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Relative formal topology: the binary positivity predicate comes first

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In this paper we introduce relative formal topology so that we can deal constructively with the closed subsets of a topological space as well as with the open ones. In fact, within standard formal topology, the cover relation allows the definition of the formal opens, which are supposed to act as the formal counterparts of the open subsets, and within balanced formal topology, the binary positivity predicate allows the definition of the formal closed subsets, which are supposed to act as the formal counterparts of the closed subsets. However, these approaches are only fully satisfactory (according to the criterion introduced by the author in Valentini (2005)) if we can provide an adequate formalisation of the cover relation and the positivity predicate. But current formalisations fail in this respect since some intuitionistically valid relations between the open and closed subsets of a concrete topological space cannot be expressed. Our central result is that we can solve this problem through a generalisation of the standard cover relation together with a binary positivity predicate satisfying the positivity axiom.

1. Introduction

Formal topologies were proposed by P. Martin-Löf and G. Sambin as suitable structures for dealing with topological spaces within constructive mathematics (Sambin 1987). We will begin by recalling some basic ideas about the approach they proposed.

We write $(X, \Omega(X))$ to mean a *topological space*, that is, a structure such that X is a collection and $\Omega(X)$ is a family of subsets of X such that:

— $X \in \Omega(X)$;
— $\varnothing \in \Omega(X)$;
— if $X_1, X_2 \in \Omega(X)$, then $X_1 \cap X_2 \in \Omega(X)$;
— if $(X_i)_{i \in I}$ is a family of subsets of X such that for all $i \in I$, we have $X_i \in \Omega(X)$, then $\bigcup_{i \in I} X_i \in \Omega(X)$.

Any topological space can be presented using one of its bases, that is, a subset $\mathscr{A}$ of elements of $\Omega(X)$ such that any element of $\Omega(X)$ is obtained as a union of elements of $\mathscr{A}$. It is then trivial to see that, for any $x \in X$, there exists $C \in \mathscr{A}$ such that $x \in C$ and, for any $x \in X$ and any $C_1, C_2 \in \mathscr{A}$, if $x \in C_1 \cap C_2$, then there is $B \in \mathscr{A}$ such that $x \in B$ and $B \subseteq C_1 \cap C_2$.

Given a base $\mathscr{A}$ for a topological space $(X, \Omega(X))$, we can introduce a map ext from $\mathscr{A}$ to $\mathscr{P}(X)$ by setting $\mathsf{ext}(C) = \{x \in X \mid x \in C\}$ for any $C \in \mathscr{A}$. It is clear that the map

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$\mathsf{ext}(-)$ defined here is just the identity. However, its relevance will became clearer later when we will substitute a *concrete* base $\mathscr{A}$ for a topological space with a *formal* base A, which is just a set of names for the elements of the concrete base. From now on, we will introduce all of the following definitions using the map $\mathsf{ext}(-)$, even if it is not necessary. In this way, all the definitions we introduce can be used directly in the formal case.

The elements of $\Omega(X)$ are usually called the *open subsets* of the topological space $(X, \Omega(X))$, and using the map $\mathsf{ext}(-)$, we can deal with them immediately. Indeed, we can introduce the map Ext from $\mathscr{P}(\mathscr{A})$ to $\mathscr{P}(X)$ by setting $\mathsf{Ext}(U) \equiv \bigcup_{C \in U} \mathsf{ext}(C)$ for any subset U of the base $\mathscr{A}$ for the topological space $(X, \Omega(X))$, and it is obvious both that $\mathsf{Ext}(U)$ is an open subset, since it is the union of a family of open subsets, and that all the open subsets are equal to $\mathsf{Ext}(U)$ for some subset U of $\mathscr{A}$, since any open subset is obtained as a union of basic open subsets.

The following theorem is then obvious.

Theorem 1.1. Let $(X, \Omega(X))$ be a topological space and $\mathscr{A}$ be one of its bases. Then a subset O of X is open if and only if there exists a subset U_O of $\mathscr{A}$ such that $O = \mathsf{Ext}(U_O)$.

Proof. We have already noted that for any subsets U of the topological base $\mathscr{A}$, we have $\mathsf{Ext}(U)$ is an open subset. To prove the other implication direction, we set

$$U_O \equiv \{B \in \mathscr{A} \mid \mathsf{ext}(B) \subseteq O\}.$$

Then $O = \mathsf{Ext}(U_O)$. □

In the proof of the previous theorem we associated with any open subset O a subset U_O of basic open subsets, namely, the biggest subset W of basic open subsets such that $\mathsf{Ext}(W)$ is equal to O. We can refer to O_U as a *formal open* since it is built out of basic open subsets instead of points of X. While the open O can be equal to $\mathsf{Ext}(U)$ for many different subsets U of basic open subsets, so there is no bijective correspondence between open subsets and subsets of basic open subsets, it is clear that there is a bijective correspondence between concrete open subsets and formal open subsets.

The definition of U_O uses the *inclusion* relation among subsets of X, that is, the notion of formal open subset depends on the points of X. In order to have a purely formal definition of the formal open subsets that does not depend on the points of X, we can introduce a *cover relation* $B \lhd U$ between a basic open subset B and a subset U of basic open subsets whose intended meaning is that $\mathsf{ext}(B) \subseteq \mathsf{Ext}(U)$. Indeed, after this move, for any subset U of basic open subsets, a formal open $\lhd(U)$ can be defined by setting $\lhd(U) \equiv \{B \in \mathscr{A} \mid B \lhd U\}$.

Of course, in order for this approach to be effective, we need to formalise the cover relation as far as possible without any reference to the points in X.

Thus the cover relation can be used to define a formal counterpart of the concrete open subsets of a topological space. In this way, we will be able to express some typical topological situation in a formal way. For example, in order to state that a topological space is compact, we can say that for any subset U of $\mathscr{A}$, if $B \lhd U$ holds for all the basic open subsets $B \in \mathscr{A}$, then there exists a finite subset U_0 of U such that $B \lhd U_0$ holds for all the basic open subsets $B \in \mathscr{A}$.

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However, there are topological statements that are better expressed using the closed subsets of the topological space, or such that the statement one obtains using the open subsets is only classically equivalent to the one obtained using the closed subsets. Now it is not clear to us whether the standard point-free notions of closed or weakly closed sublocale are always adequate in this sense. So it may turn out that the ability also to define a formal counterpart of the closed subsets of a topological space will be convenient. Although we expect it to be fruitful, it should be noted that it is not known whether having a formal counterpart of closed subsets will allow constructive proofs of classical theorems that cannot be obtained using the formal cover relation alone.

From the classical point of view, a closed subset is the complement of an open subset. However, within an intuitionistic approach, the complement $\mathsf{C}(Y)$ of an open subset Y in union with Y itself does not give the whole space X, that is, it does not respect the idea of complementation. But, being the complement of an open subset is not the only definition of a closed subset one may consider since the various characterisations of a closed subset are not equivalent intuitionistically. For instance, a condition that is classically equivalent to stating that C is the complement of on open subset states that for all $x \in X$, we have $x \in C$ if and only if for all the basic open subsets $B \in \mathscr{A}$ such that $x \in B$, we have B *meets* C, in other words, the intersection between B and C contains some point (notation $B \between C$).

This definition has two advantages with respect to the standard one. First, it does not require the use of complementation. Moreover, it suggests how we can characterise a closed subset using basic open subsets alone, which is essential in our formal approach since they are the only ones for which we are going to have names. Indeed, in order to characterise the closed subset C, we just need to know the subset of the basic open subsets that meet C, as shown by the following theorem.

Theorem 1.2. Let $(X, \Omega(X))$ be a topological space and $\mathscr{A}$ be one of its bases. Then a subset C of X is closed if and only if there exists a subset F_C of $\mathscr{A}$ such that $C = \mathsf{Rest}(F_C)$ where, for any subset F of $\mathscr{A}$, we have

$$\mathsf{Rest}(F) \equiv \{x \in X \mid (\forall B \in \mathscr{A})\ x \in \mathsf{ext}(B) \to B \in F\}.$$

Proof. One can immediately check that for any subset F of the topological base $\mathscr{A}$, $\mathsf{Rest}(F)$ is a closed set. To prove the other implication direction, we set

$$F_C = \{B \in \mathscr{A} \mid \mathsf{ext}(B) \between C\}.$$

Then $C = \mathsf{Rest}(F_C)$. □

In the proof of Theorem 1.2 we associated with any closed subset C a subset F_C of basic open subsets, namely, the smallest subset W of basic open subsets such that $\mathsf{Rest}(W)$ is equal to C. By analogy with our treatment of the formal open subsets, we can refer to F_C as a *formal closed* since it is built from basic open subsets rather than points of X. One can see immediately that there is a bijective correspondence between concrete closed subsets and formal closed subsets.

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The definition of F_C uses the *meet* relation between subsets of X, so it depends on the points of X. To avoid such a dependence and get a purely formal definition of the formal closed subsets, we can observe that for any basic open subset $B \in \mathscr{A}$ and any subset F of basic open subsets, we can prove classically that

$$\mathsf{ext}(B) \between \mathsf{Rest}(F) \text{ if and only if } \neg(\mathsf{ext}(B) \subseteq \mathsf{Ext}(\mathsf{C}(F))).$$

Thus, a possible approach to the definition of the formal closed subsets that only requires us to use the cover relation is to state that for any subset F of basic open subsets, $\{B \in \mathscr{A} \mid \neg(B \lhd \mathsf{C}(F))\}$ is a formal closed.

However, if we do this, all the problems we wanted to avoid reappear since we need to use negation and complementation to get a definition that is only classically equivalent to what we really wanted. A possible way out is to introduce directly a *binary positivity predicate* $B \ltimes F$ between a basic open subset B and a subset F of basic open subsets with the aim of formalising the relation $\mathsf{ext}(B) \between \mathsf{Rest}(F)$ (Gebellato and Sambin 2002). In this way, for any subset F of basic open subsets, a formal closed $\ltimes(F)$ can be defined by setting $\ltimes(F) \equiv \{B \in \mathscr{A} \mid B \ltimes F\}$.

Of course, we now need to formalise the positivity predicate as far as possible without any reference to the points in X.

With the introduction of the binary positivity predicate, we get a more convenient language. For instance, we can now formalise the fact that a topological space is compact by saying that, from any family of closed subsets whose intersection is empty, it is possible to extract a finite sub-family whose intersection is also empty. Indeed, we can now state that a topological space is compact if and only if for any family $(F_i)_{i\in I}$ of basic open subsets, if for any finite subset J of I there exists some basic open subset $B \in \mathscr{A}$ such that $B \ltimes \bigcap_{j\in J} F_j$, then there exists $C \in \mathscr{A}$ such that $C \ltimes \bigcap_{i\in I} F_i$.

Thus two different definitions of a compact topological space are now directly available, but they are only equivalent from a classical point of view. So even if it is not clear if the presence of the binary positivity predicate is indispensable, we expect that it will be convenient in increasing the usability of the language of formal topologies.

Since the definition of the formal open subsets relies on the cover relation, and the definition of the formal closed subsets relies of the positivity predicate, the correspondence between concrete and formal open subsets and between concrete and formal closed subsets depends completely on the formalisation of these two predicates. But in the approach proposed in Gebellato and Sambin (2002), such a formalisation is insufficient in the sense that, for instance, the *positivity condition* cannot be expressed, that is, if

$$\mathsf{ext}(a) \between \mathsf{Rest}(F)$$

implies

$$\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U),$$

then

$$\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U),$$

which is characteristic of the positivity predicate and intuitionistically valid in any concrete topological space (see Section 2.1). Thus, a fundamental link between the cover relation

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and the positivity predicate is missing, so the characterisation of the formal open and closed subsets is not a faithful description of the point-wise situation.

For instance, the theorem of uniqueness of the positivity predicate, which holds for the original version of formal topology (see Curi (2004) or Maietti and Valentini (2004)), is lost. As a consequence, many different binary positivity predicates can be defined for the same cover relation, so many different families of formal closed subsets can be considered for the same family of formal opens (Valentini 2005). This breaks the connection between a concrete topological space, where the open and closed subsets are uniquely determined by the basic opens, and a formal topology, where this property does not hold. So the bijective correspondence between concrete open subsets and formal open subsets and between concrete closed subsets and formal closed subsets is not working, which means that the cover relation and the positivity predicate may have other meanings beside the intended ones.

Another symptom of the lack of a sufficient formalisation becomes evident when one considers the *formal points*, namely, the formal counterpart of the concrete points of a topological space (see Section 2.3). Indeed, just as open and closed subsets of a topological space have their formal counterpart in the form of suitable subsets of basic open subsets, a formal counterpart of the points of the topological space can also be provided by associating with any point $x \in X$, the subset $\alpha_x = \{B \in \mathscr{A} \mid x \in \mathsf{ext}(B)\}$ of all the basic open subsets that contain x.

Now, when one wants to characterise the formal points formally, that is, without any reference to the actual points in X, the roles of the cover relation and the positivity predicate seem to be unbalanced in both the original definition in Sambin (1987) and the new definition in Gebellato and Sambin (2002), even if one might well expect that the formal points determined by the open subsets and those determined by the closed subsets of the same topological space are the same. Indeed, in the original definition of formal topology in Sambin (1987), the formal points are determined by a condition on the cover relation alone. On the other hand, with the new definition in Gebellato and Sambin (2002), a condition on the binary positivity predicate alone is sufficient to define the formal points, while the condition on the cover relation is not.

