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Limit groups, positive-genus towers and measure-equivalence

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Abstract. By definition, an ω-residually free tower is positive-genus if all surfaces used in its construction are of positive-genus. We prove that every limit group is virtually a subgroup of a positive-genus, ω-residually free tower. By combining this construction with results of Gaboriau, we prove that elementarily free groups are measure-equivalent to free groups.

1. *Introduction*

Measure-equivalence was introduced by Gromov in [**10**] as a measure-theoretic analogue of quasi-isometry. After proving that two finitely generated groups are quasi-isometric if and only if they admit commuting, properly discontinuous actions by homeomorphisms on some locally compact space with *compact* fundamental domain, he went on to define measure-equivalence by requiring instead that the groups admit commuting, free, measure-preserving actions on a Lebesgue measure space with *finite measure* fundamental domain. The *uniform* lattices in a fixed, locally compact, second countable group are all quasi-isometric, whereas *all* lattices in such a group are measure-equivalent, regardless of whether they are uniform or not. For motivation and background we refer the reader to the papers of Damien Gaboriau, particularly [**7**].

Much progress has been made in distinguishing measure-equivalence classes (see, e.g., [**1, 5–8, 17**]), but there have been many fewer constructions of examples of measure-equivalent groups. The only groups whose measure-equivalence classes are completely classified are finite groups, amenable groups, and lattices in simple Lie groups of higher rank (see [**5**]). In particular, the measure-equivalence class of non-abelian free groups is still quite poorly understood: there are some obvious examples, such as the

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fundamental groups of compact surfaces of genus ≥ 2, which are also lattices in $\mathrm{SL}(2,\mathbb{R})$, but beyond this we know very little.

In [**9**], Gaboriau constructs some new examples of groups measure-equivalent to free groups, encapsulated in the following theorem.

THEOREM 1.1. (Gaboriau) *Let Σ be a compact orientable surface of positive-genus, with one boundary component. Then the iterated amalgamated product*

$$\pi_1(\Sigma) *_{\langle\partial\Sigma\rangle} \pi_1(\Sigma) *_{\langle\partial\Sigma\rangle} \cdots *_{\langle\partial\Sigma\rangle} \pi_1(\Sigma)$$

is measure-equivalent to a free group.

The amalgamated product in the above statement is an example of a *limit group* (indeed, it is an elementarily free group). Gaboriau [**9**] asked if all limit groups are measure-equivalent to free groups.

Limit groups have been studied under a variety of names and guises (see [**2, 4, 12, 13**]). The name limit group was introduced by Sela in his solution of the Tarski problem (see [**14**] *et seq.*; cf. [**12**], [**13**] *et seq.*). The *elementary theory* of a group G is the set of first-order sentences that are true in G. The Tarski problem asks which groups have the same elementary theory as the free group of rank two. The *existential theory* consists of those sentences involving only existential quantifiers. Limit groups turn out to be precisely the finitely generated groups with the same existential theory as a free group. Another class, still more closely related to free groups, is the class of *elementarily free groups*; those groups with the same elementary theory as a free group. It turns out that every elementarily free group is a (Gromov) hyperbolic limit group. However, there are hyperbolic limit groups that are *not* elementarily free.

Using results of Kharlampovich and Myasnikov [**12, 13**] and Sela [**14**], one can give more constructive definitions of limit groups and elementarily free groups. For the purposes of this paper, both classes are defined in terms of ω-residually free towers, which are in turn defined as the fundamental groups of certain complexes, inductively constructed from graphs, surfaces and tori. The limit groups are precisely the finitely generated subgroups of ω-residually free towers, but not every limit group is itself such a tower (see §2). We call a tower *positive-genus* if every surface used in its construction is of positive-genus. Our main result concerning towers is given in the following.

