Probability in the Engineering and Informational Sciences, **26**, 2012, 183–195.
doi:10.1017/S0269964811000325

STOCHASTIC SURVIVAL MODELS WITH EVENTS TRIGGERED BY EXTERNAL SHOCKS

JI HWAN CHA
Department of Statistics
Ewha Womans University, Seoul 120-750, Korea
E-mail: jhcha@ewha.ac.kr

MAXIM FINKELSTEIN
Department of Mathematical Statistics
University of the Free State
339 Bloemfontein 9300, South Africa
and
Max Planck Institute for Demographic Research
Rostock, Germany
E-mail: finkelm@ufs.ac.za

In most conventional settings, the events caused by an external shock are initiated at the moments of its occurrence. In this paper, we study *the new classes of shock models*: (i) When each shock from a nonhomogeneous Poisson processes can trigger a failure of a system not immediately, as in classical extreme shock models, but with delay of some random time. (ii) When each shock from a nonhomogeneous Poisson processes results not in an 'immediate' increment of wear, as in classical accumulated wear models, but triggers its own increasing wear process. The wear from different shocks is accumulated and the failure of a system occurs when it reaches a given boundary. We derive the corresponding survival and failure rate functions. Furthermore, we study the limiting behavior of the failure rate function where it is applicable.

1. INTRODUCTION

Consider an orderly point process (without multiple occurrences) $N(t), t \geq 0$ of some 'initiating' events (IEs) with arrival times $T_1 < T_2 < T_3 < \ldots$. Let each event from this process triggers the 'effective event' (EE), which occurs after a random time (delay) $D_i, i = 1, 2, \ldots$, since the occurrence of the corresponding IE at T_i.

© Cambridge University Press 2012 0269-9648/12 $25.00

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

Obviously, in contrast to the initial ordered sequence $T_1 < T_2 < T_3 < \ldots$, the EEs $\{T_i + D_i\}, i = 1, 2, \ldots$ are now not necessarily ordered. This setting can be encountered in many practical situations, when, for example, IEs start the process of developing the nonfatal faults in a system and we are interested in the number of these faults in $[0, t)$. Alternatively, EEs can result in fatal, terminating faults (failures) and then we are interested in the survival probability of our system. Therefore, the latter setting means that the *first EE ruins our system.* The corresponding stochastic survival model will be considered in the first part of our paper. When there are no delays, each shock (with the specified probability) results in the failure of the survived system and the described model obviously reduces to the classical extreme shock model (Gut and Hüsler, [7]; Finkelstein, [4]; Finkelstein and Marais, [5]).

The IEs can often be interpreted as some external shocks affecting a system, and for convenience, we will often use this wording (interchangeably with the "IE"). The second part of the paper will be devoted to considering the consequences of a different kind initiated by shocks. Namely, we assume that the "wear process" triggered by each shock is activated at the moment of its occurrence and continuously increases with time. The failure of a system occurs when the wear accumulated from all previous shocks reaches a given random boundary. This new setting differs from the traditional one, where the wear caused by a shock is incurred at the moment of its occurrence (see., e.g., Esary, Marshal, and Proschan [3]; Gut, [6]; Nakagawa, [9]; Cha and Finkelstein, [1,2]). It differs also from a shot-noise setting when there is a discounting factor (decreasing function) gradually annihilating the wear input of each shock since its occurrence time (Rice, [10]; Lemoine and Wenocur, [8]).

We will consider the case of the nonhomogeneous Poisson process (NHPP) of the IEs. The approach can, in principle, be applied to the case of renewal processes, but the corresponding formulas are too cumbersome. However, the obtained results for the NHPP case are in simple, closed forms that allow intuitive interpretations and proper analyses.

The paper is organized as follows. In Section 2, the corresponding lifetime distribution and the failure rate are obtained for the model with delays. The limiting behavior of the failure rate for $t \to \infty$ is analyzed for different limiting properties of the rate of the NHPP of shocks. In Section 3, a survival model with cumulative shock effect is considered (with 'elementary' increasing wear processes triggered by each shock) for the specific case of exponentially distributed boundary. Finally, in Section 4, concluding remarks are given.

