MATRIX REPRESENTATIONS OF SEMIGROUPS

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In a previous paper, the author gave necessary and sufficient conditions for the algebra of a finite semigroup S over a field $\mathfrak{F}$ of suitable characteristic to be semisimple. When this algebra is semisimple, the matrix representations of S over $\mathfrak{F}$ are completely reducible.

The present paper is concerned with the above situation. A method is given for obtaining a complete set of inequivalent irreducible representations of S in terms of those of the basic groups of the principal factors of S. The case in which S is an inverse semigroup is of special interest. It is shown that the algebra of such a semigroup over a suitable field is semisimple; the representation theory is then applied in detail.

1. *Notation.* All the semigroups with which we deal are finite; this will be assumed throughout. We shall use the notation of (2). Thus $\mathfrak{A}(S)$ denotes the algebra of a semigroup S over some specified field, and $\mathfrak{A}(T)$ the subspace of $\mathfrak{A}(S)$ spanned by the subset T of S. If S has a zero element, this may be identified with the zero of the algebra, and we obtain the 'contracted' algebra of S ((2), §3·1). A principal series for S is a proper series of ideals without a proper refinement, and the factors of such a series are called the principal factors of S. If all the principal factors are simple, then S is called semisimple; this is necessarily the case if $\mathfrak{A}(S)$ is semisimple ((2), §3·4).

Let $G(z)$ denote the group-with-zero obtained by adjoining the zero z to the group G, let $(x)_{ij}$ denote the $m \times n$ matrix over $G(z)$ with $x \in G(z)$ in the (i,j)th position and z elsewhere, and let P be an $n \times m$ matrix over $G(z)$. Define a multiplication (o) by the rule

$$(x)_{ij} \circ (y)_{kl} = (x)_{ij} P (y)_{kl} = (x p_{jk} y)_{il}.$$

Then the elements $(x)_{ij}$ form a semigroup, which we denote by $S_{mn}[G, P]$ and call a matrix semigroup over $G(z)$. We call G its basic group. If P has at least one non-zero element in each row and column, then the semigroup is said to be regular. Rees (6) has shown that every simple (finite) semigroup is isomorphic with a regular matrix semigroup over a group-with-zero, and conversely that every such matrix semigroup is simple.

Let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$, $\mathfrak{S}(\mathfrak{A})$ the set of all $m \times n$ matrices over $\mathfrak{A}$, and P a fixed $n \times m$ matrix over $\mathfrak{A}$. Then if we define a multiplication (o) in $\mathfrak{S}(\mathfrak{A})$ by the rule $A \circ B = APB$, $\mathfrak{S}(\mathfrak{A})$ becomes an algebra over $\mathfrak{F}$, which we denote by $M_{mn}[\mathfrak{A}, P]$. In particular, if $\mathfrak{A}$ has an identity element, we denote the $n \times n$ 'unit' matrix over $\mathfrak{A}$ by U_n; then $M_{nn}[\mathfrak{A}, U_n] = M_n(\mathfrak{A})$, the algebra of $n \times n$ matrices over $\mathfrak{A}$. The algebra $M_{mn}[\mathfrak{A}, P]$ is semisimple if and only if $\mathfrak{A}$ is semisimple and P is non-singular (implying that $m = n$) ((2), §4·7).

As was remarked in (2), §3·8, the contracted algebra of $S_{mn}[G, P]$ over a field $\mathfrak{F}$ is $M_{mn}[\mathfrak{A}, P]$, where $\mathfrak{A} = \mathfrak{A}(G)$. Here the element z is identified with the zero of $\mathfrak{A}$; since

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we shall always be concerned with the contracted algebra, we shall assume this throughout. It is well known that $\mathfrak{A}$ is semisimple if $\mathfrak{F}$ has characteristic zero or a prime not dividing the order of G. If P is non-singular qua matrix over $\mathfrak{A}$, then the semigroup is called a non-singular simple semigroup; the non-singularity of P depends only on the characteristic of $\mathfrak{F}$ ((2), §6·3).

Finally, if A is a matrix over an algebra $\mathfrak{A}$, and if θ is a matrix representation of $\mathfrak{A}$, then $\theta(A)$ will denote the matrix whose (i,j)th block is $\theta(a_{ij})$.