It is for these reasons that, elaborating on an old idea that we proposed at the *Types '98* conference in Munchen, we develop here a new structure, which we call *relative formal topology*, that extends the definition of formal topology in Gebellato and Sambin (2002) in a way that allows us to express the positivity condition, and hence to solve the problems described above. In support of our choice, we will show that an adequacy theorem in the style of Valentini (2006) can also be proved for these new structures.

Moreover, we intend to show in a future paper (see Maietti and Valentini (2006) for a preliminary version) how the topology induced on an open subset can be defined using standard formal topology, and how the topology induced on a closed subset can be defined using relative formal topology. In fact, we do not know if the use of relative formal topology to get a formal representation of the topology induced on a closed subset is essential. For instance, Vickers (2007) proposed a constructive solution to the same problem using closed sublocales, whose definition only requires the cover relation, but note that closed sublocales are only classically equivalent to our relative formal topologies.

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1.1. *The rules of the game*

Many of the results of this paper require impredicative or classical reasoning, even though the natural framework for the development of formal topology is constructive mathematics[†]. Thus, it will be worth spending a little time explaining the 'rules of the game'.

Our aim is to develop an expressive theory that is adequate for working constructively with topological spaces. Thus, the first requirement is that the axioms, rules, conditions, and so on, that we decide to use to define such a theory are intuitionistically valid in any concrete topological space. The second requirement is that we must be able to express all the desired topological statements in the language of the theory. Finally, we require that the theory is adequate, that is, it must be sufficient for proving all the intuitionistically valid theorems on concrete topological spaces that can be expressed in its language.

So when we are interested in results that are to be proved *within* the theory, the use of intuitionistic reasoning is strictly required; indeed, it is only in this way that we are guaranteed that all the results we obtain are intuitionistically valid topological statements.

On the other hand, when we are interested in results *about* the theory, the situation is quite different, and impredicative or classical reasoning should not be cast aside *a priori*, especially when we consider that topology is a classical and impredicative theory since it makes free use of classical set-theory and higher-order quantification.

For instance, one of our main concerns will be to prove that the theory of relative formal topologies is adequate. However, the question as to whether there is an intuitionistically acceptable proof of adequacy of an intuitionistic formalisation of topology[‡] is still open. On the other hand, the lack even of a non-constructive proof of adequacy can only make the situation less satisfactory. This is why we propose a proof of adequacy in Section 3.1 that requires impredicative reasoning, and another proof in Section 3.3.2 that requires classical reasoning.

In any case, note that a classical proof of adequacy can also be extremely useful when it points out crucial intuitionistically valid conditions, as happened in our case with the *positivity* condition.

After an adequate formalisation has been achieved, a constructive mathematician can use it as the *definition* of a topological space, and continue using purely constructive reasoning, while a classical mathematician can still use all the results obtained therein as genuine topological results.

2. Relative formal topologies

Our aim now is to find a formalisation for the cover relation and positivity predicate that solves the problems described in the Introduction.

[†] Throughout this paper we work within Martin-Löf intuitionistic type theory except when we explicitly state that a result requires impredicative or classical reasoning (Martin-Löf 1984). For this reason, we distinguish between *sets*, which are inductively defined and support the use of the axiom of choice, and *collections*, which are just collections of objects provided with an identity relation. On the other hand, we will freely use the word *subset* since, in general, a subset is not a set (Sambin and Valentini 1997).

[‡] For instance, any proof of adequacy of a theory defined using the cover relation alone must be classical (see Valentini (2006) or Stefanova and Valentini (2011)).

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2.1. *The formal approach*

In this section we define our *formal* approach to constructive topology. As we have already hinted, we want to deal with the topological space $(X, \Omega(X))$ by working only with a set A of *names* for the elements of a base $\mathscr{A}$ for $(X, \Omega(X))$, that is, by abstracting away from the fact that the elements of the base are subsets of X. Then, when we want to deal with a concrete open subset O, we will work with the subset of A consisting of the names of the basic open subsets contained in the formal open subset $U_O = \{B \in \mathscr{A} \mid B \subseteq O\}$ and, when we want to deal with the concrete closed subsets C, we will work with the subset of A consisting of the names of the basic open subsets contained in the formal closed subset $F_C = \{B \in \mathscr{A} \mid B \between C\}$. In this way we can deal with the topological space $(X, \Omega(X))$ even when X is not a set but just a collection, since we can reason on elements of the set A instead of on elements of the collections X and $\Omega(X)$ (for an analysis of the foundational motivations that suggest this approach, see the first sections of Coquand *et al.* (2003) or the more recent Sambin (2003)).

So we suppose that A is a set of names for the elements of the base $\mathscr{A}$ for the topological space $(X, \Omega(X))$ and that there exists an indexing $\mathsf{ext}(-)$, that is, a surjective map from A to the base $\mathscr{A}$, associating with each name $a \in A$ an actual subset of X in the base $\mathscr{A}$; so $\mathsf{ext}(a)$ is the *extension* of the basic open subset whose name is a. Thus, $\mathsf{ext}(-)$ is no longer the identity, as in the previous section, but a *realisation map* since it realises the names in A.

Now, in analogy with the definitions using the same names in the previous section, we can introduce the maps Ext and Rest as follows:

$$\mathsf{Ext}(U) \equiv \{x \in X \mid (\exists a \in A)\ x \in \mathsf{ext}(a) \,\&\, a \in U\}$$

and

$$\mathsf{Rest}(F) \equiv \{x \in X \mid (\forall a \in A)\ x \in \mathsf{ext}(a) \rightarrow a \in F\}.$$

It is easy to see, by mimicking the proof of Theorems 1.1 and 1.2, that because of the surjectivity of the map ext, a subset O of X is open if and only if there exists a subset U_O of A such that $O = \mathsf{Ext}(U_O)$, and a subset C of X is closed if and only if there exists a subset F_C of A such that $C = \mathsf{Rest}(F_C)$. However, as in the previous section, there may well be many different subsets U determining the same open subset and many different subsets F determining the same closed subset. Hence, there is no bijective correspondence between concrete open subsets of $(X, \Omega(X))$ and subsets of names of basic opens, and no bijective correspondence between concrete closed subsets of $(X, \Omega(X))$ and subsets of names of basic opens. If we want to obtain such bijective correspondences, and hence be able to define notions of *formal open subset* and *formal closed subset*, we are required to choose one of the various subsets U of names such that $\mathsf{Ext}(U)$ identifies the same concrete open subset and one of the various subset F of names such that $\mathsf{Rest}(F)$ identifies the same concrete closed subset.

In the previous section we have already found a solution for this problem using the cover relation and the positivity predicate. However, in order to overcome the problem of the lack of an adequate formalisation, we propose here a slight generalisation of the cover relation by introducing the *relative cover relation* $a \lhd_F U$ between an element a of

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A and two subsets F and U of A whose intended meaning is

$$a \lhd_F U \equiv \mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U),$$

in other words, the part of $\mathsf{ext}(a)$ within the closed subset $\mathsf{Rest}(F)$ is covered by the open subset $\mathsf{Ext}(U)$, together with a standard *binary positivity predicate* $a \ltimes F$ between an element a of A and a subset F of A whose intended meaning is

$$a \ltimes F \equiv \mathsf{ext}(a) \between \mathsf{Rest}(F),$$

in other words, that there is some point in the intersection between the basic open subset $\mathsf{ext}(a)$ and the closed subset $\mathsf{Rest}(F)$.

It is obvious that the relative cover relation is a generalisation of the standard cover relation $a \lhd U$. Indeed, the latter aims to formalise the fact that $\mathsf{ext}(a)$ is a subset of $\mathsf{Ext}(U)$ and $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ holds if and only if $\mathsf{ext}(a) \cap \mathsf{Rest}(A) \subseteq \mathsf{Ext}(U)$ since $\mathsf{Rest}(A) = X$. Thus we get $a \lhd U$ if and only if $a \lhd_A U$.

On the other hand, from a classical point of view, a relative cover relation can always be reduced to a standard cover relation since $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$ is classically equivalent to $\mathsf{ext}(a) \subseteq \mathsf{Ext}(F \to U)$, where

$$F \to U \equiv \{c \in A \mid \text{if } c \in F \text{ then } c \in U\}.$$

Hence, given a standard cover relation, we can define a new cover relation by setting

$$a \lhd^F U \equiv a \lhd F \to U.$$

The intended meaning is that $\lhd^F$ is the cover relation of the topological space induced by the original cover on the closed subset $\mathsf{Rest}(F)$ determined by F; indeed, $a \lhd^F U$ means that $\mathsf{ext}(a)$ is covered, in the original cover, by $\mathsf{Ext}(F \to U)$, which is classically equivalent to stating that the part of $\mathsf{ext}(a)$ within the closed $\mathsf{Rest}(F)$ is covered by $\mathsf{Ext}(U)$[†].

As we saw in the previous section, the binary positivity predicate can also be defined using a standard cover relation since $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ is classically equivalent to $\neg(\mathsf{ext}(a) \subseteq \mathsf{Ext}(\mathsf{C}(F)))$, so we can set

$$a \ltimes F \equiv \neg(a \lhd \mathsf{C}(F)).$$

Thus, a standard cover relation seems to be sufficient to get all of the results that we can obtain by using the relative cover relation and the binary positivity predicate. Indeed, we can use the usual conditions on a standard cover relation to prove the validity of all the conditions that we require on a relative formal topology according to Definition 2.1. Hence, all the results in the current paper can also be obtained in a localic approach.

The problem is that a standard two places cover relation can only be used to define a relative cover relation and a binary positivity predicate within a classical framework.

[†] According to Vickers (2007), a locale corresponds in the language of formal topology to a cover relation, and a sublocale is obtained by considering a new cover relation containing the given one. Hence, in our case, for any subset F of basic open subsets, $\lhd^F$ corresponds to a sublocale of the locale determined by $\lhd$ because $a \lhd U$ yields $a \lhd^F U$ since $U \subseteq F \to U$. In the same paper, the relationship between this notion of a sublocale and the original one in Johnstone (1982) was analysed and, because of its meaning, the kind of sublocale we have defined here is called a *closed* sublocale.

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Indeed, one can prove that in general there is no constructive, or even impredicative, way to add a positivity predicate to all cover relations (see, for instance, Curi (2010)). Moreover, any proof of adequacy of the formalisation of a standard cover relation has to be classical while, as far as we are aware, the question of whether it is possible to get a constructive proof of completeness of the formalisation of a relative cover relation and a positivity predicate satisfying positivity is still open (see, for instance, Stefanova and Valentini (2011), which analysed the case of countably presented formal topology with a unary positivity predicate, which is also known as an open locale).

Thus, even if at present we have no example of a classical topological result that can only be proved constructively within relative formal topology, our proposal is to use the intuitionistically valid conditions for a relative cover relation and a binary positivity predicate directly in the definition of relative formal topology itself in the hope that in this way we can develop topology using purely constructive reasoning, that is, using inductive and co-inductive methods (see Section 3.3).

Following the introduction of the relative cover relation, we can choose the biggest of the various subsets U of names that determine the same concrete open subset $\mathsf{Ext}(U)$ by setting

$$\lhd_A(U) \equiv \{a \in A \mid a \lhd_A U\}.$$

Indeed, $\lhd_A(U)$ is the biggest subset of names containing U such that $\mathsf{Ext}(U) = \mathsf{Ext}(\lhd_A(U))$. Borrowing the terminology from the previous section, we say that $\lhd_A(U)$ is the *formal open subset* corresponding to the concrete open subset $\mathsf{Ext}(U)$.

In a similar way, following the introduction of the binary positivity predicate, we can choose the smallest of the various subsets F of names that determine the same concrete closed subset $\mathsf{Rest}(F)$ by setting

$$\ltimes(F) \equiv \{a \in A \mid a \ltimes F\}.$$

Indeed, $\ltimes(F)$ is the smallest subset of names contained in F such that $\mathsf{Rest}(\ltimes(F)) = \mathsf{Rest}(F)$. Again borrowing the terminology of the previous section, we say that $\ltimes(F)$ is the *formal closed subset* corresponding to the concrete closed subset $\mathsf{Rest}(F)$.

However, this approach is only satisfactory if an adequate formalisation of the relative cover relation and the binary positivity predicate can be provided, since the definitions of the formal open and closed subsets are completely dependent on such predicates. In the next paragraph, we give some intuitionistically valid formal conditions on the relative formal cover and the binary positivity predicate – we will discuss the adequacy of the formalisation in Section 3.

In order to check the validity of the conditions we propose, it is worth noting that the maps Ext and Rest are monotone, and that Ext distributes over unions and Rest distributes over intersections.

$$U \subseteq \lhd_F(U) \qquad \text{(reflexivity)}$$

$$\ltimes(F) \subseteq F \qquad \text{(pos-reflexivity)}$$

$$\frac{U \subseteq \lhd_F(V)}{\lhd_F(U) \subseteq \lhd_F(V)} \qquad \text{(transitivity)}$$

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$$\frac{\ltimes(F) \subseteq G}{\ltimes(F) \subseteq \ltimes(G)} \qquad \text{(pos-transitivity)}$$

$$\frac{\ltimes(F) \between \lhd_F(U)}{\ltimes(F) \between U} \qquad \text{(compatibility)}$$

$$\frac{a \ltimes F \to a \lhd_F U}{a \lhd_F U} \qquad \text{(positivity)}$$

$$\frac{\ltimes(F) \subseteq \ltimes(G)}{\lhd_G(U) \subseteq \lhd_F(U)} \qquad (\ltimes\text{-restriction})$$

where

$$\lhd_F(U) \equiv \{c \in A \mid c \lhd_F U\}$$

and

$$\ltimes(F) \equiv \{c \in A \mid c \ltimes F\}.$$

Reflexivity, *transitivity* and *compatibility* are just the generalisations to our framework of the conditions with the same names in Gebellato and Sambin (2002). Moreover, *pos-reflexivity* and *pos-transitivity* are exactly the conditions required on the binary positivity predicate in the same paper.