2. AN EXTREME SHOCK MODEL WITH DELAYED TERMINATION

2.1. Lifetime Distribution

Consider a system subject to the nonhomogeneous Poisson process of IEs $N(t), t \geq 0$ to be called shocks. Let the rate of this process be $\nu(t)$ and the corresponding arrival times be denoted as $T_1 < T_2 < T_3 \ldots$. Assume that the ith shock is 'harmless' to the system with probability $q(T_i)$, and with probability $p(T_i)$ it triggers the failure process

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

of the system which results in its failure after a random time $D(T_i), i = 1, 2, \ldots$, where $D(t)$ is a nonnegative, semi-continuous random variable with the point mass at "0" (at each fixed t). Note that this 'point mass' at 0 opens the possibility of 'immediate failure' of the system at the occurrence of a shock, which is practically very important, and, furthermore the case of the 'full point mass' of $D(t)$ at 0 reduces to the ordinary 'extreme shock model'. Obviously, without point mass at 0, we arrive at an absolutely continuous random variable. The distributions of $D(t)$ having point masses at other values could be considered similarly.

Let $G(t,x) \equiv P(D(t) \leq x)$, $\overline{G}(t,x) \equiv 1 - G(t,x)$, and $g(t,x)$ be the Cdf, the survival function and the pdf for the 'continuous part' of $D(t)$, respectively. Then, in accordance with the terminology in the Introduction, the failure in this case is the *EE*.

First of all, we are interested in describing the lifetime of our system T_S. The corresponding conditional survival function is given by

$$P(T_S > t | N(s), 0 \leq s \leq t; D(T_1), D(T_2), \ldots, D(T_{N(t)}); J_1, J_2, \ldots, J_{N(t)}) = \prod_{i=1}^{N(t)} (J_i + (1 - J_i) I(D(T_i) > t - T_i)), \quad \textbf{(1)}$$

where the indicators are defined as

$$I(D(T_i) > t - T_i) = \begin{cases} 1, & \text{if } D(T_i) > t - T_i \\ 0, & \text{otherwise} \end{cases},$$

$$J_i = \begin{cases} 1, \text{if the } i\text{th shock does not trigger the subsequent failure process,} \\ 0, \text{otherwise.} \end{cases}$$

Assume the following conditions regarding 'conditional independence':

(i) Given the shock process, $D(T_i), i = 1, 2, \ldots$, are mutually independent.

(ii) Given the shock process, $J_i, i = 1, 2, \ldots$, are mutually independent. (It means that whether each shock triggers the failure process of the system or not is 'independently determined'.)

(iii) Given the shock process, $\{D(T_i), i = 1, 2, \ldots\}$ and $\{J_i, i = 1, 2, \ldots\}$, are mutually independent.

Integrating out all conditional random quantities in (1) under the basic assumptions described above results in the following theorem.

THEOREM 1: *Suppose that* $\nu(+0) > 0$. *Then*

$$P(T_S \geq t) = \exp\left\{ - \int_0^t G(x; t - x) p(x) \nu(x) dx \right\}, \; t \geq 0,$$

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

and the failure rate function of the system is

$$\lambda_S(t) = \int_0^t g(x; t-x)p(x)\nu(x)dx + G(t,0)p(t)\nu(t), t \geq 0.$$

PROOF: Given the assumptions, we can directly 'integrate out' J_i's and D_i's and define the corresponding probability in the following way:

$$P(T_S > t | N(s), 0 \leq s \leq t) = \prod_{i=1}^{N(t)} (q(T_i) + p(T_i)\overline{G}(T_i, t-T_i)).$$

Therefore,

$$\begin{aligned} P(T_S > t) &= E\left[\prod_{i=1}^{N(t)}(q(T_i) + p(T_i)\overline{G}(T_i, t-T_i))\right] \\ &= E\left[E\left[\prod_{i=1}^{N(t)}(q(T_i) + p(T_i)\overline{G}(T_i, t-T_i))|N(t)\right]\right]. \end{aligned} \quad \textbf{(2)}$$

