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# The Largest Minimum Codegree of a 3-Graph Without a Generalized 4-Cycle

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A 3-graph is said to contain a generalized 4-cycle if it contains 4 edges $A, B, C, D$ such that $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$. We show that a 3-graph in which every pair of vertices is contained in at least 4 edges must contain a generalized 4-cycle. When the number of vertices, $n$, is equivalent to 1 or 5 modulo 20, this result is optimum, in the sense that for such $n$ there are 3-graphs where every pair of vertices is contained in 3 edges but which do not contain a generalized 4-cycle.

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## 1. Introduction

Erdős [1] posed the following problem: Determine the maximum number of $r$-sets chosen from $[n]$ so that all disjoint pairs of $r$-sets have distinct unions. In other words, determine the maximum number of edges of an $r$-graph that does not contain four distinct edges $A, B, C, D$ such that $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$. For $r = 2$ the problem is that of determining the maximum number of edges in a graph with no 4-cycle, which was asked by Erdős in 1938 and has has been well studied since. Erdős, Rényi and Sós [2] solved the problem asymptotically, proving that the maximum is $\left(\frac{1}{2} + o(1)\right)n^{3/2}$, and Füredi [3] determined the exact maximum for infinitely many $n$.

When $r = 3$ there is only one (up to isomorphism) 3-graph with four distinct edges $A, B, C, D$ such that $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$, namely the 3-graph on $\{a, b, c, d, e, f\}$ with edges $\{abe\}$, $\{abf\}$, $\{cde\}$ and $\{cdf\}$, which we shall denote by $C_4^3$. (Note that this notation is also used in [7] and [8], but in other literature $C_4^3$ may be used to denote a different generalization of a 4-cycle than that given here.) For $r \geqslant 2$ we let $\mathcal{C}_4^r$ denote the family of $r$-graphs which have four distinct edges $A, B, C, D$ such that $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$. Note that $\mathcal{C}_4^3$ consists of just the single 3-graph $C_4^3$.

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For a family of $r$-graphs $\mathcal{F}$, we let $\mathrm{ex}(n, \mathcal{F})$ denote the maximum number of edges an $r$-graph on $n$ vertices can have if does not contain any member of $\mathcal{F}$ as a a subgraph. If $\mathcal{F}$ contains just a single graph $F$ we write $\mathrm{ex}(n, F)$ instead. Erdős's problem can now be stated as determining the value of $\mathrm{ex}(n, \mathcal{C}_4^r)$.

Füredi [4] obtained the following result for $r \geqslant 3$:

$$\binom{n-1}{r-1} + \left\lfloor \frac{n-1}{r} \right\rfloor \leqslant \mathrm{ex}(n, \mathcal{C}_4^r) < \frac{7}{2}\binom{n}{r-1}.$$

The lower bound is attained by an $r$-graph on a vertex set $V$ of size $n$ with all edges containing a fixed vertex $x$ (that is, a star with centre $x$) plus a maximal set of disjoint edges on $V - x$. For $r = 3$ and $n \equiv 1, 5 \mod 20$ he gave another construction which improves on the previous one.

**Example 1.1 ([4], Example 1.3).** Let $S$ be a Steiner system $S(2, 5, n)$, that is, $S$ is a 5-graph on $[n]$ with the property that every pair of vertices is contained in exactly one edge. If we replace each edge of $S$ with the $\binom{5}{3}$ triples contained in that edge then the resulting 3-graph has $\binom{n}{2}$ edges but no copy of $C_4^3$. Hanani [5, 6] showed that such Steiner systems exist if and only if $n \equiv 1, 5 \mod 20$.

Based on his results, Füredi made the following conjecture.

**Conjecture 1.2 ([4], Conjecture 1.4).** *If $r \geqslant 3$ and $n$ is sufficiently large, then*

$$\mathrm{ex}(n, \mathcal{C}_4^r) \leqslant \binom{n}{r-1}.$$

*Moreover, for $r \geqslant 4$ and $n \geqslant 1$,*

$$\mathrm{ex}(n, \mathcal{C}_4^r) = \binom{n-1}{r-1} + \left\lfloor \frac{n-1}{r} \right\rfloor.$$

The current best result for the case $r = 3$ is given by Pikhurko and Verstraëte.

