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A topological lens for a measure-preserving system

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Abstract. We introduce a functor which associates to every measure-preserving system $(X, \mathcal{B}, \mu, T)$ a topological system $(C_2(\mu), \tilde{T})$ defined on the space of twofold couplings of μ, called the topological lens of T. We show that often the topological lens 'magnifies' the basic measure dynamical properties of T in terms of the corresponding topological properties of $\tilde{T}$. Some of our main results are as follows: (i) T is weakly mixing if and only if $\tilde{T}$ is topologically transitive (if and only if it is topologically weakly mixing); (ii) T has zero entropy if and only if $\tilde{T}$ has zero topological entropy, and T has positive entropy if and only if $\tilde{T}$ has infinite topological entropy; (iii) for T a K-system, the topological lens is a P-system (i.e. it is topologically transitive and the set of periodic points is dense; such systems are also called chaotic in the sense of Devaney).

0. *Introduction*

Ergodic theory and topological dynamics are two branches of the theory of dynamical systems. The first deals with groups acting on a probability measure space in a measure-preserving way; the second, with the action of groups on compact spaces as groups of homeomorphisms. Some of the terminology used in both branches is almost the same. One speaks of transitivity, ergodicity, weak and strong mixing, distality, rigidity, etc, both in ergodic theory and in topological dynamics. Even more surprising is the fact that major theorems in both areas read almost the same. To mention one conspicuous example, compare the statement of Furstenberg's theorem, identifying topologically distal dynamical systems as inverse limit of isometric extensions [**6**], with Zimmer's theorem, characterizing measure distal systems (i.e. systems having a separating sieve) as systems admitting Furstenberg's towers of (measure) isometric extensions [**18, 19**].

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We refer the reader to the article 'On the interplay between measurable and topological dynamics' [**1**] for a comprehensive review of this subject.

In the present work we restrict our attention to the classical case where the acting group is the group of integers $\mathbb{Z}$. We denote by T the measure-preserving transformation (or the homeomorphism) which corresponds to $1 \in \mathbb{Z}$ and write $(X, \mathcal{B}, \mu, T)$ (or (X, T)) for the general measure-preserving (or topological compact metric) system we study.

Given a topological system (X, T) we denote by $M(X)$ the compact convex set of probability measures on X and by $M_T(X)$ the compact convex subset (in fact the simplex) of T-invariant measures. By a classical theorem of Krylov and Bogoliouboff this is always a non-vacuous set. When $M_T(X)$ consists of a single measure, say $M_T(X) = \{\mu\}$, we say that (X, T) is uniquely ergodic. In the general case, each element μ of $M_T(X)$ defines a measure dynamical system $(X, \mathcal{B}_X, \mu, T)$ ($\mathcal{B}_X$ denotes the Borel σ-field of the topological space X). We say that $\mu \in M(X)$ is *full* when supp $\mu = X$.

Part of the mystery of the elusive connection between the two theories is removed by the following theorem, whose proof is straightforward; one only has to note that, for a full measure μ, $\mu(U) > 0$ whenever U is a non-empty open set.

THEOREM 1. *Let (X, T) be a topological dynamical system, $\mu \in M_T(X)$ a full measure; then (X, T) is topologically transitive, weakly mixing, topologically mixing if the measure-preserving system $(X, \mathcal{B}_X, \mu, T)$ is ergodic, weakly mixing, mixing, respectively.*

A substantial part of modern ergodic theory deals with the converse situation. One starts with a given measure ergodic system $(X, \mathcal{B}, \mu, T)$ and then looks for a *topological model*; i.e. a topological system (Y, S) and a measure $\nu \in M_S(Y)$ such that the systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{B}_Y, \nu, S)$ are isomorphic, and such that the topological system (Y, S) has some special properties like being minimal or uniquely ergodic. The prototype for this kind of statements is the famous Jewett–Krieger theorem which guarantees the existence of a uniquely ergodic model for any ergodic system.

In the present work we offer a novel perspective on the investigation of the connection between measure and topological systems. We introduce a functor which associates to every measure-preserving system $(X, \mathcal{B}, \mu, T)$ a topological dynamical system $(C_2(\mu), \tilde{T})$ defined on the space of twofold couplings of μ, called the *topological lens* of T. More specifically, a *coupling* of μ is a probability measure ξ on $X \times X$ with both marginals equal to μ. We equip $C_2(\mu)$ with the (compact metrizable) weak* topology and define $\tilde{T} : C_2(\mu) \to C_2(\mu)$ by the formula

$$\tilde{T}(\xi)(A \times B) = \xi(T^{-1}A \times T^{-1}B), \quad A, B \in \mathcal{B},$$

that is, $\tilde{T}(\xi) = (T \times T)_*(\xi)$. As we will show, the topological lens usually 'magnifies' the basic measure dynamical properties of T in terms of the corresponding topological properties of $\tilde{T}$.

Briefly our main results are as follows: (i) T is weakly mixing if and only if $\tilde{T}$ is topologically transitive (if and only if it is topologically weakly mixing); (ii) T has zero entropy if and only if $\tilde{T}$ has zero topological entropy, and T has positive entropy if and only if $\tilde{T}$ has infinite topological entropy; (iii) T is rigid if and only if $\tilde{T}$ is pointwise recurrent (if and only if $\tilde{T}$ is uniformly rigid); (iv) distality of $\tilde{T}$ implies that T has a discrete spectrum; (v) for the Bernoulli system T of infinite entropy, the topological lens

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$(C_2(\mu), \tilde{T})$ is a universal system both topologically and measure-theoretically in the sense that every metric compact topologically transitive, as well as every measure-preserving system, appears as a subsystem of $C_2(\mu)$; (vi) for T a K-system, the topological lens is a P-system (i.e. it is topologically transitive and the set of periodic points is dense; such systems are also called chaotic in the sense of Devaney); (vii) finally, for many zero entropy measure systems (including the generic automorphism $T \in \mathrm{Aut}(\mu)$) the set of periodic points of the topological lens $\tilde{T}$ is closed and nowhere dense.

1. *The space of couplings*

Assume that $(X, \mathcal{B}, \mu)$ is a standard probability Borel space and that μ has no atoms. By $C_2(X, \mu)$ (or $C_2(\mu)$) we denote the space of *2-couplings* of $(X, \mathcal{B}, \mu)$, that is the space of probability measures on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ with projections μ on both coordinates. The formula

$$\langle J_\rho(1_A), 1_B\rangle_{L^2(\mu)} = \langle 1_A, 1_B\rangle_{L^2(\rho)}, \quad A, B \in \mathcal{B} \tag{1}$$

establishes a one-to-one correspondence $\rho \mapsto J_\rho$ between $C_2(\mu)$ and the space $\mathcal{J}(\mu)$ of doubly stochastic operators (Markov operators) on $L^2(X, \mathcal{B}, \mu)$ (the operator: $L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ is called *doubly stochastic* if it is positive and $J(1) = J^*(1) = 1$; note that necessarily $\|J\| = 1$). With respect to the weak operator topology, $\mathcal{J}(\mu)$ forms a compact semitopological (metrizable) semigroup, where multiplication is defined by composition: $J_{\rho_1 \circ \rho_2} = J_{\rho_1} \circ J_{\rho_2}$. Via the above identification we will view $C_2(\mu)$ as a compact semitopological semigroup. To see what topology on $C_2(\mu)$ is involved recall that a metric compatible with the weak topology on $\mathcal{J}(\mu)$ is given by

$$d(J, J') = \sum_{i,j=0}^{\infty} \frac{1}{2^{i+j}} |\langle Jf_i, f_j\rangle - \langle J'f_i, f_j\rangle|, \tag{2}$$

where $\{f_i\}_{i\geq 0}$ is a dense subset of $L^2(X, \mathcal{B}, \mu)$ and $\|f_i\| = 1$ for $i \geq 0$. Observe that in the weak topology on $C_2(\mu)$ we have

$$\rho_n \to \rho \text{ iff } \rho_n(A \times B) \to \rho(A \times B) \text{ for all } A, B \in \mathcal{B}.$$

It follows that a basis for open sets in our compact semitopological semigroup is given by the family of sets of the form

$$U(\alpha, \varepsilon, P) = \{\rho \in C_2(\mu) : |\rho(A_i \times A_j) - p_{ij}| < \varepsilon\}, \tag{3}$$

where $\alpha = \{A_1, \ldots, A_k\}$ is a Borel partition of X, $\varepsilon > 0$ and $P = [p_{ij}]_{i,j=1}^k$ is a non-negative matrix such that the sum of elements in the i-row is equal to $\mu(A_i)$ and the sum of elements in the j-column is equal to $\mu(A_j)$, $1 \leq i, j \leq k$.

Denote by $\mathrm{Aut}(\mu) = \mathrm{Aut}(X, \mathcal{B}, \mu)$ the group of automorphisms of $(X, \mathcal{B}, \mu)$. Notice that $\mathrm{Aut}(X, \mathcal{B}, \mu)$ naturally embeds into $C_2(\mu)$ where the embedding is given by $S \mapsto \mu_S$. Here μ_S stands for the graph measure given by S, i.e. $\mu_S(A \times B) = \mu(S^{-1}A \cap B)$ for each $A, B \in \mathcal{B}$. The Markov operator corresponding to μ_S is equal to U_S (the Koopman operator associated to S), where $U_S(f) = f \circ S$ for each $f \in L^2(X, \mathcal{B}, \mu)$. Once μ is understood, we will also use the notation Δ_S for μ_S. The embedding $S \mapsto \Delta_S$ is also topological, as on the group of automorphisms considered as a (closed) subset of $\mathcal{U}(L^2(X, \mathcal{B}, \mu))$, the group of unitary operators on $L^2(X, \mathcal{B}, \mu)$, the weak and the strong operator topologies coincide. It follows from a theorem of Kuratowski that the set of graph couplings, that is the image of the embedding $\mathrm{Aut}(\mu) \subset C_2(\mu)$, is a G_δ subset of $C_2(\mu)$. It is not hard to

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check that it is also dense in $C_2(\mu)$ (see e.g. §3.3 below). We denote this dense G_δ subset of graph couplings by $C_{gr}(\mu)$, but often we will regard $\mathrm{Aut}(\mu)$, via this embedding, as a subset of $C_2(\mu)$.

Suppose $\mathcal{A} \subset \mathcal{B}$ is a T-invariant sub-σ-algebra. We can then consider the factor system $(X/\mathcal{A}, \mathcal{A}, \mu)$ (instead of μ we sometimes write $\mu_{\mathcal{A}}$ if we want to emphasize that we consider the quotient system), here $X/\mathcal{A}$ stands for classes of points of X that cannot be separated by sets of $\mathcal{A}$. If $\lambda \in C_2(\mu_{\mathcal{A}})$ then we can lift it to an element of $C_2(\mu)$, denoted by $\widehat{\lambda}$ and called the *relatively independent extension of* λ, by setting

$$\int_{X\times X} f \otimes g \, d\widehat{\lambda} = \int_{X/\mathcal{A}\times X/\mathcal{A}} E(f|\mathcal{A})(\overline{x}) E(g|\mathcal{A})(\overline{y}) \, d\lambda(\overline{x}, \overline{y})$$

for $f, g \in L^2(X, \mathcal{B}, \mu)$. Denoting by $i_{L^2(\mathcal{A})}$ the natural embedding of $L^2(\mathcal{A})$ into $L^2(\mathcal{B})$, by a simple calculation we have the following.

LEMMA 1. $J_{\widehat{\lambda}} = i_{L^2(\mathcal{A})} \circ J_\lambda \circ E(\cdot|\mathcal{A})$.

It follows immediately that the following holds.

LEMMA 2. *The map* $C_2(\mu_{\mathcal{A}}) \ni \lambda \mapsto \widehat{\lambda} \in C_2(\mu)$ *is a continuous monomorphism of semitopological semigroups; i.e. it is continuous, one-to-one and* $(\lambda \circ \rho)\widehat{\ } = \widehat{\lambda} \circ \widehat{\rho}$. *In particular this map is an embedding of the topological system* $C_2(\mu_{\mathcal{A}})$ *into* $C_2(\mu)$.

Let T be an element of $\mathrm{Aut}(\mu)$. By $J_2(T) \subset C_2(\mu)$ we denote the set of *2-self-joinings* of T. Recall that a 2-self-joining of T is just an element of $C_2(\mu)$ which is $T \times T$-invariant. Equivalently, $\rho \in C_2(\mu)$ is a 2-self-joining of T if and only if $J_\rho \circ U_T = U_T \circ J_\rho$. From this it easily follows that $J_2(T)$ is a closed subsemigroup of $C_2(\mu)$. When T is ergodic, then $J_2^e(T)$, the set of ergodic 2-self-joinings, is non-empty and it coincides with the collection of extremal points of the simplex $J_2(T)$. (Warning: $C_2(\mu)$ is not a simplex.) When $\mathcal{A} \subset \mathcal{B}$ is a T-invariant sub-σ-algebra then the quotient action of T on $(X/\mathcal{A}, \mathcal{A}, \mu)$ is called a *factor* of T; we will often write $T|_{\mathcal{A}}$ to denote this action. It is easy to see that if $\lambda \in J_2(T|_{\mathcal{A}})$ then $\widehat{\lambda} \in J_2(T)$ and therefore Lemma 2 is also true in the context of self-joinings.