On the other hand, *positivity* and $\ltimes$-*restriction* are new and deserve a little explanation.

The *positivity* condition states that it is possible to reason by cases on $a \ltimes F$ when it is a matter of showing that $a \lhd_F U$ holds. Thus, its interest is more logical than topological since it brings a bit of classical reasoning into the realm of an intuitionistically valid approach to topology.

On the other hand, $\ltimes$-*restriction* shows that two subsets F and G determine the same *concrete* closed subset (that is, $a \lhd_G U$ holds if and only if $a \lhd_F U$ holds) if they determine the same *formal* closed subset, namely, $\ltimes(F) = \ltimes(G)$ (see Section 4 for a strengthening of this condition that shows how to glue together the cover relation on different closed subsets).

If, with no loss of generality, we add the requirement that the base $\mathscr{A}$ is closed under finite intersection[†], we can assume that there is an infimum operation $\wedge$ defined on the set A of the names of the basic opens, which is just going to be interpreted in the intersection of basic open subsets, that is, for any $a, b \in A$, we have $\mathsf{ext}(a \wedge b) = \mathsf{ext}(a) \cap \mathsf{ext}(b)$. It is easy to check that in this case $\wedge$ satisfies the following equalities:

$$a \wedge a = a \qquad \text{(identity)}$$

$$a \wedge b = b \wedge a \qquad \text{(commutativity)}$$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c\,. \qquad \text{(associativity)}$$

[†] It is easy to check that if a base $\mathscr{A}$ is indexed by the set A then the (topologically equivalent) base obtained by closing $\mathscr{A}$ under finite intersection can be indexed by the set of the lists on A.

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And that the following conditions are valid:

$$\frac{a \lhd_F U}{a \wedge b \lhd_F U} \qquad \frac{b \lhd_F U}{a \wedge b \lhd_F U} \tag{$\lhd$-left}$$

$$\frac{a \wedge b \bowtie F}{a \bowtie F} \qquad \frac{a \wedge b \bowtie F}{b \bowtie F} \tag{$\bowtie$-left}$$

$$\lhd_F(U) \cap \lhd_F(V) = \lhd_F(U \wedge V) \tag{$\lhd$-right}$$

where

$$U \wedge V \equiv \{u \wedge v \in A \mid u \in U \text{ and } v \in V\}.$$

Thus we arrive at the following definition of a *relative formal topology*.

Definition 2.1 (rf-topology). An rf-*topology* is a structure $(A, \wedge, \lhd, \bowtie)$ such that $(A, \wedge)$ is a semi-lattice, $\lhd$ is a relation, called a *relative cover*, between elements and pair of subsets of A, and $\bowtie$ is a relation, called a *binary positivity predicate*, between elements and subsets of A, satisfying *reflexivity*, *transitivity*, $\lhd$-*left*, $\lhd$-*right*, *pos-reflexivity*, *pos-transitivity*, $\bowtie$-*left*, *compatibility*, *positivity* and $\bowtie$-*restriction*.

In the following sections we will show that there are plenty of examples of rf-topologies.

2.2. *Immediate consequences of the basic conditions*

We can show that the conditions that an rf-topology satisfies are relevant by showing some of their consequences. Note that these consequences are completely formal, that is, they follow directly from the required conditions without any reference to the points of a concrete topological space.

Our first lemma is borrowed from Gebellato and Sambin (2002), but adapted to our framework. The proof is immediate from *reflexivity*, *transitivity*, *pos-reflexivity* and *pos-transitivity*.

Lemma 2.2. Let $(A, \wedge, \lhd, \bowtie)$ be an rf-topology. Then, for any subset F of A, $\lhd_F$ is a closure operator, that is, it satisifes *reflexivity* and, for any $U, V \subseteq A$,

$$\frac{U \subseteq V}{\lhd_F(U) \subseteq \lhd_F(V)} \tag{monotonicity}$$

$$\lhd_F(U) = \lhd_F(\lhd_F(U)) \tag{idempotency}$$

and $\bowtie$ is an interior operator, that is, it satisifes *pos-reflexivity* and, for any $F, G \subseteq A$,

$$\frac{F \subseteq G}{\bowtie(F) \subseteq \bowtie(G)} \tag{pos-monotonicity}$$

$$\bowtie(F) = \bowtie(\bowtie(F)). \tag{pos-idempotency}$$

We then obtain the following corollary, which shows a formal way of expressing the fact that F and $\bowtie(F)$ determine the same concrete closed subset.

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Corollary 2.3. Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology. Then the following conditions are valid for any subsets F, G and U of A,

$$\frac{F \subseteq G}{\lhd_G(U) \subseteq \lhd_F(U)} \qquad \text{(sub-restriction)}$$

$$\lhd_{\ltimes(F)}(U) = \lhd_F(U). \qquad \text{(pos-equivalence)}$$

Proof. *Sub-restriction* is an immediate consequence of $\ltimes$-*restriction* and *pos-monotonicity* while *pos-equivalence* follows from $\ltimes$-*restriction* and *pos-idempotency*. □

Moreover, we have properties that are specific to relative formal topologies and their *positivity* condition.

Lemma 2.4. For any $a \in A$ and $F, U \subseteq A$:

(1) If $a \lhd_F U$, then $a \lhd_F U^{+_F}$, where $U^{+_F} \equiv U \cap \ltimes(F)$.
(2) (positivity axiom) $a \lhd_F a^{+_F}$, where $a^{+_F} \equiv \{a\} \cap \ltimes(F)$.
(3) If $a \lhd_G F \to U$, then $a \lhd_{G \cap F} U$.
(4) If $a \lhd_A F \to U$, then $a \lhd_F U$.
(5) $a \lhd_F F$.

Proof.

(1) From *transitivity*, in order to prove the first point, it is sufficient to prove that $U \lhd_F U^{+_F}$. To this end, we assume that $c \in U$ and $c \ltimes F$. So $c \in U^{+_F}$ and thus $c \lhd_F U^{+_F}$. So $c \ltimes F \to c \lhd_F U^{+_F}$ and thus $c \lhd_F U^{+_F}$ follows by *positivity*.
(2) This is immediate since $a \lhd_F \{a\}$ holds by *reflexivity*.
(3) First note that $a \lhd_G F \to U$ implies $a \lhd_{G \cap F} F \to U$ by *sub-restriction*. Then, from (1), we get $a \lhd_{G \cap F} (F \to U)^{+_{G \cap F}}$, so to conclude by *transitivity* it is sufficient to show that $(F \to U)^{+_{G \cap F}} \lhd_{G \cap F} U$. So, we let $c \in (F \to U)^{+_{G \cap F}}$, that is, $c \ltimes (G \cap F)$ and $c \in F$ implies $c \in U$. Then $c \in (G \cap F)$ follows by *pos-reflexivity*, which gives $c \in F$, so we get $c \in U$ by logic, which gives $c \lhd_{G \cap F} U$ by *reflexivity*.
(4) This is immediate by putting $G \equiv A$ in (3).
(5) This is immediate from the observation that $F \to F = A$, so $a \lhd_A F \to F$ holds by *reflexivity* and thus $a \lhd_F F$ by (3). □

We will see that the *positivity axiom* is in fact equivalent to the *positivity* condition (*cf.* Section 3.3.2).

An important consequence of the chosen conditions is that the positivity predicate is uniquely determined by the relative cover relation. Thus, by using relative formal topology, we solve the problem that we pointed out in the Introduction on the possibility of having the same family of formal open subsets with different families of formal closed subsets.

The proof of this result is analogous to the proof of uniqueness of the unary positivity predicate in formal topology (see Curi (2004) or Maietti and Valentini (2004)).

Theorem 2.5 (Pos-uniqueness). Let $(A, \wedge, \lhd, \ltimes^1)$ and $(A, \wedge, \lhd, \ltimes^2)$ be two rf-topologies with the same cover relation. Then $\ltimes^1$ and $\ltimes^2$ are equivalent.

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Proof. By the *positivity axiom*, for every $a \in A$, we have $a \lhd_F a^{+^2_F}$, where $a^{+^2_F}$ is a shorthand for the subset $\{x \in A \mid x = a \;\&\; x \ltimes^2 F\}$. We now assume $a \ltimes^1 F$. Then, by *compatibility*, there exists $x \in a^{+^2_F}$ such that $x \ltimes^1 F$ holds. But $x \in a^{+^2_F}$ means that both $x = a$ and $x \ltimes^2 F$ hold, so $a \ltimes^2 F$ follows.

We can prove the other implication direction in a completely analogous way. □

It is interesting that the *positivity axiom* is sufficient to generalise an observation made by Peter Aczel for standard formal topology to the case of rf-topology.

Lemma 2.6. Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology. Then $a \ltimes F$ holds if and only if, for all $U \subseteq A$, we have $a \lhd_F U$ implies that U is inhabited.

Proof. Suppose that for all $U \subseteq A$, we have $a \lhd_F U$ implies that U is inhabited. Then, by instantiating with the subset a^{+_F}, we get that if $a \lhd_F a^{+_F}$, then a^{+_F} is inhabited. Then a^{+_F} follows by logic because of the *positivity axiom*, so we get $a \ltimes F$ by intuitionistic logic.

On the other hand, if $a \ltimes F$ and $a \lhd_F U$, then, by *compatibility*, we obtain

$$(\exists u \in A)\; u \in U \;\&\; u \ltimes F,$$

which gives $(\exists u \in A)\; u \in U$ by logic. □

Note that we cannot use

$$(\forall U \subseteq A)\; a \lhd_F U \rightarrow (\exists u \in A)\; u \in U$$

as an impredicative definition of a positivity predicate since in order to prove that it satisfies *compatibility* and *positivity*, we would have to use the fact that it is equivalent to a positivity predicate $\ltimes$ that we are supposed to have already; in particular, we cannot get a direct proof of its equivalence to $\ltimes$ using the uniqueness theorem for the positivity predicate.

2.3. *Formal points of an* rf-*topology*

The collection of concrete points of a topological space $(X, \Omega(X))$ has some properties that can be expressed in the language of rf-topology. Indeed, let x be a concrete point in X and suppose that $\mathscr{A}$ is a base closed under intersection for the topology $\Omega(X)$. Then, in order to deal with the point x by using only the elements of the base $\mathscr{A}$, we can consider the collection $\alpha_x \equiv \{C \in \mathscr{A} \mid x \in C\}$, that is, the filter of basic opens that contain x. From a topological point of view, α_x is a good representation of the point x since one can distinguish only those points that are separated by a (basic) open subset.

Now we get that α_x is not empty since x is contained in some basic open subset and, after the condition of closure under intersection for our base, we get

$$\frac{B \in \alpha_x \quad C \in \alpha_x}{B \cap C \in \alpha_x}. \qquad \text{(point convergence)}$$

Thus, if we write $\alpha_y \wedge \alpha_z$ to mean the subset $\{B \cap C \in \mathscr{A} \mid B \in \alpha_y, C \in \alpha_z\}$, we get $\alpha_x \wedge \alpha_x \subseteq \alpha_x$, and thus $\alpha_x = \alpha_x \wedge \alpha_x$ since for any basic open C, $C \cap C = C$ holds trivially.

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Moreover, if U and F are subsets of $\mathscr{A}$, then

$$\frac{C \in \alpha_x \quad \alpha_x \subseteq F \quad \mathsf{ext}(C) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)}{(\exists B \in U)\ B \in \alpha_x}. \qquad \text{(relative splitness)}$$

Indeed, $C \in \alpha_x$ means that $x \in \mathsf{ext}(C)$ since, when we are dealing with a concrete topological space, the map ext is just the identity. Moreover $\alpha_x \subseteq F$ means that $x \in \mathsf{Rest}(F)$ since $x \in \mathsf{Rest}(F)$ if and only if for all $B \in \mathscr{A}$, we have $x \in \mathsf{ext}(B)$ implies $B \in F$, that is, $B \in \alpha_x$ implies $B \in F$, and thus if and only if $\alpha_x \subseteq F$. Thus, $\mathsf{ext}(C) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$ implies $x \in \mathsf{Ext}(U)$, so there is $B \in U$ such that $x \in \mathsf{ext}(B)$, that is, $B \in \alpha_x$.

Finally, if F is a subset of $\mathscr{A}$, then

$$\frac{C \in \alpha_x \quad \alpha_x \subseteq F}{\mathsf{ext}(C) \between \mathsf{Rest}(F)}. \qquad \text{(point positivity)}$$

Indeed, $C \in \alpha_x$ means that $x \in \mathsf{ext}(C)$, and $\alpha_x \subseteq F$ means that $x \in \mathsf{Rest}(F)$, so we have found a point that belongs to both $\mathsf{ext}(C)$ and $\mathsf{Rest}(F)$, that is, $\mathsf{ext}(C) \between \mathsf{Rest}(F)$ holds.

