Matter for Debate

What makes a good Proof without Words?

GERRY LEVERSHA

A man is watching his son Robin play with a peasant boy, slightly older than he is but without much formal education, in the grounds of a Tuscan villa he has rented. The latter, Guido, has used a stick to draw a couple of diagrams in the sand.

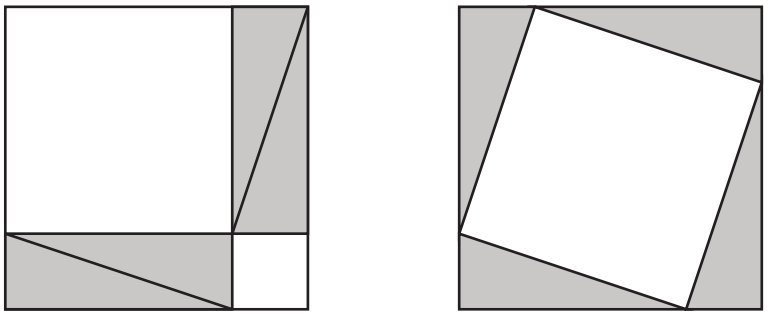


FIGURE 1

The father asks Guido what he is doing. Rather anxiously, as if there might something wrong in sketching like this, the boy explains.

> "You see," he said, "it seemed to me so beautiful. Because those squares" – he pointed at the two small squares in the first figure – "are just as big as this one." And, indicating the square on the hypotenuse in the second diagram, he looked up with a deprecating smile. I nodded. "Yes, it's very beautiful," I said – "it's very beautiful indeed." An expression of delighted relief appeared on his face; he laughed with pleasure. "You see, it's like this," he went on, eager to initiate me into the glorious secret he had discovered. "You cut these two long squares" – he meant the rectangles – "into two slices. And then there are four slices, all just the same, because, because – oh, I ought to have said that before – because these long squares are the same, because those lines, you see ...". "But I want a train," protested Robin.

This is an extract from *Young Archimedes*, a short story written in 1943 by Aldous Huxley (with no diagram). The father recognises that Guido has discovered for himself a proof of the theorem attributed to Pythagoras which the Greek might have used himself. A variation on this method, using pieces of slate, is illustrated by Jacob Bronowski on the island of Samos in Episode 5 of his TV series The *Ascent of Man*.

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This is an example of a *Proof without Words* (often abbreviated to PWW). Martin Gardner, writing in the *Scientific American* in 1973, used the term 'look-see diagram' for this, describing it economically as 'a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance'. It is not unusual for a mathematical argument to be accompanied by explanatory diagrams, but in this discussion I am focusing on PWWs where the visual argument is sufficient on its own.

A search online reveals that the *Gazette* has published about fifty such submissions, the earliest dating from about 1997. However, they were appearing in US journals some twenty years earlier than this, and in 1993 Roger Nelsen wrote a justly celebrated book [1], which gathered together some of his favourite examples of the genre. Since then there have been two more volumes [2, 3], the latest of which was reviewed in the pages of the *Gazette* [4]. Nelsen is eager to point out that such renditions have a very long history, going back to ancient China, classical Greece and 12th century India, like the example above.

What is a Proof without Words?

So what makes something a PWW? What sort of standards should be applied in deciding whether or not to publish a new example?

There are some obvious criteria for such a decision, such as:

- it should be original;
- it should *illuminate* the result so that it is not just a technical exercise;
- it should illustrate the result pictorially;
- it should not require any words or equations.

Let us return to Guido's drawing. He can see clearly that he has proved that two areas are equal, simply by observing that both of them are obtained by removing four triangles from a square. There is no need to label these or force the point. More interestingly, he realises that Pythagoras' theorem is not about an equation of the type $a^2 + b^2 = c^2$ but about areas. In fact, he is *re-presenting* the areas in question so that they can easily be manipulated. I think that this technique of representation is crucial to producing a PWW proof.

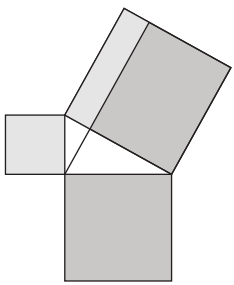


FIGURE 2

The same interpretation of Pythagoras as a result about areas is crucial to the canonical geometrical proof of Pythagoras, the so-called *Bride's Chair*.