Extending the notion of 2-self-joinings we define the set $J_n(T)$ ($n \geq 1$ or even $n = \infty$) of *n-self-joinings* of T. These are the $T^{\times n}$-invariant probability measures on $(X^n, \mathcal{B}^{\otimes n})$ all of whose one-dimensional marginals are equal to μ.

For a more about joinings we refer to [**7**] and the list of references thereof.

2. *The topological lens of an automorphism*

Given $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ consider the $\mathbb{Z}$-action $\widetilde{T}$ on $\mathcal{J}(\mu)$ defined by conjugation:

$$\widetilde{T}(J) = U_T^{-1} \circ J \circ U_T, \tag{4}$$

where U_T is the Koopman operator associated to T. We will call this action the *topological lens of* T. Notice that $\widetilde{T}$ is a homeomorphism of $\mathcal{J}(\mu)$. Since

$$\langle \widetilde{T}(J)1_A, 1_B\rangle_{L^2(\mu)} = \langle J \circ U_T 1_A, U_T 1_B\rangle_{L^2(\mu)} = \langle J 1_{T^{-1}A}, 1_{T^{-1}B}\rangle_{L^2(\mu)},$$

the corresponding action on $C_2(\mu)$, which we also denote by $\widetilde{T}$, is given by (see (1))

$$\widetilde{T}(\rho)(A \times B) = \rho(T^{-1}A \times T^{-1}B) \quad \text{for each } A, B \in \mathcal{B}. \tag{5}$$

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It is easy to see that

$$\widetilde{T^n} = \widetilde{T}^n \tag{6}$$

for each $n \in \mathbb{Z}$. Moreover if T_1 (acting on $(X_1, \mathcal{B}_1, \mu_1)$) is a factor of T, then

$$\widetilde{T_1} \text{ is a topological factor of } \widetilde{T}. \tag{7}$$

In fact, if $\theta : (X, \mathcal{B}, \mu) \to (X_1, \mathcal{B}_1, \mu_1)$ satisfies $\theta \circ T = T_1 \circ \theta$, then the map $\widetilde{\theta}(J) = V_\theta^* \circ J \circ V_\theta$ $(V_\theta : L^2(X_1, \mu_1) \to L^2(X, \mu), \quad V_\theta(f_1) = f_1 \circ \theta)$ is the corresponding continuous homomorphism of topological dynamical systems. Equivalently, if $\rho \in C_2(\mu)$ then

$$\widetilde{\theta}(\rho)(A_1 \times B_1) = \rho(\theta^{-1}A_1 \times \theta^{-1}B_1)$$

for each $A_1, B_1 \in \mathcal{B}_1$. In order to show that $\widetilde{\theta}$ is onto we use the relative independent extension construction (see Lemma 2 above).

Of course for every $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ the dense G_δ subset $C_{gr}(\mu) \subset C_2(\mu)$ is $\widetilde{T}$-invariant. Since the action of $\widetilde{T}$ on $C_{gr}(\mu)$ is isomorphic to conjugation by T on $\mathrm{Aut}(\mu)$, and as the group $\mathrm{Aut}(\mu)$ is algebraically simple [**5**], it follows that the homomorphism $T \mapsto \widetilde{T}$ from $\mathrm{Aut}(\mu)$ into the group $\mathrm{Homeo}(C_2(\mu))$ is an isomorphism. It can also be shown directly that the center of $\mathrm{Aut}(\mu)$ consists of the identity alone. A more difficult question is whether there are $S, T \in \mathrm{Aut}(\mu)$ which are not conjugate (that is, the measure-preserving systems $(X, \mathcal{B}, \mu, S)$ and $(X, \mathcal{B}, \mu, T)$ are not isomorphic) while their topological lenses $(C_2(\mu), \widetilde{S})$ and $(C_2(\mu), \widetilde{T})$ are isomorphic as topological systems. It is not hard to see that this cannot happen for ergodic rotations (see Proposition 2 below). However, it is very likely that the answer is yes. In fact, it seems that the argument in the proof of Theorem 8 below might be refined to show that any two measure-theoretical Bernoulli systems have isomorphic lenses.

Notice that every (non-trivial) topological lens has many fixed points (in particular, $\widetilde{T}$ has many minimal subsets); indeed the fixed point set of $\widetilde{T}$ coincides with the set of joinings,

$$\mathrm{Fix}(\widetilde{T}) = J_2(T). \tag{8}$$

Notice that $J_2(T)$ is a closed $\widetilde{T}$-invariant subset of $C_2(\mu)$ whose interior is empty (to see this, given $\rho \in J_2(T)$ and $\varepsilon > 0$ take the first $\eta \in C_2(\mu)$ which is not a 2-self-joining and then consider $\rho_\varepsilon = (1 - \varepsilon)\rho + \varepsilon\eta$ which cannot be a 2-self-joining for $\varepsilon > 0$). Similarly, periodic points for the lens correspond to 2-self-joinings of powers of T. Thus the set of periodic points for $\widetilde{T}$, which is just $\bigcup_{n \geq 1} J_2(T^n)$, is $\widetilde{T}$-invariant and meager.

The aim of this paper is to investigate how topological properties of the lens $(C_2(\mu), \widetilde{T})$ reflect ergodic properties of T. *Unless we say explicitly otherwise we usually assume that T is ergodic.*

Before we begin our study of topological lenses we briefly discuss a simpler topological system on $\mathcal{J}(\mu)$ given by the one sided composition t_T:

$$t_T(J) := J \circ U_T \quad \text{for } J \in \mathcal{J}(\mu);$$

the corresponding action on $C_2(\mu)$ which is given by the formula

$$t_T(\rho)(A \times B) = \rho(T^{-1}A \times B) \quad \text{for each } A, B \in \mathcal{B}$$

we will also denote by t_T. We will now argue that this system is particularly simple from the dynamical point of view. Indeed, because of one-sided continuity of the composition

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of Markov operators (both left and right), the enveloping semigroup of t_T consists solely of continuous maps t_J, where J belongs to the weak closure of the group generated by U_T. It follows that the topological system $(t_T, C_2(\mu))$ is weakly almost periodic (WAP); see [**4, 7**]. For such systems the closure of each orbit contains exactly one minimal subsystem and this unique minimal system is a compact monothetic topological group. In fact, we can describe these minimal (sub)systems quite precisely.

Denote by $\mathcal{K} \subset \mathcal{B}$ the sub-σ-algebra corresponding to the Kronecker factor of T. Put

$$\widehat{J_2^e(\mathcal{K})} = \{\widehat{\lambda} : \lambda \in J_2^e(T|_{\mathcal{K}}, \mu_{\mathcal{K}})\}$$

and $\widehat{\mathcal{J}^e(\mathcal{K})}$ for the corresponding set of Markov operators.

PROPOSITION 1. *There exists a sequence* (n_i) *of density* 1 *such that for each* $\rho \in C_2(\mu)$ *the limit points along subsequences* (n_{i_k}) *of* $t_T^{n_{i_k}}(\rho)$ *are of the form* $\rho \circ \widehat{\lambda}$ *for some* $\lambda \in J_2^e(T|_{\mathcal{K}}, \mu_{\mathcal{K}})$.

Proof. The result holds trivially if the system T has a discrete spectrum; indeed, recall that in this case the set of Markov operators corresponding to $J_2^e(T)$ is closed and equal to the closure of powers of U_T which, in turn, consists of the Markov operators U_S where S commutes with T. Take for (n_i) the sequence of all natural numbers.

Next assume that T has a partly continuous spectrum. There exists a sequence (n_i) of density 1 such that if we denote

$$L^2(X, \mathcal{B}, \mu) = L^2(\mathcal{K}) \oplus F \tag{9}$$

then for each $f, g \in F$

$$\langle U_T^{n_i} f, g\rangle \to 0 \quad \text{when } i \to \infty.$$

Since $T|_{\mathcal{K}}$ has a discrete spectrum, we can choose a subsequence (n_{i_k}) of (n_i) such that $U_{T|_{\mathcal{K}}}^{n_{i_k}} \to V$, with $V \in \mathcal{J}(T|_{\mathcal{K}})$ and $V = J_\lambda$ for some $\lambda \in J_2^e(T|_{\mathcal{K}})$. All we need to show is that

$$U_T^{n_{i_k}} = t_T^{n_{i_k}}(\mathrm{Id}) \to J_{\hat{\lambda}}.$$

With no loss of generality we can assume $U_T^{n_{i_k}} \to J$. Note that J preserves the decomposition (9); a weak limit of powers of U_T preserves a weakly closed U_T-invariant subspace. For $f \in L^2(X, \mathcal{B}, \mu)$ and $g \in F$, by decomposing $f = f_1 + f_2$, where $f_1 \in L^2(\mathcal{K})$ and $f_2 \in F$, we have

$$\int_X f_2 \circ T^{n_{i_k}} \cdot g \, d\mu \to 0$$

by the property of (n_i), hence

$$\int_X f \circ T^{n_{i_k}} \cdot g \, d\mu \to \int_X J(f) \cdot g \, d\mu = \int_X J(f_1) \cdot g \, d\mu.$$

Thus $\mathrm{Im}(J) \subset L^2(\mathcal{K})$ or, more precisely, $J(L^2(\mathcal{K})) \subset L^2(\mathcal{K})$ and $J(L^2(\mathcal{K})^\perp) = \{0\}$. Hence, $J = i_{L^2(\mathcal{K})} \circ J_\lambda \circ E(\cdot|\mathcal{K}) = J_{\hat{\lambda}}$ by Lemma 1. □

Notice that the action of $t_{T|_{\mathcal{K}}}$ on $J_2^e(T|_{\mathcal{K}})$ just goes back to the action of $T|_{\mathcal{K}}$ on $(X/\mathcal{K}, \mathcal{K}, \mu)$ if we recall that $T|_{\mathcal{K}}$ is isomorphic to a minimal rotation on a compact monothetic group $X/\mathcal{K}$ (endowed with its normalized Haar measure $\mu_{\mathcal{K}}$). Therefore t_T acting on $\widehat{J_2^e(\mathcal{K})}$ is also minimal, and for each $J \in \mathcal{J}(T)$ the only minimal subset contained in the orbit closure of J is equal to $J \circ \widehat{\mathcal{J}^e(\mathcal{K})}$.

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Remark 1. Let S be a homeomorphism of a compact metric space M. Let $A \subset M$ be closed. We say that A is a *quasi-attractor* if there exists a sequence (n_i) of density one such that for each $x \in M$, every limit point of the sequence $S^{n_i}x$ lies in A. We claim that, in such a case, each invariant measure is concentrated on A and each minimal subset is contained in A. In fact, suppose that μ is ergodic for S and let $x \in M$ be a generic point for μ. Suppose that $\mu(M\backslash A) > 0$. Choose a compact subset $C \subset M\backslash A$ for which $\mu(C) > 0$ and then a continuous function f with $0 \le f \le 1$ such that $f = 1$ on C and $f = 0$ on A. Since $f = 0$ on A and (n_i) has density 1,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(S^k x) \to 0.$$

Hence $\int_M f\, d\mu = 0$, contradicting the fact that $\int_M f\, d\mu \ge \int_C f\, d\mu = \mu(C) > 0$.

We now observe that $J \circ \widehat{\mathcal{J}^e(\mathcal{K})}$ is a quasi-attractor in the closure of the orbit of $J \in \mathcal{J}(\mu)$. Finally note that if T is weakly mixing, then all the sets $J \circ \widehat{\mathcal{J}^e(\mathcal{K})}$ collapse into one point ($\widehat{J_2^e(\mathcal{K})} = \{\mu \otimes \mu\}$), and we obtain just one attracting fixed point for the whole action t_T on $C_2(\mu)$.

In contrast, the conjugation by $T \in \mathrm{Aut}(\mu)$ on $\mathcal{J}(\mu)$, that is, the action of $\widetilde{T}$ on $C_2(\mu)$, is usually not WAP. This will be amply demonstrated in the rest of this work, but here is a simple first example.

