Canad. Math. Bull. Vol. **64** (4), 2021, pp. 779–786
http://dx.doi.org/10.4153/S000843952000082X
© Canadian Mathematical Society 2020

A note on the phase retrieval of holomorphic functions

Rolando Perez III

Abstract. We prove that if f and g are holomorphic functions on an open connected domain, with the same moduli on two intersecting segments, then $f = g$ up to the multiplication of a unimodular constant, provided the segments make an angle that is an irrational multiple of π. We also prove that if f and g are functions in the Nevanlinna class, and if $|f| = |g|$ on the unit circle and on a circle inside the unit disc, then $f = g$ up to the multiplication of a unimodular constant.

1 Introduction

The study of phase retrieval involves the recovery of a function f in some function space from given data about the magnitude of f (phaseless information) and other assumptions on f, where these other assumptions can be in terms of some transform of f. Phase retrieval problems are widely studied because of their physical applications in fields of science and engineering. The most natural question asked on phase retrieval is about the uniqueness of the solution. However, phase retrieval problems generally have large solution sets, so additional assumptions related to the known data are usually added to reduce the solution set, or to consequently force the uniqueness of the solution. We refer the reader to the survey articles by Grohs et al. [6] and Klibanov et al. [11] for a more general perspective on the phase retrieval problem, together with some examples and further references.

Phase retrieval problems have been formulated in both finite and infinite-dimensional cases. In turn, they have been solved using a diverse array of techniques, which include the use of tools from complex analysis. In some cases, the problem shifts to the complex analytic scenario by holomorphic extensions or by integral transforms. For instance, consider f and g to be two band-limited L^2 functions; then from the Paley-Wiener theorem, they are entire functions of finite type. Complex analysis was the key tool used by Akutowicz [1, 2] (and a few years later independently by Walther [15] and Hofstetter [8]) to determine all band-limited functions g such that $|g| = |f|$. We now enumerate some further work that used complex analytic tools. Grohs et al. [7] considered the recovery of a function in a modulation space from phaseless Gabor measurements, where they considered the short-time Fourier

Received by the editors September 3, 2020; revised September 29, 2020.
Published online on Cambridge Core October 8, 2020.
The author was supported by the CHED-PhilFrance scholarship from Campus France and the Commission of Higher Education (CHED), Philippines.
AMS subject classification: **30D05**, **30H10**, **30H15**, 94A12.
Keywords: **Phase retrieval, holomorphic functions**.

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

transform and used the Poisson–Jensen formula in their estimates. Waldspurger et al. [14] solved a continuous case of the recovery of an L^2 function from the modulus of its wavelet transform by using Fourier transforms and holomorphic extensions to the upper half-plane. Moreover, McDonald [12] has extended the work of Akutowicz to cover entire functions of finite genus. The solutions were characterized with the help of Hadamard factorization. In [9], Jaming also used the Hadamard factorization to show that an entire function of finite order can be reconstructed from its modulus on two lines, where these lines intersect at an angle that is an irrational multiple of π. Bodmann et al. [4] then used this result with conformal mappings to show that a polynomial of degree at most $n-1$ can be determined by its magnitude at $4n-4$ well-chosen points in the complex plane. More information on phase retrieval problems from holomorphic measurements can be found in [6].

Our aim in this paper is to generalize some uniqueness results on the phase retrieval of holomorphic functions on the unit disc. Indeed, in [10], we considered the phase retrieval problem: given $f \in L^2(\mathbb{R})$ such that its Fourier transform satisfies an exponential decay condition, find all functions $g \in L^2(\mathbb{R})$ such that $|f| = |g|$ on $\mathbb{R}$ and with its Fourier transform satisfying the same exponential decay condition as f. Using a Paley–Wiener theorem and a conformal map, the problem can be translated to the Hardy space on the unit disc. By the inner-outer factorization, the explicit form of the solution was obtained. One of our motivations stems from one of the coupled phase retrieval problems from [10, Lemma 4.5], which states that for f and g in the Hardy space on the unit disc with $|f| = |g|$ on $(-1,1)$ and on some segment inside the disc, g can be obtained uniquely from f. For our first result, we extend this uniqueness result to holomorphic functions on open connected domains.