2. *Representations of non-singular simple semigroups.*

2·1. THEOREM. *Let $\{\gamma_r;\ r = 1, \ldots, k\}$ be a complete set of inequivalent irreducible representations of the semisimple algebra $\mathfrak{A}$ over $\mathfrak{F}$, and let P be a non-singular $n \times n$ matrix over $\mathfrak{A}$. Define the mapping γ'_r on $M_{nn}[\mathfrak{A}, P]$ by the rule $\gamma'_r(X) = \gamma_r(XP)$. Then $\{\gamma'_r;\ r = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of $M_{nn}[\mathfrak{A}, P]$ over $\mathfrak{F}$.*

Proof. There is a natural (1, 1) correspondence between the ideals of $\mathfrak{A}$ and those of $M_n(\mathfrak{A})$. In particular, if $\mathfrak{M}_r$ is the simple ideal of $\mathfrak{A}$ corresponding to the representation γ_r, then $M_n(\mathfrak{M}_r)$, which is the set of all $n \times n$ matrices over $\mathfrak{M}_r$, is a simple ideal of $M_n(\mathfrak{A})$ (see (2), §4·5) and $X \to \gamma_r(X)$ is the irreducible representation of $M_n(\mathfrak{A})$ derived from it. We denote this also by γ_r. It follows that $\{\gamma_r;\ r = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of $M_n(\mathfrak{A})$.

Let θ denote the mapping $X \to XP$ of $M_{nn}[\mathfrak{A}, P]$ into $M_n(\mathfrak{A})$. Then

$$\gamma'_r(X)\,\gamma'_r(Y) = \gamma_r(X^\theta)\,\gamma_r(Y^\theta) = \gamma_r\{(X \circ Y)^\theta\} = \gamma'_r(X \circ Y)$$

for any $X,\ Y \in M_{nn}[\mathfrak{A}, P]$, and so γ'_r is a representation. Now since P is non-singular, θ is an isomorphism of $M_{nn}[\mathfrak{A}, P]$ onto $M_n(\mathfrak{A})$; the irreducibility of γ'_r and the inequivalence of $\gamma'_r, \gamma'_s\ (r \neq s)$ follow immediately. Hence $\{\gamma'_r;\ r = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of $M_{nn}[\mathfrak{A}, P]$ over $\mathfrak{F}$.

The representations of a matrix semigroup over a group-with-zero have been discussed by Clifford (1). However, when the semigroup is non-singular, the above result enables us to find a complete set of irreducible representations in a very simple form.

2·2. COROLLARY. *Let $S = S_{nn}[G, P]$, let $\{\gamma_r;\ r = 1, \ldots, k\}$ be a complete set of inequivalent irreducible representations of G over a field $\mathfrak{F}$ of characteristic zero or a prime not dividing the order of G, and let P be non-singular qua matrix over $\mathfrak{A} = \mathfrak{A}(G)$. Define γ'_r by the rule*

$$\gamma'_r\{(x)_{ij}\} = \gamma_r\{(x)_{ij}P\} = \sum_{s=1}^{n} \gamma_r\{(xp_{js})_{is}\} \qquad (2\cdot2, 1)$$

for all $(x)_{ij} \in S$. Then $\{\gamma'_r;\ r = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$.

It should be noted that, since the representations are of the contracted algebra $M_{nn}[\mathfrak{A}, P]$, the identical representation of S (which maps all elements, including the zero, onto the identity of $\mathfrak{F}$) is not included in the above set.

2·3. We define the *character* χ of a representation γ of a semigroup S over a field of characteristic zero by the relation $\chi(a) = \text{trace } \gamma(a)$ for all $a \in S$.

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Let S, $\gamma_r, \gamma'_r (r = 1, \ldots, k)$ be as in 2·2, and let χ_r, χ'_r be the characters of γ_r, γ'_r respectively. Then from (2·2, 1) we have

$$\chi'_r\{(x)_{ij}\} = \chi_r(xp_{ji}). \tag{2·3, 1}$$

Let $\{C_t; t = 1, \ldots, h\}$ denote the conjugacy classes of G. To these we adjoin the class C_0 consisting of the zero of $\mathfrak{A}(G)$. Define the classes $C'_t (t = 0, \ldots, h)$ of $S_{nn}[G, P]$ by the rule that $(x)_{ij} \in C'_t$ if and only if $xp_{ji} \in C_t$. It then follows from (2·3, 1) that elements in the same class of S have the same character in every representation of S.