Example. Let $G = \mathrm{SL}(2, \mathbb{R})$ and $\Gamma \subset G$ be a discrete cocompact subgroup. The *geodesic* and *horocycle* flows on the compact homogeneous space $X = G/\Gamma$ are the restrictions of the left-multiplication G-action to the subgroups

$$\left\{A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R}\right\} \quad \text{and} \quad \left\{B_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R}\right\},$$

respectively. These flows preserve the normalized Haar measure λ on X. Now the commutation relations

$$A_t B_s A_t^{-1} = \begin{pmatrix} 1 & se^{2t} \\ 0 & 1 \end{pmatrix},$$

applied to the corresponding unitary operators on $L_2(X, \lambda)$, show that in the weak operator topology $\lim_{t\to-\infty} A_t B_s A_t^{-1} = \mathrm{Id}$, while, by mixing of the horocycle flow, $\lim_{t\to\infty} A_t B_s A_t^{-1} = P$, where P is the projection onto the subspace of constant functions. Thus in $\mathcal{J}(\lambda)$ the orbit closure of, say, B_1 under conjugation by A_1 is isomorphic to the two-point compactification of $\mathbb{Z}$. In particular, there are in this orbit closure two distinct minimal sets $\{\mathrm{Id}\}$ and $\{P\}$, and therefore it is not WAP.

3. *Recurrence, distality and weak mixing*

In this section we impose certain topological conditions on the lens-like recurrence, distality and transitivity, and examine their implication for the corresponding measure-preserving transformation.

3.1. *Pointwise recurrence and rigidity.*

Let us recall some basic definitions. Let R be a homeomorphism of a compact space Z. Then R is called *pointwise recurrent* if for every

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$z \in Z$ and $\varepsilon > 0$ the orbit of z returns to the ε-ball centered at z infinitely often. If there is an increasing sequence (n_i) of integers such that $R^{n_i} \to \mathrm{Id}$ uniformly (pointwise) then R is called *uniformly rigid* (*rigid*).

An automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is called *rigid* if for some increasing sequence (n_i) of integers, $U_T^{n_i} \to \mathrm{Id}$ in the strong operator topology on the bounded linear operators on $L^2(X, \mathcal{B}, \mu)$.

LEMMA 3. *If $(C_2(\mu), \tilde{T})$ is pointwise recurrent then $(X, \mathcal{B}, \mu, T)$ is rigid.*

Proof. Let $\alpha = \{A_1, \ldots, A_k\}$ be a finite measurable partition of X such that the numbers $a_i = \mu(A_i)$ are all positive and distinct ($a_i \neq a_j$ whenever $i \neq j$). Fix $n \in \mathbb{N}$ and set $t_{ij} = \mu(T^{-n}A_i \cap A_j)$. Thus $\sum_{i=1}^k a_i = 1$ and for each i, $\sum_{j=1}^k t_{ij} = a_i > 0$.

Let $\xi = \xi_\alpha$ be the measure on $X \times X$ defined by

$$\xi = \sum_{i=1}^k a_i \mu_{A_i} \otimes \mu_{A_i} \quad \text{where } \mu_{A_i}(B) = \frac{\mu(B \cap A_i)}{\mu(A_i)}, \ 1 \le i \le k.$$

If we fix $B \in \mathcal{B}$ then

$$\xi(B \times X) = \xi(X \times B) = \sum_{i=1}^k a_i \frac{\mu(A_i \cap B)}{\mu(A_i)} = \mu(B),$$

so $\xi \in C_2(\mu)$. Notice that $\xi(\bigcup_{j=1}^k A_j \times A_j) = 1$.

Fix an $\varepsilon > 0$. Since $\widetilde{T}$ is pointwise recurrent, we can assume that, for some $n \ge 1$, $\tilde{T}^n\xi$ is so close to ξ that

$$\left| \tilde{T}^n \xi \left(\bigcup_{j=1}^k A_j \times A_j \right) - 1 \right| < \varepsilon.$$

If we now put $b_{ij} = (\mu(T^{-n}A_j \cap A_i))/a_i$ then $\sum_{j=1}^k b_{ij} = 1$ and

$$\begin{aligned} \tilde{T}^n \xi \left(\bigcup_{j=1}^k A_j \times A_j \right) &= \sum_{j=1}^k \sum_{i=1}^k a_i \mu_{A_i} \otimes \mu_{A_i}(T^{-n}A_j \times T^{-n}A_j) \\ &= \sum_{i=1}^k a_i \sum_{j=1}^k \left(\frac{\mu(T^{-n}A_j \cap A_i)}{a_i} \right)^2 \\ &= \sum_{i=1}^k a_i \sum_{j=1}^k \left(\frac{t_{ij}}{a_i} \right)^2 = \sum_{i=1}^k a_i \sum_{j=1}^k b_{ij}^2, \end{aligned}$$

so

$$\left| \sum_{i=1}^k a_i \sum_{j=1}^k b_{ij}^2 - 1 \right| < \varepsilon.$$

If ε is sufficiently small then this implies that for all $1 \le i \le k$, $|\sum_{j=1}^k b_{ij}^2 - 1| < \varepsilon'$ or, equivalently, $\sum_{j \neq j'} b_{ij} b_{ij'} < \varepsilon'$. This, in turn, means that, given i, for only one j_i we have $|t_{ij_i} - a_i| < \varepsilon''$. The map $i \mapsto j_i$ is one-to-one. However the numbers a_i are distinct, so if for each i

$$|\mu(T^{-n}A_i \cap A_{ij_i}) - \mu(A_i)| < \varepsilon''$$

then $j_i = i$. It follows that $j_i = i$ for each $1 \le i \le k$, that is, $|\mu(T^n A_i \cap A_i) - a_i| < \varepsilon''$. This proves the rigidity of $(X, \mathcal{B}, \mu, T)$ since $\varepsilon'' \to 0$ when $\varepsilon \to 0$. □

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THEOREM 2. *The following conditions on $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ are equivalent.*
(i) *T is rigid.*
(ii) *$\widetilde{T}$ is pointwise recurrent.*
(iii) *$\widetilde{T}$ is rigid.*
(iv) *$\widetilde{T}$ is uniformly rigid.*

Proof. To prove that (i) implies (iv), let $\{f_k\}_{k\geq 0}$ be a dense set of L^2-functions all of norm 1 (see (2)). Let $J \in \mathcal{J}(\mu)$ and fix $\varepsilon > 0$. Choose $N \geq 1$ so that $\sum_{k,l=N}^{\infty}(1/2^{k+l}) < \varepsilon/8$. Since T is rigid we can find $m \geq 1$ such that

$$\sum_{k,l=0}^{N-1} \frac{1}{2^{k+l}}(\|U_T^m f_k - f_k\| + \|U_T^m f_l - f_l\|) < \varepsilon/2.$$

We now have

$$\begin{aligned}
&\sum_{k,l\geq 0} \frac{1}{2^{k+l}}|\langle J \circ U_T^m f_k, U_T^m f_l\rangle - \langle J f_k, f_l\rangle| \\
&\quad \leq \sum_{k,l\geq 0} \frac{1}{2^{k+l}}(|\langle J \circ U_T^m f_k, U_T^m f_l\rangle - \langle J f_k, U_T^m f_l\rangle| + |\langle J f_k, U_T^m f_l\rangle - \langle J f_k, f_l\rangle|) \\
&\quad \leq \sum_{k,l=0}^{N-1} \frac{1}{2^{k+l}}(\|U_T^m f_k - f_k\| + \|U_T^m f_l - f_l\|) \\
&\qquad + \sum_{k,l\geq N} \frac{1}{2^{k+l}}(\|U_T^m f_k - f_k\| + \|U_T^m f_l - f_l\|)
\end{aligned}$$

and therefore

$$\sum_{k,l\geq 0} \frac{1}{2^{k+l}}|\langle \widetilde{T}^m(J) f_k, f_l\rangle - \langle J f_k, f_l\rangle| < \varepsilon/2 + \varepsilon/8 < \varepsilon.$$

The uniform rigidity of $\widetilde{T}$ follows.

The implications (iv) implies (iii) implies (ii) are true for every dynamical system and finally the implication (ii) implies (i) is Lemma 3. □

3.2. *The discrete spectrum case and distality of the topological lens.* Suppose T is ergodic and has a discrete spectrum. In this case the set $\{U_T^n : n \in \mathbb{Z}\}$ is relatively compact in the strong operator topology, and moreover each limit point of this set is of the form U_S, where $S \in C(T)$. As we have seen in the proof of Theorem 2 this implies that whenever $J \in \mathcal{J}(\mu)$ and $U_T^{n_k} \to U_S$ then $\widetilde{T}^{n_k}(J) \to U_S^{-1} \circ J \circ U_S$. It follows that any pointwise limit Θ of powers of $\widetilde{T}$, that is any element of the Ellis semigroup of $\widetilde{T}$, is also a conjugation. Hence the Ellis semigroup of $\widetilde{T}$ is a group of homeomorphisms. This fact implies that $\widetilde{T}$ is equicontinuous, that is, the family $\{\widetilde{T}^n : n \in \mathbb{Z}\}$ is equicontinuous. Moreover each minimal subset (which must be the closure of an orbit) is of the form

$$\{U_S \circ J \circ U_{S^{-1}} : S \in C(T)\}$$

for some $J \in \mathcal{J}(\mu)$.

Equicontinuous systems are special examples of distal systems. Recall that a homeomorphism R of a compact metric space Z is *distal* if for every pair (z_1, z_2) of distinct points of Z the closure of its orbit (via $R \times R$) is disjoint from the diagonal

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$\Delta_Z = \{(z, z) : z \in Z\}$ in $Z \times Z$. Every distal system has a decomposition into minimal components, in other words every point $z \in Z$ is almost periodic (uniformly recurrent) that is, the set of return times of z to any fixed neighborhood has bounded gaps. (For more details see e.g. [**7**].)

THEOREM 3. *Suppose that T is ergodic and that $\widetilde{T}$ on $C_2(\mu)$ is distal. Then T has a discrete spectrum. Therefore if $\widetilde{T}$ is distal then it is equicontinuous.*

Proof. Using the notation of Lemma 3, by our assumption of distality, we obtain that for a finite measurable partition $\alpha = \{A_1, \ldots, A_k\}$ of X with the property that the k numbers $a_i = \mu(A_i)$ are distinct and positive, the corresponding measure,

$$\xi_\alpha = \sum_{i=1}^{k} a_i \mu_{A_i} \times \mu_{A_i} \quad \text{where } \mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)},$$

is a uniformly recurrent point of the topological system $(C_2(\mu), \tilde{T})$. However, as we have seen in the proof of that lemma, this implies that the set of recurrence times for the sets A_i $(i = 1, 2, \ldots, k)$, i.e. the set

$$\{n \in \mathbb{Z} : |\mu(T^n A_i \cap A_i) - \mu(A_i)| < \varepsilon,\ 1 \le i \le k\},$$

has bounded gaps or equivalently the set $\{n \in \mathbb{Z} : \|U_T^{n_i} 1_{A_i} - 1_{A_i}\|_2 < \sqrt{2}\varepsilon;\ i = 1, 2, \ldots, k\}$ has bounded gaps. It follows that for each $\varepsilon > 0$ the set $\{U_T^n(1_A) : n \in \mathbb{Z}\}$ admits a finite ε-net, which means that its L^2-closure is compact. In turn this implies that for every k-tuple of real numbers $(c_1, c_2, \ldots, c_k)$ the $L^2(\mu)$-function

$$\sum_{i=1}^{k} c_i \chi_{A_i}$$

also has a compact U_T-orbit and therefore T has a discrete spectrum (see [**13**] or notice that in the terminology of [**6**] we obtained a dense set of compact functions in $L^2(\mu)$). This completes the proof of the theorem. □

In particular, for T with a non-discrete spectrum there are always non-trivial proximal pairs in the topological lens.

PROPOSITION 2. *The topological lenses of aperiodic ergodic rotations are topologically conjugate if and only if the rotations are conjugate as measure-preserving systems.*

Proof. We first point out the following facts concerning an aperiodic ergodic rotation $T \in \mathrm{Aut}(\mu)$.

(1) Each minimal subset of $\widetilde{T}$ is of the form $Y_J = \{U_S \circ J \circ U_{S^{-1}} : S \in C(T)\}$ for $J \in \mathcal{J}(\mu)$.

(2) Assume additionally (but with no loss in generality) that $Tx = x + x_0$, i.e. T is a uniquely ergodic rotation on a compact monothetic (metric) group X. Then T is topologically conjugate to the translation t_T by T on $C(T)$.

(3) Under the assumption in fact (2), $\widetilde{T}|_{Y_J}$ is a topological factor of T. Indeed, the map

$$C(T) \ni S \mapsto U_S \circ J \circ U_{S^{-1}} \in Y_J$$

is equivariant (between t_T and $\widetilde{T}|_{Y_J}$).

(4) If J does not commute with any $S \in C(T) \backslash \{\mathrm{Id}\}$ then the above map is an isomorphism.