On the other hand, our next objective is to improve the result of Boche et al. [3, Theorem 3], which states that given functions f and g in the Hardy space on the disc without singular inner part, if $|f| = |g|$ on the unit circle and on a smaller circle inside the unit circle, then g can be uniquely determined from f. Moreover, they also showed by construction that the Blaschke product associated with g can be uniquely recovered by its modulus on a smaller circle inside the unit circle. For our other result, we extend this uniqueness result to all functions in the Nevanlinna class, regardless of the presence of the singular parts.

This work is organized as follows. Section 2 includes a quick review of definitions and results related to spaces of holomorphic functions on the disc, and the statements of our results. Section 3 is devoted to the proofs of our results.

2 Preliminaries and Statement of Results

2.1 The Nevanlinna Class and the Hardy Space on the Disc

Let $\mathbb{D}$ be the unit disc and let $\mathbb{T} := \partial\mathbb{D}$ be its boundary. We denote by $D(a,r)$ the disc centered at $a \in \mathbb{C}$ with radius $r > 0$, $r\mathbb{D} = D(0,r)$ and $r\mathbb{T} = \partial D(0,r)$. We denote by $H^\infty(\mathbb{D})$ the space of bounded holomorphic functions on $\mathbb{D}$. The Nevanlinna class is defined by

$$\mathcal{N} = \left\{ \varphi \in \mathrm{Hol}(\mathbb{D}) : \varphi = \frac{f}{g},\ f, g \in H^\infty(\mathbb{D}) \right\}.$$

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

Note that the radial limit given by $\varphi^*(\zeta) = \lim_{r\to 1} \varphi(r\zeta)$ exists almost everywhere in $\mathbb{T}$ and $\log|\varphi^*| \in L^1(\mathbb{T})$. For $\varphi \in \mathcal{N}$, φ has a factorization [5, Theorem 2.9] of the form

$$\varphi = \frac{e^{i\gamma} B_\varphi S_{\nu_1} O_\varphi}{S_{\nu_2}}, \tag{2.1}$$

where $e^{i\gamma} \in \mathbb{T}$, B_φ is the Blaschke product formed from the zeros of φ, S_{ν_1} and S_{ν_2} are singular inner functions, and O_φ is the outer part of φ. Here, the Blaschke product is defined for all $z \in \mathbb{D}$ as

$$B_\varphi(z) = z^k \prod_{\alpha\in\Lambda} \frac{\alpha}{|\alpha|} \frac{\alpha - z}{1 - \bar{\alpha} z},$$

where Λ is the set of nonzero zeros of φ counted with multiplicity, which satisfy the Blaschke condition $\sum_{\alpha\in\Lambda}(1 - |\alpha|) < \infty$. The singular inner function is given by

$$S_\nu(z) = \exp\Big(\int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} \, \mathrm{d}\nu(\zeta) \Big),$$

where ν is a finite positive singular measure on $\mathbb{T}$. Finally, the outer part of φ is given by

$$O_\varphi(z) = \exp\Big(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|\varphi^*(\zeta)| \, |\mathrm{d}\zeta| \Big).$$

It is easy to see that for $f \in \mathcal{N}$, if $f(z_0) = 0$, then $\frac{f(z)}{z - z_0} \in \mathcal{N}$. We also recall the subclass of $\mathcal{N}$ called the *Smirnov class*, defined by

$$\mathcal{N}^+ = \Big\{ \varphi \in \mathrm{Hol}(\mathbb{D}) : \varphi = \frac{f}{g},\ f, g \in H^\infty(\mathbb{D}),\ g \text{ is outer} \Big\}.$$

Furthermore, the Generalized Maximum Principle [13, Section 3.3.1, (g)] states that if $\varphi \in \mathcal{N}^+$ with radial limit $\varphi^* \in L^p(\mathbb{T})$, then φ belongs to classical Hardy space on the disc $H^p(\mathbb{D})$, $1 \le p \le \infty$. Note that every function in $H^p(\mathbb{D})$ has a factorization as in (2.1) with $S_{\nu_2} = 1$, and this factorization is unique.