Now suppose that $\mathfrak{F}$ is a splitting field for G of characteristic zero. Then the characters of G satisfy certain well-known orthogonality relations. Similar relations hold for the characters of S. Thus, for example, we have

$$\sum_{a \in s} \chi'_r(a)\, \chi'_s(a) = \delta_{r\hat{s}} fg, \tag{2·3, 2}$$

where f is the number of non-zero idempotents in S, g is the order of G, and $\gamma_{\hat{s}}$ is the representation of G contragredient to γ_s; for

$$\begin{aligned}
\sum_{a \in S} \chi'_r(a)\, \chi'_s(a) &= \sum_{i,j=1}^{n} \sum_{x \in G} \chi'_r\{(x)_{ij}\}\, \chi'_s\{(x)_{ij}\} \\
&= \sum_{i,j=1}^{n} \sum_{x \in G} \chi_r(xp_{ji})\, \chi_s(xp_{ji}) \quad \text{from (2·3, 1)} \\
&= \sum_{\substack{\text{all pairs } i,j \text{ for} \\ \text{which } p_{ji} \neq 0}} \{ \sum_{y \in G} \chi_r(y)\, \chi_{\hat{s}}(y) \} \\
&= f \sum_{y \in G} \chi_r(y)\, \chi_{\hat{s}}(y^{-1}) \\
&= \delta_{r\hat{s}} fg,
\end{aligned}$$

from the orthogonality relation for the characters of G.

3. *The representations of a general semigroup whose algebra is semisimple.* Suppose that S is a semigroup of known structure, and that the algebra of S over a field $\mathfrak{F}$ is semisimple; this implies in particular that S is semisimple. Can we give a method for obtaining all the matrix representations of S over $\mathfrak{F}$?

3·1. THEOREM. *Let S be a semisimple semigroup whose algebra $\mathfrak{A}(S)$, over a field $\mathfrak{F}$ of characteristic zero or a prime not dividing the orders of any of the basic groups of the principal factors of S, is semisimple. Let*†

$$\mathfrak{A}(S) = \mathfrak{A}(S_1) \supset \mathfrak{A}(S_2) \supset \ldots \supset \mathfrak{A}(S_n) \supset \mathfrak{A}(S_{n+1}) = (0)$$

be the series of ideals of $\mathfrak{A}(S)$ corresponding to the principal series

$$S = S_1 \supset S_2 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset$$

of S, and let e_i be the identity element of the algebra $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1}) (i = 1, \ldots, n)$. Let $\{\gamma'_{ir}; r = 1, \ldots, k_i\}$ be a complete set of inequivalent irreducible representations of $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})$ over $\mathfrak{F}$. Define γ^*_{ir} on S by the rule

$$\gamma^*_{ir}(x) = \gamma'_{ir}(x^\theta e_i), \tag{3.1, 1}$$

† We use the convention that $\mathfrak{A}(\emptyset) = (0)$.

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where θ is the natural homomorphism of S onto $S - S_{i+1}$. Then

$$\{\gamma^*_{ir};\ r = 1, \ldots, k_i;\ i = 1, \ldots, n\}$$

is a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$.

Proof. We first note that since $\mathfrak{A}(S)$ is semisimple, then so also is $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})$ ((2), §3·3); hence in particular the element e_i exists ($i = 1, \ldots, n$). Furthermore, $\mathfrak{A}(S) - \mathfrak{A}(S_{i+1})$ may be regarded as the contracted algebra of $S - S_{i+1}$, and hence θ may be extended to be the natural homomorphism of $\mathfrak{A}(S)$ onto $\mathfrak{A}(S) - \mathfrak{A}(S_{i+1})$.

Let x, y be any elements of S. Then

$$\begin{aligned}
\gamma^*_{ir}(x)\,\gamma^*_{ir}(y) &= \gamma'_{ir}(x^\theta e_i)\,\gamma'_{ir}(y^\theta e_i),\\
&= \gamma'_{ir}(x^\theta e_i y^\theta e_i), \quad \text{since } \gamma'_{ir} \text{ is a representation of } \mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1}),\\
&= \gamma'_{ir}(x^\theta y^\theta e_i), \quad \text{since } e_i(y^\theta e_i) = y^\theta e_i,\\
&= \gamma'_{ir}\{(xy)^\theta e_i\} = \gamma^*_{ir}(xy).
\end{aligned}$$

Hence γ^*_{ir} is a representation of S.