The proof consists of showing that the two light rectangles and the two dark rectangles are equal in area (in pairs), and this implies the Pythagorean theorem.

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Actually this is more insightful than Guido's proof, since it shows precisely how to partition the large square into the two smaller ones. But is it a PWW? I think it would be very difficult to maintain that it is, since there are a number of steps in showing that the areas in question are equal. You need to identify some congruent triangles and use the fact that triangles on the same base and of the same height are equal in area. Both these steps are elementary, but that does not mean you can assume that they are obvious. Once you start 'adding reasons' (to use that phrase so beloved of GCSE examiners) this reverts to being a normal geometrical demonstration.

The same is true *a fortiori* of some other familiar proofs of Pythagoras, including the one which splits the triangle in two using an altitude from the right-angle, identifies similar triangles and uses a little algebraic manipulation. Now you could of course replace the similarity steps and the algebra by producing supplementary diagrams showing enlargements.

This is done skilfully in [2, p. 7], and so here the quality of the representation merits its validity as a PWW. I am, however, sceptical about the amount of algebra in [1, p. 7], which is included for historical reasons, as it is ascribed to a President of the USA.

Geometry is, after all, about figures, so there is *always* a visual element to proof, but simply omitting the details from a proof does not make it into a PWW. Consider, for example, the two diagrams in Figure 3.

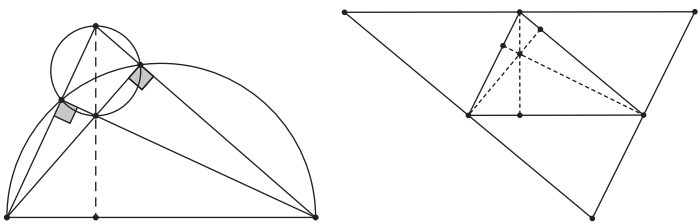


FIGURE 3

They both incorporate a proof that the altitudes of a triangle are concurrent, but I am sceptical that either can be viewed as a PWW. This brings back the matter of representation—in order to qualify, the result must be exhibited in a way which allows the purely visual element to act as a proof.

How much text is acceptable?

How rigidly should we apply the principle that PWW is really 'without words'? Nelsen seems quite strict about this and he takes care over the labelling of diagrams so that they do contain all the information needed to understand how they work.

A common use for a PWW is to explain an algebraic process, such as completing the square. Figure 4 is an example of this from [1, p. 19]. There is no need for any text apart from the title: $x^2 + ax = \left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$.

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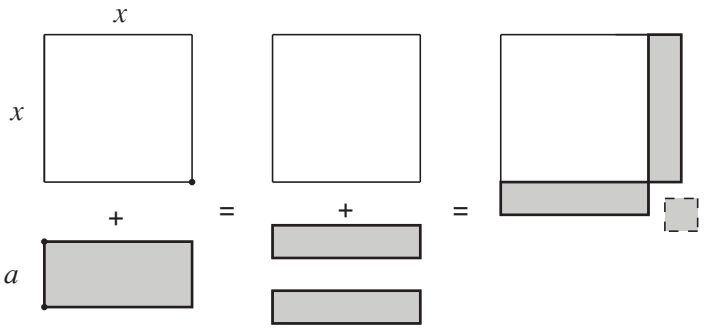
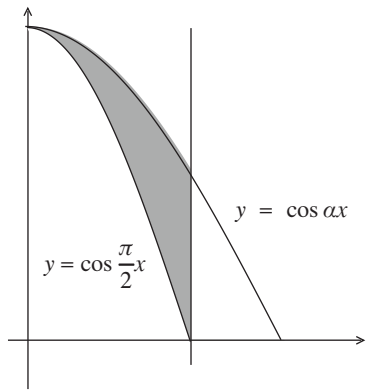


FIGURE 4

There are clearly steps here to be understood, but any reasonably focussed reader will be able to see why the parts of the diagram on the left and right represent the expressions on the left and right side of the equation. Note that shading helps to emphasise parts of the Figure which are moved around, and the little square with dashed perimeter explains the subtraction of the final term in the equation.

I do not think the same could be said about this proposal for a PWW.