2.2 Statement of Results

We begin with a simple observation. Let ω, Ω be open connected sets such that $\omega \subset \Omega$, let $f, g \in \mathrm{Hol}(\Omega)$ and suppose that $|f| = |g|$ on ω. Then, for some $c \in \mathbb{T}$, $g = cf$ on Ω. Indeed, we can assume that $|f| = |g|$ on a closed disc $\overline{D}$; hence, f and g have the same zeros with the same multiplicities on $\overline{D}$. Consequently, $F = f/g$ is a holomorphic function on D and $|F| = 1$. Therefore, $0 = \Delta|F^2| = |F'|^2$ on D, and hence $F = c$ for some $c \in \mathbb{T}$.

In the same spirit when ω is not open, we have shown in the paper [10, Lemma 4.5] that if $f, g \in H^2(\mathbb{D})$ and

$$|f(z)| = |g(z)|, \qquad z \in (-1, 1) \cup e^{i\alpha}(-1, 1)$$

where $\alpha \notin \pi\mathbb{Q}$, then f and g are equal up to the multiplication of a unimodular constant. For this result, uniqueness was established by showing that the Blaschke products, singular inner parts, and outer parts of f and g are equal. Our first result

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

consists in showing that this is true for arbitrary holomorphic functions in an open connected domain.

Theorem 2.1 *Let Ω be an open connected domain. Let $f, g \in \mathrm{Hol}(\Omega)$ and suppose that*

$$|f(z)| = |g(z)|, \qquad z \in I \cup I_\alpha, \tag{2.2}$$

where I and I_α are segments inside Ω, I_α is the α-rotation of I about the midpoint of I, and $\alpha \notin \pi\mathbb{Q}$. Then $g(z) = cf(z)$ for all $z \in \Omega$ and for some $c \in \mathbb{T}$.

Let us now see what is happening if segments are replaced by circles. To do so, recall that if f and g are outer functions in $\mathcal{N}$ such that $|f| = |g|$ almost everywhere on $\mathbb{T}$, then f is equal to g up to the multiplication of a unimodular constant. Now, Boche et al. [3, Theorem 3] solved a more general problem: if $f, g \in H^1(\mathbb{D})$ have no singular parts (*i.e.*, $f = B_f O_f$, $g = B_g O_g$) and $|f| = |g|$ almost everywhere on $\mathbb{T}$ and $|f| = |g|$ on $\rho\mathbb{T}$ for some $0 < \rho < 1$, then g is uniquely determined by f. The heart of their proof is the explicit construction of the Blaschke product associated with g, as the equality of the outer parts immediately follow. For our next result, with the same equalities of the moduli on the aformentioned circles, we improve the result by Boche et al. by showing that uniqueness holds for all functions in $\mathcal{N}$. We emphasize that in this result, we can either have the presence or the absence of the singular inner part.

Theorem 2.2 *Let $f, g \in \mathcal{N}$ and let $\rho \in (0,1)$. If*

$$|f(\zeta)| = |g(\zeta)|, \quad \text{a.e. } \zeta \in \mathbb{T} \quad \text{and} \quad |f(z)| = |g(z)|, \quad z \in \rho\mathbb{T}, \tag{2.3}$$

then $g(z) = cf(z)$ for all $z \in \mathbb{D}$ and for some $c \in \mathbb{T}$.

3 Proofs and Remarks

In this section, we present the proofs of our results, and some immediate consequences.

3.1 Proof of Theorem 2.1

Observe that replacing $f(z)$ by $f(z_0 + rze^{i\beta})$ with z_0, r, β appropriately chosen, we can assume that

- $(1+\varepsilon)\overline{\mathbb{D}} \subset \Omega$ for $\varepsilon > 0$,
- $I = (-1,1)$, $I_\alpha = e^{i\alpha}(-1,1)$.

Note that now $f, g \in H^2(\mathbb{D})$ so that we could apply [10, Lemma 4.5] and obtain $g = cf$, for some $c \in \mathbb{T}$. We will give an alternative simpler proof.

Note that, as the zeros of f and g are isolated, by choosing r small enough, we can assume that they have at most one zero in $(1+\varepsilon)\overline{\mathbb{D}}$, which is at 0. We can write

$$f(z) = z^k e^{\varphi(z)} \qquad \text{and} \qquad g(z) = z^l e^{\psi(z)}, \qquad z \in \overline{\mathbb{D}}, \tag{3.1}$$

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

where $\varphi, \psi \in \mathrm{Hol}((1+\varepsilon)\mathbb{D})$ and k, l are nonnegative integers. As $|f(x)| = |g(x)|$ for $x \in (-1,1)$, we conclude that $k = l$.