It is well known that the joint distribution of $T_1, T_2, \ldots, T_n$ given $N(t) = n$ is the same as the joint distribution of order statistics $T'_{(1)} \leq T'_{(2)} \leq \ldots \leq T'_{(n)}$ of i.i.d. random variables $T'_1, T'_2, \ldots, T'_n$, where the p.d.f. of the common distribution of T'_j's is given by $\nu(x)/m(t), 0 \leq x \leq t$, where $m(t) \equiv E(N(t)) = \int_0^t \nu(x)\,dx$:

$$(T_1, T_2, \ldots, T_n | N(t) = n) =^d (T'_{(1)}, T'_{(2)}, \ldots, T'_{(n)}).$$

Then

$$\begin{aligned} &E\left[\prod_{i=1}^{N(t)}(q(T_i) + p(T_i)\overline{G}(T_i, t-T_i))|N(t) = n\right] \\ &= E\left[\prod_{i=1}^{n}(q(T'_{(i)}) + p(T'_{(i)})\overline{G}(T_{(i)}, t-T'_{(i)}))\right] \\ &= E\left[\prod_{i=1}^{n}(q(T'_i) + p(T'_i)\overline{G}(T'_i, t-T'_i))\right] \\ &= (E[q(T'_i) + p(T'_i)\overline{G}(T'_i, t-T'_i)])^n \\ &= \left(\frac{1}{m(t)}\int_0^t (q(x) + p(x)\overline{G}(x; t-x))\,\nu(x)dx\right)^n. \end{aligned} \quad \textbf{(3)}$$

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

From Eqs. (2) and (3),

$$P(T_S > t) = \sum_{n=0}^{\infty} \left(\frac{1}{m(t)} \int_0^t (q(x) + p(x)\overline{G}(x, t-x))\nu(x)dx \right)^n \frac{m(t)^n}{n!} e^{-m(t)}$$

$$= e^{-m(t)} \cdot \exp\left\{ \int_0^t (q(x) + p(x)\overline{G}(x, t-x))\, \nu(x)dx \right\}$$

$$= \exp\left\{ \int_0^t q(x)\nu(x)dx + \int_0^t \overline{G}(x; t-x)p(x)\nu(x)dx - \int_0^t \nu(x)dx \right\}$$

$$= \exp\left\{ -\int_0^t G(x, t-x)p(x)\nu(x)dx \right\}.$$

Therefore, by Leibnitz rule, the failure rate function of the system, $\lambda_S(t)$, is given in the following meaningful and rather simple form:

$$\lambda_S(t) = \int_0^t g(x; t-x)p(x)\nu(x)dx + G(t, 0)p(t)\nu(t), t \geq 0. \quad \textbf{(4)}$$

■

Remark 1: It has been recognized that many queuing systems are most appropriately described by nonstationary queuing models, in which "both the arrival and service rates" are functions of time. Therefore, there is a growing literature on methods for calculating time-dependent performance measures in these models. It can be seen that in a queuing context, when $G(t, x) = G(x)$, our system's failure time can be interpreted as the first departure time in the $M_t/G/\infty$ system that starts empty (Ross, [11]). The model considered in this section is a time variant of it and, accordingly, the failure time can be interpreted in terms of the first departure time from the corresponding $M_t/G_t/\infty$ system that starts empty.

Remark 2: Speaking formally, the split of effects to effective and ineffective shocks does not add any mathematical complexity due to the NHPP nature of the arrival process. This means that the result would be the same if we had only one type of effects and the NHPP with the rate function $p(t)\nu(t)$. However, from the practical point of view and keeping in mind that we are generalizing here the classical extreme shock model with two types of effects, this splitting seems to be reasonable. Furthermore, we can consider the case of multi-type delayed consequences of shocks ($n > 1$), where the shock that occurs at time t causes the delayed (with distribution $G_i(t, x)$) effect of type i with probability $p_i(t)$, whereas the probability of 'no effect' is $1 - \sum_{i=1}^n p_i(t)$. Obviously, this model is the same as the single-type model with $G(t, x) = \sum_{i=1}^n p_i^*(t)G_i(t, x)$ and $p(t) = \sum_{i=1}^n p_i(t)$, where $p_i^*(t) = p_i(t)/\sum_{i=1}^n p_i(t)$. Therefore, similar to Theorem 1,

$$P(T_S \geq t) = \exp\left\{ -\int_0^t \left(\sum_{i=1}^n p_i(x)G_i(x, t-x) \right) \nu(x)dx \right\},\ t \geq 0$$