This suggests that the formal counterpart of a concrete point should be a non-empty subset α of the set of names A that is a fixed point of the $\wedge(-) \equiv - \wedge -$ operator, that is, such that $\wedge(\alpha) = \alpha \wedge \alpha = \alpha$ holds, and that satisfies

$$\frac{c \in \alpha \quad \alpha \subseteq F \quad c \lhd_F U}{(\exists b \in U)\ b \in \alpha} \qquad \text{(relative splitness)}$$

$$\frac{c \in \alpha \quad \alpha \subseteq F}{c \ltimes F}. \qquad \text{(point positivity)}$$

However, the next lemma shows that the last two conditions are both equivalent to

$$\ltimes(\alpha) = \alpha, \qquad (\ltimes\text{-fixpoint})$$

that is, α is a fixed-point of the map $\ltimes(-)$.

Lemma 2.7. Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology. Then, *relative splitness*, *point positivity* and $\ltimes$-*fixpoint* are equivalent conditions.

Proof. We will first prove that *relative splitness* implies *point positivity*. Indeed, by the *positivity axiom*, we have $c \lhd_F c^{+_F}$, so, by *relative splitness*, there exists some element $b \in c^{+_F}$ such that $b \in \alpha$. But $b \in c^{+_F}$ gives $c \ltimes F$ by logic.

We can now prove that *point positivity* implies $\ltimes$-*fixpoint*. Indeed, $\ltimes(\alpha) \subseteq \alpha$ holds by *pos-reflexivity*, so we only have to prove the other inclusion, which is immediate by *point positivity* since $\alpha \subseteq \alpha$ holds trivially.

Finally, we will prove that $\ltimes$-*fixpoint* implies *relative splitness*. Now, $c \in \alpha$ implies $c \in \ltimes(\alpha)$ by $\ltimes$-*fixpoint*. Moreover, $\alpha \subseteq F$ and $c \lhd_F U$ imply $c \lhd_\alpha U$ by *sub-restriction*, so the conclusion of *relative splitness* follows from *compatibility* and then *pos-reflexivity*. □

This suggests the following definition.

Definition 2.8 (Formal point). Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology. Then a non-empty subset α of A is a *formal point* if it is a fixed point of both the operator $\wedge(-)$ and the operator $\ltimes(-)$.

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Hence, apart from being a fixed point of the operator $\wedge(-)$, which is a condition on the elements of A, the definition of a formal point depends *only* on the binary positivity predicate and not (directly) on the cover relation.

Note, however, that in the definition of a formal point we could have equivalently used a condition using the relative cover relation alone, that is, *relative splitness*, instead of $\ltimes$-*fixedpoint*. Thus, relative formal topology solves the problem we pointed out in the Introduction concerning the different characterisations of the formal points obtained by considering either the open subsets or the closed subsets of the same topological space.

It is interesting that in this way the problem of finding formal points is reduced to a well-known problem, namely, the problem of finding the common solutions of two monotone operators; indeed, it is easy to check that $\wedge(-)$ is monotone and we have already noted that the positivity operator is also monotone (see Tarski (1955) or Merryfiled and Stein (1999) for the problem of finding the common fixed points of monotone operators, and Section 3.2 later in the current paper for a specific solution).

It is possible to associate with any rf-topology $(A, \wedge, \lhd, \ltimes)$ the collection $\mathsf{Pt}_{\ltimes}(A)$ of the formal points of A, and we can then immediately turn such a collection into a topological space by specifying a base and a map $\mathsf{ext}^{\mathsf{Pt}_{\ltimes}}$. So, following the usual approach due to Stone, we set $\mathsf{ext}^{\mathsf{Pt}_{\ltimes}}(c) = \{\alpha \in \mathsf{Pt}_{\ltimes}(A) \mid c \in \alpha\}$ for any $c \in A$, and then take as the base for a topology on $\mathsf{Pt}_{\ltimes}(A)$ the set-indexed collection $\mathscr{B}^{\mathsf{Pt}_{\ltimes}} = \{\mathsf{ext}^{\mathsf{Pt}_{\ltimes}}(c) \mid c \in A\}$.

If we extend the definitions of the maps Ext and Rest to the case of the topological space on $\mathsf{Pt}_{\ltimes}(A)$, we can express in a formal way the condition that a formal point belongs to an open or closed subset of this topological space.

Theorem 2.9. Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology and α be a formal point of A. Then

$$\alpha \in \mathsf{Ext}^{\mathsf{Pt}_{\ltimes}}(U) \text{ iff } (\exists u \in A)\, u \in \alpha \,\&\, u \in U \text{ iff } \alpha \between U$$
$$\alpha \in \mathsf{Rest}^{\mathsf{Pt}_{\ltimes}}(F) \text{ iff } (\forall c \in A)\, c \in \alpha \to c \in F \text{ iff } \alpha \subseteq F.$$

Proof. The proof is immediate. □

This simple result is important since it can be used to check directly, namely, without any categorical characterisation, the correctness of the definition of the topology induced on an open or closed subset of a given formal topology. Indeed, the relative formal topology induced on an open or closed subset will consider the same set of names for basic open subsets and the same infimum operation of the original relative formal topology; the novelty with respect to the case of the whole space is going to be in the definition of the relative cover relation and the binary positivity predicate. Hence, we can check if we have found the correct definition for the topology induced on the open subset determined by U by simply checking whether the formal points of the new topology are the formal points of the original one that meet U; while to check if we have found the correct definition of the topology induced on the closed subset determined by F, we just have to check whether the formal points of the new topology are the formal points of the original one included in F.

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3. In search of completeness

We have now shown that the conditions we have identified for the relative cover relation and the binary positivity predicate are intuitionistically valid and are sufficient for solving the problems we pointed out in the Introduction, but an important issue is whether they are in fact sufficient to characterise them, in other words, whether a completeness (that is, adequacy) theorem can be proved in the same way as we did in Valentini (2006) for countably presented inductively generated formal topologies.

The main lesson we learned from the completeness proof in Valentini (2006) was that completeness means completeness with respect to the topological space on the collection of the formal points, that is, completeness means spatiality. Indeed, there is no need to check that $a \lhd_F U$ if and only if $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$ and $a \ltimes F$ if and only if $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ for any indexing $\mathsf{ext}(-)$ on any possible topological space since it is sufficient to check that this holds with respect to the topological space $\mathsf{Pt}_{\ltimes}(A)$ and the indexing $\mathsf{ext}^{\mathsf{Pt}_{\ltimes}}(-)$ (see Valentini (2006, Section 2, in particular, Theorem 2.2)). Thus, proving a completeness theorem means proving that $a \ltimes F$ holds if and only if

$$\mathsf{ext}^{\mathsf{Pt}_{\ltimes}}(a) \between \mathsf{Rest}^{\mathsf{Pt}_{\ltimes}}(F)$$

and that $a \lhd_F U$ holds if and only if

$$\mathsf{ext}^{\mathsf{Pt}_{\ltimes}}(a) \cap \mathsf{Rest}^{\mathsf{Pt}_{\ltimes}}(F) \subseteq \mathsf{Ext}^{\mathsf{Pt}_{\ltimes}}(U).$$

As an immediate application of this result, we can show that any complete formalisation of constructive topology cannot exclude the *positivity* condition. First, we observe that *point positivity* is equivalent to the following condition:

$$\frac{c \in \alpha \quad \alpha \subseteq F}{\alpha \between c^{+_F}}. \qquad \text{(semantical positivity axiom)}$$

Indeed, if $c \in \alpha$ and $\alpha \subseteq F$, then $c \ltimes F$ follows by *point positivity*, so $c \in c^{+_F}$ holds and we then get $\alpha \between c^{+_F}$ through another use of the hypothesis $c \in \alpha$. On the other hand, if $\alpha \subseteq F$ and $c \in \alpha$, then $\alpha \between c^{+_F}$ follows by the *semantic positivity axiom*, and we then have $c \ltimes F$ by logic.

Thus, the *semantical positivity axiom* holds for any formal point α since *point positivity* does, and this shows that for any $c \in A$, we have

$$\mathsf{ext}^{\mathsf{Pt}}(c) \cap \mathsf{Rest}^{\mathsf{Pt}}(F) \subseteq \mathsf{Ext}^{\mathsf{Pt}}(c^{+_F})$$

since

$$\alpha \in \mathsf{ext}^{\mathsf{Pt}}(c) \text{ if and only if } c \in \alpha,$$

and

$$\alpha \in \mathsf{Rest}^{\mathsf{Pt}}(F) \text{ if and only if } \alpha \subseteq F$$

and

$$\alpha \in \mathsf{Ext}^{\mathsf{Pt}}(c^{+_F}) \text{ if and only if } \alpha \between c^{+_F}.$$

Thus, if we want spatiality to hold for a certain relative formal topology, the *positivity axiom*, that is, $c \lhd_F c^{+_F}$, must also hold.

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3.1. *Impredicative completeness*

In this section we show that, starting from any binary predicate $\mathsf{Pos}(a, F)$ on a set of names A, endowed with an infimum operation $\wedge$, satisfying *pos-reflexivity* and *pos-transitivity*, it is possible to define impredicatively a positivity predicate $\ltimes$ and a relative cover relation $\lhd$ that are spatial and such that the topological space $\mathsf{Pt}_{\ltimes}(A)$ coincides with the topological space $\mathsf{Pt}_{\mathsf{Pos}}(A)$ on the collection of the Pos-formal points, namely, the non-empty subsets of A that are both fixed points of $\wedge(-)$ and $\mathsf{Pos}(-)$.

Indeed, suppose we are given such a predicate Pos on a set A endowed with infimum operation $\wedge$. Then, provided we use an impredicative quantification on subsets of A, we can define a positivity predicate $\ltimes$ by setting

$$a \ltimes F \equiv \mathsf{ext}^{\mathsf{Pt}_{\mathsf{Pos}}}(a) \between \mathsf{Rest}^{\mathsf{Pt}_{\mathsf{Pos}}}(F), \tag{impPos}$$

that is, $a \ltimes F$ holds if and only if there is some Pos-formal point α in $\mathsf{Pt}_{\mathsf{Pos}}(A)$ such that $a \in \alpha \subseteq F$.

The following lemma shows how the predicates Pos and $\ltimes$ are related.

Lemma 3.1. Let $(A, \wedge)$ be a semi-lattice and Pos be a predicate satisfying *pos-reflexivity* and *pos-transitivity*. Then, $a \ltimes F$ implies $a \in \mathsf{Pos}(F)$.

Proof. Suppose $a \ltimes F$. Then there exists a Pos-formal point $\alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A)$ such that $a \in \alpha \subseteq F$, so $a \in \alpha = \mathsf{Pos}(\alpha) \subseteq \mathsf{Pos}(F)$ since α is a fixed point for Pos and Pos satisfies *pos-monotonicity*, which is a consequence of *pos-reflexivity* and *pos-transitivity*. So $a \in \mathsf{Pos}(F)$. □

In general, the other implication direction does not hold: consider, for instance, the case in which there is no Pos-formal point in $\mathsf{Pt}_{\mathsf{Pos}}(A)$. Thus, $\ltimes$ chooses within the relation $\mathsf{Pos}(a, F)$ the couples (a, F) that are in fact supported by a Pos-formal point.

The main observation is now stated in the following theorem.

Theorem 3.2. The collections $\mathsf{Pt}_{\ltimes}(A)$ and $\mathsf{Pt}_{\mathsf{Pos}}(A)$ coincide.

Proof. First note that the properties of being non-empty and a fixed point for the map $\wedge(-)$ depend only on the set A and the operation $\wedge$ on elements of A. Thus we have only to show that α is a fixed point of the map $\ltimes(-)$ if and only if it is a fixed point of the map $\mathsf{Pos}(-)$. But $\mathsf{Pos}(\alpha) \subseteq \alpha$ holds by *pos-reflexivity* of Pos, and $\ltimes(\alpha) \subseteq \alpha$ is obvious from the definition. Thus we have to show that if $\alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A)$, then $\alpha \subseteq \ltimes(\alpha)$, and that if $\alpha \in \mathsf{Pt}_{\ltimes}(A)$, then $\alpha \subseteq \mathsf{Pos}(\alpha)$.

So we suppose that $\alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A)$ and $a \in \alpha$. So there exists a Pos-formal point β in $\mathsf{Pt}_{\mathsf{Pos}}(A)$, namely, α itself, such that $a \in \beta \subseteq \alpha$, so $a \ltimes \alpha$ holds, that is, $a \in \ltimes(\alpha)$. On the other hand, if $\alpha \in \mathsf{Pt}_{\ltimes}(A)$, then $a \in \alpha$ implies that $a \in \ltimes(\alpha)$, that is, there exists a Pos-formal point $\beta \in \mathsf{Pt}_{\mathsf{Pos}}(A)$ such that $a \in \beta \subseteq \alpha$. So $a \in \beta = \mathsf{Pos}(\beta) \subseteq \mathsf{Pos}(\alpha)$ follows by *pos-monotonicity* and the fact that β is a fixed point for Pos. □

The following is an immediate corollary of this theorem.

Corollary 3.3 ($\ltimes$-spatiality). Let the predicate $\ltimes$ be defined as above. Then $a \ltimes F$ if and only there exists a formal point $\alpha \in \mathsf{Pt}_{\ltimes}(A)$ such that $a \in \alpha \subseteq F$.

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A useful observation is that $\ltimes$-spatiality makes the meaning of the $\ltimes(-)$ operator explicit, *viz*.,

$$\ltimes(F) = \bigcup\{\alpha \in \mathsf{Pt}_{\ltimes}(A) \mid \alpha \subseteq F\}.$$

The relative cover relation can also be defined using the Pos-formal points (or, equivalently, the formal points in $\mathsf{Pt}_{\ltimes}(A)$) through an impredicative definition

$$a \lhd_F U \equiv \mathsf{ext}^{\mathsf{Pt}_{\mathsf{Pos}}}(a) \cap \mathsf{Rest}^{\mathsf{Pt}_{\mathsf{Pos}}}(F) \subseteq \mathsf{Ext}^{\mathsf{Pt}_{\mathsf{Pos}}}(U). \qquad \text{(impCov)}$$

In other words, $a \lhd_F U$ holds if and only if, for any $\alpha \in \mathsf{Pt}_{\ltimes}(A)$, we have $a \in \alpha \subseteq F$ implies $\alpha \between U$.