It remains to show that the zero-free factors of f and g are equal up to a unimodular constant. First, we note that (2.2) is equivalent to

$$\mathrm{Re}\,\varphi(t) = \mathrm{Re}\,\psi(t) \qquad \text{and} \qquad \mathrm{Re}\,\varphi(te^{i\alpha}) = \mathrm{Re}\,\psi(te^{i\alpha}), \qquad t \in (-1,1).$$

Since $\varphi, \psi \in \mathrm{Hol}(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$, $\mathrm{Re}\,\varphi$ and $\mathrm{Re}\,\psi$ are harmonic and

$$\varphi(z) = \sum_{n\geq 0} |z|^n \widehat{\varphi}(n) e^{in\theta} \qquad \text{and} \qquad \psi(z) = \sum_{n\geq 0} |z|^n \widehat{\psi}(n) e^{in\theta} \tag{3.2}$$

for $z = |z|e^{i\theta} \in \mathbb{D}$. It follows that

$$\mathrm{Re}\,\varphi(z) = \mathrm{Re}\,\widehat{\varphi}(0) + \sum_{n\in\mathbb{N}} |z|^n \frac{\widehat{\varphi}(n)e^{in\theta} + \overline{\widehat{\varphi}(n)}e^{-in\theta}}{2},$$

$$\mathrm{Re}\,\psi(z) = \mathrm{Re}\,\widehat{\psi}(0) + \sum_{n\in\mathbb{N}} |z|^n \frac{\widehat{\psi}(n)e^{in\theta} + \overline{\widehat{\psi}(n)}e^{-in\theta}}{2}.$$

Thus, (2.2) and (3.1) together with the sums above imply that $\mathrm{Re}\,\varphi(t) = \mathrm{Re}\,\psi(t)$ for $t \in (-1,1)$, if and only if

$$\mathrm{Re}\,\widehat{\varphi}(n) = \mathrm{Re}\,\widehat{\psi}(n), \qquad n \in \mathbb{N},$$

and $\mathrm{Re}\,\varphi(te^{i\alpha}) = \mathrm{Re}\,\psi(te^{i\alpha})$ if and only if

$$\mathrm{Re}\,\widehat{\varphi}(0) = \mathrm{Re}\widehat{\psi}(0) \qquad \text{and} \qquad \mathrm{Re}\,(\widehat{\varphi}(n)e^{in\alpha}) = \mathrm{Re}\,(\widehat{\psi}(n)e^{in\alpha})$$

for all $n \in \mathbb{N}$. In other words,

$$\begin{cases} \mathrm{Re}\,(\widehat{\varphi}(n) - \widehat{\psi}(n)) = \langle \widehat{\varphi}(n) - \widehat{\psi}(n), 1\rangle_{\mathbb{C}} = 0, \\ \mathrm{Re}\,((\widehat{\varphi}(n) - \widehat{\psi}(n))e^{in\alpha}) = \langle \widehat{\varphi}(n) - \widehat{\psi}(n), e^{-in\alpha}\rangle_{\mathbb{C}} = 0, \end{cases} \tag{3.3}$$

for all $n \in \mathbb{N}$. Since $\alpha \notin \pi\mathbb{Q}$, $\{1, e^{-in\alpha}\}$ is a basis for $\mathbb{C}$ when $n \neq 0$, so by (3.3), we have $\widehat{\varphi}(n) = \widehat{\psi}(n)$. On the other hand, as $\mathrm{Re}\,\widehat{\varphi}(0) = \mathrm{Re}\,\widehat{\psi}(0)$, there exists $\lambda \in \mathbb{R}$ such that $\widehat{\psi}(0) = \widehat{\varphi}(0) + i\lambda$. It follows from (3.2) that $\psi = \varphi + i\lambda$, thus $g(z) = e^{i\lambda} f(z)$ for all $z \in \mathbb{D} \subset \Omega$. As Ω is connected and $f, g \in \mathrm{Hol}(\Omega)$, this implies that $g(z) = e^{i\lambda} f(z)$ also holds on Ω.