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

and

$$\lambda_S(t) = \int_0^t \left(\sum_{i=1}^n p_i(x) g_i(x, t-x) \right) \nu(x) dx + \left(\sum_{i=1}^n p_i(t) G_i(t, 0) \right) \nu(t).$$

2.2. Limiting Behavior

In this subsection, we study the limiting behavior of the failure rate $\lambda_S(t)$. Without loss of generality, assume that $p(t)$ and $\nu(t)$ are continuous functions with $p(t) > 0$, for all $t \geq 0$. We further assume that

$$D(t) \to D(\infty) \equiv D \text{ in distribution as } t \to \infty,$$

where $D(t)$, $t > 0$, and D are semi-continuous random variables with supports $[0, \infty)$ and point masses at "0", and

$$\frac{g(t,x)}{g(0,x)} \quad \text{is bounded for all} \quad t, x \geq 0. \tag{5}$$

Obviously, Eq. (5) is a rather weak condition. The simplest example of a continuous random variable is

$$g(t,x) = (1/\mu(t)) \exp\{-(1/\mu(t))x\}, \quad x \geq 0, \tag{6}$$

where $\mu(t)$ is continuous and decreasing with

$$\mu(t) \to \mu, \text{ as } t \to \infty.$$

Observe that the distribution which corresponds to the pdf in (6) is an exponential distribution for each fixed t.

THEOREM 2: *In addition to the assumptions of Theorem 1, let* $\lim_{t\to\infty} p(t) \equiv p(\infty)$ *and* $\lim_{t\to\infty} \nu(t) \equiv \nu(\infty) \leq \infty$ *exist. Then*

$$\lim_{t\to\infty} \lambda_S(t) \equiv p(\infty)\nu(\infty).$$

PROOF: Suppose first that $\nu(t) \leq \nu_0 < \infty$, for all $t \geq 0$. This implies that there exists a positive constant $l_0 < \infty$ such that $p(t)\nu(t) \leq l_0$, for all $t \geq 0$. Observe that the first term of $\lambda_S(t)$ in (4) can be rewritten as

$$\int_0^t p(t-x)\nu(t-x)g(t-x;x)dx$$
$$= \int_0^\infty p(t-x)\nu(t-x)\frac{g(t-x,x)}{g(0,x)} I_{(0,t)}(x) \cdot g(0,x)dx,$$

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

where $I_{(0,t)}(x) = 1$ if $x \in (0,t)$; and $I_{(0,t)}(x) = 0$ otherwise. Then, clearly, there also exists l_1 such that

$$p(t-x)\nu(t-x)\frac{g(t-x,x)}{g(0,x)}I_{(0,t)}(x) \leq l_1 < \infty \text{ for all } t,x \geq 0,$$

as $g(t-x,x)/g(0,x)$ is bounded for all $t,x \geq 0$. Applying now Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}\lim_{t\to\infty}\lambda_S(t) &= \lim_{t\to\infty}\left[\int_0^\infty p(t-x)\nu(t-x)I_{(0,t)}(x)\cdot g(t-x,x)dx + G(t,0)p(t)\nu(t)\right]\\ &= \int_0^\infty \lim_{t\to\infty}(p(t-x)\nu(t-x)I_{(0,t)}(x)\cdot g(t-x,x))dx\\ &\quad + G(\infty,0)p(\infty)\nu(\infty)\\ &= \int_0^\infty p(\infty)\nu(\infty)g(\infty,x)dx + G(\infty,0)p(\infty)\nu(\infty)\\ &= p(\infty)\nu(\infty),\end{aligned}$$

where $g(\infty,x)$ is the pdf of the continuous part of $D(\infty) \equiv D$.