THEOREM A. (Theorem 2.11) *Every limit group L is virtually a subgroup of a positive-genus, ω-residually free tower. If L is elementarily free, then the tower can be chosen to be elementarily free (hyperbolic).*

In general, it is necessary to pass to a finite-index subgroup in order to embed a limit group in a positive-genus tower. For example, if L is constructed by amalgamating the fundamental group of a punctured sphere along its boundary components with another limit group L' in such a way that the resulting two-vertex graph-of-groups decomposition of L is the cyclic Jaco, Shalen, Johannson (JSJ) decomposition, then L does not embed in a positive-genus tower.

Using Theorem A and the results of [**9**], we deduce the following theorem, which partially answers Gaboriau's question.

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THEOREM B. (Theorem 3.6) *Every elementarily free group is measure-equivalent to a free group.*

This paper is organized as follows. In §1 we construct various finite-index subgroups of towers and prove Theorem A. In §2 we recapitulate some useful results of [**9**] and prove that elementarily free groups are measure-equivalent to free groups. In §3 we discuss methods of attacking the case of all limit groups.

2. *Limit groups*

2.1. *ω-residually free towers.*

For an introduction to the theory of limit groups see, for instance, [**3**] or [**4**].

Definition 2.1. An *ω-residually free tower space of height* 0, denoted X_0, is a one-point union of finitely many compact graphs, n-tori, and closed hyperbolic surfaces of Euler characteristic less than -1.

An *ω-residually free tower space of height* h, denoted X_h, is obtained from an ω-residually free tower space X_{h-1} of height $h-1$ by attaching one of two sorts of blocks.

(1) *Quadratic block.* Let Σ be a connected compact hyperbolic surface with boundary, with each component either a punctured torus or having $\chi \leq -2$. Then X_h is the quotient of $X_{h-1} \sqcup \Sigma$ obtained by identifying the boundary components of Σ with homotopically non-trivial curves on X_{h-1}, in such a way that there exists a retraction $\rho : X_h \to X_{h-1}$. The retraction is also required to satisfy the property that $\rho_*(\pi_1(\Sigma))$ be non-abelian.

(2) *Abelian block.* Let T be an n-torus, and fix a coordinate circle γ. Fix a loop c in X_{h-1} that generates a maximal abelian subgroup in $\pi_1(X_{h-1})$. Then X_h is the quotient of $X_h \sqcup (S^1 \times [0,1]) \sqcup T$ obtained by identifying $S^1 \times \{0\}$ with c, and $S^1 \times \{1\}$ with γ.

Attaching an abelian block is also often called *extending a centralizer*.

An ω-residually free tower space is called *hyperbolic* if no tori are used in its construction, i.e. there are no tori in X_0 and no abelian blocks.

Definition 2.2. An (*ω-residually free*) *tower of height* h, denoted L_h, is the fundamental group of an ω-residually free tower space of height h.

The following deep theorem from [**16**] (cf. [**12**] *et seq.*) will, for our purposes, serve as a definition of elementarily free groups.

THEOREM 2.3. *A group is* elementarily free *if and only if it is the fundamental group of a hyperbolic ω-residually free tower space.*

Towers are examples of limit groups. Another theorem of Sela [**15**] and Kharlampovich and Myasnikov [**13**] will serve as a definition of limit groups.

THEOREM 2.4. *A group is a* limit group *if and only if it is a finitely generated subgroup of an ω-residually free tower.*

A little care is needed here. While every limit group embeds in a tower, there exist hyperbolic limit groups that do not embed in hyperbolic towers.

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Improving on Theorem 2.4, Kharlampovich and Myasnikov proved that every limit group embeds in a tower in which no surface pieces are used at level 0 and no quadratic blocks are used ([**13**]; cf. [**4**]).

We will need the following weak form of the well-known fact that limit groups are precisely the finitely generated ω-residually free groups.

LEMMA 2.5. *Limit groups are* residually free*; that is, if L is a limit group and $g \in L - \{1\}$ then there exists a homomorphism to a free group $f : L \to F$ with $f(g) \neq 1$.*

A key feature of the definition of a tower is the retraction $\rho : X_h \to X_{h-1}$. If the block at height h is abelian, the retraction simply projects T onto the coordinate circle γ, and thus to c. In both cases, ρ induces a retraction $\rho_* : L_h \to L_{h-1}$ on the level of fundamental groups.