**Theorem 1.3 ([8], Theorem 2).** *For every $n$,*

$$\mathrm{ex}(n, C_4^3) \leqslant \frac{13}{9}\binom{n}{2}.$$

Given an $r$-graph $G$, we define the neighbourhood of a set of $k$ ($1 \leqslant k \leqslant r-1$) vertices $S$, denoted by $\Gamma_G(S)$, to be the collection of sets of vertices $T \in V(G)^{(r-k)}$ such that $S \cup T \in E(G)$. The *degree* of $S$ is then $d_G(S) = |\Gamma_G(S)|$. (The subscript is sometimes dropped if the context is clear.) If $k = r - 1$ then this quantity is called the *codegree* and the minimum codegree $C(G)$ of an $r$-graph $G$ is the minimum codegree over all sets of size $r - 1$ in $V(G)$. Given a family of $r$-graphs $\mathcal{F}$, we define the extremal function $\mathrm{coex}(n, \mathcal{F})$ to be the maximum value of $C(G)$ over all $r$-graphs $G$ on $n$ vertices that do not contain any member of $\mathcal{F}$ as a subgraph.

A problem related to that of Erdős is the following question asked by Mubayi [7]: What is the minimum value of $C(G)$ that will force a $C_4^3$? In other words, determine $\mathrm{coex}(n, C_4^3)$. Example 1.1

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shows us that when $n \equiv 1, 5 \mod 20$, we can have a 3-graph which is $C_4^3$-free and has minimum codegree 3. On the other hand, we know that if a 3-graph $G$ has

$$C(G) > \frac{3\mathrm{ex}(n, C_4^3)}{\binom{n}{2}},$$

then it has more than $\mathrm{ex}(n, C_4^3)$ edges and so must contain a copy of $C_4^3$. Using Theorem 1.3, this gives us the bound $\mathrm{coex}(n, C_4^3) \leqslant 4$. We prove the following result.

**Theorem 1.4.** *If $H$ is a 3-graph in which every pair of vertices is in at least 4 edges, then $H$ contains a copy of $C_4^3$. That is, for every $n$,*

$$\mathrm{coex}(n, C_4^3) \leqslant 3.$$

Example 1.1 shows that the inequality in the theorem above is tight if $n \equiv 1, 5 \mod 20$. We remark that a proof of Conjecture 1.2 for the case $r = 3$ would imply Theorem 1.4, and in fact the bound $\mathrm{ex}(n, C_4^3) < \frac{4}{3}\binom{n}{2}$ would suffice.

## 2. Proof of Theorem 1.4

We begin with a short overview of the proof. Consider first the special case that $H$ is a 3-graph in which every pair has codegree exactly 4. If $H$ has no copy of $C_4^3$ then the common neighbourhood $\Gamma(x) \cap \Gamma(y)$ of any pair of vertices $x, y$ cannot contain a pair of vertex-disjoint 2-edges, and hence must either be a triangle or a star (all 2-edges have a common vertex). If it is a star, with centre $w$ say, then it cannot contain more than 4 leaves since they are contained in the neighbourhood of the pair $xw$. So, by counting the number of pairs of edges in $H$ which share 2 vertices, we obtain the following contradiction:

$$\binom{4}{2}\binom{n}{2} = \sum_{xy \in V(H)^{(2)}} \binom{d(xy)}{2} = \sum_{xy \in V(H)^{(2)}} |\Gamma(x) \cap \Gamma(y)| \leqslant 4\binom{n}{2}.$$

(For a set $S$ we use the notation $S^{(t)}$ to mean the set of all subsets of $S$ of size $t$.)

Given this, we take our $C_4^3$-free 3-graph $H$ and modify it into another $C_4^3$-free 3-graph $H^\star$ in which many of the codegrees are exactly 4. We then use similar counting arguments to the one above to derive a contradiction.

We shall now prove Theorem 1.4 for each $n \geqslant 1$. Suppose for a contradiction that we have a counterexample and let $H$ be an edge-minimal one. Thus $H$ is a $C_4^3$-free 3-graph with minimum codegree at least 4 having the property that every edge contains a pair of codegree exactly 4. Let $n = |V(H)|$. We shall now attempt to derive a contradiction.

First we show that $H$ must contain a certain subgraph. Denote by $K(1, 2, 5)$ the complete tripartite 3-graph with class sizes 1, 2 and 5.

**Claim 2.1.** *$H$ contains a copy of $K(1, 2, 5)$.*

**Proof.** Arguing as above, we have that for any pair of vertices $x, y$ in $V(H)$ the 2-graph $\Gamma(x) \cap \Gamma(y)$ cannot contain a pair of vertex-disjoint 2-edges: otherwise we would get a copy of $C_4^3$.