Consider now the case when $\lim_{t\to\infty}\nu(t) \equiv \nu(\infty) = \infty$. Then, for any $L > 0$, there exists t_0 such that

$$p(t)\nu(t) > L, \text{ for all } t > t_0.$$

Hence, for $t > t_0$,

$$\begin{aligned}\int_0^t p(x)\nu(x)g(x;t-x)dx &\geq \int_{t_0}^t p(x)\nu(x)g(x,t-x)dx > L\int_{t_0}^t g(x,t-x)dx\\ &= L\int_0^{t-t_0} g(t-x,x)dx = L\int_0^\infty I_{(0,t-t_0)}(x)\cdot g(t-x,x)dx\\ &= L\int_0^\infty I_{(0,t-t_0)}(x)\frac{g(t-x,x)}{g(0,x)}\cdot g(0,x)dx.\end{aligned}$$

Clearly, $I_{(0,t-t_0)}(x) \leq 1$, for all $t,x \geq 0$, and thus

$$I_{(0,t-t_0)}(x)\frac{g(t-x,x)}{g(0,x)} \text{ is bounded for all } t,x \geq 0,$$

which allows the application of Lebesgue's Dominated Convergence Theorem and

$$\begin{aligned}\liminf_{t\to\infty}\int_0^t p(x)\nu(x)g(t-x)dx &> \lim_{t\to\infty} L\int_0^\infty I_{(0,t-t_0)}(x)g(t-x,x)dx\\ &= L\int_0^\infty \lim_{t\to\infty}[I_{(0,t-t_0)}(x)g(t-x,x)]dx = L.\end{aligned}$$

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

As $L > 0$ is arbitrary, we can now conclude that

$$\lim_{t\to\infty} \lambda_S(t) = \infty,$$

which completes the proof. ■

Remark 3: The intuitive meaning of Theorem 2 is as follows: In the ordinary extreme shock model, $\lambda_S(t) = p(t)\nu(t)$ and, therefore, Theorem 2 trivially holds. Now, we are looking at the perturbed extreme shock model, where the perturbation is executed by considering the corresponding delays. It is clear that this effect vanishes as $D(t)$ becomes negligible (in some suitable probabilistic sense) compared with t, when $t \to \infty$. Specifically, when $\nu(t) = \nu, p(t) = p, \forall t \geq 0, G(t,x) = G(x)$, it follows from (4) that

$$\lambda_S(t) = p\nu G(t), t \geq 0.$$

The function $\lambda_S(t)$ is strictly increasing from 0 to $p\nu$, as $t \to \infty$, that is, to the failure rate of time to termination in the ordinary extreme shocks model.

Remark 4: If $D(t)$ is stochastically decreasing with t in the sense of the likelihood ratio order (Ross [11], Shaked and Shanthikumar [12]),

$$D(t_1) \geq_{lr} D(t_2), \text{ for } t_1 < t_2,$$

then the condition in (5) can be replaced by a simpler one

$$\frac{g(t,0)}{g(0,0)} \text{ is bounded for all } t \geq 0.$$

Observe that $g(t,x)$ defined in (6) is obviously admissible.

Note that the stochastically decreasing $D(t)$ describes deterioration of our system in the following sense: the time to failure since the occurrence of the IE stochastically decreases (nonincreases) with t (the system is getting 'weaker'). As most of engineering systems are ageing, this assumption is quite natural.

3. CUMULATIVE SHOCK MODEL WITH INITIATED WEAR PROCESSES

Consider now a cumulative model for IEs (shocks), where the accumulated wear can result in a system failure when it reaches the given boundary. Our setting that follows is different from the conventional one. In the conventional setting, the wear caused by a shock is incurred at the moment of the corresponding shock (Cha and Finkelstein, [1,2]). In our model, however, the "wear process", triggered by a shock, is activated at the moment of a shocks' occurrence and continuously increases with time, which is somehow related to the setting of the previous section (see also Remark 4).