An ω-residually free tower space X_h has a natural graph-of-spaces† decomposition Γ_X, with two vertex spaces, namely X_{h-1} and the block at height h; the edge spaces are circles. We will often use the retraction to pull finite covers back from X_{h-1} to X_h. It is worth noting that such pullbacks inherit a similar graph-of-spaces decomposition from X_h.

LEMMA 2.6. *Let X be a CW-complex with a graph-of-spaces decomposition Γ_X, such that there is a retraction $\rho : X \to X'$ to a vertex space. Let $Y' \to X'$ be a connected covering of degree d, and let $Y \to X$ be the connected covering obtained by pulling back along ρ; that is, $\pi_1(Y) = \rho_*^{-1}(\pi_1(Y'))$. Then the following conditions apply:*

(1) *$Y \to X$ is of degree d and inherits a graph-of-spaces decomposition Γ_Y;*
(2) *the pre-image of X' in Y is a (connected) vertex space of Γ_Y naturally homeomorphic to Y';*
(3) *$Y \to X$ extends $Y' \to X'$, and Y inherits a retraction to Y' covering ρ.*

A tower L_h inherits, by the Seifert–van Kampen theorem, a graph-of-groups decomposition Γ_L from the graph-of-spaces decomposition Γ_X of the associated ω-residually free tower space X_h.

2.2. *Positive-genus towers.* The purpose of this section is to prove that, up to finite index, the quadratic blocks in Definition 2.1 can be assumed to have positive-genus. A compact, connected surface Σ with Euler characteristic $\chi(\Sigma)$ and $b(\Sigma)$ boundary components is of *positive-genus* if $\chi(\Sigma) + b(\Sigma) \leq 0$. Note that, in particular, all finite covers of such Σ also have positive-genus.

Definition 2.7. An ω-residually free tower space is *positive-genus* if every quadratic block used in its construction is of positive-genus. A tower is *positive-genus* if it is the fundamental group of a positive-genus, ω-residually free tower space.

Our strategy for obtaining positive-genus quadratic blocks is to identify connected p-sheeted coverings (here p is a prime number) that restrict to a p-sheeted covering on each boundary component. We achieve this by passing to a finite-index subgroup of the tower that admits a map to $\mathbb{Z}/p\mathbb{Z}$ which maps each attaching loop of the top quadratic

† By convention, our graphs of spaces are connected and have connected vertex and edge spaces.

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block non-trivially. In particular, we must arrange for the attaching loops to become non-trivial in homology.

LEMMA 2.8. *If G is a finitely generated residually free group and $g_1, \ldots, g_m$ is a finite collection of elements of G, then there exists a finite-index subgroup $K \subset G$ so that, whenever $k \in K$ is a power of a conjugate of some g_i, k has infinite order in $H_1(K)$.*

Proof. Let F denote the free group of rank two. Since L is residually free, for each i there exists a homomorphism $f_i : L \to F$ with $f_i(g_i) \neq 1$. By Hall's theorem [**11**], there exists a finite-index subgroup $F_i \subset F$ containing $f_i(g_i)$, such that $f_i(g_i)$ is primitive in $H_1(F_i)$. Let K be the finite-index subgroup $\bigcap_i f_i^{-1}(F_i)$ of G. If $k \in K$ is a power of a conjugate of g_i then, since $f_i(g_i)$ has infinite order in $H_1(F_i)$, it follows that k has infinite order in $H_1(K)$. □

We can now construct a map to $\mathbb{Z}/p\mathbb{Z}$ as required.