If $0 < \alpha < \frac{\pi}{2}$, then $\sin\alpha \geqslant \frac{2\alpha}{\pi}$.



Hence $\int_0^1 \left(\cos\alpha x - \cos\frac{\pi x}{2}\right) dx \geqslant 0$ and so $\sin\alpha \geqslant \frac{2\alpha}{\pi}$.

FIGURE 5

The calculus at the end of this proposal is correct (although with the stated range the inequality should be strict), but it involves a trigonometrical integration, which surely invalidates its status as a proof without words, Indeed, Figure 5 is entirely redundant, since it is obvious from the nature of the cosine function that the integrand is positive.

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However, Figure 6, which is much simpler, is a perfectly adequate PWW of the same result.

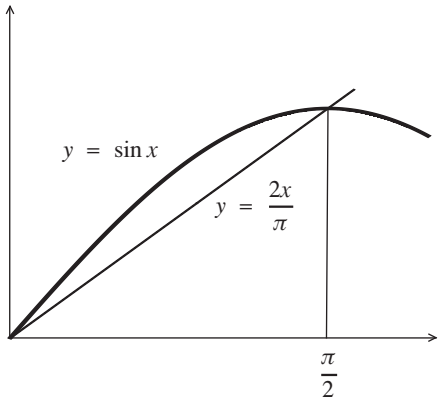


FIGURE 6

This involves nothing but the fact that the sine function is convex in the chosen range.

How do we deal with generality?

Many results which are appropriate for visual proofs are true for any positive integer. A PWW, of course, can represent only *particular* positive integers. It is important, then, that the reader can see how the diagram will work in any case: in other words, that the representation is generic.

A well-known visual argument which (I hope) most teachers use in their classes demonstrates the formula for triangular numbers: $T_n = \frac{1}{2}n(n + 1)$.

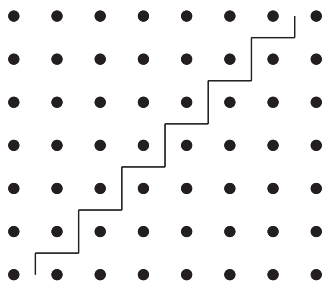


FIGURE 7

Figure 7 shows the case $n = 7$ but the idea is readily generalisable to any positive integer.

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One identity which cries out for a visual proof is $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$. It is straightforward to evaluate the two sides algebraically, but I have always felt that such an argument does not really explain what is going on. There are plenty of excellent demonstrations in the Nelsen books, some of which involve three-dimensional representations. Here is a particularly ingenious argument from the *Gazette* [5]. You get two proofs for the price of one.

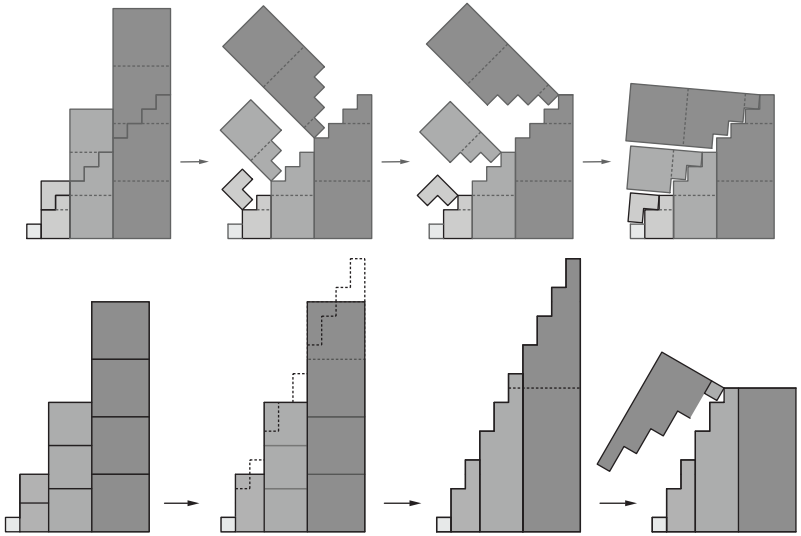


FIGURE 8

The sequence of diagrams in Figure 8 is a model of exposition and there should be no doubt in the reader's mind that the method is generic.