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(5) There is always some $J \in \mathcal{J}(\mu)$ which does not commute with any $S \in C(T)\backslash\{\mathrm{Id}\}$, e.g. take $J = R$, where $R \in \mathrm{Aut}(\mu)$ is weakly mixing with $C(R) = \{R^n : n \in \mathbb{Z}\}$.

Now assume that for aperiodic ergodic rotations T_1 and T_2 we have a topological conjugacy $\Phi : (C_2(\mu), \widetilde{T}_1) \to (C_2(\mu), \widetilde{T}_2)$. Then Φ sends minimal subsets onto minimal subsets and moreover every minimal subset has a minimal preimage. It now follows, in view of the facts (1)–(5), that T_1 and T_2 are weakly topologically isomorphic (i.e. each is a topological factor of the other), hence are isomorphic (a well-known fact). Of course, this implies that they are also measure-theoretically isomorphic. □

3.3. *Weak mixing of a system versus topological transitivity of its lens.* It is well-known that invertible elements are dense in the set $C_2(\mu)$ (see [**8**]). However we will need a slightly more concrete result saying that a special family of interval exchange transformations is dense in $C_2(\lambda_{[0,1]})$, where $\lambda_{[0,1]}$ stands for Lebesgue measure on [0, 1].

Thus we now assume (with no loss in generality) that $X = [0, 1]$, $\mu = \lambda_{[0,1]}$. Let $\alpha = \{I_1, I_2 \ldots, I_k\}$ be the partition of [0, 1] into k intervals of equal length. Fix $\varepsilon > 0$ and P as in (3). We also assume that $p_{ij} \in \mathbb{Q}$ and let

$$p_{ij} = \frac{m_{ij}}{L}, \quad m_{ij} \in \mathbb{N},\ L \in \mathbb{N}\backslash\{0\},$$

for all $i, j = 1, \ldots, k$. Divide each interval I_i into L subintervals J_{ij} of equal length. We are now going to define a transformation $S \in \mathrm{Aut}([0, 1], \lambda_{[0,1]})$ which will be an element of $U(\alpha, \epsilon, P)$. It will be defined as an interval exchange automorphism; i.e. each J_{ij} will be mapped by S onto an interval $J_{\sigma(i,j)}$ via a map of the form $x \mapsto x + \beta_{ij}$ (defined on J_{ij}), where σ is a suitable bijection of $\{(i, j) : 1 \le i \le k, 1 \le j \le L\}$. In fact, we will group some consecutive J_{ij} into longer subintervals and then permute these new subintervals. Therefore if we 'visualize' the graph of S in $[0, 1] \times [0, 1]$ as given by the diagonals of some little subsquares, we only need to say what is this family of subsquares. We begin by taking the subsquares

$$(J_{11} \cup \cdots \cup J_{1,km_{11}}) \times (J_{11} \cup \cdots \cup J_{1,km_{11}}),$$
$$(J_{1,km_{11}+1} \cup \cdots \cup J_{1,km_{11}+km_{12}}) \times (J_{21} \cup \cdots \cup J_{2,km_{12}}), \ldots,$$
$$(J_{1,km_{11}+km_{12}+\cdots+km_{1,k-1}+1} \cup \cdots \cup J_{1,km_{11}+km_{12}+\cdots+km_{1k}}) \times (J_{k1} \cup \cdots \cup J_{1,km_{1k}})$$

which define S on I_1 (since $\sum_{j=1}^{k} m_{1j}/L = 1/k$, $\sum_{j=1}^{k} km_{1j} = L$). To define S on I_2 we choose the following subsquares:

$$(J_{21} \cup \cdots \cup J_{2,km_{21}}) \times (J_{1,km_{11}+1} \cup \cdots \cup J_{1,km_{11}+km_{21}}),$$
$$(J_{2,km_{21}+1} \cup \cdots \cup J_{2,km_{21}+km_{22}}) \times (J_{2,km_{12}+1} \cup \cdots \cup J_{2,km_{12}+km_{22}}), \ldots$$
$$(J_{2,km_{21}+\cdots+km_{2,k-1}+1} \cup \cdots \cup J_{2,km_{21}+\cdots+km_{2,k-1}+km_{2k}})$$
$$\times (J_{k,km_{1k}+1} \cup \cdots \cup J_{k,km_{1k}+km_{2k}}).$$

We keep going with this procedure of choosing 'the first possible' square, defining S on I_3, then through all the remaining intervals. This construction is correct since, for each $a, b = 1, \ldots, k$,

$$\sum_{j=1}^{k} \frac{km_{aj}}{L} = 1, \quad \sum_{i=1}^{k} \frac{km_{ib}}{L} = 1.$$

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Given $k \geq 1$, let $\mathcal{S}_k$ stand for the family of automorphisms of $([0, 1], \lambda)$ given by dividing $[0, 1]$ into k intervals of equal length and then permuting them according to a permutation π of $\{1, 2, \ldots, k\}$. By the above reasoning we have proved the following proposition.

PROPOSITION 3. *The family* $\bigcup_{k\geq 1} \mathcal{S}_k$ *is dense in* $C_2(\lambda_{[0,1]})$.

Remark 2. It has been proved by Kechris and Rosendal in [**11**] that there exists a residual set of $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, such that $\widetilde{T}|\mathrm{Aut}(X, \mathcal{B}, \mu)$, that is, conjugation by T, is topologically transitive on $\mathrm{Aut}(X, \mathcal{B}, \mu)$. Hence by the above remark we also have that for a residual set of $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, the corresponding $\widetilde{T}$ is transitive on $C_2(\lambda_{[0,1]})$. As the set of weakly mixing transformations is residual in $\mathrm{Aut}(X, \mathcal{B}, \mu)$, it follows that for a residual set of weakly mixing transformations T, conjugation by T is transitive. It turns out, however, that this property is in fact a characterization of weak mixing as the theorem below shows.

THEOREM 4. *Assume that* $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ *is ergodic. Then the following conditions are equivalent.*

(i) T *is weakly mixing.*
(ii) $\widetilde{T}$ *is transitive.*
(iii) $\widetilde{T}$ *is topologically weakly mixing.*

Proof.

(ii) $\Rightarrow$ (i) Denote by (X_1, μ_1, T_1) the Kronecker factor of (X, μ, T). In view of (7), $\widetilde{T_1}$ is a factor of $\widetilde{T}$ and since by assumption $\widetilde{T}$ is transitive, so is $\widetilde{T_1}$. Since $\widetilde{T_1}$ is WAP it has exactly one minimal set. However, if it is not the trivial one point system, it has at least two distinct fixed points, $\mu_1 \times \mu_1$ and Δ_{μ_1}. Thus $\widetilde{T_1}$ is trivial, hence T is weakly mixing.

(i)$\Rightarrow$(ii) For a partition $\alpha = \{A_1, \ldots, A_k\}$ with $\mu(A_j) = 1/k$ for $j = 1, \ldots, k$, a permutation η of $\{1, \ldots, k\}$ and an $\varepsilon > 0$ set

$$V(\alpha, \eta, \varepsilon) = \left\{\xi \in C_2(\mu) : \left|\xi(A_i \times A_{\eta(i)}) - \frac{1}{k}\right| < \varepsilon\right\}.$$

CLAIM. *Given two permutations* π, σ *of* $\{1, \ldots, k\}$ *there exists an* $n \geq 1$ *such that*

$$\widetilde{T}^{-n}(V(\alpha, \sigma, \varepsilon)) \cap V(\alpha, \pi, 2\varepsilon') \neq \emptyset, \tag{10}$$

where $\varepsilon' > 0$ *is made precise below.*

Indeed, by the weak mixing property of T there exists $n \geq 1$ such that

$$\left|\mu(T^n A_i \cap A_j) - \frac{1}{k^2}\right| < \varepsilon \quad \text{for all } i, j = 1, \ldots, k. \tag{11}$$

Let

$$\Delta_\eta = \bigcup_{i=1}^{k} A_i \times A_{\eta(i)} \quad \text{where } \eta = \sigma, \pi.$$

For a fixed $1 \leq i \leq k$ let $\overline{B}^i_s = T^{-n} A_s \cap A_i$, $s \geq 1$. Then $\beta_i = \{\overline{B}^i_1, \ldots, \overline{B}^i_k\}$ is a partition of A_i and by (11)

$$\left|\mu(\overline{B}^i_s) - \frac{1}{k^2}\right| < \varepsilon \quad \text{for } s = 1, \ldots, k.$$

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We now replace the partitions β_i by partitions for which all atoms B_s^i, $1 \le s \le k$, have measure $1/k^2$ and moreover

$$\sum_{i,s=1}^{k} \mu(\overline{B}_s^i \triangle B_s^i) < \varepsilon', \tag{12}$$

where $\varepsilon' = \varepsilon'(\varepsilon, k)$ and $\varepsilon' \to 0$ when $\varepsilon \to 0$. Now choose any $\xi \in C_2(\mu)$ such that

$$\xi(B_s^i \times B_{\pi(s)}^{\sigma(i)}) = \frac{1}{k^2}$$

for all i, $s = 1, \ldots, k$ (for example we can take the measure $\sum_{i,s=1}^{k}(1/k^2)\mu_{B_s^i} \otimes \mu_{B_{\pi(s)}^{\sigma(i)}}$, see the proof of Lemma 3). Notice that the measure ξ is supported by the union $\bigcup_{i,s=1}^{k} B_s^i \times B_{\pi(s)}^{\sigma(i)}$, and in particular

$$\xi(B_s^i \times B_t^j) = 0 \quad \text{unless } j = \sigma(i),\ t = \pi(s). \tag{13}$$

We will now check that the following hold.

(i) $\xi \in V(\alpha, \sigma, \varepsilon)$;

(ii) $\widetilde{T}^n \xi \in V(\alpha, \pi, 2\varepsilon')$.

Indeed, as $\bigcup_{s=1}^{k} B_s^i = A_i$ for each $1 \le i \le k$, and (13) holds,

$$\xi(A_i \times A_{\sigma(i)}) = \xi\left(\bigcup_{s=1}^{k} B_s^i \times B_{\pi(s)}^{\sigma(i)}\right) = \sum_{s=1}^{k} \xi(B_s^i \times B_{\pi(s)}^{\sigma(i)}) = k \cdot \frac{1}{k^2} = \frac{1}{k}$$

and thus $\xi \in V(\alpha, \sigma, \varepsilon)$. Moreover, $\xi(\bigcup_{i=1}^{k} A_i \times A_{\sigma(i)}) = 1$. In order to check (ii), consider

$$\begin{aligned}(\widetilde{T}^n \xi)(A_s \times A_t) &= \xi\left(\bigcup_{i=1}^{k}(A_i \cap T^{-n}A_s) \times \bigcup_{j=1}^{k}(A_j \times T^{-n}A_t)\right) \\ &= \xi\left(\bigcup_{i,j=1}^{k} \overline{B}_s^i \times \overline{B}_t^j\right) = \sum_{i,j=1}^{k} \xi(\overline{B}_s^i \times \overline{B}_t^j).\end{aligned}$$

We have

$$\begin{aligned}\left|\sum_{i,j=1}^{k} \xi(\overline{B}_s^i \times \overline{B}_t^j) - \sum_{i,j=1}^{k} \xi(B_s^i \times B_t^j)\right| &\le \sum_{i,j=1}^{k} |\xi(\overline{B}_s^i \times \overline{B}_t^j) - \xi(B_s^i \times B_t^j)| \\ &\le \sum_{i,j=1}^{k} (\mu(\overline{B}_s^i \triangle B_s^i) + \mu(\overline{B}_t^j \triangle B_t^j)) < 2\varepsilon'.\end{aligned}$$

In particular,

$$\left|(\widetilde{T}^n \xi)(A_s \times A_{\pi(s)}) - \sum_{i,j=1}^{k} \xi(B_s^i \times B_{\pi(s)}^j)\right| < 2\varepsilon'.$$

However, in view of (13),

$$\sum_{i,j=1}^{k} \xi(B_s^i \times B_{\pi(s)}^j) = \sum_{i=1}^{k} \xi(B_s^i \times B_{\pi(s)}^{\sigma(i)}) = \frac{1}{k}$$

and the proof of the claim is complete.

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Let U_1, U_2 be any non-empty open subsets of $C_2(\mu)$. By Proposition 3 there are a partition $\alpha = \{A_1, \ldots, A_k\}$ of X with $\mu(A_i) = 1/k$, $i = 1, \ldots, k$, $\varepsilon > 0$ and permutations π, σ such that

$$V(\alpha, \sigma, \varepsilon) \subset U_1, \quad V(\alpha, \pi, 2\varepsilon') \subset U_2.$$

By the above claim there exists $n \geq 1$ such that

$$\widetilde{T}^{-n} V(\alpha, \sigma, \varepsilon) \cap V(\alpha, \pi, 2\varepsilon') \neq \emptyset$$

and therefore $\widetilde{T}^{-n} U_1 \cap U_2 \neq \emptyset$, which completes the proof of this part of the theorem.