3.2 Proof of Theorem 2.2

We begin with a simple observation. If $z_0 \in \rho\mathbb{T}$ is a zero of f, we write $f(z) = (z - z_0)^k \widetilde{f}(z)$ and $g(z) = (z - z_0)^j \widetilde{g}(z)$ with nonnegative integers j, k and $\tilde{f}, \widetilde{g} \in \mathcal{N}$ nonvanishing at z_0. Then $|f| = |g|$ on $\rho\mathbb{T}$ reads

$$|(z - z_0)^k \tilde{f}(z)| = |(z - z_0)^j \tilde{g}(z)|, \qquad z \in \rho\mathbb{T},$$

and this implies that $k = j$. Therefore, f and g have the same zeros on $\rho\mathbb{T}$ with the same multiplicities. We can thus write $f = Pf_1$ and $g = Pg_1$ with P a polynomial that

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

has all the zeros in $\rho\mathbb{T}$ and $f_1, g_1 \in \mathcal{N}$ nonvanishing on $\rho\mathbb{T}$. Then $|f_1| = |g_1|$ on $\rho\mathbb{T} \cup \mathbb{T}$. In other words, we can assume that f and g do not vanish on $\rho\mathbb{T}$.

Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ be the zeros of f and g on $\rho\mathbb{D}$, respectively, counted with multiplicities. For all $z \in \mathbb{C}$, write

$$P_f(z) = \prod_{i=1}^{n} \frac{\rho(z - a_i)}{\rho^2 - \bar{a}_i z} \qquad \text{and} \qquad P_g(z) = \prod_{i=1}^{m} \frac{\rho(z - b_i)}{\rho^2 - \bar{b}_i z}.$$

Notice first that if $z \in \rho\mathbb{T}$, $|P_f(z)| = |P_g(z)| = 1$. Further, $\frac{f}{P_f}$ and $\frac{g}{P_g}$ do not vanish in $\rho\mathbb{D}$. By the Poisson–Jensen formula, for all $z \in \rho\mathbb{D}$, we have

$$\begin{aligned} \log\left|\frac{f(z)}{P_f(z)}\right| &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\Big(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z}\Big) \log|f(\rho e^{i\theta})|\, \mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\Big(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z}\Big) \log|g(\rho e^{i\theta})|\, \mathrm{d}\theta \\ &= \log\Big|\frac{g(z)}{P_g(z)}\Big|. \end{aligned}$$

But then $\left|\frac{g(z)}{P_g(z)}\right| = \left|\frac{f(z)}{P_f(z)}\right|$ for all $z \in \rho\mathbb{D}$. Thus, we get that there is some $\widetilde{c} \in \mathbb{T}$ such that $\frac{g(z)}{P_g(z)} = \widetilde{c}\frac{f(z)}{P_f(z)}$ for all $z \in \rho\mathbb{D}$. Finally, as an identity between holomorphic functions on $\mathbb{D}$, this is valid for all $z \in \mathbb{D}$. In particular, taking $z = re^{i\theta}$ and $r \to 1$, we get $\frac{g(z)}{P_g(z)} = \widetilde{c}\frac{f(z)}{P_f(z)}$ for almost every $z \in \mathbb{T}$. By (2.3), we have $\left|\frac{1}{P_f}\right| = \left|\frac{1}{P_g}\right|$ on $\mathbb{T}$ as well. Hence, by the following lemma, we get $\frac{1}{P_f} = \frac{c}{P_g}$ for some $c \in \mathbb{T}$, which implies $g = c\widetilde{c}f$.

Lemma 3.1 *Let F and G be meromorphic on $\mathbb{C}$ without poles on $\mathbb{T} \cup \rho\mathbb{T}$ for $0 < \rho \neq 1$. Suppose*

$$|F(\zeta)| = |G(\zeta)| \qquad \text{and} \qquad |F(\rho\zeta)| = |G(\rho\zeta)|, \qquad \zeta \in \mathbb{T}. \tag{3.4}$$

Then F and G have the same zeros and poles in $\mathbb{C}\backslash\{0\}$, with the same multiplicities. In particular, if F and G are rational functions that satisfy (3.4), then $G = cF$ outside the poles with $c \in \mathbb{T}$.