Now γ^*_{ir} is irreducible, since it induces an irreducible representation of $S_i - S_{i+1}$, namely, γ'_{ir}. Further, it is clear that if $i \neq j$ or if $r \neq s$ then γ^*_{ir}, γ^*_{js} are inequivalent. But by (2), §3·5 the number of inequivalent irreducible representations of S is $\sum_{i=1}^{n} k_i$, and the result then follows.

The representations γ'_{ir} ($r = 1, \ldots, k_i$) may be obtained by the process given in 2·2. We must first, however, choose an isomorphism ϕ_i of $S_i - S_{i+1}$ onto some non-singular matrix semigroup $S_{n_i n_i}[G_i, P_i]$, say. This may be extended to an isomorphism of $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})$ onto $M_{n_i n_i}[\mathfrak{A}_i, P_i]$, where $\mathfrak{A}_i = \mathfrak{A}(G_i)$. The identity element e_i of $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})$ is then just the inverse image of P_i^{-1} under ϕ_i.

We now go on to discuss an important class of semigroups for which the above process can be somewhat simplified.

4. *Application to inverse semigroups.* An *inverse semigroup* (Preston (4)), or *generalized group* (Vagner (7)), is a semigroup S in which

(i) the equation $axa = a$ has a solution for any given $a \in S$, and

(ii) all idempotents commute.

It follows that the equations $axa = a$, $xax = x$ have a unique common solution,† and we denote this by a^{-1}. Hence if e is an idempotent, then $e^{-1} = e$. It may also be shown that

$$(ab)^{-1} = b^{-1}a^{-1}. \tag{4·1}$$

4·2. LEMMA. *A (finite) simple inverse semigroup is isomorphic with a matrix semigroup of the form $S_{nn}[G, U_n]$.*

Proof. Let S be a simple semigroup. Then $S \cong S_{mn}[G, P] = S'$, say, for some m, n, G, P. It is readily seen that the non-zero idempotents of S' are the elements $(p_{ji}^{-1})_{ij}$, where $p_{ji} \in G$. Now suppose that S is also an inverse semigroup, and let $p_{ji}, p_{ki} \in G$. Then we require that $(p_{ji}^{-1})_{ij}\circ(p_{ki}^{-1})_{ik}{}' = (p_{ki}^{-1})_{ik}\circ(p_{ji}^{-1})_{ij}$; that is, $(p_{ki}^{-1})_{ik} = (p_{ji}^{-1})_{ij}$. Hence there is exactly one non-zero element in each row of P. (There is at least one, since

† Conversely, if S has this property, then it is an inverse semigroup (see (3)).

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S' is regular.) Similarly, there is exactly one non-zero element in each column of P. Thus $m = n$, and P is a 'monomial' matrix over $G(0)$. In particular, P is non-singular, and P^{-1} is also a monomial matrix. Finally, from the relation $P^{-1}PU_n = U_n$ we have $S' \cong S_{nn}[G, U_n]$ ((6), Theorem 2·91).

Conversely, every semigroup of the form $S_{nn}[G, U_n]$ is a simple inverse semigroup. As was pointed out by Clifford ((1), §8), such a semigroup is obtained by adjoining a zero element to a Brandt groupoid. A simple inverse semigroup with zero will thus be termed a *Brandt semigroup*.

We also note the following result.

4·3. COROLLARY. *A simple inverse semigroup without zero is a group.*

This is immediate from the above proof.

Now consider a general inverse semigroup S. Since each ideal and difference semigroup of an inverse semigroup is again an inverse semigroup ((3), Corollary 3), it follows that each principal factor of S is an inverse semigroup, and hence in particular must be simple; that is, S is semisimple. We then have

4·4. THEOREM. *Let S be an inverse semigroup, and $\mathfrak{F}$ a field of characteristic zero or a prime not dividing the orders of any of the basic groups of the principal factors of S. Then $\mathfrak{A}(S)$ is semisimple.*

Proof. The principal factors of S other than the kernel are Brandt semigroups, and by 4·3 the kernel is a group. Now the contracted algebra of $S_{nn}[G, U_n]$ is $M_{nn}[\mathfrak{A}, U_n]$, where $\mathfrak{A} = \mathfrak{A}(G)$, and this is semisimple if $\mathfrak{A}$ is semisimple. Hence, with the above restriction on the characteristic of $\mathfrak{F}$, the algebra of each principal factor is semisimple, and so $\mathfrak{A}(S)$ is semisimple ((2), §3·3).