(i) $\Rightarrow$ (iii) In order to obtain topological weak mixing we repeat the arguments used for the proof of transitivity but now with

$$\widetilde{T}^{-n} V(\alpha, \sigma_i, \varepsilon) \cap V(\alpha, \pi_i, 2\varepsilon') \neq \emptyset$$

for $i = 1, 2$. □

Remark 3. Recall that the map $S \mapsto \mu_S$ is a homeomorphism from the Polish group $\mathrm{Aut}(X, \mathcal{B}, \mu)$ onto a dense G_δ subset of $C_2(\mu)$ which is $\widetilde{T}$-invariant. Moreover it intertwines $\overline{T}$ and $\widetilde{T}$, where by $\overline{T}$ we denote the action of T on $\mathrm{Aut}(X, \mathcal{B}, \mu)$ by conjugation (see [**8**]). We conclude that T is weakly mixing if and only if $\overline{T}$ is topologically transitive (if and only if it is topologically weakly mixing) on the Polish space $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

For $T, S \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ denote by $\langle T, S \rangle$ the closed subgroup generated by T and S.

COROLLARY 1. *If T is weakly mixing then*

$$\mathcal{S} = \{S \in \mathrm{Aut}(X, \mathcal{B}, \mu) : \langle T, S \rangle = \mathrm{Aut}(X, \mathcal{B}, \mu)\}$$

is a residual subset of $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

Proof. By Theorem 4

$$\{S \in \mathrm{Aut}(X, \mathcal{B}, \mu) : \{T^{-n} S T^n : n \in \mathbb{Z}\} \text{ is dense in } \mathrm{Aut}(X, \mathcal{B}, \mu)\}$$

is dense G_δ and it is clearly a subset of $\mathcal{S}$. □

Remark 4. The proof of Theorem 4 shows also that if T is mixing then $\widetilde{T}$ is topologically mixing.

4. *Periodic points of the lens*

Recall that a topologically transitive dynamical system (Y, S) is a *P-system* (or is *chaotic in the sense of Devaney*) if the set of S-periodic points is dense in Y (see [**9**]).

Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. We recall that an element $\xi \in C_2(\mu)$ is a fixed point for $\tilde{T}$ ($\tilde{T}\xi = \xi$) if and only if ξ is a self-joining for T, i.e. $\xi \in J_2(T)$. Thus $\xi \in C_2(\mu)$ is a periodic point for $\tilde{T}$ ($\tilde{T}^n \xi = \xi$ for some $n \geq 1$) if and only if $\xi \in J_2(T^n)$. Set

$$J^n(T) = \{\xi \in C_2(\mu) : \tilde{T}^n \xi = \xi\}, \quad J^\infty(T) = \bigcup \{J^n(T) : n \geq 1\}.$$

Similarly we let

$$C^n(T) = \{S \in \mathrm{Aut}(X, \mathcal{B}, \mu) : ST^n = T^n S\}, \quad C^\infty(T) = \bigcup \{C^n(T) : n \geq 1\}.$$

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(Thus $C^1(T) = C(T)$ is the centralizer of T in $\mathrm{Aut}(X, \mathcal{B}, \mu)$, and, for each $n \geq 1$, $C^n(T) = C(T^n)$.) Identifying a transformation $S \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ with its graph measure Δ_S (the image of μ under the map $x \mapsto (x, Sx)$ of X into $X \times X$), we can think of $\mathrm{Aut}(X, \mathcal{B}, \mu)$ as a dense G_δ subset of $C_2(\mu)$. In particular, viewed in this way, $C^n(T)$ is a subset of $J^n(T)$ for $n = 1, 2, \ldots$.

4.1. *The topological lens of a K-transformation is a P-system.*

THEOREM 5. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system.*

(1) *If $(X, \mathcal{B}, \mu, T)$ is Bernoulli then the set $C^\infty(T)$ is dense in* $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

(2) *If $(X, \mathcal{B}, \mu, T)$ is a K-system then the set $J^\infty(T)$ is dense in $C_2(\mu)$. Hence, if $(X, \mathcal{B}, \mu, T)$ is a K-system then the topological system $(C_2(\mu), \tilde{T})$ is a P-system.*

Proof. Let T be a Bernoulli transformation. Given a measurable partition of X, $\mathcal{A} = \{A_0, A_1, \ldots, A_{d-1}\}$, into d sets of equal measure, and an $\varepsilon > 0$, we would like to find an n_0 and $S \in C(T^{n_0}) = C^{n_0}(T)$ such that

$$\mu(SA_i \triangle A_{i+1}) < \varepsilon \quad \text{for } i = 0, \ldots, d-1 \ (\mathrm{mod}\ d). \tag{14}$$

Step 1. As T is K, there exists n_0 so that the partitions $\{T^{jn_0}\mathcal{A}\}_{j\in\mathbb{Z}}$ are δ-independent. Here, δ is chosen small enough so that, by the Ornstein version of Sinai's theorem (see [**15**, Lemma 5]), there exists a measurable partition $\hat{\mathcal{A}} = \{\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{d-1}\}$ of X into d sets, with $\mu(\hat{A}_i) = 1/d$, such that

$$\begin{gathered} \text{the partitions } \{T^{jn_0}\hat{\mathcal{A}}\}_{j\in\mathbb{Z}} \text{ are independent,} \\ \sum_{i=0}^{d-1} \mu(A_i \triangle \hat{A}_i) < \frac{\varepsilon}{100}. \end{gathered} \tag{15}$$

Step 2. Next use Thouvenot's relative version of Sinai's theorem (see [**12, 17**]) to get a complementary partition $\mathcal{P} = \{P_0, P_1, \ldots, P_{\ell-1}\}$ such that

$$\begin{gathered} \text{the partitions } \{T^{jn_0}\mathcal{P}\}_{j\in\mathbb{Z}} \text{ are independent,} \\ \bigvee_{-\infty}^{\infty} T^{jn_0}\hat{\mathcal{A}} \perp \bigvee_{-\infty}^{\infty} T^{jn_0}\mathcal{P}, \\ H(\mathcal{P}) + H(\hat{\mathcal{A}}) = h(X, T^{n_0}). \end{gathered} \tag{16}$$

Step 3. Of course T^{n_0} is also Bernoulli and Ornstein's isomorphism theorem (see [**16**, Proposition 11, p. 31]) says that the partition $\mathcal{P} \vee \hat{\mathcal{A}}$ can be modified by an arbitrarily small amount so as to produce partitions $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{A}}$ with the properties

$$\begin{gathered} \text{the partitions } \{T^{jn_0}(\tilde{\mathcal{P}} \vee \tilde{\mathcal{A}})\}_{j\in\mathbb{Z}} \text{ are independent,} \\ \bigvee_{-\infty}^{\infty} T^{jn_0}\tilde{\mathcal{A}} \perp \bigvee_{-\infty}^{\infty} T^{jn_0}\tilde{\mathcal{P}}, \\ \bigvee_{-\infty}^{\infty} T^{jn_0}(\tilde{\mathcal{P}} \vee \tilde{\mathcal{A}}) \text{ is the full Borel } \sigma\text{-algebra of } X, \\ \sum_{i=0}^{d-1} \mu(\tilde{A}_i \triangle \hat{A}_i) < \frac{\varepsilon}{100}. \end{gathered} \tag{17}$$

Step 4. Now we can view T^{n_0} as a shift on $d + \ell$ symbols $\{\alpha_i : i \in \{0, \ldots, d-1\}\}$ and $\{\beta_i : i \in \{0, \ldots, \ell - 1\}\}$, with the product measure. On the phase space we define

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a transformation S by the map

$$\begin{pmatrix} \dots, & \alpha_{-1}, & \alpha_0, & \alpha_1, \dots \\ \dots, & \beta_{-1}, & \beta_0, & \beta_1, \dots \end{pmatrix} \mapsto \begin{pmatrix} \dots, & \alpha_{-1}+1, & \alpha_0+1, & \alpha_1+1, \dots \\ \dots, & \beta_{-1}, & \beta_0, & \beta_1, \dots \end{pmatrix} \times (\text{mod } d).$$

This map is measure preserving, commutes with the shift, and has the property

$$S\tilde{A}_i = \tilde{A}_{i+1} \quad \text{for } i = 0, \dots, d-1 \text{ (mod } d). \tag{18}$$

Taking into account the formula (16), (17) and (18), we obtain (14). This proves part 1 of the theorem.

To prove part 2 it suffices to show that, given a K-transformation T, a partition $\mathcal{A} = \{A_0, A_1, \dots, A_{d-1}\}$ of X into d sets with $\mu(A_i) = 1/d$, and $\varepsilon > 0$, there are n_0 and a self-joining λ of T^{n_0} such that

$$\lambda\left(\bigcup_{i=0}^{d-1} A_i \times A_{i+1}\right) > 1 - \varepsilon \text{ (mod } d). \tag{19}$$

Step 5. As in Step 1 above, there exists n_0 so that the partitions $\{T^{jn_0}\mathcal{A}\}_{j\in\mathbb{Z}}$ are δ-independent, and an application of Ornstein's theorem yields a measurable partition $\hat{\mathcal{A}} = \{\hat{A}_0, \hat{A}_1, \dots, \hat{A}_{d-1}\}$ of X into d-sets with $\mu(\hat{A}_i) = 1/d$, such that

$$\text{the partitions } \{T^{jn_0}\hat{\mathcal{A}}\}_{j\in\mathbb{Z}} \text{ are independent,}$$
$$\sum_{i=0}^{d-1} \mu(A_i \triangle \hat{A}_i) < \frac{\varepsilon}{100}.$$

Step 6. Now, for the Bernoulli d-shift corresponding to T^{n_0} and the independent partition $\hat{\mathcal{A}}$ there is an automorphism $\hat{S}$ cyclically permuting the sets $\{\hat{A}_i : i = 1, 2, \dots, d-1\}$. This defines a graph joining $\Delta_{\hat{S}}$ on the Bernoulli factor defined by $\bigvee_{-\infty}^{\infty} T^{jn_0}\hat{\mathcal{A}}$. The joining $\Delta_{\hat{S}}$ gives measure close to 1 to the set $\bigcup_{i=0}^{d-1} A_i \times A_{i+1}$ since it gives measure 1 to the set $\bigcup_{i=0}^{d-1} \hat{A}_i \times \hat{A}_{i+1}$. Finally lift $\Delta_{\hat{S}}$ to a self-joining λ of T^{n_0} on $X \times X$ to get (19). □

4.2. *Zero entropy and P-systems.* We have been unable to find T with zero entropy such that $(\widetilde{T}, C_2(\mu))$ is a P-system (in general, there are P-systems with zero topological entropy, see [**9**]). A clear case where the periodic points of the lens fail to be dense is when the equality

$$J^\infty(T) = J^1(T) = J_2(T) \tag{20}$$

holds, as in this case the set of periodic points is closed and nowhere dense (see the remarks following equation (8) in §2). It has been shown in [**10**, Theorem 6.1] that the equality (20) holds for all weakly mixing twofold simple systems. (Although [**10**, Theorem 6.1] states merely that $C(T^k) = C(T)$ for all $k \neq 0$, the proof actually yields the stronger result $J^e(T^k) = J^e(T) = C(T^k) \cup \{\mu \times \mu\} = C(T) \cup \{\mu \times \mu\}$. In particular (20) holds.)

Notice also that if S is a root of T satisfying (20) then the set of S-periodic points is non-dense as well. Indeed, assume that $S^k = T$ for some $k \geq 2$. Take $r \geq 1$ and $J \in \mathcal{J}(\mu)$ so that $J \circ U_{S^r} = U_{S^r} \circ J$. Then $U_{S^{kr}} \circ J = J \circ U_{S^{kr}}$ or, equivalently, $U_{T^r} \circ J = J \circ U_{T^r}$ and therefore $J^\infty(S) \subset J_2(T)$.

The equality (20) is also satisfied when

$$T^n \text{ has a simple spectrum for each } n \geq 1. \tag{21}$$

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In fact, if a unitary operator $V \in U(H)$ of a separable Hilbert space H has a simple spectrum then each bounded linear operator $W \in L(H)$ commuting with V is a (weak) limit of polynomials in V and therefore the semigroup of such W is commutative. It follows that when $S \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ has a simple spectrum the semigroup $\{J_\lambda : \lambda \in J_2(S)\}$ is also commutative. We clearly have $J_2(T) \subset J_2(T^n)$ for each $n \geq 1$. Assume now that $J \in \mathcal{J}(\mu)$ and for some n, $J \circ U_{T^n} = U_{T^n} \circ J$. Since $U_T \circ U_{T^n} = U_{T^n} \circ U_T$ and U_{T^n} has a simple spectrum, $J \circ U_T = U_T \circ J$, and therefore $J_2(T) = J_2(T^n)$, so (20) follows. Notice that the property (21) is closed under taking both powers and roots (indeed, for the latter just observe that whenever S^{km} has a simple spectrum then both S^k and S^m have simple spectra).