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Hence $\Gamma(x) \cap \Gamma(y)$ must be either a triangle or a star. If $\Gamma(x) \cap \Gamma(y)$ contained a star with 5 leaves then we would have a copy of $K(1,2,5)$ in $H$. Hence we may assume that $|\Gamma(x) \cap \Gamma(y)| \leqslant 4$ for all pairs of vertices $x, y$. But now, by counting pairs of edges which share 2 vertices, we have

$$\binom{4}{2}\binom{n}{2} \leqslant \sum_{xy \in V(H)^{(2)}} \binom{d(xy)}{2} = \sum_{xy \in V(H)^{(2)}} |\Gamma(x) \cap \Gamma(y)| \leqslant 4\binom{n}{2},$$

which is a contradiction. □

Consider a copy of $K(1,2,5)$, say with vertices $a, b_1, b_2, c_1 \ldots, c_5$ and edges $ab_ic_j$ for all $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant 5$. Depending on the codegree of the pair $ac_1$, we have one of the two following cases.

(I) The codegree of $ac_1$ is 4, so we have two distinct edges $e_1, e_2$ ($ab_1c_1$ and $ab_2c_1$) which share a pair of vertices ($ac_1$) of codegree 4 and which each contain a pair of vertices ($ab_1$ or $ab_2$) of codegree greater than 4.

(II) The codegree $ac_1$ is greater than 4, so we have an edge $e$ ($ab_1c_1$) which contains two pairs of vertices ($ab_1$ and $ac_1$) of codegree greater than 4. (Note that the other pair $b_1c_1$ must have codegree 4 by the minimality of $H$.)

Let $A = V(H)$. By transforming $H$ we create a series of 3-graphs

$$H_0, H_1, H_2, \ldots$$

via the following algorithm.

**Algorithm.**

- Let $H_0 = H$.
- For $i \geqslant 0$, if all pairs of $A^{(2)}$ have codegree exactly 4 in $H_i$ then the series terminates. Otherwise we have some edge $xyz$ in $E(H_i) \cap A^{(3)}$, where either

  (a) $d_{H_i}(xy) = 4$ and $d_{H_i}(xz) \geqslant d_{H_i}(yz) > 4$, or

  (b) $d_{H_i}(xy) = d_{H_i}(xz) = 4$ and $d_{H_i}(yz) > 4$.

  If case (a) occurs then we construct $H_{i+1}$ from $H_i$ by deleting the edge $xyz$ and adding a new vertex $u$ and a new edge $uxy$. If case (b) occurs then we delete the edge $xyz$ and add two new vertices $u, v$ and two new edges $uxy, vxz$.

We make a few observations about this process. At each iteration of the algorithm we delete an edge from the original graph $H$ and add either one or two new edges and vertices. Note that the the codegrees of pairs in $A^{(2)}$ are always at least 4 in each graph $H_i$ and the sum of these codegrees decreases at every iteration, so the process must terminate after a finite number of steps. In each graph $H_i$, if a pair of vertices has codegree greater than 4 then its neighbourhood must be contained entirely in $A$, which explains why one of (a) or (b) must occur if there is a pair of codegree greater than 4 in $A^{(2)}$.

If case (I) occurs then we shall elect that the edge $e_1$ is to be deleted first in the algorithm and that the edge $e_2$ is to be deleted second (which is a valid thing to do since each contains a pair of codegree greater than 4). If case (II) occurs then we choose the edge $e$ to be deleted first in the algorithm.

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We denote by $H^\star$ the final 3-graph of the sequence. All pairs of $A^{(2)}$ have codegree exactly 4 in $H^\star$. Let $B = V(H^\star)\backslash A$.

**Claim 2.2.** *$H^\star$ satisfies the following:*

(i) *$H^\star$ is $C_4^3$-free,*
(ii) $e(H^\star) = \frac{4}{3}\binom{n}{2} + \frac{2|B|}{3}$.

**Proof.** For (i), note that $C_4^3$ is 2-regular (every vertex has degree 2) but every vertex of $B$ has degree exactly 1 in $H^\star$. Hence every copy of $C_4^3$ in $H^\star$ must be contained in $A$. However, no such copy of $C_4^3$ can exist since since $H^\star[A]$ (the 3-graph with vertex set $A$ and edge set $E(H^\star) \cap A^{(3)}$) is a subgraph of $H$.