How much detail is needed?

On the whole, a good PWW should be self-contained. However, there are cases where some knowledge of the subject is assumed.

For example, the use of Ferrers diagrams in the theory of partitions is common, and in the argument below it is assumed that the reader is familiar with this representation. This argument, which shows that the number of partitions of n into k parts is equal to the number of partitions of n with greatest part k, comes from [6, p. 55].

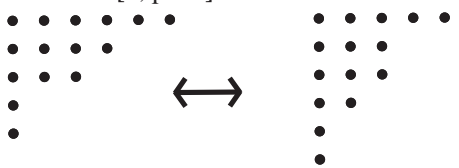


FIGURE 9

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Both diagrams in Figure 8 show partitions of 15. The partition on the left is 6 4 3 1 1 and it has 5 parts. That on the right is 5 3 3 2 1 1 and the greatest part is 5. The double-headed arrow indicates that there is a one-to-one correspondence between such partitions. This may take a little time to spot, but what has happened is that rows and columns have been swapped. It should be clear that this is, indeed, a bijection, and the result has been proved.

Are there hidden assumptions?

Consider the following PWW argument.

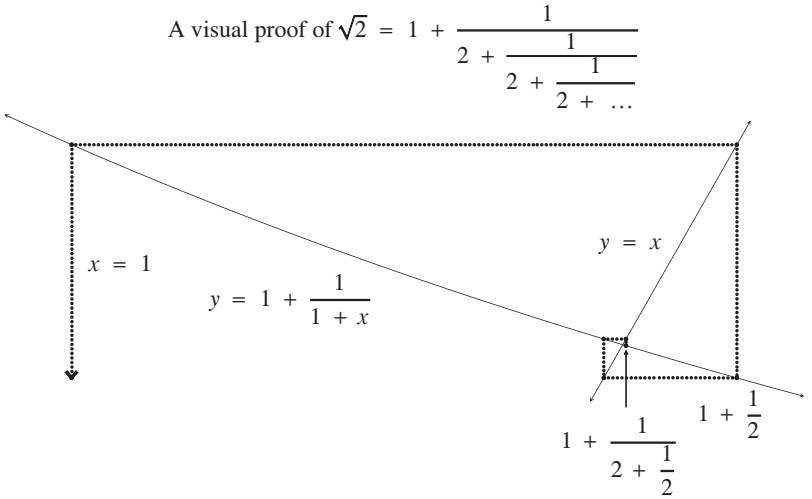


FIGURE 10

This was submitted to me with a hand-drawn diagram but no explanation of what was going on. Now the continued fraction for $\sqrt{2}$ is correct and there is an elementary proof that it takes this form (if the process converges). Figure 10 shows the cobweb diagram associated with the recursion $x_{n+1} = 1 + \dfrac{1}{1 + x_n}$ with $x_0 = 1$. The two labelled points are x_1 and x_2. The limit of this process is claimed to be $\sqrt{2}$.

This is an intriguing idea, but how much is left for the reader to supply? It seems to me that there are three vital steps which are not properly addressed. They are

- why is $\sqrt{2}$ the limit of the process?
- why does the sequence converge?
- why is the sequence of points in the diagram the sequence of convergents of the continued fraction?

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None of these steps is very hard. The first is simply a matter of finding the x coordinate of the intersection of $y = 1 + \dfrac{1}{1+x}$. The second is because, in the range of values considered, the absolute value of the gradient of $y = 1 + \dfrac{1}{1+x}$ is less than 1. It could be argued that this is pretty obvious from the diagram.

The third point needs a little more work. As we move horizontally, from curve to line, the coordinates change from $\left(a, 1 + \dfrac{1}{1+a}\right)$ to $\left(a + \dfrac{1}{1+a}, 1 + \dfrac{1}{1+a}\right)$, and as we move vertically, from line to curve, this becomes $\left(a + \dfrac{1}{1+a}, 1 + \dfrac{1}{2 + \dfrac{1}{1+a}}\right)$. Hence in a cycle, the x-coordinate changes from a to $1 + \dfrac{1}{2 + \dfrac{1}{1+a}}$, and so, as we began with $a = 1$, we see that the continued fraction is developed.