LEMMA 2.9. *Let K be a finitely generated group and let $k_1, \ldots, k_n$ be a collection of elements in K that are all of infinite order in $H_1(K)$. Then, for all sufficiently large primes p, there exists a homomorphism $\varphi : G \to \mathbb{Z}/p\mathbb{Z}$ for which $\varphi(k_j)$ is non-trivial for all j.*

Proof. Since $\mathbb{Z}^n$ is ω-residually free, there exists a homomorphism $K \to \mathbb{Z}$ under which each k_j has non-trivial image. Choose a prime p that does not divide any of the images of the k_j in $\mathbb{Z}$. In particular, each k_j has non-trivial image under the composition

$$\varphi : K \to H_1(K) \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}.$$ □

We shall apply the preceding lemmas to the height $h-1$ subspace X_{h-1} and pull back to the full tower to obtain the positive-genus cover that we seek.

Recall that, for X a topological space, $c : S^1 \to X$ a loop, and $Y \to X$ a covering map, the *elevations* of c to Y are the minimal connected covers $\hat{S}^1 \to S^1$ such that $\hat{S}^1 \to X$ lifts to Y. Fixing basepoints, it follows from standard covering space theory that $\pi_1(\hat{S}^1)$ is the pre-image of $\pi_1(Y)$ in $\pi_1(S^1)$.

PROPOSITION 2.10. *Let X_h be an ω-residually free tower space constructed by attaching a quadratic block Σ to a space X_{h-1} of height $h-1$. Then there exists a connected cover $Z_h \to X_h$ with an inherited graph-of-spaces decomposition Γ_Z, with one vertex space a connected cover $Z_{h-1} \to X_{h-1}$, and the remaining vertex spaces connected covers $\bar{\Sigma}_i \to \Sigma$, where each $\bar{\Sigma}_i$ has positive-genus. The retraction $\rho : X_h \to X_{h-1}$ pulls back to a retraction $Z_h \to Z_{h-1}$.*

Proof. Let $c_1, \ldots, c_m$ be the images of the boundary curves of Σ in X_{h-1}. Since $\pi_1(X_{h-1})$ is residually free, the finite-index subgroup provided by Lemma 2.8 corresponds to a finite-sheeted covering $Y_{h-1} \to X_{h-1}$ so that if $d_1, \ldots, d_n$ are the elevations of the c_i, each d_j is of infinite order in homology. Let $Y_h \to X_h$ be the covering obtained by pulling back along the retraction ρ, with a graph-of-spaces decomposition Γ_Y.

By Lemma 2.9, there exists a homomorphism $\varphi : \pi_1(Y_{h-1}) \to \mathbb{Z}/p\mathbb{Z}$ with $\varphi(d_j) \neq 0$ for each j. Let $Z_{h-1} \to Y_{h-1}$ be the covering corresponding to the kernel of φ. Finally, pull the covering $Z_{h-1} \to Y_{h-1}$ back along the retraction $Y_h \to Y_{h-1}$ to give a covering $Z_h \to Y_h$, with a graph-of-spaces decomposition Γ_Z.

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The key point to observe is that each edge space of Γ_Y is only covered by one edge space of Γ_Z. Indeed, $Z_h \to Y_h$ is a covering of degree p, but the image in $\mathbb{Z}/p\mathbb{Z}$ of every cycle d_j has order p, so an elevation of it to Z_h covers d_j with degree p. Thus d_j only has one elevation to Z_h.

It follows that the underlying graphs of Γ_Y and Γ_Z are the same. Consider a surface vertex $\bar{\Sigma}_i$ of Γ_Z, covering a surface vertex Σ_i of Γ_Y. By construction $b(\bar{\Sigma}_i) = b(\Sigma_i)$, so we have

$$\chi(\bar{\Sigma}_i) + b(\bar{\Sigma}_i) = p\,\chi(\Sigma_i) + b(\Sigma_i).$$

Since $\chi(\Sigma_i) \leq -1$ and $\chi(\Sigma_i) + b(\Sigma_i) \leq 2$, it follows that $\bar{\Sigma}_i$ has positive-genus for $p \geq 3$. □

The above result is all we need to prove that elementarily free groups are measure-equivalent to free groups. However, it is perhaps more cleanly expressed in terms of the following theorem, which we believe to be of independent interest.

THEOREM 2.11. *Every limit group L has a finite-index subgroup M that is a subgroup of a positive-genus tower P. If L is elementarily free then P can be taken to be elementarily free (hyperbolic).*

By Theorem 2.4 it suffices to prove the theorem for towers. More precisely, we prove the following.