THEOREM 6.

(1) *The generic transformation $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ has the property that for every $n \in \mathbb{Z}\backslash\{0\}$, T^n is of rank one. Thus the generic T satisfies condition (21) and in particular its set of periodic points is closed and nowhere dense. Thus the topological lens is not a P-system.*

(2) *The equality (20) holds for all weakly mixing 2-fold simple systems. In particular, for such systems the lens is never a P-system.*

(3) *If T is a Gaussian system with a simple spectrum then it satisfies condition (21); in particular $\widetilde{T}$ is not a P-system.*

Proof.

(1) It is well-known, and not hard to check, that the set $\mathcal{R}$ of rank-one transformations is a dense G_δ subset of the Polish group $G = \mathrm{Aut}(\mu)$. It is also well-known that a rank-one transformation has a simple spectrum (see e.g. [**7**, Theorem 16.5]). For each integer $k \geq 1$, let $\pi_k : \mathrm{Aut}(\mu) \to \mathrm{Aut}(\mu)$ be the map $T \mapsto T^k$. Clearly π_k is a continuous map and it follows that the set $\mathcal{S}_k := \pi_k^{-1}(\mathcal{R}) = \{T \in G : T^k \in \mathcal{R}\}$ is a G_δ subset of G. Clearly $\mathcal{R}$, and therefore also $\mathcal{S}_k$, are conjugation invariant and, as $\mathcal{S}_k$ contains ergodic transformations (e.g. an irrational rotation), it follows, by Halmos' theorem, that $\mathcal{S}_k$ is a dense G_δ subset of G. In fact, it can be shown that $\mathcal{S}_k \subset \mathcal{R}$ (if T^k is rank one then so is T) but in any case we can take $\mathcal{R}_\infty = \mathcal{R} \cap \bigcap_{k \geq 1} \mathcal{S}_k$ as the required generic (in fact, dense G_δ) set of transformations, since $T \in \mathcal{R}$ if and only if $T^{-1} \in \mathcal{R}$.

(2) This has been shown in [**10**].

(3) Since T^n is a (generalized) Gaussian system whose maximal spectral multiplicity is bounded by n, it follows that T^n has a simple spectrum. In fact, the maximal spectral multiplicity of a Gaussian system is either 1 or ∞ (see [**2, 14**]). □

Of course every Kronecker (that is, ergodic discrete spectrum) system is simple, but in this class we encounter both types of behavior.

For an irrational rotation R_α on the circle $\mathbb{T}$, we have $C(R_\alpha) = C(R_\alpha^n) = C^n(R_\alpha) = \{R_\beta : \beta \in \mathbb{T}\}$ for all $n \neq 0$. Thus, by simplicity, also $J_1(R_\alpha) = J_n(R_\alpha)$ for all $n \neq 0$. Of course, $\widetilde{T}$ is not a P-system because of the absence of transitivity.

The situation changes when T is not totally ergodic. Considering the dyadic adding machine (X, μ, T), with $X = \{0, 1\}^{\mathbb{N}}$ and $Tx = x + \mathbf{1}$, where $\mathbf{1} = (1, 0, 0, \ldots)$, one can easily check that, given any permutation π of $\mathbb{Z}_{2^n} = \{0, 1, \ldots, 2^n - 1\}$, the corresponding homeomorphism S_π defined on X by permuting the first 2^n coordinates, commutes with T^n

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which do not affect the first 2^n coordinates of elements of X. It follows that the set $C^\infty(T)$ is dense in $\mathrm{Aut}(X, \mathcal{B}, \mu)$; i.e. the periodic points are dense in $(C_2(\mu), \tilde{T})$. Note however that $(C_2(\mu), \tilde{T})$ is not topologically transitive and therefore not a P-system.

5. *Invariant measures on the topological lens and quasi-factors*

In this section we will examine the relation between $\widetilde{T}$-invariant probability measures on the topological lens and quasi-factors. For completeness we will repeat some arguments from [**7**] concerning the basic connection between quasi-factors and self-joinings—this has to be checked because we must drop the assumption, made in [**7**], of ergodicity of the barycenter. Given a standard Borel space $(Y, \mathcal{C})$, we denote by $M(Y)$ the set of probability measures on it. Recall that $M(Y)$ carries a natural Borel structure which is generated by the evaluation maps $\phi_A : \mu \mapsto \mu(A),\ A \in \mathcal{C}$.

Assume now that T is an automorphism (not necessarily ergodic) of a standard probability Borel space $(X, \mathcal{B}, \mu)$. Let $P \in QF(T, \mu)$ be a *quasi-factor* of the system $(T, X, \mathcal{B}, \mu)$, i.e. $P \in M(M(X))$ is T-invariant for the natural induced action of T on measures, and its barycenter $\int_{M(X)} \nu \, dP(\nu)$ equals μ. Consider also $M_{\mathrm{sym}}(X^{\mathbb{Z}})$, the set of probability measures on $X^{\mathbb{Z}}$ which are invariant under the group $\mathcal{S}$ of all finite permutations. Let $\phi : M(M(X)) \to M_{\mathrm{sym}}(X^{\mathbb{Z}})$ be defined by the formula

$$\phi(P) = \overline{P}^{\infty} := \int_{M(X)} (\cdots \otimes \theta \otimes \theta \otimes \cdots) \, dP(\theta). \tag{22}$$

By the de Finetti–Hewitt–Savage theorem, ϕ is an affine isomorphism of these two convex sets (simplices). Now consider $J_{\mathrm{sym},\infty}(T, \mu)$, the set of infinite symmetric self-joinings of $(T, X, \mathcal{B}, \mu)$ (it is a closed subset of $M_{\mathrm{sym}}(X^{\mathbb{Z}})$ but, since μ is not necessarily ergodic, in general this set is not a simplex). We have

$$\phi(QF(T, \mu)) = J_{\mathrm{sym},\infty}(T, \mu). \tag{23}$$

In fact, if $P \in QF(T, \mu)$, then clearly $\phi(P) \in M_{\mathrm{sym}}(X^{\mathbb{Z}})$ and the fact that its one-dimensional marginals equal μ follows from the barycenter condition on P. In order to see that $\phi(P)$ is T^∞-invariant note that

$$\begin{aligned} \phi(P) \circ T^\infty &= \int_{M(X)} (\cdots \otimes \theta \otimes \theta \otimes \cdots) \, dP \circ T(\theta) \\ &= \int_{M(X)} (\cdots \otimes \theta \otimes \theta \otimes \cdots) \, dP(\theta) = \phi(P). \end{aligned}$$

Finally, if $\rho \in J_{\mathrm{sym},\infty}(T, \mu)$ then, by the de Finetti–Hewitt–Savage theorem, its $\mathcal{S}$-ergodic decomposition is of the form

$$\rho = \int_{M(X)} (\cdots \otimes \theta \otimes \theta \otimes \cdots) \, dP(\theta).$$

Clearly $P \in QF(T, \mu)$ and $\phi(P) = \rho$. Thus ϕ is onto and (23) is established. Note that this shows that (22) is in fact the $\mathcal{S}$-ergodic decomposition of $\overline{P}^{\infty}$.

PROPOSITION 4. *The quasi-factor $(T, M(X), P)$ is isomorphic (as a dynamical system) to the factor of $(T^\infty, X^{\mathbb{Z}}, \overline{P}^{\infty})$ given by the σ-algebra of symmetric sets (i.e. $\mathcal{S}$-invariant sets).*

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Proof. Consider the map $\Phi : (T^\infty, X^{\mathbb{Z}}, \overline{P}^\infty) \to (T, M(X), P)$ given in the following way. Take first the $\mathcal{S}$-ergodic decomposition of $\overline{P}^\infty$:

$$\overline{P}^\infty = \int_{M(X)} (\cdots \otimes \theta \otimes \theta \otimes \cdots)\, dP(\theta).$$

Now, to $\overline{x} \in X^{\mathbb{Z}}$ which belongs to exactly one ergodic component (the support of some $\cdots \otimes \theta \otimes \cdots$) we associate the measure $\theta = \theta_{\overline{x}}$. All we need to show is that this map establishes a homomorphism. (Since it is constant on atoms of the partition given by the $\mathcal{S}$-ergodic decomposition, the preimage of $\mathcal{B}(M(X))$ will be exactly the σ-algebra of symmetric sets.) Given a Borel subset $\Lambda \subset M(X)$ we have

$$(\cdots \otimes \theta \otimes \theta \otimes \cdots)(\Phi^{-1}(\Lambda)) = \begin{cases} 1 & \text{when } \theta = \theta_{\overline{x}} \text{ and } \overline{x} \in \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

and therefore

$$\overline{P}^\infty(\Phi^{-1}(\Lambda)) = P(\Lambda).$$

Finally, the map Φ is clearly equivariant. □

We will now discuss the ergodic case.

COROLLARY 2. *Assume that μ and $P \in QF(T, \mu)$ are ergodic. Then there exists $\rho \in J^e_\infty(T, \mu)$ such that the factor given by the σ-algebra of symmetric sets of the system $(T^\infty, X^{\mathbb{Z}}, \rho)$ is isomorphic to the dynamical system $(T, M(X), P)$.*

Proof. By our assumption, the σ-algebra of symmetric sets, which (as a factor of $(T^\infty, X^{\mathbb{Z}}, \overline{P}^\infty)$) is isomorphic to $(T, M(X), P)$, is ergodic. It follows that the same σ-algebra is a factor of almost every T^∞-ergodic component of $\overline{P}^\infty$. Since μ itself is ergodic, $J_\infty(T, \mu)$ is a simplex, so the ergodic components of $\overline{P}^\infty$ are self-joinings. Any such ergodic self-joining will serve as ρ. □

Remark 5. The family of dynamical systems given by ergodic quasi-factors of an ergodic automorphism T is hence contained in

$$\{(T^\infty, X/\mathcal{B}^{\otimes\infty}_{\text{sym}}, \mathcal{B}^{\otimes\infty}_{\text{sym}}, \lambda) : \lambda \in J^e_\infty(T, \mu)\}.$$

Let us come back to the topological lens $\widetilde{T}$ of an automorphism T. We will relate $\widetilde{T}$-invariant measures to quasi-factors of T. Notice that $C_2(\mu) \subset M(X \times X)$ is a closed (compact) subset in the weak-$*$ topology, hence each $T \times T$-invariant Borel measure on $C_2(\mu)$ is also a $T \times T$-invariant Borel measure on $M(X \times X)$. Let $P \in M(C_2(\mu))$ be $\widetilde{T}$-invariant. If we let $\lambda = \int_{M(X\times X)} \nu\, dP(\nu)$ be the barycenter of P, then $\lambda \in J_2(\mu)$ and $(C_2(\mu), P, \widetilde{T})$ is a quasi-factor of $(X \times X, \lambda, T \times T)$. Notice that even if P is ergodic, λ need not be ergodic. (As an easy example consider P being the Dirac measure at a non-ergodic self-joining of T.) However, as we have shown above, the arguments from [**7**] about the characterization of quasi-factors as factors of infinite symmetric self-joinings go through, and therefore the dynamical system $(C_2(\mu), P, \widetilde{T})$ is a factor of an infinite self-joining of the system $(X \times X, \lambda, T \times T)$ (moreover this infinite self-joining is symmetric, that is, it is an invariant measure for the group of finite permutations of $(X \times X)^{\mathbb{Z}}$). It follows directly that $(\widetilde{T}, P)$ is a factor of an infinite self-joining of (X, μ, T) and since $(C_2(\mu), P, \widetilde{T})$ is ergodic, it is also a factor of an infinite ergodic self-joining of (X, μ, T).

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COROLLARY 3. *Assume that $(T, X, \mathcal{B}, \mu)$ is ergodic and let $P \in M_{\widetilde{T}}(C_2(\mu))$ be ergodic. Let $\lambda \in J_2(T, \mu)$ denote the barycenter of P. Then $P \in QF(T \times T, \lambda)$ and there exists $\rho \in J^e_\infty(T, \mu)$ such that $(C_2(\mu), P, \widetilde{T})$ is isomorphic to the factor of $(X^{\mathbb{Z}}, \rho, T^\infty)$ given by the σ-algebra of sets invariant under finite permutations of consecutive pairs of coordinates. If additionally λ is ergodic, then $\rho \in J^e_\infty(T \times T, \lambda)$ (the identification is given by grouping every two consecutive coordinates).*

Proof. Apply Proposition 4 and notice that $\overline{P}^\infty$ can also be seen as an element of $J_\infty(T, \mu)$ and that μ is ergodic. □

5.1. *The entropy of $\widetilde{T}$.* Let us recall that the class of zero entropy systems is closed under taking joinings, since each marginal factor is included in the Pinsker factor of the system given by the joining. Clearly this class is also closed under taking factors. Therefore using Corollary 3 and the variational principle we obtain the following.