Thus, given a predicate Pos satisfying *pos-reflexivity* and *pos-transitivity*, and, in particular, given any rf-topology, we can always refine it to a spatial positivity predicate $\ltimes$ and a spatial cover relation $\lhd$ using impredicative definitions, and in this way obtain a complete rf-topology describing the same topological space.

Thus, given any set A endowed with an infimum operation $\wedge$, any binary predicate Pos satisfying *pos-reflexivity* and *pos-transitivity* is sufficient to define through impredicative mathematics a spatial rf-topology on the collection of the formal points determined by Pos. However, the real challenge comes when we want to work within a predicative framework (see Section 3.3).

3.2. *Algebraic completeness*

The problem of defining a positivity predicate can also be viewed as the problem of finding the solution of a system of two equations. Indeed, assume we have a semi-lattice $(A, \wedge)$. Then we want to find a positivity predicate, namely, a map $\ltimes : \mathscr{P}(A) \to \mathscr{P}(A)$, and a collection $\mathsf{Pt}_{\ltimes}(A)$ of formal points on A such that the following conditions hold:

— for any $F \subseteq A$, we have

$$\ltimes(F) = \bigcup\{\alpha \in \mathsf{Pt}_{\ltimes}(A) \mid \alpha \subseteq F\};$$

— $\alpha \in \mathsf{Pt}_{\ltimes}(A)$ if and only if α is a non-empty subset of A such that $\wedge(\alpha) = \alpha$ and $\ltimes(\alpha) = \alpha$.

Now note that, independently of the collection $\mathsf{Pt}_{\ltimes}(A)$ of the formal points that we are considering, the first condition guarantees that the positivity predicate satisfies both *pos-reflexivity* and *pos-transitivity*. Indeed, we can prove the following lemma[†].

Lemma 3.4. Let A be a set and H be any collection of subsets of A. Then the predicate $\ltimes_H(-)$ defined by setting $\ltimes_H(F) = \bigcup\{\alpha \in H \mid \alpha \subseteq F\}$ satisfies *pos-reflexivity* and *pos-transitivity*.

Proof. *Pos-reflexivity*, that is, $\ltimes_H(F) \subseteq F$, holds trivially. To prove *pos-transitivity*, that is, that $\ltimes_H(F) \subseteq G$ implies $\ltimes_H(F) \subseteq \ltimes_H(G)$, it is sufficient to note that for any $\alpha \in H$, if $\alpha \subseteq F$, then $\alpha \subseteq \ltimes_H(F)$, which is immediate. Indeed, following this observation, if $a \in \ltimes_H(F)$, there exists a subset $\alpha \in H$ such that $a \in \alpha \subseteq F$, so $\alpha \subseteq \ltimes_H(F)$, so $\ltimes_H(F) \subseteq G$ implies $\alpha \subseteq G$, and thus $a \in \alpha \subseteq \ltimes_H(G)$. □

[†] The same lemma was proved independently in Sambin and Trentinaglia (2005).

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Note that in order to find a solution for the two equations above, the collection $\mathsf{Pt}_{\ltimes}(A)$ of the formal points must be a subset of the collection $\mathsf{Fix}_{\wedge}$ of the non-empty fixed-points of the operator $\wedge(-)$ mapping a subset α into $\alpha \wedge \alpha$. So if we consider *any* subset H of $\mathsf{Fix}_{\wedge}$, we can define a positivity predicate $\ltimes_{\mathsf{H}}$ by setting $\ltimes_{\mathsf{H}}(F) = \bigcup\{\alpha \in \mathsf{H} \mid \alpha \subseteq F\}$ since Lemma 3.4 shows that $\ltimes_{\mathsf{H}}$ satisfies both *pos-reflexivity* and *pos-transitivity*.

We want to show that $\ltimes_{\mathsf{H}}$ is spatial with respect to the collection $\mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A)$ of the $\ltimes_{\mathsf{H}}$-formal points, namely, the non-empty subsets of A that are fixed-points of $\wedge(-)$ and $\ltimes_{\mathsf{H}}(-)$.

Theorem 3.5 ($\ltimes_H$-spatiality). Let $(A, \wedge)$ be a semi-lattice, $\mathsf{Fix}_{\wedge}$ be the collection of the non-empty fixed-points of the $\wedge(-)$ operator, H be a subset of $\mathsf{Fix}_{\wedge}$ and $\mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A)$ be the collection of the $\ltimes_{\mathsf{H}}$-formal points of A. Then, for any $F \subseteq A$, we have

$$\ltimes_{\mathsf{H}}(F) = \bigcup\{\alpha \in \mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A) \mid \alpha \subseteq F\}.$$

Proof. It is clear that H is a subset of $\mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A)$ since for any $\alpha \in \mathsf{H}$, we have

$$\alpha = \bigcup\{\beta \in \mathsf{H} \mid \beta \subseteq \alpha\} \equiv \ltimes_{\mathsf{H}}(\alpha).$$

So

$$\ltimes_{\mathsf{H}}(F) \subseteq \bigcup\{\alpha \in \mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A) \mid \alpha \subseteq F\}.$$

But other $\ltimes_{\mathsf{H}}$-formal points can exist in $\mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A)$ that do not belong to H. However, the presence of other $\ltimes_{\mathsf{H}}$-formal points does not change the definition of $\ltimes_{\mathsf{H}}(F)$ since any $\ltimes_{\mathsf{H}}$-formal point $\beta \in \mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A)$ contained in F is a fixed-point of $\ltimes_{\mathsf{H}}$, in other words,

$$\beta = \ltimes_{\mathsf{H}}(\beta) = \bigcup\{\gamma \in H \mid \gamma \subseteq \beta\},$$

so the contribution of β to the definition of $\bigcup\{\alpha \in \mathsf{Pt}_{\ltimes_{\mathsf{H}}}(A) \mid \alpha \subseteq F\}$ has already been taken into consideration since any $\ltimes_{\mathsf{H}}$-formal point $\gamma \in H$ contained in β is also contained in F. □

Thus we have proved that for any semi-lattice $(A, \wedge)$, we have a spatial positivity predicate for any subset of the collection $\mathsf{Fix}_{\wedge}$ (of course, it is quite possible that two different subsets define the same positivity predicate). A much more interesting question is solving the problem when other specific conditions are required for the solution, such as that some couple (a, F) belongs to the positivity predicate (see Section 3.3.2).

3.3. *Predicative completeness*

It is easy to devise a predicate Pos satisfying both *pos-reflexivity* and *pos-transitivity* using a predicative definition (see, for instance, Section 3.3.2). One can then use the approach we described in Section 3.1 to get a spatial positivity predicate and a spatial relative cover relation. The question we want to address now is how to provide in a predicative way a predicate Pos that is already spatial: in this way we can avoid the impredicative definitions *impPos* and *impCov*.

3.3.1. *The concrete case.* The first solution to this problem is through a topological space defined over a *set* X of points whose base closed under intersection is indexed by a

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map ext on a set A, in other words, the case of a topological basic pair (Gebellato and Sambin 2002).

Indeed, in this case we can define a positivity predicate Pos satisfying *pos-reflexivity* and *pos-transitivity* by setting

$$\mathsf{Pos}(a, F) \equiv (\exists x \in X)\ a \in \alpha_x \subseteq F$$

where $\alpha_x = \{c \in A \mid x \in \mathsf{ext}(c)\}$, and we can prove the following lemma, which, with Lemma 3.1, shows that Pos is equivalent to the predicate $\ltimes$ defined according to the definition *impPos*.

Lemma 3.6. Let Pos be a predicate defined as above. Then, if $\mathsf{Pos}(a, F)$,

$$(\exists \alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A))\ a \in \alpha \subseteq F.$$

Proof. If $\mathsf{Pos}(a, F)$, there exists $x \in X$ such that $a \in \alpha_x \subseteq F$ and it is easy to check that $\alpha_x \in \mathsf{Pt}_{\mathsf{Pos}}(A)$. □

We can also avoid the impredicative definition of the relative cover relation since the following lemma shows that we can just set

$$\mathsf{Cov}(a, F, U) \equiv (\forall x \in X)\ a \in \alpha_x \subseteq F \rightarrow \alpha_x \between U$$

to get a cover relation equivalent to the cover relation $\lhd$ defined according to the definition *impCov*.

Lemma 3.7. Let Cov be a predicate defined as above. Then $\mathsf{Cov}(a, F, U)$ holds if and only if $(\forall \alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A))\ a \in \alpha \subseteq F \rightarrow \alpha \between U$.

Proof. The right to left implication is trivial since α_x is a formal point in $\mathsf{Pt}_{\mathsf{Pos}}(A)$ for any $x \in X$, so we get that if $a \in \alpha_x \subseteq F$, then $\alpha_x \between U$.

To prove the left to right implication, we suppose $\alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A)$ and $a \in \alpha \subseteq F$ to show that $\alpha \between U$. Now, $\alpha \in \mathsf{Pt}_{\mathsf{Pos}}(A)$ implies $\alpha = \mathsf{Pos}(\alpha)$, so $a \in \alpha$ implies $a \in \mathsf{Pos}(\alpha)$, that is, there exists $x \in X$ such that $a \in \alpha_x \subseteq \alpha$. So $\alpha \subseteq F$ implies $\alpha_x \subseteq F$, and thus $\mathsf{Cov}(a, F, U)$ implies $\alpha_x \between U$. But $\alpha_x \subseteq \alpha$, so $\alpha \between U$. □

3.3.2. *The inductively generated case.* A second predicative way to define a positivity predicate satisfying *pos-reflexivity* and *pos-transitivity* is to use a co-inductive generation as in Sambin *et al.* (2003) and Valentini (2006). The novelty is that we now have to deal with a new and more general kind of axiom. Indeed, in the usual approach to the inductive generation of a formal topology (see Coquand *et al.* (2003)), the set of axioms is specified by means of a set $I(a)$ of indexes for any $a \in A$ and a family of subsets $C(a, i)$ of A whose intended meaning is the specification of the relevant subsets amongst those covering a.

However, in the approach we are considering here, the axioms have to specify when a basic open is covered by an open subset *within a closed subset*, so, for any basic open a, we need a set of indexes $I(a)$ and two families of subsets $C(a, i)$ and $D(a, i)$ of A whose

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intended meaning is to specify that the part of the basic open (determined by) a within the closed (determined by) $D(a,i)$ is covered by the open (determined by) $C(a,i)$.

Indeed, it is only in this way that we can show impredicatively that every rf-topology is inductively generated since we can just set

$$\begin{aligned} I(a) &= \{\langle F, U\rangle \mid a \lhd_F U\} \\ C(a,\langle F,U\rangle) &= U \\ D(a,\langle F,U\rangle) &= F. \end{aligned}$$

Note that what we are doing here is different from the standard approach, which aims to define the topology induced on a closed subset (see, for instance, Maietti and Valentini (2006) or Vickers (2007)), even if we can get such a result as an easy corollary by considering the case in which all the axioms are *total*, that is, such that $D(a,i) = A$ for any $a \in A$ and $i \in I(a)$. In fact, our aim is to provide the topological space such that all the topologies induced on the closed subsets match the given set of axioms.

Thus, we arrive at the following definition[†].

Definition 3.8 (Inductively generated rf-topology). An *inductively generated* rf-*topology* is a structure $(A, \wedge, I, C, D)$ where

— $(A, \wedge)$ is a semi-lattice,
— $I(a)$ is a set (of indexes) for any $a \in A$, and
— $C(a,i)$ and $D(a,i)$ are subsets of A, for any $a \in A$ and $i \in I(a)$,

such that the following conditions hold:

— there exists $i \in I(a \wedge b)$ such that $C(a \wedge b, i) = \{a\}$ and $D(a \wedge b, i) = A$
— there exists $j \in I(a \wedge b)$ such that $C(a \wedge b, j) = \{b\}$ and $D(a \wedge b, j) = A$
— there exists $k \in I(a)$ such that $C(a,k) = \{a \wedge a\}$ and $D(a,k) = A$
— for any $i \in I(b)$ there exists $j \in I(a \wedge b)$ such that $C(a \wedge b, j) \subseteq \{a \wedge b\} \wedge C(b,i)$ and $D(b,i) \subseteq D(a \wedge b, j)$.

Bearing in mind the intended meaning of the axioms, the required conditions for an inductively generated rf-topology are quite natural (see also Section 3.3.4). Indeed, the first two conditions just state that

$$\mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{ext}(a)$$

and

$$\mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{ext}(b),$$

while the third states that

$$\mathsf{ext}(a) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(a).$$

Finally, the fourth condition states that if

$$\mathsf{ext}(b) \cap \mathsf{Rest}(D(b,i)) \subseteq \mathsf{Ext}(C(b,i))$$

[†] The name *inductively generated* rf-*topology* for this new structure arises from the fact that in any inductively generated rf-topology it is possible to define inductively a binary positivity predicate and a relative cover relation satisfying all the conditions for an rf-topology.