Proof Let F be meromorphic on $\mathbb{C}$ and $z_0 \in \mathbb{C}$. Write $F(z) = (z - z_0)^k \widetilde{F}(z)$ and for some $k \in \mathbb{Z}$ with $\widetilde{F}(z_0) \neq 0$. Define the multiplicity $k := m_F(z_0)$, where $k > 0$ if z_0 is a zero of F, and $k < 0$ if z_0 is a pole of F. In particular, for meromorphic functions F_1, F_2 on $\mathbb{C}$, we have $m_{F_1F_2} = m_{F_1} + m_{F_2}$, $m_{F_1/F_2} = m_{F_1} - m_{F_2}$ and if $F_1 = F_2$ in the neighborhood of a pole z_0, then $m_{F_1} = m_{F_2}$.

First, note that if $\rho > 1$, we replace F and G by $f(z/\rho)$ and $g(z/\rho)$, and replace ρ by $1/\rho < 1$. We thus assume that $\rho < 1$. Observe that (3.4) is equivalent to

$$F(z)\overline{F\Big(\frac{1}{\bar{z}}\Big)} = G(z)\overline{G\Big(\frac{1}{\bar{z}}\Big)}, \qquad z \in \mathbb{T}$$

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

and

$$F(z)\overline{F\Big(\frac{\rho^2}{\bar z}\Big)} = G(z)\overline{G\Big(\frac{\rho^2}{\bar z}\Big)}, \qquad z \in \rho\mathbb{T}.$$

As an identity between meromorphic functions in $\mathbb{C}$, these equations are also valid for $z \in \mathbb{C}$ not a pole of any of the functions involved. As poles are isolated, we have for $z \neq 0$,

$$m_F(z) + m_F\Big(\frac{1}{\bar z}\Big) = m_G(z) + m_G\Big(\frac{1}{\bar z}\Big), \tag{3.5}$$

$$m_F(z) + m_F\Big(\frac{\rho^2}{\bar z}\Big) = m_G(z) + m_G\Big(\frac{\rho^2}{\bar z}\Big). \tag{3.6}$$

Now, (3.5) gives

$$m_F(z) - m_G(z) = m_G\Big(\frac{1}{\bar z}\Big) - m_F\Big(\frac{1}{\bar z}\Big) = m_F(\rho^2 z) - m_G(\rho^2 z)$$

with (3.6) applied to $1/\bar z$, for $z \in \mathbb{C} \setminus \{0\}$. If $z_0 \neq 0$ is such that $m_F(z_0) \neq m_G(z_0)$, then

$$\begin{aligned} 0 \neq m_F(z_0) - m_G(z_0) &= m_F(\rho^2 z_0) - m_G(\rho^2 z_0) = \cdots \\ &= m_F(\rho^{2k} z_0) - m_G(\rho^{2k} z_0) = m_{F/G}(\rho^{2k} z_0) \end{aligned}$$

for all $k \in \mathbb{N}$. But then F/G is meromorphic and either has $\rho^{2k} z_0$ as a zero ($m_{F/G}(\rho^{2k} z_0) > 0$) or as a pole ($m_{F/G}(\rho^{2k} z_0) < 0$) for every k. Letting $k \to \infty$, we have $\rho^{2k} z_0 \to 0$. As $z_0 \neq 0$, this contradicts the fact that zeros and poles of F/G are isolated. Hence, F and G have the same nonzero zeros and poles with the same multiplicities.

Furthermore, if F and G are rational functions, then they have same zeros and poles in $\mathbb{C} \setminus \{0\}$, thus there exists $c \in \mathbb{T}$ and $m \in \mathbb{Z}$ such that $G = cz^m F$. But then (3.4) implies that

- on one hand, $|F(\rho e^{it})| = |c|\rho^m |F(\rho e^{it})|$ for all t, thus $|c|\rho^m = 1$.
- on the other hand, $|F(e^{it})| = |c||F(e^{it})|$, thus $|c| = 1$.

As $\rho < 1$, this implies $m = 0$ and then $G = cF$. □

Corollary 3.2 *Let F, G be two meromorphic functions in $\mathbb{C}$ with no pole at 0 such that*

$$|F(\zeta)| = |G(\zeta)| \qquad \textit{and} \qquad |F(\rho\zeta)| = |G(\rho\zeta)|, \qquad \zeta \in \mathbb{T},\ 0 < \rho \neq 1.$$

Then there exists $c \in \mathbb{T}$ such that $G = cF$.

Proof As seen in the previous proof, we can assume that $\rho < 1$.