PROPOSITION 2.12. *Let L_h be a tower of height h. Then there exists a finite-index subgroup $M_h \subset L_h$ that embeds into a positive-genus tower P_h. If L_h is elementarily free then P_h can be taken to be elementarily free. If $A \subset M_h$ is a maximal abelian subgroup then A is also maximal abelian in P_h.*

Proof. The proof is by induction on height. By definition, every level 0 tower is positive-genus. Consider L_h, the fundamental group of an ω-residually free tower space X_h of height h, obtained as usual by attaching a block to a height $h-1$ space X_{h-1} with fundamental group L_{h-1}.

First we consider the case of a quadratic block Σ. By induction, L_{h-1} has a finite-index subgroup M_{h-1} that is a subgroup of a positive-genus tower P_{h-1}. By Proposition 2.10, L_h has a finite-index subgroup $K_h = \pi_1(Z_h)$ with graph-of-groups decomposition Γ_K, with one vertex labelled by K_{h-1} a finite-index subgroup of L_{h-1} and the remaining vertices labelled by the fundamental groups of surfaces of positive-genus, amalgamated with K_{h-1} along boundary components. Set $M_h = \rho_*^{-1}(K_{h-1} \cap M_{h-1})$. Then M_h inherits a graph-of-groups decomposition Γ_M, with one vertex labelled by $K_{h-1} \cap M_{h-1}$ and the remainder by fundamental groups of surfaces of positive-genus, amalgamated with $K_{h-1} \cap M_{h-1}$ along boundary components. The retraction $\rho_* : L_h \to L_{h-1}$ restricts to a retraction $M_h \to K_{h-1} \cap M_{h-1}$. Enlarge Γ_M to Γ_P by replacing $K_{h-1} \cap M_{h-1}$ with P_{h-1}. Extending ρ_* by the identity on P_{h-1}, it is clear that $P_h = \pi_1(\Gamma_P)$ is a positive-genus tower.

The case of an abelian block T is similar. By induction, there exists a finite-index subgroup $M_{h-1} \subset L_{h-1}$ that embeds in a positive-genus tower P_{h-1}. The pullback $M_h = \rho_*^{-1}(M_{h-1})$ inherits a graph-of-groups decomposition Γ_M, with one vertex labelled

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by M_{h-1} and the remainder by finitely generated free abelian groups. Each abelian vertex has a coordinate factor amalgamated with a cyclic maximal abelian subgroup of M_{h-1}. Enlarge Γ_M to Γ_P by replacing M_{h-1} by P_{h-1}. Since cyclic maximal abelian subgroup of M_{h-1} are maximal abelian in P_{h-1}, the resulting fundamental group $P_h = \pi_1(\Gamma_P)$ is a positive-genus tower.

It remains to show that a maximal abelian subgroup A of M_h is maximal abelian in P_h. For this we need a Bass–Serre theory. The result follows by induction on height if A is elliptic in Γ_P, so assume that A is not elliptic and let $g \in P_h$ be such that $[g, a] = 1$ for all $a \in A$.

Consider the Bass–Serre tree T_P of Γ_P. Let v be a vertex of T_P stabilized by P_{h-1}. Certain edges $\{e_i\}$ adjoining v correspond to the cosets of $(M_h : C) \subset (P_h : C)$ for some edge group C of Γ_P. Consider the subspace $T_M \subset T_P$ given by the M-translates of v and the e_i. Then T_M is connected since the normal form in Γ_M of an $m \in M_h$ specifies a path from v to mv and furthermore, by construction, T_M is in fact the Bass–Serre tree of Γ_M.

Let $l \subset T_P$ be the unique line stabilized by A. Fix an edge e in l. Then ge is an edge of l, so lies in T_M. There is only one M_h-orbit of e in T_M, so there exists $m \in M_h$ such that $me = ge$. The stabilizer of e lies in M_h, so it follows that $g \in M_h$. Since A was maximal abelian in M_h, we obtain $g \in A$. □

3. *Measure-equivalence*

We are now in a position to use the results of [**9**] to prove that elementarily free groups are measure-equivalent to free groups.