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then

$$\mathsf{ext}(a) \cap \mathsf{ext}(b) \cap \mathsf{Rest}(D(a \wedge b, j)) \subseteq \mathsf{Ext}(C(a \wedge b, j))$$

for some index $j \in I(a \wedge b)$ such that

$$\mathsf{Ext}(C(a \wedge b, j)) \subseteq \mathsf{ext}(a \wedge b) \cap \mathsf{Ext}(C(b, i))$$

and

$$\mathsf{Rest}(D(b, i)) \subseteq \mathsf{Rest}(D(a \wedge b, j)).$$

When we are provided with a set of axioms, we have to specify how to obtain a positivity predicate and a cover relation using them. We shall now adapt the procedure adopted in the case of formal topologies to our framework, so we propose the following co-inductive rules to generate the positivity predicate:

$$\frac{a \ltimes F}{a \in F} \qquad \text{(pos-reflexivity)}$$

$$\frac{a \ltimes F \quad i \in I(a) \quad \ltimes(F) \subseteq D(a, i)}{(\exists y \in C(a, i))\ y \ltimes F}. \qquad \text{(pos-infinity)}$$

It is not difficult to check that our rules define the following monotone operator $\tau(-)$ on subsets of A:

$$\tau(X) = F \ \cap \{x \in A \mid (\forall i \in I(x))\ X \subseteq D(x, i) \rightarrow (\exists y \in C(x, i))\ y \in X\}.$$

Hence τ has a maximum fixed point by the Tarski fixed-point theorem (but a fully predicative proof of this result can be provided as in Maietti and Valentini (2004)), so, in fact, our rules define a subset, namely, $\ltimes(F)$.

We can now check that the predicate defined in this way satisfies the conditions we have identified.

Lemma 3.9. Let $(A, \wedge, I, C, D)$ be an inductively generated rf-topology and $\ltimes$ be the positivity predicate defined by the co-inductive rules above. Then we have $\ltimes$ satisfies *pos-reflexivity*, *pos-transitivity* and $\ltimes$-*left*.

Proof. *Pos-reflexivity* is built into the rules, while the validity of *pos-transitivity* can be proved by co-induction by showing that if $\ltimes(F) \subseteq G$, then $\ltimes(F)$ satisfies the co-inductive rules that define $\ltimes(G)$. So we have to show that if $a \in \ltimes(F)$, then $a \in G$, which follows immediately from $\ltimes(F) \subseteq G$ and the fact that $\ltimes(F)$ satisfies *pos-infinity*, which is obvious. Finally, if $a \wedge b \ltimes F$, then both $a \ltimes F$ and $b \ltimes F$ follow. Indeed, according to the definition of inductively generated rf-topology, there are $i, j \in I(a \wedge b)$ such that:

$$\begin{aligned} C(a \wedge b, i) &= \{a\} \\ D(a \wedge b, i) &= A \\ C(a \wedge b, j) &= \{b\} \\ D(a \wedge b, i) &= A. \end{aligned}$$

Hence, by *pos-infinitity*, we get that there exists $y \in \{a\}$ and $z \in \{b\}$ such that $y \ltimes F$ and $z \ltimes F$, in other words, $a \ltimes F$ and $b \ltimes F$ hold. □

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After the positivity predicate has been defined by co-induction, we can also introduce a relative cover relation by induction using the following rules:

$$\frac{a \in U}{a \lhd_F U} \qquad \text{(reflexivity)}$$

$$\frac{i \in I(a) \quad \ltimes(F) \subseteq D(a,i) \quad C(a,i) \lhd_F U}{a \lhd_F U} \qquad \text{(infinity)}$$

$$\frac{a^{+_F} \lhd_F U}{a \lhd_F U} \qquad \text{(pos-infinity)}$$

In this case too, the rules define a monotone operator:

$$\begin{aligned}\tau(X) = \; & U \cup \\ & \{x \in A \mid (\exists i \in I(x)) \;\; \ltimes(F) \subseteq D(x,i) \;\wedge\; (\forall y \in C(x,i)) \; y \in X\} \cup \\ & \{x \in A \mid x^{+_F} \subseteq X\}.\end{aligned}$$

Hence τ has a minimum fixed point by the Tarski fixed-point theorem, so our rules determine a subset, namely, $\lhd_F(U)$.

Now we can show that the relative cover relation defined by induction in this way and the positivity predicate defined above by co-induction satisfy all of the conditions for an rf-topology.

Lemma 3.10. Let $(A, \wedge, I, C, D)$ be an inductively generated rf-topology, $\ltimes$ be the positivity predicate defined by the co-inductive rules above and $\lhd$ be the relative cover relation defined by the inductive rules above. Then they satisfy *reflexivity*, *transitivity*, *compatibility*, *positivity*, $\ltimes$*-restriction*, $\lhd$*-left* and $\lhd$*-right* and, moreover, for any index $i \in I(a)$, we have $a \lhd_{D(a,i)} C(a,i)$ holds.

Proof. Apart from *reflexivity*, which is immediate, the proof of the validity of almost all the other conditions is by induction and is an adaptation to our framework of similar results in Coquand *et al.* (2003) and Valentini (2006), so we will only give the completely new proofs here.

Positivity: The validity of the positivity condition can be obtained by showing that the *positivity axiom*, which is built into our rules, yields the *positivity* condition. Indeed, we can prove first that if $a \ltimes F \to a \lhd_F U$, then $a^{+_F} \lhd_F U$, so we can conclude $a \lhd_F U$ by *pos-infinity*, which is in fact equivalent to the positivity axiom. So we assume that $x \in a^{+_F}$, that is, $x \ltimes F$ and $x = a$. Then $a \ltimes F$ follows by logic, so $a \ltimes F \to a \lhd_F U$ implies $a \lhd_F U$, which, using the assumption $x = a$ again, gives $x \lhd_F U$.

$\ltimes$-restriction: We have to show that $\ltimes(F) \subseteq \ltimes(G)$ and $a \lhd_G U$ imply $a \lhd_F U$. The proof is by induction on the length of the derivation of $a \lhd_G U$:

— Case: $a \lhd_G U$ because $a \in U$.
We get $a \lhd_F U$ trivially by *reflexivity*.

— Case: $a \lhd_G U$ because there exists $i \in I(a)$ such that $\ltimes(G) \subseteq D(a,i)$ and $C(a,i) \lhd_G U$.

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We immediately get $C(a,i) \lhd_F U$ by the induction hypothesis. Moreover, $\ltimes(F) \subseteq \ltimes(G)$, so $\ltimes(F) \subseteq D(a,i)$ holds, and we get $a \lhd_F U$ by an instance of *infinity* on the same axiom.

— Case: $a \lhd_G U$ because $a^{+_G} \lhd_G U$.
Then, by the induction hypothesis, we immediately get $a^{+_G} \lhd_F U$ and can conclude by *pos-infinity* since it is trivial to check that if $\ltimes(F) \subseteq \ltimes(G)$, then $a^{+_F} \subseteq a^{+_G}$. □

3.3.3. *Completeness of inductively generated* rf*-topologies.* In the previous section we proved that an inductively generated rf-topology induces both a cover relation and a positivity predicate, so one is led to ask whether the positivity predicate $\ltimes$ obtained in this way is spatial.

The first step in answering this question is to adapt the notion of a formal point to the case of inductively generated rf-topologies.

Theorem 3.11. Let $(A, \wedge, I, C, D)$ be an inductively generated rf-topology. Then a non-empty subset α of A is a formal point if and only if the following conditions hold

$$\frac{a \in \alpha \quad b \in \alpha}{a \wedge b \in \alpha} \qquad (\wedge\text{-closure})$$

$$\frac{a \in \alpha \quad i \in I(a) \quad \alpha \subseteq D(a,i)}{(\exists c \in C(a,i))\ c \in \alpha}. \qquad (\text{ax-completeness})$$

Proof. It is easy to check that any formal point satisfies $\wedge$-*closure* (since a formal point is a fixed point for $\wedge(-)$) and *ax-completeness* (since it is a fixed point for $\ltimes(-)$).

Moreover, any non-empty subset α of A satisfying *ax-completeness* also satisfies $\ltimes$-*fixpoint*. Indeed, one inclusion is immediate since $\ltimes(\alpha) \subseteq \alpha$ holds by *pos-reflexivity*. So we only need to prove that $\alpha \subseteq \ltimes(\alpha)$. We can get this result by co-induction by showing that α satisfies the co-inductive conditions for $\ltimes(\alpha)$. First, it is clear that if $a \in \alpha$, then $a \in \alpha$. Moreover, if $a \in \alpha$, $i \in I(a)$ and $\alpha \subseteq D(a,i)$, we get $(\exists y \in C(a,i))\ y \in \alpha$ by *ax-completeness*. Finally, from $\wedge$-*closure* and the idempotency of $\wedge$, we get that α is a fixed point of $\wedge(-)$ and hence is a formal point. □

Returning to the completeness of the axiomatisation of a positivity predicate defined by co-induction, the problem cannot be solved in the general case since there are inductively generated rf-topologies with no formal point. However, a solution can be found when the set of axioms I, C and D is countable, that is, when the set $\Sigma(A,I)$ of the couples $\langle a,i\rangle$ such that $a \in A$ and $i \in I(a)$ is countable.

Indeed, given any set of axioms I, C and D, we can provide a new set of axioms for an inductively generated rf-topology that is classically equivalent to the one inductively generated by the original set of axioms[†]. The main idea for getting this result is to define

[†] The proof of the spatiality theorem (Theorem 3.15) requires classical logic anyway, so using it here to substitute the original set of axioms with another does not change the required foundational framework.

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the new set of axioms by putting

$$\begin{aligned} I'(a) &\equiv I(a) \\ C'(a,i) &\equiv D(a,i) \to C(a,i) \\ D'(a.i) &\equiv A. \end{aligned}$$

It is then easy to verify, using classical logic, that if (A, I, C, D) is an inductively generated rf-topology, then (A, I', C', D') is too, and, it is obvious that if the original set of axioms is countable, then the new one is also countable.

Moreover, according to the classical equivalence between rf-topologies and formal topologies we pointed out in Section 2.1, we can prove the following theorem.

Theorem 3.12. Let $(A, \wedge, I, C, D)$ be an inductively generated rf-topology and define the inductively generated rf-topology $(A, \wedge, I', C', D')$ as above. Then the positivity predicate $\ltimes$ of $(A, \wedge, I, C, D)$ is classically equivalent to the positivity predicate $\ltimes'$ of $(A, \wedge, I', C', D')$.

Proof. The proof is by co-induction for both implication directions:

— $a \ltimes F$ implies $a \ltimes' F$:

It is sufficient to prove that $\ltimes(F)$ satisfies the co-inductive conditions for $\ltimes'(F)$. The validity of *pos-reflexivity* is trivial. Moreover, if $a \in \ltimes(F)$ and $i \in I'(a)$, then $i \in I(a)$, and we can argue classically by cases:

- $\ltimes(F) \subseteq D(a,i)$:
  Hence there exists $y \in C(a,i)$, so $y \in D(a,i) \to C(a,i)$, such that $y \in \ltimes(F)$.
- $\ltimes(F) \not\subseteq D(a,i)$:
  Hence there exists $y \in \ltimes(F)$ such that $y \notin D(a,i)$, so $y \in D(a,i) \to C(a,i)$.

— $a \ltimes' F$ implies $a \ltimes F$:

Again, the validity of *pos-reflexivity* is trivial, and if we assume that $a \ltimes' F$, $i \in I(a)$ and $\ltimes'(F) \subseteq D(a,i)$, then, from the first two assumptions, we get that there exists $y \in D(a,i) \to C(a,i)$ such that $y \in \ltimes'(F)$, so the third assumption shows that $y \in D(a,i)$, and we can conclude that $y \in C(a,i)$ by logic. □

We have now proved that, from a classical point of view, the two sets of axioms generate the same positivity predicate, so they also define the same collection of formal points since formal points depend only on the positivity predicate. But the second set of axioms is a set of total axioms, namely, for any $a \in A$ and $i \in I(a)$, we have $D(a,i) = A$, so we can adapt the spatiality theorem in Valentini (2006) to our case here. It should be no surprise that we have to move to total axioms to prove the spatiality theorem since formal points are determined by the full topological space and do not depend on the topology induced on the closed subsets.

We will now recall a definition we will need to state the spatiality theorem.

Definition 3.13 (Up-complete subset). Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology and F be a subset of A. Then F is *up-complete* if the following condition holds for any $a \in A$ and $U \subseteq A$:

$$\frac{a \in F \quad a \lhd_A U}{(\exists u \in U)\ u \in F}.$$

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The most relevant example of an up-complete subset for our purposes is any fixed point of the $\ltimes(-)$ operator. Indeed, we can prove the following lemma.

Lemma 3.14. Let $(A, \wedge, \lhd, \ltimes)$ be an rf-topology and F be a subset of A. We then have that if $F = \ltimes(F)$, then F is up-complete.

Proof. We suppose that $a \in F$, so $a \in \ltimes(F)$ since $F \subseteq \ltimes(F)$, and thus if $a \lhd_A U$, we can show there exists $u \in U$ such that $u \in \ltimes(F)$ by using *sub-restriction* first in order to get $a \lhd_F U$, and then *compatibility*. So we have $u \in F$ since $\ltimes(F) \subseteq F$. □

The main result is now the following theorem, which we borrow from Valentini (2006, Theorem 2.4) since we are working with total axioms (note that classical logic along with a use of the axiom of choice is required to prove this theorem). We will repeat the proof here to make the current paper more self-contained.