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

Denote by $W(t,u)$ the random wear incurred in u units of time after a single shock (IE) that has occurred at time t. Let $W(t,0) \equiv 0$, for all $t \geq 0$. Assume that $W(t,u)$ is stochastically increasing in t and u, that is,

$$W(t_1,u) \leq_{st} W(t_2,u) \quad \text{for all } t_2 > t_1 > 0 \quad \text{and} \quad \text{for all } u > 0;$$

and

$$W(t,u_1) \leq_{st} W(t,u_2) \quad \text{for all } u_2 > u_1 > 0 \quad \text{and} \quad \text{for all } t > 0.$$

An example for this type of $W(t,u)$ is the gamma process, with the pdf for $W(t,u)$ given by

$$f(w,t,u) = \frac{\beta^{\alpha(t,u)} \cdot w^{\alpha(t,u)-1} \exp\{-\beta w\}}{\Gamma(\alpha(t,u))}, \qquad w \geq 0,$$

where $\alpha(t,0) = 0$, for all $t \geq 0$, and $\alpha(t,u)$ is strictly increasing in both t and u.

If all shocks from the initial process trigger wear, then the accumulated wear from all shocks in $[0,t)$ is

$$W(t) = \sum_{i=0}^{N(t)} W(T_i, t - T_i),$$

which can be considered as a general form of the shot-noise process. Distinct from the previous sections (but our model is also different), assume that each shock with probability $p(t)$ results in an immediate failure (termination), otherwise with probability $q(t)$ it triggers the wear process in the way described above. The failure also occurs when the accumulated wear reaches the random boundary R and we are interested in obtaining the distribution of T_S—the time to failure.

Similar to Cha and Finkelstein [2], and using the notation of the previous sections, the corresponding conditional survival probability for this model can be written as

$$\begin{aligned} &P(T_S > t | N(s), 0 \leq s \leq t; W(T_i, t - T_i), \qquad i = 1,2,\ldots,N(t); R) \\ &\quad = \prod_{i=0}^{N(t)} q(T_i) \cdot I\left(\sum_{i=0}^{N(t)} W(T_i, t - T_i) \leq R\right). \end{aligned}$$

Assume that R is the exponentially distributed (with parameter λ) random variable.

THEOREM 3: *Let the shock process be NHPP with rate $\nu(t)$ and suppose that $\nu(+0) > 0$. Then*

$$P(T_S \geq t) = \exp\left\{-\int_0^t \nu(x)dx + \int_0^t M_{W(x,t-x)}(-\lambda) \cdot q(x)\, \nu(x)dx\right\}, \qquad t \geq 0,$$

and the corresponding failure rate function is

$$\lambda_S(t) = p(t)\nu(t) - \int_0^t \frac{d}{dt}(M_{W(x,t-x)}(-\lambda))q(x)\, \nu(x)dx, \qquad t \geq 0,$$

where $M_{W(t,u)}(\cdot)$ is the mgf of $W(t,u)$ (for fixed t and u).

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

PROOF: Given the assumptions, we can directly 'integrate out' the variable R and define the corresponding probability in the following way:

$$\begin{aligned} &P(T_S > t | N(s), 0 \le s \le t; W(T_i, t - T_i), i = 1, 2, \ldots, N(t)) \\ &= \left(\prod_{i=0}^{N(t)} q(T_i) \right) \cdot \exp \left\{ - \int_0^{\sum_{i=0}^{N(t)} W(T_i, t - T_i)} \lambda \, du \right\} \\ &= \exp \left\{ -\lambda \sum_{i=1}^{N(t)} W(T_i, t - T_i) + \sum_{i=1}^{N(t)} \ln q(T_i) \right\}. \end{aligned}$$

Thus the survival function can be obtained as

$$P(T_S > t) = E \left[E \left[\exp \left\{ -\lambda \sum_{i=1}^{N(t)} W(T_i, t - T_i) + \sum_{i=1}^{N(t)} \ln q(T_i) \right\} | N(t) \right] \right]. \quad \textbf{(7)}$$

Then, following the same procedure described in the proof of Theorem 1,

$$\begin{aligned} &E \left[\exp \left\{ -\lambda \sum_{i=1}^{N(t)} W(T_i, t - T_i) + \sum_{i=1}^{N(t)} \ln q(T_i) \right\} | N(t) = n \right] \\ &= (E[\exp\{-\lambda W(T_1', t - T_1') + \ln q(T_1')\}])^n. \end{aligned}$$