3.1. *Definition and properties.*

Definition 3.1. Two countable groups G_1, G_2 are *measure-equivalent* if there exist commuting, measure-preserving, (essentially) free actions on some measure space (Ω, m), such that the action of G_i admits a finite measure fundamental domain. Write

$$G_1 \overset{\text{ME}}{\sim} G_2.$$

The standard examples of measure-equivalent groups are commensurable groups and lattices in the same locally compact second countable group. We will not use the definition of measure-equivalence directly, but deduce our result from the following properties.

THEOREM 3.2. (**P**$_{\mathbf{ME}}$**7** in [**9**]) *If G_1 and G_2 are measure-equivalent to a free group then so is $G_1 * G_2$.*

THEOREM 3.3. (**P**$_{\mathbf{ME}}$**9** in [**9**]) *If G is measure-equivalent to a free group and $H \subset G$ is a subgroup then H is measure-equivalent to a free group.*

Theorem 1.1 is a special case of the following more general result of Gaboriau.

THEOREM 3.4. (Corollary 3.19 of [**9**]) *Consider a countable group G measure-equivalent to a free group, and $C \subset G$ an infinite cyclic subgroup. If Σ is a compact orientable surface of positive-genus with a single boundary component then $G *_{C=\langle \partial \Sigma \rangle} \pi_1(\Sigma)$ is also measure-equivalent to a free group.*

We generalize Theorem 3.4 to the case of multiple boundary components.

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COROLLARY 3.5. *Consider a path-connected space X with $G = \pi_1(X)$ measure-equivalent to a free group. Let Σ be a compact, orientable surface of positive-genus with non-empty boundary. Let X' be the quotient of $X \sqcup \Sigma$ obtained by identifying the boundary curves of Σ with loops in X that generate infinite cyclic subgroups of $\pi_1(X)$. Then $\pi_1(X')$ is measure-equivalent to a free group.*

Proof. By cutting Σ along a certain simple closed curve γ, we can decompose it as $\Sigma_1 \cup_\gamma \Sigma_2$, where Σ_1 is planar and Σ_2 is of positive-genus and has one boundary component. The space X' acquires a similar decomposition as $X_1 \cup_\gamma \Sigma_2$, where X_1 is obtained from X by amalgamating loops on X with all of the boundary curves of Σ_1 except γ. Note that Σ_1 deformation retracts onto the graph formed by the boundary circles $c_1, \ldots, c_n$ other than γ, together with a disjoint collection of arcs α_j $(j = 2, \ldots, n)$ connecting c_1 to c_j. This deformation retraction extends to a deformation retraction of X_1 onto the union of X and the arcs α_j. It follows from Theorem 3.2 that $\pi_1(X_1) \cong \pi_1(X) * F_{n-1}$ is measure-equivalent to a free group. Thus $\pi_1(X') = \pi_1(X_1) *_{\langle \partial \Sigma_2 \rangle} \pi_1(\Sigma_2)$ is measure-equivalent to a free group, by Theorem 3.4. □

We are now ready to prove that elementarily free groups are measure-equivalent to free groups.

3.2. *Elementarily free groups.*

THEOREM 3.6. *Every elementarily free group is measure-equivalent to a free group.*

Proof. By Theorem 2.11, it suffices to prove the result for positive-genus, elementarily free groups.

At height 0, X_0 is a one-point union of graphs and hyperbolic surfaces. Hyperbolic surface groups are lattices in $\mathrm{PSL}_2(\mathbb{R})$, so are measure-equivalent to a free group. Thus, by Theorem 3.2, $\pi_1(X_0)$ is measure-equivalent to a free group.

At height h, assume that X_h is obtained as usual by gluing a surface Σ to X_{h-1}. By induction, $\pi_1(X_{h-1})$ is measure-equivalent to a free group. There are two cases to consider.