Now we prove condition (ii). Since all pairs of $A^{(2)}$ have codegree 4, all pairs of $B^{(2)}$ have codegree 0 and every vertex of $B$ has degree 1, we have

$$3e(H^\star) = \sum_{xy \in A^{(2)}} d(xy) + 2\sum_{x \in B} d(x) = 4\binom{n}{2} + 2|B|. \qquad \square$$

**Claim 2.3.** *We have that*

- $|B| \leqslant \frac{2}{3}\binom{n}{2}$ *if case* (I) *occurs, and*
- $|B| < \frac{2}{3}\binom{n}{2}$ *if case* (II) *occurs.*

**Proof.** Note first that $e(H) \geqslant e(H^\star) - \frac{|B|}{2}$, since in the algorithm when we delete an edge in $H_i$ we create at most two new ones for $H_{i+1}$. Hence

$$e(H) \geqslant \frac{4}{3}\binom{n}{2} + \frac{2|B|}{3} - \frac{|B|}{2} \geqslant \frac{4}{3}\binom{n}{2} + \frac{|B|}{6}.$$

By Theorem 1.3 we have that $e(H) \leqslant \frac{13}{9}\binom{n}{2}$, from which we deduce that $|B| \leqslant \frac{2}{3}\binom{n}{2}$.

If case (II) occurs then we do case (a) of the algorithm to create $H_1$. In particular $H_1$ is obtained from $H$ by deleting $e$ and adding just one extra edge, and so we have the strict inequality $e(H) > e(H^\star) - \frac{|B|}{2}$ instead. The result $|B| < \frac{2}{3}\binom{n}{2}$ then follows as before. $\square$

**Claim 2.4.** *We have that*

- $|B| \geqslant \frac{2}{3}\binom{n}{2}$ *if case* (II) *occurs, and*
- $|B| > \frac{2}{3}\binom{n}{2}$ *if case* (I) *occurs.*

**Proof.** Let $Q_2$ be the number of (not necessarily induced) subgraphs of order 4 in $H^\star$ with exactly 2 edges or, equivalently, the number of pairs of edges that share 2 vertices. Then

$$Q_2 = \sum_{xy \in V(H^\star)^{(2)}} \binom{d(xy)}{2} = 6\binom{n}{2},$$

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since all pairs of $V(H^\star)^{(2)}\backslash A^{(2)}$ have codegree at most 1 and all pairs of $A^{(2)}$ have codegree 4. We also have that

$$\begin{aligned} Q_2 &= \sum_{xy\in V(H^\star)^{(2)}} |\Gamma(x)\cap\Gamma(y)| \\ &\leqslant \sum_{xy\in A^{(2)}} |\Gamma(x)\cap\Gamma(y)| + \sum_{x\in B}\sum_{y\in V(H^\star)-x} |\Gamma(x)\cap\Gamma(y)|. \end{aligned}$$

Again, for any pair of vertices $x, y$ in $V(H^\star)$ the 2-graph $\Gamma(x)\cap\Gamma(y)$ cannot contain a pair of vertex-disjoint 2-edges and so $\Gamma(x)\cap\Gamma(y)$ must be either a triangle or a star. In the former case $|\Gamma(x)\cap\Gamma(y)| = 3$ and in the latter $|\Gamma(x)\cap\Gamma(y)| \leqslant 4$, since the leaves of the star are contained in the neighbourhood of the pair $xz$ where $z$ is the star's centre. If $x \in B$ then

$$\sum_{y\in V(H^\star)-x} |\Gamma(x)\cap\Gamma(y)| = 3,$$

since $x$ has degree 1 and the pair that forms an edge with $x$ has codegree 4. Therefore

$$Q_2 \leqslant 4\binom{n}{2} + 3|B|.$$

Combining this with $Q_2 = 6\binom{n}{2}$ yields the result $|B| \geqslant \frac{2}{3}\binom{n}{2}$.

If case (I) occurs, then by considering what happens in the first two iterations of our algorithm, we see that $H^\star$ must contain a pair of vertices $u, v \in B$ for which $|\Gamma(u)\cap\Gamma(v)| = 1$. Hence

$$Q_2 < \sum_{xy\in A^{(2)}} |\Gamma(x)\cap\Gamma(y)| + \sum_{x\in B}\sum_{y\in V(H^\star)-x} |\Gamma(x)\cap\Gamma(y)|,$$

from which we argue as before to deduce that $|B| > \frac{2}{3}\binom{n}{2}$. □

Combining Claims 2.3 and 2.4 gives us a contradiction, which completes the proof of Theorem 1.4.

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