambda})$ in mesh points can be proven to be zero as

$$\lim_{t\to\infty}\tilde{E} = 0 \tag{B6}$$

which implies every solution in mesh point satisfies

$$\lim_{t\to\infty}\tilde{u}\left(x(1),y(1)\,|\,\underline{\theta}\right) = u\left(x(1),y(1)\right),$$
$$\lim_{t\to\infty}\tilde{u}\left(x(1),y(2)\,|\,\underline{\theta}\right) = u\left(x(1),y(2)\right), \cdots,$$
$$\lim_{t\to\infty}\tilde{u}\left(x(\alpha_x),y(\alpha_y)\,|\,\underline{\theta}\right) = u\left(x(\alpha_x),y(\alpha_y)\right),$$
$$\lim_{t\to\infty}\tilde{v}\left(x(1),y(1)\,|\,\underline{\lambda}\right) = v\left(x(1),y(1)\right),$$
$$\lim_{t\to\infty}\tilde{v}\left(x(1),y(2)\,|\,\underline{\lambda}\right) = v\left(x(1),y(2)\right), \cdots,$$
$$\text{and } \lim_{t\to\infty}\tilde{v}\left(x(\alpha_x),y(\alpha_y)\,|\,\underline{\lambda}\right) = v\left(x(\alpha_x),y(\alpha_y)\right). \quad \text{QED}$$

C. Proof of Theorem 3

The convergences of the proposed fuzzy solutions to the exact solutions based on Theorem 1, Theorem 2 and Assumption 2 for all points (x, y), *i.e.*, $\alpha_x \to \infty$ and $\alpha_y \to \infty$ can be proven as the following.

From (A1), the approximation upper bound between the exact solution $u(x, y)$ and the proposed fuzzy solution $\tilde{u}(x,y\,|\,\underline{\theta})$ can be expressed as below:

$$\left|u(x,y)-\tilde{u}(x,y\,|\,\underline{\theta})\right| \le 0.83452(S_{ux\infty}\Delta_x + S_{uy\infty}\Delta_y) + \max|e_u^{ij}| \tag{C1}$$

From the facts that $S_{ux\infty}$ and $S_{uy\infty}$ are bounded functions, we take the limit for both sides of (C1) via letting $\alpha_x \to \infty$ and $\alpha_y \to \infty$, *i.e.*, $\Delta_x \to 0$ and $\Delta_y \to 0$. Then, the approximation error between $u(x, y)$ and $\tilde{u}(x,y\,|\,\underline{\theta})$ with respect to infinite numbers of mesh points are bounded as

$$\lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\left|u(x,y)-\tilde{u}(x,y\,|\,\underline{\theta})\right|$$
$$= \left|u(x,y) - \lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\tilde{u}(x,y\,|\,\underline{\theta})\right| \le \lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\max_{ij}|e_u^{ij}| \tag{C2}$$

From Eq. (5), the standard deviations π_i^x and π_j^y will converge to zero with infinite numbers of mesh points ($\alpha_x \to \infty$, $\alpha_y \to \infty$), and the Gaussian type membership functions in Eq. (4) are degenerated into the singleton types. Suppose with the help of adaptive laws in Theorem 1, a set of $\theta_{ij} = u(m_i^x, m_j^y)$ can be found such that $e^{ij} = 0$, when α_x, $\alpha_y \to \infty$, and the following result can be derived from (C2)

$$\left|u(x,y) - \lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\tilde{u}(x,y\,|\,\underline{\theta})\right| = 0 \tag{C3}$$

i.e., $\lim_{\alpha_x\to\infty}\lim_{\alpha_y\to\infty}\tilde{u}(x,y\,|\,\underline{\theta}) = u(x,y)$

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Similarity, it is easy to prove $\lim_{\alpha_x \to \infty} \lim_{\alpha_y \to \infty} \tilde{v}(x, y \mid \underline{\lambda}) = v(x, y)$ with the same manner as the above. Then, Theorem 3 is proven. QED

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