If Σ is orientable, then the result is given by Corollary 3.5.

If Σ is non-orientable, then it has an orientable double cover $\Sigma' \to \Sigma$ of positive-genus, with twice the number of boundary components. The amalgam of Σ' with two disjoint copies of X_{h-1} gives a double cover $X'_h \to X_h$. Identify a point in each copy of X_{h-1} to create a space Y. By Proposition 3.2, $\pi_1(X_{h-1} \vee X_{h-1}) = \pi_1(X_{h-1}) * \pi_1(X_{h-1})$ is measure-equivalent to a free group.

We have built Y by gluing the orientable surface of positive-genus Σ' to $X_{h-1} \vee X_{h-1}$, and each boundary component of Σ' defines an element of infinite order in one of the free factors of $\pi_1(X_{h-1} \vee X_{h-1})$. It follows by Corollary 3.5 that $\pi_1(Y)$ is measure-equivalent to a free group. Since $\pi_1(Y) \cong \pi_1(X'_h) * \mathbb{Z}$, the result follows from Theorem 3.3. □

4. *The case of arbitrary limit groups*

In the light of Theorem 3.3, to show that all limit groups are measure-equivalent to free groups it would suffice to prove that ω-residually free towers are measure-equivalent to free groups. Even the case of $F_C = F *_{C=Z} \mathbb{Z}^n$, where C is a maximal cyclic subgroup of F

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and Z is a direct factor of $\mathbb{Z}^n$, seems non-trivial. The methods of the proof of Theorem 3.4 in [**9**] suggest a possible approach.

Let (X, μ) be a probability measure space, and consider an essentially free measure-preserving action α of the group G on X. The *orbit relation* of the action is the equivalence relation given by the orbits of G, and is denoted $\mathcal{R}_\alpha$. There is a notion of free products for equivalence relations, motivated by the normal form for free products of groups.

Definition 4.1. Consider measured equivalence relations $\mathcal{R}, \mathcal{A}$ and $\mathcal{B}$ on X. Write $\mathcal{R} = \mathcal{A} * \mathcal{B}$ if:

(1) $\mathcal{R}$ is generated by $\mathcal{A}$ and $\mathcal{B}$;

(2) for all $p \geq 1$, for almost all $2p$-tuples $(x_i)_{i \in \mathbb{Z}/2p\mathbb{Z}}$ such that

$$x_{2j-1} \sim_{\mathcal{A}} x_{2j} \sim_{\mathcal{B}} x_{2j+1}$$

for each j, one has $x_i = x_{i+1}$ for some i.

Gaboriau [**9**] defines a subgroup $H \subset G$ to be a *measure-free factor* if there exists a free probability-measure-preserving action α of G and a subrelation $\mathcal{S}$ of $\mathcal{R}_\alpha$ so that

$$\mathcal{R}_\alpha = \mathcal{R}_{\alpha|H} * \mathcal{S}.$$

It follows from the normal form theorem for free products that free factors are measure-free factors. In [**9**], Gaboriau constructs a non-trivial example: the boundary circle of a positive-genus orientable surface with one boundary component generates a measure-free factor. Theorem 3.4 is a special case of the following theorem.

THEOREM 4.2. (Theorems 3.13 and 3.17 of [**9**]) *If G and G' are measure-equivalent to free groups, $C \subset G$ and $C' \subset G'$ are infinite cyclic subgroups, and C is a measure-free factor in G, then $G *_{C=C'} G'$ is measure-equivalent to a free group.*

At present, the only non-trivial C for which we know that $F_C \overset{\text{ME}}{\sim} F_2$ is that given by Gaboriau's example, in which C is generated by a boundary component of an orientable surface. It is natural to ask if each maximal cyclic subgroup of F is a measure-free factor. If this were so, then it would follow from Theorem 4.2 that every F_C is measure-equivalent to a free group. It is also natural to generalize the question to towers, and ask if each maximal abelian cyclic subgroup of a tower is a measure-free factor. Again, if so, it would follow that every limit group is measure-equivalent to a free group.

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