THEOREM 7. *If T has zero entropy, then the topological entropy of $\widetilde{T}$ is zero.*

We will now show that also the converse is true. In fact a stronger result holds.

THEOREM 8. *If $h(T) > 0$ then $h_{\text{top}}(\widetilde{T}) = \infty$.*

Proof. Step 1. Assume first that T acting on $(X, \mathcal{B}, \mu)$ is Bernoulli $(1/2, 1/2)$ with an independent generator (A, A^c), i.e. $\mu(A) = 1/2$ and the family $\{T^k A, T^k A^c : k \in \mathbb{Z}\}$ is independent. Consider the map $F : C_2(\mu) \to [0, 1/2]^{\mathbb{N}}$ given by

$$F(\lambda) = (\widetilde{T}^n(\lambda)(A \times A))_{n \geq 0} = (\lambda(T^{-n}A \times T^{-n}A))_{n \geq 0},$$

where on the space $[0, 1/2]^{\mathbb{N}}$ we consider the one-sided shift S: $S((x_i)_{i \geq 0}) = (y_j)_{j \geq 0}$, $y_j = x_{j+1}$. Clearly, F is an equivariant continuous map. We will show that F is onto. To this end let us first notice that all we need to show is that for each $n \geq 1$, each block $(b_0, b_1, \ldots, b_{n-1}) \in \{0, 1/2\}^n$, there exists $\lambda \in C_2(\mu)$ such that $F(\lambda)[0, n-1] = (b_0, \ldots, b_{n-1})$. In fact, $\{0, 1/2\}^n$ is the set of extremal points of the convex set $[0, 1/2]^n$, and once we have 'realizations' of the extremal points we use the fact that the map $\lambda \mapsto F(\lambda)[0, \ldots, n-1]$ is affine. The proof goes by induction on n. For a 'realization' of a block $B = (b_0)$ of length 1 we take λ as the graph coupling given by an isomorphism of the space $(X, \mathcal{B}, \mu)$, where the isomorphism is obtained from two measure-preserving maps (isomorphisms)

$$A_0 \to A_{2b_0+1}, \quad A_1 \to A_{2b_0},$$

where $A_0 = A$ and $A_1 = A^c$, and addition is modulo 2. Now take $n \geq 1$ and $B = (b_0, \ldots, b_{n-1})$ and suppose that the graph coupling given by an isomorphism of the space $(X, \mathcal{B}, \mu)$ obtained from 2^{n-1} measure-preserving maps (isomorphisms)

$$A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-n+2}A_{i_{n-2}} \to A_{j_0} \cap T^{-1}A_{j_1} \cap \cdots \cap T^{-n+2}A_{j_{n-2}}$$

'realizes' the block $(b_0, \ldots, b_{n-2})$. A coupling λ 'realizing' the block B is then obtained as the graph coupling given by an isomorphism of the space $(X, \mathcal{B}, \mu)$ obtained from 2^n measure-preserving maps (isomorphisms)

$$\begin{aligned} &A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-n+2}A_{i_{n-2}} \cap T^{-n+1}A_{i_{n-1}} \\ &\quad \to A_{j_0} \cap T^{-1}A_{j_1} \cap \cdots \cap T^{-n+2}A_{j_{n-2}} \cap T^{-n+1}A_{i_{n-1}+b_{n-1}+1}. \end{aligned}$$

It follows that the shift S is a topological factor of $\widetilde{T}$, so $h(\widetilde{T}) = +\infty$.

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Step 2. Suppose $h(T) > 0$. By replacing T with T^m if necessary, we can assume that the entropy of T is larger than log 2. Then, by Sinai's theorem, T has a factor T_1 which is Bernoulli $(1/2, 1/2)$. Now $h(\widetilde{T}_1) = +\infty$, so $h(\widetilde{T})$ is also infinite. □

5.2. *Invariant measures for $\tilde{T}$ supported on* $\mathrm{Aut}(\mu)$. We start with the following general setup.

Example. Let Y be a compact abelian second countable topological group with normalized Haar measure λ. Let $T : Y \to Y$ be a continuous automorphism. We recall that the measure λ is preserved by T; i.e. $T \in \mathrm{Aut}(Y, \lambda)$.

With each $z \in Y$ we associate the λ-preserving invertible transformation $R_z : Y \to Y$ defined by $R_z(y) = y + z$. We then have $T \circ R_z \circ T^{-1}(y) = T \circ R_z(T^{-1}y) = T(T^{-1}y + z) = y + Tz = R_{Tz}(y)$ for all $y, z \in Y$, so that

$$T \circ R_z \circ T^{-1} = R_{Tz}.$$

Thus the map $\phi : z \mapsto R_z$ is a topological isomorphism of the compact topological dynamical system (Y, T) into the (Polish) dynamical system $(\mathrm{Aut}(\lambda), \tilde{T})$, where $\tilde{T}$ denotes conjugation by $T \in \mathrm{Aut}(\lambda)$.

Next consider the simplex $M_T(Y)$ of T-invariant Borel probability measures on Y. For an element $\nu \in M_T(Y)$ let $P_\nu = \phi_*(\nu)$ be the push-forward image of ν in the space of probability measures on $\mathrm{Aut}(\lambda)$ under ϕ. It then follows that ϕ is an isomorphism of measure dynamical systems

$$\phi : (Y, \nu, T) \to (\mathrm{Aut}(\lambda), P_\nu, \tilde{T}).$$

Identifying $\mathrm{Aut}(\lambda)$ with its image in $C_2(\mu)$ under the canonical embedding, we obtain the measure-theoretical isomorphism

$$\phi : (Y, \nu, T) \to (C_2(\mu), P_\nu, \tilde{T}).$$

Now a particular instance of the above example will establish the following surprising result. Let $Y = \mathbb{T}^{\mathbb{Z}}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group. Let T be the shift transformation on Y and let λ be the product measure $\lambda = \lambda_0^{\mathbb{Z}}$, where λ_0 is normalized Lebesgue measure on $\mathbb{T}$. We consider Y also as a compact topological group and observe that: (i) λ is the normalized Haar measure on Y; and (ii) T is a continuous automorphism of the compact group Y.

THEOREM 9. *The topological lens $(C_2(\lambda), \tilde{T})$ is universal both topologically and measure theoretically.*

(1) *Every metric compact topologically transitive system appears as a subsystem of the G_δ dense $\tilde{T}$-invariant subset* $\mathrm{Aut}(\lambda) \subset C_2(\lambda)$.

(2) *Let $(\Omega, \mathcal{A}, \nu, S)$ be an ergodic system. There exists a $\tilde{T}$-invariant ergodic probability measure P_ν on* $\mathrm{Aut}(\lambda)$ *such that the corresponding dynamical system $(C_2(\mu), P_\nu, \tilde{T})$ is measure-theoretically isomorphic to $(\Omega, \mathcal{A}, \nu, S)$.*

Proof. Both claims follow immediately from the example above applied to the group $Y = \mathbb{T}^{\mathbb{Z}}$, and the fact that the topological Bernoulli system (Y, T) is universal both for compact metrizable topologically transitive systems as well as for ergodic measure-preserving systems. □

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Remark 6. In [**3**] Danilenko introduced the notion of near simplicity. An ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ is *twofold nearly simple* if each twofold ergodic self-joining λ of μ is either the product measure $\mu \times \mu$ or it is an integral of graph couplings:

$$\lambda = \int_{\mathrm{Aut}(X,\mathcal{B},\mu)} \mu_S \, dP(S),$$

where P is a $\widetilde{T}$-invariant measure. For all the examples of twofold near-simple maps given in [**3**], the measure P is supported on a finite set. Danilenko shows that, at least when P is supported on a finite set, the system $(X \times X, \lambda, T \times T)$ is isomorphic to the product of the original system $(X, \mathcal{B}, \mu, T)$ and the system $(\mathrm{Aut}(X, \mathcal{B}, \mu), P, \widetilde{T})$. It will be interesting to find near simple systems where P is a continuous measure.

5.3. *$\widetilde{T}$-invariant measures when T is measure-theoretically distal.* The computation of the simplex $M_{\widetilde{T}}(C_2(\mu))$ for a general transformation T seems to be a difficult task. If $(X, \mathcal{B}, \mu, T)$ is distal, then by Corollary 3 and the fact that joinings of distal systems are distal (see e.g. [**7**, Theorem 10.19]), $(C_2(\mu), P, \widetilde{T})$ is (measure-theoretically) distal for each ergodic $P \in M_{\widetilde{T}}(C_2(\mu))$. In this section we will consider mainly the particularly simple situation of an isometric extension of a Kronecker system.

Suppose first that T is a Kronecker system, that is, it is ergodic and has a discrete spectrum (and therefore is isomorphic to an ergodic translation on a compact metric monothetic group). Assume moreover that $P \in M_{\widetilde{T}}(C_2(\mu))$ is ergodic for $\widetilde{T}$. Then there exists $J = J_\rho \in \mathcal{J}(\mu)$ such that P is supported on the set

$$Y_J := \{U_S \circ J \circ U_S^{-1} : S \in C(T)\}$$

(see the beginning of §3.2). Consider the map $\kappa : C(T) \to Y_J$ defined by

$$\kappa : S \mapsto U_S \circ J \circ U_S^{-1} \tag{24}$$

(notice that it is one-to-one unless for some $\mathrm{Id} \neq R \in C(T)$ the measure ρ is $R \times R$-invariant; all measures which are $R \times R$-invariant are easy to describe). If on $C(T)$ we consider the action by the translation of T then the system we obtain is isomorphic to the original system (X, T), in particular it is uniquely ergodic and has a discrete spectrum. The system $(Y_J, \widetilde{T})$ is a topological factor of T and hence $(C_2(\mu), P, \widetilde{T})$ has also a discrete spectrum. In this way we fully described ergodic invariant measures for Kronecker systems.

Let now T be an arbitrary ergodic automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. Let G be a compact metric abelian group and let $\varphi : X \to G$ be a measurable map (a cocycle). Define $T_\varphi : X \times G \to X \times G$ by the formula

$$T_\varphi(x, g) = (Tx, g + \varphi(x))$$

and observe that T_φ preserves the product measure $\mu \otimes \lambda_G$, where λ_G is the normalized Haar measure on G. We will assume that T_φ is ergodic and has the same eigenfunctions as T. (The latter assumption does not restrict the generality of Proposition 5 below since, by enlarging the σ-algebra $\mathcal{B}$ so that the additional eigenfunctions become $\mathcal{B}$-measurable, we can achieve this situation.) Under these assumptions there exists a subsequence (n_i) of density 1 such that $U_{T_\varphi}^{n_i} \to 0$ weakly on the space

$$L^2(X \times G, \mu \otimes \lambda_G) \ominus (L^2(X, \mathcal{B}, \mu) \otimes 1_G).$$

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Given $\chi \in \widehat{G}$ ($\widehat{G}$ stands for the dual of G), we put

$$V_{T,\varphi,\chi}(f) = \chi(\varphi) \cdot f \circ T$$

for each $f \in L^2(X, \mathcal{B}, \mu)$. Then for each $\chi \neq 1$

$$V^{n_i}_{T,\varphi,\chi} \to 0 \quad \text{weakly on } L^2(X, \mathcal{B}, \mu)$$

or equivalently

$$\int_X \chi(\varphi^{(n_i)}) \cdot f \circ T^{n_i} \cdot g \, d\mu \to 0 \tag{25}$$

for each $f, g \in L^2(X, \mathcal{B}, \mu)$.

Suppose now that $\xi \in C_2(\mu \otimes \lambda_G)$ satisfies the condition $\xi|_{X\times X} = \Delta_X$. Identifying the two copies of X we can assume that $\xi \in M(X \times G \times G)$ and the projections of ξ on the two 'copies' of $X \times G$ equal $\mu \otimes \lambda_G$. Instead of $T_\varphi \times T_\varphi$ we must consider $T_{\varphi\times\varphi}$. The disintegration of ξ over μ has the form $\xi = \int_X \delta_x \otimes \xi^x \, d\mu(x)$, where ξ^x are probability measures on $G \times G$. Since for each $A \in \mathcal{B}$, $B \in \mathcal{B}(G)$

$$\int_A \lambda_G(B) \, d\mu(x) = \mu(A)\lambda_G(B) = \xi(A \times B \times G) = \int_A \xi^x(B \times G) \, \mu(x),$$

we have $\lambda_G(B) = \xi^x(B \times G)$. Similarly $\lambda_G(B) = \xi^x(G \times B)$, whence $\xi^x \in C_2(\lambda_G)$ for μ-almost every $x \in X$.