We now apply the representation theory developed in the previous section. The matrix representations of a Brandt semigroup have been discussed by Clifford ((1), §8): we may, however, deduce his result from 2·2.

4·5. THEOREM. *Let $S = S_{nn}[G, U_n]$, and let $\mathfrak{F}$ be a field of characteristic zero or a prime not dividing the order of G. Let $\{\gamma_r;\ r = 1, \ldots, k\}$ be a complete set of inequivalent irreducible representations of G over $\mathfrak{F}$. Define γ'_r on S by the rule that*

$$\gamma'_r\{(x)_{ij}\} = \gamma_r\{(x)_{ij}\}$$

for all $(x)_{ij} \in S$. Then $\{\gamma'_r;\ r = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$.

In particular, if γ_1 is the identical representation of G, then γ'_1 is the representation of S by the set of n^2 'matrix units'.

Before proceeding to the case of the general inverse semigroup, we note the following:

4·6. LEMMA. *Let S be a (finite) inverse semigroup with a principal series*

$$S = S_1 \supset S_2 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset,$$

and let e and f be distinct non-zero idempotents of the principal factor

$$S_i - S_{i+1} \quad (1 \leqslant i \leqslant n-1).$$

Let a be any element of $S - S_{i+1}$. Then if ae and af are non-zero in $S_i - S_{i+1}$, they lie in distinct minimal left and right ideals of $S_i - S_{i+1}$.

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Proof. Suppose ae and af are non-zero and belong to the same minimal left ideal L of $S_i - S_{i+1}$. Then since $S_i - S_{i+1}$ is a Brandt semigroup, there exists a unique non-zero idempotent g in L (see 4·2), and $(ae)g = ae$, $(af)g = af$. But the product of two distinct idempotents of a Brandt semigroup is zero, and so we must have $e = g = f$, which contradicts our hypothesis.

Now suppose that ae and af are non-zero and belong to the same minimal right ideal R of $S_i - S_{i+1}$. Then there exists a unique non-zero idempotent h in R, and so we have $h = (ae)(ae)^{-1} = (af)(af)^{-1}$. But by (4·1) $(ae)^{-1} = e^{-1}a^{-1} = ea^{-1}$, and $(af)^{-1} = fa^{-1}$. Hence $aea^{-1} = h = afa^{-1}$, and so $aea^{-1}a = afa^{-1}a$. But $a^{-1}a$ is an idempotent, and all idempotents of $S - S_{i+1}$ commute. Hence $aa^{-1}ae = aa^{-1}af$, and thus $ae = af$. We then proceed as in the first part of the proof, and obtain the contradiction $e = f$.

4·7. THEOREM. *Let S be an inverse semigroup. Let*

$$S = S_1 \supset S_2 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset$$

be any principal series for S, and let $\{e_{ij};\ j = 1, \ldots, m_i\}$ be the set of non-zero idempotents of $S_i - S_{i+1}$. (We take $m_n = 1$ and e_{n1} to be the identity of the group S_n.) Let $\mathfrak{F}$ be a field of characteristic zero or a prime not dividing the orders of any of the basic groups of any of the principal factors of S, and let $\{\gamma'_{ir};\ r = 1, \ldots, k_i\}$ be a complete set of inequivalent irreducible representations of $S_i - S_{i+1}$ over $\mathfrak{F}$ $(i = 1, \ldots, n)$. Define γ^*_{ir} on S by the rule

$$\gamma^*_{ir}(x) = \sum_{j=1}^{m_i} \gamma'_{ir}(x^\theta e_{ij}), \tag{4·7, 1}$$

where θ is the natural homomorphism of S onto $S - S_{i+1}$. Then

$$\{\gamma^*_{ir};\ r = 1, \ldots, k_i;\ i = 1, \ldots, n\}$$

is a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$.

Proof. From 4·4, $\mathfrak{A}(S)$ is semisimple. Since $S_i - S_{i+1}$ is a Brandt semigroup $(i = 1, \ldots, n-1)$, and since $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})$ may be regarded as the contracted algebra of $S_i - S_{i+1}$ over $\mathfrak{F}$, the identity element e_i of $\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})$ is just $\sum_{j=1}^{m_i} e_{ij}$. Hence

$$\begin{aligned} \sum_{j=1}^{m_i} \gamma'_{ir}(x^\theta e_{ij}) &= \gamma'_{ir}\left(\sum_{j=1}^{m_i} x^\theta e_{ij}\right) \\ &= \gamma'_{ir}(x^\theta e_i), \end{aligned}$$

and this remains true for $i = n$. The result then follows from 3·1.