Observe that,

$$E[\exp\{-\lambda W(T_1', t - T_1') + \ln q(T_1')\}] = \frac{1}{m(t)} \int_0^t (q(x) M_{W(x,t-x)}(-\lambda)) \nu(x) dx.$$

Hence,

$$\begin{aligned} &E \left[\exp \left\{ -\lambda \sum_{i=1}^{N(t)} W(T_i, t - T_i) + \sum_{i=1}^{N(t)} \ln q(T_i) \right\} | N(t) = n \right] \\ &= \left(\frac{1}{m(t)} \int_0^t (q(x) M_{W(x,t-x)}(-\lambda)) \, \nu(x) dx \right)^n. \end{aligned} \quad \textbf{(8)}$$

From Eq. (7) and (8),

$$P(T_S > t) = \exp \left\{ - \int_0^t \nu(x) dx + \int_0^t M_{W(x,t-x)}(-\lambda) \cdot q(x) \, \nu(x) dx \right\}.$$

Therefore, by Leibnitz rule, the failure rate function of the system, $\lambda_S(t)$, is given by

$$\begin{aligned} \lambda_S(t) &= (1 - M_{W(t,0)}(-\lambda) \cdot q(t)) \nu(t) - \int_0^t \frac{d}{dt} (M_{W(x,t-x)}(-\lambda)) q(x) \, \nu(x) dx \\ &= p(t) \nu(t) - \int_0^t \frac{d}{dt} (M_{W(x,t-x)}(-\lambda)) q(x) \, \nu(x) dx, \end{aligned}$$

which completes the proof. ■

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

Similar to Cha and Finkelstein [1], the case of the *deterministic boundary* can be also considered but the result is more cumbersome.

Let, for simplicity, $\lim_{t\to\infty} \nu(t) \equiv \nu(\infty) \equiv \nu_0 < \infty, \nu_0 > 0; p(t) \equiv p, q(t) \equiv q$. It is clear from general considerations that $\lim_{t\to\infty} \lambda_S(t) = \lim_{t\to\infty} \nu(t) = \nu_0$ monotonically approaching the limit from below. Indeed, consider a system that had survived in $[0, t)$, which means that the next interval $[t, t + dt)$ starts with the same 'resource' R, as the boundary is exponentially distributed. Due to the fact that all previous nonfatal shocks accumulate wear and all triggered wear processes are increasing, as t increases ($W(t) \to \infty$ as $t \to \infty$), the resource R is 'consumed more intensively' with time. This obviously means that the probability of failure in $[t, t + dt)$ is increasing in t and therefore, $\lambda_S(t)$ is increasing. Eventually, when $t \to \infty$, every triggering shock in the limit becomes fatal, which means that $\lim_{t\to\infty} \lambda_S(t) = \lim_{t\to\infty} \nu(t) = \nu_0$. The following example illustrates these considerations.

Example 1: Suppose that $W(t, u)$ follows the gamma process, that is, the pdf for $W(t, u)$ is given by

$$f(w; t, u) = \frac{\beta^{\alpha(t,u)} \cdot w^{\alpha(t,u)-1} \exp\{-\beta w\}}{\Gamma(\alpha(t, u))}, \quad w \geq 0,$$

where $\alpha(t, 0) = 0$, for all $t \geq 0$, and $\alpha(t, u)$ is strictly increasing in both t and u. Then

$$M_{W(x,t-x)}(-\lambda) = \left(\frac{\beta}{\beta + \lambda}\right)^{\alpha(x,t-x)},$$

and

$$\frac{d}{dt}(M_{W(x,t-x)}(-\lambda)) = \frac{d}{dt}(\alpha(x, t - x)) \ln\left(\frac{\beta}{\beta + \lambda}\right)\left(\frac{\beta}{\beta + \lambda}\right)^{\alpha(x,t-x)}.$$