Theorem 3.15. Let $(A, \wedge, I', C', D')$ be an inductively generated rf-topology with a countable set of total axioms, let a be an element of A, and let F be an up-complete subset of A that contains a. Then there exists a formal point α such that $a \in \alpha$ and $\alpha \subseteq F$.

Proof. Since the collection of the axioms is indexed by the set of the natural numbers, we can build a function f from the natural numbers to the set $\Sigma(A, I)$ of the couples $\langle a, i\rangle$, for $a \in A$ and $i \in I(a)$, such that each couple appears a countable number of times in the image of f. Now consider the following inductive definition of a sequence $c_0, \ldots, c_n, \ldots$ of elements of A:

$$\begin{aligned} c_0 &= a \\ c_{n+1} &= \begin{cases} \mathsf{i}(W_n \cap F) & \text{if } c_n \lhd_A C(\pi_1(f(n)), \pi_2(f(n))) \\ c_n & \text{otherwise} \end{cases} \end{aligned}$$

where

$$W_n \equiv c_n \wedge C(\pi_1(f(n)), \pi_2(f(n))) \equiv \{c_n \wedge u \in A \mid u \in C(\pi_1(f(n)), \pi_2(f(n)))\}$$

and $\mathsf{i}(W \cap F)$ is an element in the intersection between W and F[†]. It is not difficult to prove by induction that for any $n \geqslant 0$, we have $c_n \in F$ and, provided

$$c_n \lhd_A C(\pi_1(f(n)), \pi_2(f(n)))$$

holds, $W_n \between F$ also holds. Indeed, $c_0 = a \in F$ holds by assumption and

$$c_0 \lhd_A C(\pi_1(f(0)), \pi_2(f(0)))$$

implies

$$c_0 \lhd_A c_0 \wedge C(\pi_1(f(0)), \pi_2(f(0))) = W_0$$

by $\lhd$-*right*, so $W_0 \between F$ follows since F is up-complete. Moreover, if we assume, by the induction hypothesis both that $c_n \in F$ and that

$$c_n \lhd_A C(\pi_1(f(n)), \pi_2(f(n)))$$

[†] Provided $W \between F$ holds, the map $\mathsf{i}(-)$ can be defined using the axiom of choice.

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implies $W_n \between F$, then $c_{n+1} \in F$ holds: indeed, if

$$c_n \lhd_A C(\pi_1(f(n)), \pi_2(f(n))),$$

we get $W_n \between F$ by the induction hypothesis, so c_{n+1} is an element of both W_n and F; otherwise c_{n+1} is equal to c_n and thus belongs to F by hypothesis. Finally,

$$c_n \lhd_A C(\pi_1(f(n)), \pi_2(f(n)))$$

implies

$$c_n \lhd_A c_n \wedge C(\pi_1(f(n)), \pi_2(f(n))) = W_n$$

by $\lhd$-*right*, so $W_n \between F$ follows in this case too since F is up-complete. Moreover, one can see immediately that $c_{n+1} \lhd_A c_n$ for any $n \geqslant 0$ because either $c_{n+1} = c_n$ or $c_{n+1} = c_n \wedge u$ for some $u \in C(\pi_1(f(n)), \pi_2(f(n)))$. Now consider the filter

$$\alpha_n \equiv \uparrow c_n \equiv \{x \in A \mid c_n \lhd_A x\}$$

generated by c_n for any $n \geqslant 0$, and put

$$\alpha \equiv \bigcup_{n \in \omega} \alpha_n.$$

Then α is the formal point we are looking for. Indeed, $\alpha \subseteq F$ since $\alpha_i \subseteq F$ for each $i \geqslant 0$ because $c_i \in F$ and F is up-complete. Moreover, $a \in \alpha$ since $a \in \alpha_0 \subseteq \alpha$. We have to show now that α is a formal point. First, α is not empty since we have proved that $a \in \alpha$. Now note that for any $i \geqslant 0$, we have $\alpha_i \subseteq \alpha_{i+1}$ since $c_{i+1} \lhd_A c_i$, so, supposing $b, d \in \alpha$, we get that there exists k such that $b, d \in \alpha_k$, that is, $c_k \lhd_A b$ and $c_k \lhd_A d$, so $c_k \lhd_A b \wedge d$ follows by $\lhd$-*right*. So $b \wedge d \in \alpha_k$ and thus $b \wedge d \in \alpha$. Finally, if $b \in \alpha$ and $i \in I(b)$, then $b \in \alpha_k$ for some natural number k, that is, $c_k \lhd_A b$, so we get $c_k \lhd_A C(b, i)$ by *transitivity* since $i \in I(b)$ implies $b \lhd_A C(b, i)$. Moreover, as a consequence of the assumption of countability of the set of axioms, there is a natural number $h \geqslant k$ such that $f(h) = \langle b, i \rangle$, so $c_h \lhd_A c_k$, which, together with $c_k \lhd_A C(b, i)$, which we proved above, implies $c_h \lhd_A C(b, i)$ by *transitivity*. So $c_{h+1} \equiv \mathrm{i}(W_h \cap F)$ is an element such that $c_{h+1} \equiv c_h \wedge u$ for some $u \in C(b, i)$, and thus $c_{h+1} \lhd_A u$, so $u \in \alpha_{h+1}$ and we can conclude that $u \in \alpha$, that is, $C(b, i) \between \alpha$. □

We have the following corollary immediately.

Corollary 3.16 ($\ltimes'$-spatiality). Let $(A, \wedge, I', C', D')$ be an inductively generated rf-topology with a countable set of total axioms. Then $a \ltimes' F$ holds if and only if there exists a formal point α such that $a \in \alpha \subseteq F$.

Proof. From *pos-idempotency* and Lemma 3.14, we know that $\ltimes'(F)$ is an up-complete subset of A. The previous theorem then shows that if $a \ltimes' F$, there exists a formal point α such that $a \in \alpha \subseteq \ltimes'(F) \subseteq F$.

The other implication direction is trivial since $\alpha \subseteq F$ implies $\alpha = \ltimes'(\alpha) \subseteq \ltimes'(F)$ by $\ltimes$-*fixpoint* and *pos-monotonicity*, so $a \in \alpha$ implies $a \in \ltimes'(F)$. □

It is obvious from this corollary that we also have the spatiality of the original positivity predicate since $\ltimes$ is equivalent to $\ltimes'$ and $\mathsf{Pt}_{\ltimes}(A) = \mathsf{Pt}_{\ltimes'}(A)$.

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To complete our analysis, we will now show what happens with the relative cover relation. We begin by proving the following theorem.

Theorem 3.17. We let $(A, \wedge, I, C, D)$ be an inductively generated rf-topology and define the inductively generated rf-topology $(A, \wedge, I', C', D')$ as above. Then the relative cover relation $\lhd$ of $(A, \wedge, I, C, D)$ is classically equivalent to the relative cover relation $\lhd'$ of $(A, \wedge, I', C', D')$.

Proof. We can develop a proof by induction for both implication directions:

— To prove the first implication:
  - Case: $a \lhd'_F U$ because $a \in U$.
    In this case we trivially have $a \lhd_F U$ by *reflexivity*.
  - Case: $a \lhd'_F U$ because there exists $i \in I(a)$ such that $D(a,i) \to C(a,i) \lhd'_F U$.
    By the induction hypothesis, we have $D(a,i) \to C(a,i) \lhd_F U$, so we can conclude by *infinity* since $C(a,i) \lhd_F U$ because for any $c \in C(a,i)$, we have $c \in D(a,i) \to C(a,i)$ (note that we do not use the hypothesis $\ltimes(F) \subseteq D(a,i)$).
  - Case: $a \lhd'_F U$ because $a^{+'_F} \lhd'_F U$.
    By the induction hypothesis, we have $a^{+'_F} \lhd_F U$, but $a^{+'_F} = a^{+_F}$ since we have already proved that $\ltimes = \ltimes'$. So $a^{+_F} \lhd_F U$ and we can conclude $a \lhd_F U$ by *pos-infinity*.

— We can also prove the other implication direction using classical logic. In fact the proof is completely analogous to the previous one when the inductive step in the derivation of $a \lhd_F U$ is due to *reflexivity* or *pos-infinity*. So we just consider the remaining case:
  - Case: $a \lhd_F U$ because there is $i \in I(a)$ such that we have $\ltimes(F) \subseteq D(a,i)$ and $C(a,i) \lhd_F U$.
    Then, in order to conclude by *infinity* that $a \lhd'_F U$, we have to prove that $D(a,i) \to C(a,i) \lhd'_F U$. So we let $c \in D(a,i) \to C(a,i)$ and argue by cases:
    - $c \in D(a,i)$:
      So we have $c \in C(a,i)$, and can conclude $c \lhd'_F U$ since, by the induction hypothesis, $C(a,i) \lhd'_F U$.
    - $c \notin D(a,i)$:
      So $c \notin \ltimes(F)$, since $\ltimes(F) \subseteq D(a,i)$, and we get $c \notin \ltimes'(F)$, since $\ltimes = \ltimes'$, which yields $c \lhd'_F U$ by *positivity*. □

Hence, in order to conclude that $\lhd$ is spatial, it is sufficient to prove that $\lhd'$ is spatial since $\mathsf{Pt}_{\ltimes}(A) = \mathsf{Pt}_{\ltimes'}(A)$. We begin with two lemmas.

Lemma 3.18. Let $(A, \wedge, \lhd', \ltimes')$ be an rf-topology, and U and F be subsets of A. Then $G_{F,U} \equiv \{x \in A \mid \neg(x \lhd'_F U)\}$ is up-complete.

Proof. The proof is classical. We assume that $c \in G_{F,U}$, so $\neg(c \lhd'_F U)$ holds. We now suppose $c \lhd'_A W$ and assume that $w \lhd'_F U$ for all $w \in W$. Then we get $c \lhd'_F W$ by *sub-restriction*, so $c \lhd'_F U$ follows by *transitivity*, which gives a contradiction. Hence, it is not the case that $w \lhd'_F U$ for all $w \in W$, so, by classical logic, there exists $w \in W$ such that $\neg(w \lhd'_F U)$, that is, $w \in G_{F,U}$. □

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Lemma 3.19. Let $(A, \wedge, \lhd', \ltimes')$ be an rf-topology and U and F be subsets of A. Then $G_{F,U} \equiv \{x \in A \mid \neg(x \lhd'_F U)\}$ is a subset of F.

Proof. We again use a classical proof. Suppose $c \in G_{F,U}$ and $c \notin F$ in order to get a contradiction. Then $\neg(c \ltimes F)$ follows by *pos-reflexivity*, so we get $(c \ltimes F) \to (c \lhd'_F U)$ by logic. So $c \lhd'_F U$ follows by *positivity*, which yields the required contradiction with the assumption $c \in G_{F,U}$. □

We can now prove the spatiality theorem for countably presented total relative formal topologies.

Theorem 3.20 ($\lhd'$-spatiality). Let $(A, \wedge, \lhd', \ltimes')$ be an rf-topology with a countable set of total axioms, a be an element of A, and U and F be subsets of A. Then if, for any formal point β, we have that $a \in \beta$ and $\beta \subseteq F$ imply that there exists $u \in U$ such that $u \in \beta$, then $a \lhd'_F U$.

Proof. This proof is also classical. We assume $\neg(a \lhd'_F U)$ holds in order to get a contradiction. To begin, we again consider the subset

$$G_{F,U} = \{x \in A \mid \neg(x \lhd'_F U)\}.$$

By Lemma 3.18, $G_{F,U}$ is up-complete and, from the assumption, it clearly contains a. Thus, by Theorem 3.15, there exists a formal point α such that $a \in \alpha$ and $\alpha \subseteq G_{F,U}$. So, by Lemma 3.19, $\alpha \subseteq F$ since $G_{F,U} \subseteq F$. Hence, by the assumption in the statement of the theorem, there exists some element $u \in U$ such that $u \in \alpha$. Then, by *reflexivity*, we get $u \lhd'_F U$, but $u \in \alpha \subseteq G_{F,U}$ also holds, which implies $u \in G_{F,U}$, that is, we have $\neg(u \lhd'_F U)$. □

Thus, we have proved classically that $a \lhd'_F U$ if and only if, for any formal point β, $\beta \in \mathsf{ext}^{\mathsf{Pt}}(a) \cap \mathsf{Rest}^{\mathsf{Pt}}(F)$ implies $\beta \in \mathsf{Ext}^{\mathsf{Pt}}(U)$, in other words, the spatiality theorem. It is worth recalling here that no constructive proof of the spatiality theorem can be given (see the last section of Valentini (2006)).

3.3.4. *Generability of representable* rf*-topologies.* In this section we want to show that many rf-topologies can be generated by induction. In fact, we will show that all *representable* rf*-topologies* are inductively generated rf-topologies. We will begin by adapting the definition of a representable topology from Coquand *et al.* (2003) to our framework.

Definition 3.21 (Representable rf-topology). An rf-topology $(A, \wedge, \lhd, \ltimes)$ is *representable* if there exist a set X and a map $\mathsf{ext} : A \to \mathscr{P}(X)$ such that

$$\begin{aligned} a \lhd_F U &\equiv \mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U) \\ a \ltimes F &\equiv \mathsf{ext}(a) \between \mathsf{Rest}(F) \\ \mathsf{ext}(a \wedge b) &= \mathsf{ext}(a) \cap \mathsf{ext}(b). \end{aligned}$$

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Now, to prove the desired result, we have to show that for each representable rf-topology, there is a set of axioms I, C and D such that $a \ltimes^{\mathsf{gen}} F$ if and only if

$$\mathsf{ext}(a) \between \mathsf{Rest}(F),$$

and $a \lhd_F^{\mathsf{gen}} U$ if and only if

$$\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U),$$

where we write $a \ltimes^{\mathsf{gen}} F$ to mean the positivity predicate co-inductively generated from such a set of axioms and $a \lhd_F^{\mathsf{gen}} U$ to mean the inductively generated cover relation.