The previous lemma then shows that F and G have the same nonzero poles in $\mathbb{C}$. Note that there is at most a finite number of such poles in a neighborhood of $\mathbb{D}$. We can factor them out and write $F = \widetilde F/P$, $G = \widetilde G/P$ with $\widetilde F, \widetilde G$ having no pole in a neighborhood of $\mathbb{D}$ and P a polynomial. But then $\widetilde F, \widetilde G \in \mathcal{N}$ and $|\widetilde F| = |\widetilde G|$ on $\mathbb{T} \cup \rho\mathbb{T}$. The previous theorem shows that there exists $c \in \mathbb{T}$ such that $\widetilde G = c\widetilde F$, thus $G = cF$ on $\mathbb{D}$ and thus on $\mathbb{C}$. ■

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press

Acknowledgement The author wants to thank his supervisors at the Université de Bordeaux, Philippe Jaming and Karim Kellay, for their very helpful comments and suggestions.

References

[1] E. J. Akutowicz, *On the determination of the phase of a Fourier integral. I*. Trans. Amer. Math. Soc. **83**(1956), 179–192. http://dx.doi.org/10.2307/1992910
[2] E. J. Akutowicz, *On the determination of the phase of a Fourier integral. II*. Proc. Amer. Math. Soc. **8**(1957), 234–238. http://dx.doi.org/10.2307/2033718
[3] H. Boche, N. Li, and V. Pohl, *Phase retrieval in spaces of analytic functions on the unit disk*. In: 2017 International Conference on Sampling Theory and Applications (SampTA), Tallin, 2017, pp. 336–340. http://dx.doi.org/10.1109/SAMPTA.2017.8024411
[4] B. Bodmann and N. Hammen, *Stable phase retrieval with low redundancy frames*. Adv. Comput. Math. **41**(2015), 317–331. http://dx.doi.org/10.1007/s10444-014-9359-y
[5] P. Duren, *The theory of H^p spaces*. Academic Press, New York, 1970.
[6] P. Grohs, S. Koppensteiner, and M. Rathmair, *Phase retrieval: uniqueness and stability*. SIAM Rev. **62**(2020), 301–350. http://dx.doi.org/10.1137/19M1256865
[7] P. Grohs and M. Rathmair, *Stable Gabor phase retrieval and spectral clustering*. Comm. Pure Appl. Math. **72**(2019), 981–1043. http://dx.doi.org/10.1002/cpa.21799
[8] E. Hofstetter, *Construction of time-limited functions with specified autocorrelation functions*. IEEE Trans. Inform. Theory **10**(1964), 119–126. http://dx.doi.org/10.1109/TIT.1964.1053648
[9] P. Jaming, *Uniqueness results in an extension of Pauli's phase retrieval problem*. Appl. Comput. Harmon. Anal. **37**(2014), 413–441. http://dx.doi.org/10.1016/j.acha.2014.01.003
[10] P. Jaming, K. Kellay, and R. Perez III, *Phase retrieval for wide band signals*. J. Fourier Anal. Appl. **26**(2020), Paper No. 54. http://dx.doi.org/10.1007/s00041-020-09767-1
[11] M. Klibanov, P. Sacks, and A. Tikhonravov, *The phase retrieval problem*. Inverse Prob. **11**(1995), 1–28.
[12] J. McDonald, *Phase retrieval and magnitude retrieval of entire functions*. J. Fourier Anal. Appl. **10**(2004), 259–267. http://dx.doi.org/10.1007/s00041-004-0973-9
[13] N. Nikolski, *Hardy spaces*. *Cambridge Studies in Advanced Mathematics*, 179, Cambridge University Press, Cambridge, UK, 2019.
[14] I. Waldspurger and S. Mallat, *Phase retrieval for the Cauchy wavelet transform*. J. Fourier Anal. Appl. **21**(2015), 1251–1309. http://dx.doi.org/10.1007/s00041-015-9403-4
[15] A. Walther, *The question of phase retrieval in optics*. Optica Acta **10**(1963), 41–49. http://dx.doi.org/10.1080/713817747

Université de Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400 Talence, France

and

Institute of Mathematics, University of the Philippines Diliman, 1101 Quezon City, Philippines
e-mail: roperez@u-bordeaux.fr

https://doi.org/10.4153/S000843952000082X Published online by Cambridge University Press