The special case in which S is a semilattice of groups has already been discussed ((2), § 9·6).

Applying 4·6 to (4·7, 1) we observe that for fixed x, i and different values of j, those elements $x^\theta e_{ij}$ which are non-zero in $S_i - S_{i+1}$ lie in different minimal left and right ideals of $S_i - S_{i+1}$. Thus if we obtain the representation γ'_{ir} as in 4·5 we see that $\gamma^*_{ir}(x)$ is a block matrix which has at most one non-zero block in each row and column. In particular, if γ'_{i1} is the representation of $S_i - S_{i+1}$ by the set of m_i^2 matrix units, then the corresponding representation γ^*_{i1} of S is a representation by $m_i \times m_i$ matrices having at most one non-zero entry in each row and column, namely, the unit element of $\mathfrak{F}$. There is a natural (1, 1) correspondence between the set of all such matrices and the set of all partial transformations† of a set of m_i objects. For let τ be a partial

† A partial transformation is a (1, 1) mapping of a subset of a set onto another subset.

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transformation mapping the subset $\mathscr{A}$ onto some other subset; then we define the corresponding matrix T by the rules

$$\text{(i)} \quad T_{ij} = 0 \quad \text{if} \quad i \notin \mathscr{A} \qquad \text{and} \qquad \text{(ii)} \quad T_{ij} = \delta_{\tau(i),j} \quad \text{if} \quad i \in \mathscr{A}.$$

The transformation mapping the empty subset onto itself corresponds to the zero matrix. Furthermore, if we define the product of two partial transformations to be the mapping product in the obvious way, then the correspondence $\tau \to T$ is an isomorphism.

The representation γ^*_{i1} may thus be considered as a representation of S by a semigroup of partial transformations. Thus we have

4·8. COROLLARY. *Corresponding to each principal factor $S_i - S_{i+1}$ of S there is a homomorphic mapping of S onto a semigroup of partial transformations.*

Representations of inverse semigroups by partial transformations have been considered by Preston (5) and Vagner (7).

It should be noted that the representation γ^*_{n1} derived from the kernel of S is the identical representation $\gamma^*_{n1}(x) = 1$ for all $x \in S$.

4·9. We conclude with some remarks on the identity element of the algebra of an inverse semigroup S.

From 4·4, the algebra of S over a field of suitably restricted characteristic is semisimple and thus contains an identity. However, such an element exists for an arbitrary field. Let E denote the semigroup of idempotents of S. Then E is a semilattice of groups of order 1, and so by 4·4 $\mathfrak{A}(E)$ is semisimple. Let u denote the identity of $\mathfrak{A}(E)$, and let a be any element of S. Then $ua = u(aa^{-1}a) = (u(aa^{-1}))\,a = aa^{-1}a = a$, and similarly $au = a$; hence u is an identity for $\mathfrak{A}(S)$.

We now obtain u in terms of the elements of E. Let $e, f \in E$. Then we write $e \geqslant f$ if $ef = f$; this defines a partial ordering of E. Let F denote the set of all elements of E which are maximal with respect to '$\geqslant$'; suppose that there are m such elements. Let s_r denote the sum of all products of r distinct elements of F ($r = 1, \ldots, m$). Then†

$$u = s_1 - s_2 + \ldots + (-1)^{m-1} s_m. \tag{4·9, 1}$$

To prove this, let $w = s_1 - s_2 + \ldots + (-1)^{m-1} s_m$; then $u - w = \prod_{f \in F} (u - f)$. Now let e be any element of E, and let f_1 be any element of F such that $f_1 \geqslant e$. Then $(u - f_1)\,e = 0$, and so, since E is commutative, $(u - w)\,e = 0$. Similarly, $e(u - w) = 0$. Thus $we = e = ew$, and so $w = u$.

We say that e *covers* f if $e > f$ and there exists no $g \in E$ such that $e > g > f$. If E is such that each element other than the zero covers exactly one other element, then it is termed a *tree*. For such a semilattice it may be shown that the identity element is expressible as the linear combination

$$u = \sum_{e \in E} (1 - n_e)\,e, \tag{4·9, 2}$$

where n_e is the number of elements of E covering e (the coefficients $(1 - n_e)$ being, of course, taken mod p in a field of characteristic p).

† This result is due to Mr R. Penrose.

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