The solution is simply to extend in a trivial way the set of axioms used in Coquand *et al.* (2003) to show that all the representable formal topologies can be generated by induction. So, we set

$$\begin{aligned} I(a) &= \{g : \mathsf{ext}(a) \to A \mid (\forall x \in \mathsf{ext}(a))\ x \in \mathsf{ext}(g(x))\} \\ C(a,g) &= \mathsf{Im}[g] \\ D(a,g) &= A. \end{aligned}$$

It is easy to check that all of the conditions in the definition of inductively generated relative formal topology are satisfied by the set of axioms we are considering here. Indeed, the first three conditions are trivially satisfied. To check the fourth, we suppose that $g \in I(b)$ and define a map $h \in I(a \wedge b)$ by setting $h(x) = a \wedge g(x)$ for any $x \in \mathsf{ext}(a \wedge b)$. It is then easy to check that it satisfies the required properties.

We can now show that representable rf-topologies can be generated by induction. To this end, we first prove the following lemma.

Lemma 3.22. Let $(A, \wedge, \lhd, \ltimes)$ be a representable rf-topology. Then, for any $a \in A$ and $F \subseteq A$, we have if $\mathsf{ext}(a) \between \mathsf{Rest}(F)$, then $a \ltimes^{\mathsf{gen}} F$.

Proof. Since $a \ltimes^{\mathsf{gen}} F$ is co-inductively generated, we can prove this theorem by co-induction by showing that the predicate $P(x) \equiv \mathsf{ext}(x) \between \mathsf{Rest}(F)$ satisfies all the rules for the co-inductive generation of the positivity predicate.

We immediately get by logic that $P(x)$, that is, $\mathsf{ext}(x) \between \mathsf{Rest}(F)$, implies $x \in F$. Moreover, if g is a function from $\mathsf{ext}(x)$ to A such that $w \in \mathsf{ext}(g(w))$ for all $w \in \mathsf{ext}(x)$, and we write w^* for the witness of $\mathsf{ext}(x) \between \mathsf{Rest}(F)$, then we have both $w^* \in \mathsf{ext}(g(w^*))$ and $w^* \in \mathsf{Rest}(F)$, in other words, there is an element $y \in \mathsf{Im}[g]$, that is, $g(w^*)$, such that $\mathsf{ext}(y) \between \mathsf{Rest}(F)$. □

The previous theorem is all we need to prove that $a \lhd_F^{\mathsf{gen}} U$ is logically equivalent to $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$. In fact, we can immediately prove the following lemma, which shows one of the required implications.

Lemma 3.23. Let $(A, \wedge, \lhd, \ltimes)$ be a representable rf-topology. Then, for any $a \in A$ and $U, F \subseteq A$, we have if $a \lhd_F^{\mathsf{gen}} U$, then $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$.

Proof. Since $a \lhd_F^{\mathsf{gen}} U$ is inductively generated, we can prove the theorem by induction:

— Case: $a \lhd_F^{\mathsf{gen}} U$ because $a \in U$.

In this case, $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$ clearly holds.

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— Case: $a \lhd_F^{\mathsf{gen}} U$ has been obtained by *infinity*.
Suppose g is a function from $\mathsf{ext}(a)$ to A such that for all $x \in \mathsf{ext}(a)$, we have $x \in \mathsf{ext}(g(x))$, and such that, by the induction hypothesis, for all $y \in \mathsf{Im}[g]$, we have $\mathsf{ext}(y) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$. Then

$$\Big(\bigcup_{y \in \mathsf{Im}[g]} \mathsf{ext}(y) \Big) \cap \mathsf{Rest}(F) = \bigcup_{y \in \mathsf{Im}[g]} (\mathsf{ext}(y) \cap \mathsf{Rest}(F)) \subseteq \mathsf{Ext}(U).$$

So we get

$$\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$$

since

$$\mathsf{ext}(a) \subseteq \bigcup_{y \in \mathsf{Im}[g]} \mathsf{ext}(y).$$

— Case: $a \lhd_F^{\mathsf{gen}} U$ has been obtained from $a^{+_F} \lhd_F^{\mathsf{gen}} U$.
Assume that $\mathsf{ext}(a) \between \mathsf{Rest}(F)$. Then, by the induction hypothesis, for any $y \in a^{+_F}$, we have $\mathsf{ext}(y) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$ holds. Moreover, by the previous lemma, $a \ltimes^{\mathsf{gen}} F$ holds, so $a \in a^{+_F}$, and we get

$$\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$$

by logic, and then that

$$(\mathsf{ext}(a) \between \mathsf{Rest}(F)) \to \mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$$

holds. But this is logically equivalent to $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$. □

We can also prove the other implication direction, but the proof requires both classical logic and the axiom of choice.

Lemma 3.24. Let $(A, \wedge, \lhd, \ltimes)$ be a representable rf-topology. Then, for any $a \in A$ and $U, F \subseteq A$, we have that if $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$, then $a \lhd_F^{\mathsf{gen}} U$.

Proof. Suppose $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$. Note that this means that for all $x \in X$, we have if $x \in \mathsf{ext}(a) \cap \mathsf{Rest}(F)$, then there exists $u \in A$ such that both $u \in U$ and $x \in \mathsf{ext}(u)$. Hence, by the axiom of choice, there exists a function f from $\mathsf{ext}(a) \cap \mathsf{Rest}(F)$ to A such that for all $x \in \mathsf{ext}(a) \cap \mathsf{Rest}(F)$, both $f(x) \in U$ and $x \in \mathsf{ext}(f(x))$. We now define a function g from $\mathsf{ext}(a)$ to A by setting

$$g(x) \equiv \mathsf{if}\ x \in \mathsf{Rest}(F)\ \mathsf{then}\ f(x)\ \mathsf{else}\ b,$$

where b is the element in A such that $x \in \mathsf{ext}(b)$ and $b \notin F$, which exists when $\neg(x \in \mathsf{Rest}(F))$ holds (classical logic is required for this). Thus, for any $x \in \mathsf{ext}(a)$, we have $x \in \mathsf{ext}(g(x))$, so g is one of the elements in $I(a)$, and $g(x) \in U$ or $g(x) \notin F$, and both these cases imply that $g(x) \lhd_F^{\mathsf{gen}} U$ because if $g(x) \in U$, we get $g(x) \lhd_F^{\mathsf{gen}} U$ by *reflexivity*, and if $g(x) \notin F$, we get $\neg(g(x) \ltimes^{\mathsf{gen}} F)$ by *pos-reflexivity* and thus

$$(g(x) \ltimes^{\mathsf{gen}} F) \to g(x) \lhd_F^{\mathsf{gen}} U$$

by logic, which gives $g(x) \lhd_F^{\mathsf{gen}} U$ by *positivity*. Thus, for any $y \in \mathsf{Im}[g]$, we have $y \lhd_F^{\mathsf{gen}} U$, and we can conclude $a \lhd_F^{\mathsf{gen}} U$ by *infinity*. □

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We can now conclude the proof of the equivalence between the co-inductively generated positivity predicate $a \ltimes^{\mathsf{gen}} F$ and $\mathsf{ext}(a) \between \mathsf{Rest}(F)$. Indeed, we have already shown that the relative cover relation $a \lhd_F^{\mathsf{gen}} U$ we obtained by induction is equivalent to the cover relation defined by $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$. Moreover, we have that the co-inductively generated predicate $a \ltimes^{\mathsf{gen}} F$ is a positivity predicate for the inductively generated relative formal cover and that $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ is a positivity predicate for the cover relation defined by $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U)$. Thus, from Theorem 2.5 showing the uniqueness of the positivity predicate, we can immediately conclude that the two positivity predicates are equivalent.

4. Final remarks

When we considered $\ltimes$-*restriction*, we said that we could strengthen it to a new valid condition showing how the topologies induced on different closed subsets can be glued together. The following lemma is useful for proving its validity.

Lemma 4.1. Let A be a set and ext be an indexing from A into a base $\mathscr{A}$ for the topological space $(X, \Omega(X))$. Then, for any element $a \in A$ and any subsets $F, G \subseteq A$, we have $\mathsf{ext}(a) \cap \mathsf{Rest}(G) \subseteq \mathsf{Rest}(F)$ holds if and only if $(\mathsf{ext}(a) \cap \mathsf{ext}(c)) \between \mathsf{Rest}(G)$ implies $\mathsf{ext}(c) \between \mathsf{Rest}(F)$ for any $c \in A$.

Proof. We suppose $\mathsf{ext}(a) \cap \mathsf{Rest}(G) \subseteq \mathsf{Rest}(F)$ and assume that $c \in A$ and $(\mathsf{ext}(a) \cap \mathsf{ext}(c)) \between \mathsf{Rest}(G)$ hold. Then there exists a point $x \in X$ such that $x \in \mathsf{ext}(a)$, $x \in \mathsf{ext}(c)$ and $x \in \mathsf{Rest}(G)$ hold. So $x \in \mathsf{Rest}(F)$ follows, and we have proved that both $x \in \mathsf{ext}(c)$ and $x \in \mathsf{Rest}(F)$, that is, $\mathsf{ext}(c) \between \mathsf{Rest}(F)$ holds.

To prove the other implication direction, we suppose that for all $c \in A$, we have $(\mathsf{ext}(a) \cap \mathsf{ext}(c)) \between \mathsf{Rest}(G)$ implies $\mathsf{ext}(c) \between \mathsf{Rest}(F)$, and assume that $x \in \mathsf{ext}(a) \cap \mathsf{Rest}(G)$ in order to show that $x \in \mathsf{Rest}(F)$. To this end, we suppose that $c \in A$ and $x \in \mathsf{ext}(c)$. Then, $x \in \mathsf{ext}(a) \cap \mathsf{ext}(c) \cap \mathsf{Rest}(G)$ follows, and, according to the assumption, we can deduce that $\mathsf{ext}(c) \between \mathsf{Rest}(F)$ holds, and $c \in F$ then follows. So we have proved that for all $c \in A$, we have if $x \in \mathsf{ext}(c)$, then $c \in F$, that is, $x \in \mathsf{Rest}(F)$ holds. □

So we can prove the validity of the following condition

$$\frac{(\forall c \in A)\ ((a \wedge c) \ltimes G) \to (c \ltimes F) \quad a \lhd_F U}{a \lhd_G U}. \tag{restriction}$$

Indeed,

$$(\forall c \in A)\ ((a \wedge c) \ltimes G) \to (c \ltimes F)$$

formally expresses the fact that for any $c \in A$, we have

$$(\mathsf{ext}(a) \cap \mathsf{ext}(c)) \between \mathsf{Rest}(G)$$

implies

$$\mathsf{ext}(c) \between \mathsf{Rest}(F)$$

and thus, by the previous lemma, that

$$\mathsf{ext}(a) \cap \mathsf{Rest}(G) \subseteq \mathsf{Rest}(F).$$

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So

$$\mathsf{ext}(a) \cap \mathsf{Rest}(G) \subseteq \mathsf{ext}(a) \cap \mathsf{Rest}(F)$$

follows. Moreover, $a \lhd_F U$ means that

$$\mathsf{ext}(a) \cap \mathsf{Rest}(F) \subseteq \mathsf{Ext}(U),$$

so

$$\mathsf{ext}(a) \cap \mathsf{Rest}(G) \subseteq \mathsf{Ext}(U)$$

follows, that is, $a \lhd_G U$ is valid.

It is now easy to show that $\ltimes$-*restriction* is a consequence of *restriction* since

$$\ltimes(G) \subseteq \ltimes(F)$$

implies

$$(\forall c \in A)\ ((a \wedge c) \ltimes G) \rightarrow (c \ltimes F)$$

for any $a \in A$.

Moreover, we can prove that *restriction* holds for all the rf-topologies we studied in Section 3 that satisfy completeness.

Finally, it is useful to compare our rf-topologies with the other structures we have mentioned.

First, any rf-topology determines a standard formal topology because the cover relation and the unary positivity predicate are just the *trace* of the relative cover relation and the binary positivity predicate on the whole set A. Indeed, by setting $a \lhd U \equiv a \lhd_A U$ and $\mathsf{Pos}(a) \equiv a \ltimes A$, some of the conditions in Definition 2.1, namely, *pos-reflexivity*, *pos-transitivity* and $\ltimes$-*restriction*, become trivialities, while the others just become the usual conditions on a cover relation and a positivity predicate of a formal topology.

Moreover, one can classically define an rf-topology from any formal topology by setting

$$\begin{aligned} a \lhd_F U &\equiv a \lhd F \rightarrow U \\ a \ltimes F &\equiv \neg(a \lhd F \rightarrow \varnothing), \end{aligned}$$

and it is easy to verify that the trace of such an rf-topology on the whole set A is equivalent to the original formal topology because of the uniqueness of the positivity predicate (Maietti and Valentini 2006).

Gebellato and Sambin (2002) defined a *balanced formal topology* as a formal topology equipped with a binary positivity predicate such that the following condition is satisfied:

$$\frac{a \ltimes F \quad a \lhd_A U}{(\exists u \in U)\ u \ltimes F}. \qquad \text{(compatibility)}$$

The proof that *compatibility* holds for any rf-topology is trivial, so any rf-topology is a balanced topology. On the other hand, the *positivity* condition is not expressible in the context of balanced formal topology.

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