Let $\nu(t) = \nu, q(t) = q, t \geq 0; \alpha(t, u) = \alpha u, t, u \geq 0$. Then

$$\begin{aligned}\int_0^t \frac{d}{dt}(M_{W(x,t-x)}(-\lambda)) \cdot q(x)\, \nu(x) dx &= \int_0^t \alpha \cdot \ln\left(\frac{\beta}{\beta + \lambda}\right) \cdot \left(\frac{\beta}{\beta + \lambda}\right)^{\alpha(t-x)} q\, \nu\, dx \\ &= \int_0^{\alpha t} \ln\left(\frac{\beta}{\beta + \lambda}\right) \cdot \left(\frac{\beta}{\beta + \lambda}\right)^{x} q\, \nu\, dx \\ &= q\nu\left(\left(\frac{\beta}{\beta + \lambda}\right)^{\alpha t} - 1\right).\end{aligned}$$

Therefore, we have

$$\lambda_S(t) = p\nu + q\nu\left(1 - \left(\frac{\beta}{\beta + \lambda}\right)^{\alpha t}\right), \quad t \geq 0$$

and

$$\lim_{t\to\infty} \lambda_S(t) \equiv \nu,$$

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

which means, as already stated above for a general setting, that every triggering shock in the limit becomes fatal.

Remark 5: Clearly, the model considered in this section can be modified in a straightforward way to the case when a shock arriving at time t triggers the wear processes with delay $D(t)$ (as in Section 2).

4. CONCLUDING REMARKS

One can find a lot of different shock models in the vast literature on this topic (see, e.g., Nakagawa, 2007 and references therein). These models usually deal with systems that are subject to shocks of random magnitudes at random times.

Traditionally, one distinguishes between two major types: cumulative shock models (systems break down because of a cumulative effect) and extreme shock models (systems break down because of one single large shock). A combination of these models was investigated by Gut and Hüsler [7], where the failures were due either to a cumulative effect, or to a single, fatal shock.

In this paper, we study two new shock models. The first one generalizes the extreme shock model to the case when each shock from a nonhomogeneous Poisson processes can be fatal only after some time delay. In the second model, each shock from a nonhomogeneous Poisson processes results not in an 'immediate' increment of wear, as in classical accumulated wear models, but triggers its own increasing wear process. The wear from different shocks is accumulated and the failure of a system occurs when it reaches a given boundary. We derive the corresponding survival probabilities and failure rates in a rather simple, meaningful form that allows probabilistic analysis and prompts for further generalizations.

Acknowledgments

We would like to thank two anonymous reviewers for their extremely helpful and profound comments. The work of the first author was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0093827). The work of the second author was supported by the NRF (National Research Foundation of South Africa) grant FA2006040700002.

References

1. Cha, J.H. & Finkelstein, M. (2009). On a terminating shock process with independent wear increments. *Journal of Applied Probability* 46: 353–362.
2. Cha, J.H. & Finkelstein, M. (2011). On new classes of extreme shock models and some generalizations. *Journal of Applied Probability* 48: 258–270.
3. Esary, J.D., Marshal, A.W. & Proschan, F. (1973). Shock models and wear processes. *Annals of Probability* 1: 627–649.
4. Finkelstein, M. (2008). *Failure rate modelling for reliability and risk*. London: Springer.
5. Finkelstein, M. & Marais, F. (2010). On terminating Poisson processes in some shock models. *Reliability Engineering and System Safety* 95: 874–879.

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press

6. Gut, A. (1990). Cumulated shock models. *Advances in Applied Probability* 22: 504–507.
7. Gut, A. & Hüsler, J. (2005) Realistic variation of shock models. *Statistics and Probability Letters* 74: 187–204.
8. Lemoine, A.J. & Wenocur, M.L. (1986). A note on shot-noise and reliability modeling. *Operations Research* 34: 320–323.
9. Nakagawa, T. (2007). *Shock and damage models in reliability theory*. London: Springer.
10. Rice, J. (1977). On generalized shot noise. *Advances in Applied Probability* 9: 553–565.
11. Ross, S.M. (1996). *Stochastic processes*. New York: John Wiley.
12. Shaked, M. & Shanthikumar, J. (2007). *Stochastic orders*. New York: Springer.

https://doi.org/10.1017/S0269964811000325 Published online by Cambridge University Press