For $\chi, \eta \in \widehat{G}$ we have

$$\begin{aligned}
&\int_{X\times G\times G} f(x)\chi(g)\eta(h) \, d(T^{n_i}_{\varphi\times\varphi})_*(\xi)(x, g, h) \\
&\quad = \int_{X\times G\times G} f(T^{n_i}x)\chi(g + \varphi^{(n_i)}(x))\eta(h + \varphi^{(n_i)}(x)) \, d\xi(x, g, h) \\
&\quad = \int_X (\chi \cdot \eta)(\varphi^{(n_i)}(x)) f(T^{n_i}x) \left(\int_{\{x\}\times G\times G} \chi(g)\eta(h) \, d\xi^x(g, h) \right) d\mu(x).
\end{aligned}$$

If $\chi \neq \overline{\eta}$ then, by (25), the limit of the latter expression, when $i \to \infty$, is zero. If $\chi = \overline{\eta}$, assuming additionally that $T^{n_i} \to S \in C(T)$ and that $(T^{n_i}_{\varphi\times\varphi})_*(\xi) \to \widetilde{\xi}$, we obtain as the limit

$$\int_{X\times G\times G} f(x)\chi(g)\eta(h) \, d\widetilde{\xi}(x, g, h) = \int_{X\times G\times G} f(Sx)\chi(g)\overline{\chi(h)} \, d\xi(x, g, h).$$

Thus the coupling $\widetilde{\xi}$ has the form $\widetilde{\xi} = \int_X \delta_{Sx} \otimes \widetilde{\xi}^x \, d\mu(x)$, where for μ-almost every $x \in X$ the measure $\widetilde{\xi}^x$ on $G \times G$ has the following Fourier coefficients:

$$\widehat{\widetilde{\xi}^x}(\chi, \eta) = \begin{cases} 0 & \text{if } \chi \neq \overline{\eta}, \\ \displaystyle\int_{G\times G} \chi(g - h) \, d\xi^{S^{-1}x}(g, h) & \text{if } \chi = \overline{\eta}. \end{cases} \tag{26}$$

In order to gain a better picture of the limit measure and its dynamics we perform the following change of coordinates. Let

$$m : G \times G \to G, \quad m(g, h) = g - h$$

and

$$\overline{m} : X \times G \times G \to X \times G \times G, \quad \overline{m}(x, g, h) = (x, g - h, h).$$

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Set $\nu^x := m_*(\widetilde{\xi}^x)$ and $\nu := \overline{m}_*(\widetilde{\xi})$. We then have

$$\nu = \int_X \delta_{Sx} \otimes \nu^x \otimes \lambda_G \, d\mu(x). \tag{27}$$

In fact, denoting the measure on the right-hand side of (27) by ξ_1, we have

$$\int_{X\times G\times G} f(x)\chi(g)\eta(h)\, d\xi_1(x, g, h)$$
$$= \int_X f(Sx)\left(\int_{\{x\}\times G\times G} \chi(g)\eta(h)\, d\nu^x(g)\, d\lambda_G(h)\right) d\mu(x) = 0$$

whenever $\eta \neq 1$ (indeed, $\chi \neq \overline{(\overline{\chi}\cdot\eta)}$ if and only if $\eta \neq 1$). For $\eta = 1$ we obtain

$$\int_{X\times G\times G} f(x)\chi(g)\eta(h)\, d\xi_1(x, g, h)$$
$$= \int_X f(Sx)\left(\int_{\{x\}\times G} \chi(g)\, d\nu^x(g)\right) d\mu(x)$$
$$= \int_X f(Sx)\left(\int_{\{x\}\times G\times G} \chi(g-h)\, d\tilde{\xi}^x(g, h)\right) d\mu(x)$$

and we see that indeed $\overline{m}_*(\widetilde{\xi}) = \xi_1$. We also note that

$$\overline{m}\circ T_{\varphi\times\varphi} = T_{0\times\varphi}\circ\overline{m},$$

where $T_0 = T \times \mathrm{Id}_G$.

From now on we assume that T is an ergodic rotation (so that $C(T)$ is a compact group in the strong operator topology). We have proved that in this case, under the change of coordinates, the limit points along subsequences of (n_i) are measures of the form

$$\int_X \delta_{Sx} \otimes \nu^x \otimes \lambda_G \, d\mu(x) = \theta \otimes \lambda_G,$$

where $\theta \in M_\mu(X \times G)$ and the latter is the subset of $M(X \times G)$ consisting of measures whose projection on X is μ. In the new coordinate system the automorphism $T_{\varphi\times\varphi}$ is transformed to $T_{0\times\varphi}$. Now from the fact that (n_i) has density 1 (see Remark 1) and the fact that the action of $T_{0\times\varphi}$ preserves the set $M_\mu(X \times G) \times \{\lambda_G\}$, it follows that each invariant measure P for $\widetilde{T}_{0\times\varphi}$ is concentrated on $M_\mu(X \times G) \times \{\lambda_G\}$. Thus we only need to consider the set of ergodic $\widetilde{(T \times \mathrm{Id}_G)}$-invariant measures on $M_\mu(X \times G)$. Since the action of $T \times \mathrm{Id}_G$ on $X \times G$ is equicontinuous, we conclude that the system $(C_2(\mu \otimes \lambda_G), P, \widetilde{T}_\varphi)$ has a discrete spectrum.

Given $R \in C(T)$, write

$$Z_R = \{\rho \in C_2(\mu \otimes \lambda_G) : \rho|_{X\times X} = \mu_R\}.$$

Notice that Z_R is closed and $\widetilde{T}_\varphi$-invariant. By the above we have proved the following result.

PROPOSITION 5. *If T has a discrete spectrum and $\varphi : X \to G$ is a cocycle taking values in a compact metric abelian group G such that T_φ is ergodic then for each ergodic $\widetilde{T}_\varphi$-invariant measure P concentrated on Z_{Id} the resulting system $(Z_{\mathrm{Id}}, P, \widetilde{T}_\varphi)$ has a discrete spectrum.*

Remark 7. Let us note that the difficulty in the proof of Proposition 5 arises from the fact that the cocycle $\varphi \times \varphi$ is not ergodic. Indeed, we will shortly argue that whenever T has

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a discrete spectrum and $\varphi : X \to G$ is ergodic (i.e. T_φ is ergodic) then the action of T_φ on $M_\mu(X \times G)$ has only one invariant measure and, moreover, this measure is the Dirac measure supported on $\mu \otimes \lambda_G$.

First notice that by the Stone–Weierstrass theorem the set of continuous functions of the form $f \otimes \chi$, where $f \in C(X)$ and $\chi \in \widehat{G}$, is linearly dense in $C(X \times G)$ and therefore to check weak convergence of a sequence of measures in $M_\mu(X \times G)$ we only need to test this on functions of the form $f \otimes \chi$. Assume that (n_i) has density 1 and that $U_{T_\varphi}^{n_i}$ converges (weakly) to zero in the orthocomplement of $L^2(X, \mu) \otimes 1_G$. Take $\xi \in M_\mu(X \times G)$. Suppose that along a subsequence of (n_i), which we still denote by (n_i), we have $(T_\varphi^{n_i})_*(\xi) \to \widetilde{\xi}$. Then a calculation as above shows that for $\chi \neq 1$

$$\int_{X \times G} f(x)\chi(g)\, d\widetilde{\xi}(x, g) = 0,$$

and we conclude that $\widetilde{\xi} = \mu \otimes \lambda_G$.

Suppose now that the assumptions of Proposition 5 are fulfilled and assume moreover that $R \in C(T)$ is such that the cocycle

$$\varphi \times \varphi \circ R : X \to G \times G \quad \text{is ergodic.} \tag{28}$$

We consider $\widetilde{T}_\varphi$ on Z_R. Identifying the two copies of X we see that the action of $\widetilde{T}_\varphi$ on Z_R can be identified with a subsystem of the action of $T_{\varphi\times\varphi\circ R}$ on $M_\mu(X \times G \times G)$. In view of Remark 7, which by (28) we may apply to $\varphi \times \varphi \circ R$, we obtain the following.

PROPOSITION 6. *Assume that T has a discrete spectrum and that $\varphi : X \to G$ is a cocycle taking values in a compact metric abelian group G such that T_φ is ergodic. Then for each $R \in C(T)$ such that $\varphi \times \varphi \circ R$ is ergodic the system $(Z_R, \widetilde{T}_\varphi)$ has a quasi-attracting point. In particular it has only one invariant measure (which is the Dirac measure concentrated at the unique self-joining of T_φ projecting on μ_R).*

Notice however that the two propositions above do not describe all the ergodic invariant measures for $\widetilde{T}_\varphi$ in the case of T having a discrete spectrum. In particular we do not know whether an ergodic measure P (for $\widetilde{T}_\varphi$) which projects on the natural Haar measure of Y_J, where the map κ (see (24)) is one-to-one, yields also a system with a discrete spectrum.

We will show next that going up one further step in the distal echelon, namely taking a further isometric extension, one can already obtain a transformation $\overline{T}$ preserving a measure $\overline{\mu}$ for which there exists an ergodic probability measure $P \in M_{\widetilde{\overline{T}}}(C_2(\overline{\mu}))$ such that the measure distal system $(C_2(\overline{\mu}), P, \widetilde{\overline{T}})$ has a mixed spectrum.

Let $Tx = x + \alpha$ be an irrational rotation on $\mathbb{T}$, let $\varphi : \mathbb{T} \to \mathbb{T}$ be given by $\varphi(x) = x$ and let $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $T_\varphi(x, y) = (x + \alpha, x + y)$. Let $Rx = x + \beta$, where α, β and 1 are independent over the rational numbers. Then the cocycle $\varphi \times \varphi \circ R$ is ergodic and the skew product $T_{\varphi\times\varphi\circ R}$ has a partly continuous spectrum (notice however that its discrete part is bigger than the one given by the rotation by α). Now Proposition 6 applies and we know that there exists only one (trivial) $\widetilde{T}_\varphi$-invariant measure on Z_R.

Let now $\overline{T}$ (an extension of T_φ) on $\mathbb{T}^3$ be given by

$$\overline{T}(x, y, z) = (x + \alpha, x + y, x + y + z),$$

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and set $\overline{\mu} = \lambda_{\mathbb{T}} \otimes \lambda_{\mathbb{T}} \otimes \lambda_{\mathbb{T}}$. Denote by $S_{a,b,c} \in \mathrm{Aut}(\overline{\mu})$ the rotation by $(a, b, c) \in \mathbb{T}^3$. We have

$$\overline{T} \circ S_{a,b,c} \circ \overline{T}^{-1} = S_{a,a+b,a+b+c}. \tag{29}$$

In other words the action of $\widetilde{\overline{T}}$ on the set of graph measures given by $S_{a,b,c}$, $(a, b, c) \in \mathbb{T}^3$, is isomorphic to the action W on $\mathbb{T}^3$ given by

$$W(a, b, c) = (a, a + b, a + b + c).$$

Now, the homeomorphism W has many invariant tori $\mathbb{T}_a$ $(a \in \mathbb{T})$, where $\mathbb{T}_a = \{(a, b, c) : b, c \in \mathbb{T}\}$. Identifying $\mathbb{T}_a$ with $\mathbb{T}^2$, the action of $W = W_a$ on $\mathbb{T}_a$ becomes $W_a(b, c) = (b + a, b + c + a)$. If now $a = \beta$ is irrational then W_β is isomorphic to the T_φ we started with (with β taking the place of α; the constant cocycle β is a coboundary). Thus we have proved the following.

PROPOSITION 7. *For the transformation* $\overline{T}(x, y, z) = (x + \alpha, x + y, x + y + z)$ *as above there are* $\widetilde{\overline{T}}$*-invariant measures* P *such that the system* $(C_2(\lambda_{\mathbb{T}}^{\otimes 3}), P, \widetilde{\overline{T}})$ *is ergodic and has a partly continuous spectrum. In fact, for each* $\beta \in \mathbb{T}$ *there exists* P_β *such that the system* $(C_2(\lambda_{\mathbb{T}}^{\otimes 3}), P_\beta, \widetilde{\overline{T}})$ *is isomorphic to the affine transformation* $(x, y) \mapsto (x + \beta, x + y)$ *of* $\mathbb{T}^2$.

We conclude with the following intriguing problem.

Problem 1. Construct an ergodic system of zero entropy for which the topological lens is a P-system. In view of Theorem 4 this amounts to finding a weakly mixing system $(X, \mathcal{B}, \mu, T)$ with zero entropy such that the periodic points of the topological lens are dense.

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