My view was that there is far too much here for the reader to supply, and so, despite the idea being interesting, I rejected it on the basis that it did not qualify as a PWW.

I think this is an extreme case of something which is all too common with PWW arguments which depend on graphs. Figure 11 is an example.

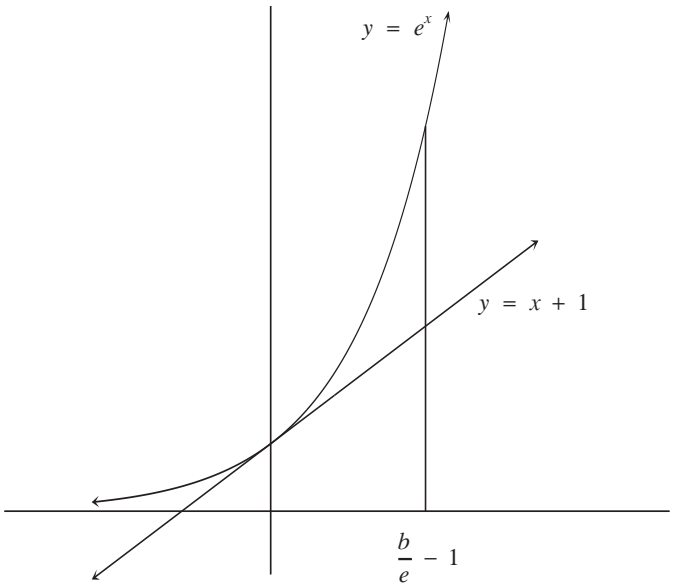


FIGURE 11

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The aim of this is to show that, if $e < b$ then $b^e < e^b$, with the well-known corollary that $\pi^e < e^\pi$.

The difficulty here is that the diagram does not really add anything to the argument. It assumes that the line $y = x + 1$ is tangent to the curve $y = e^x$ at the point $(0, 1)$, which is readily provable by knowing how to differentiate the exponential function, and then assumes that the reader can finish the argument; we have $\frac{b}{e} < \exp\left(\frac{b}{e} - 1\right)$ and the result follows. Moreover, we do not need the condition $e < b$, since the conclusion is true so long as $b \neq e$. There is an argument in [1, p. 59] which shows that if $e \leqslant a < b$ then $b^a < a^b$ using the convexity of the log function, but this also needs more text than is advisable. It is not surprising that several *Gazette* articles which discussed this and related problems [7, 8] did not even include graphs. The problem with arguments which rely on the shape of graphs is that they sometimes simply *restate* the problem rather than solve it. There is an element of 'Trust me and my graph-plotter' here.

PWW as part of a proof

The focus of this discussion has been stand-alone visual proofs, but it is also possible, and indeed desirable, to use this approach to focus on part of a much longer argument. An example which springs to mind concerns Fermat's two square theorem: an odd prime p is a sum of two squares if, and only if, $p \equiv 1 \pmod 4$. The 'only if' part of this result is trivial, so we need to show that if $p \equiv 1 \pmod 4$ then there are positive integers x, y such that $p = x^2 + y^2$. As p is odd, x and y are of different parity, so we can write this as $p = x^2 + 4z^2$.

The original proof of this was quite involved, but in 1971 a much more elementary argument was discovered by Roger Heath-Brown (see [9, pp.17-22]). The key is to consider the related equation $p = x^2 + 4yz$ and to show that this has an *odd number* of solutions. It is clear that y and z can be interchanged in the equation for any x, so this implies that there is a further solution with $y = z$, and this is what we need to finish the proof of Fermat's theorem. This idea was written up in a 'one-sentence' version by Don Zagier in 1984, but it uses the language of linear involutions and is rather opaque. This is where the visual representation comes in.

Consider the case $p = 29$. The triples (x, y, z) are now easily listed as follows

$$(1, 1, 7), (1, 7, 1), (3, 5, 1), (3, 1, 5), (5, 1, 1)$$

with the final triple showing that $29 = 5^2 + 4 \times 1 \times 1 = 5^2 + 2^2$. We now represent the x^2 part of each triple as a square and the $4yz$ as four rectangles (distinguishing between the cases y, z and z, y) and finally attaching the rectangles to the square to form a 'windmill'. The five windmills which result from this process are shown in Figure 11.

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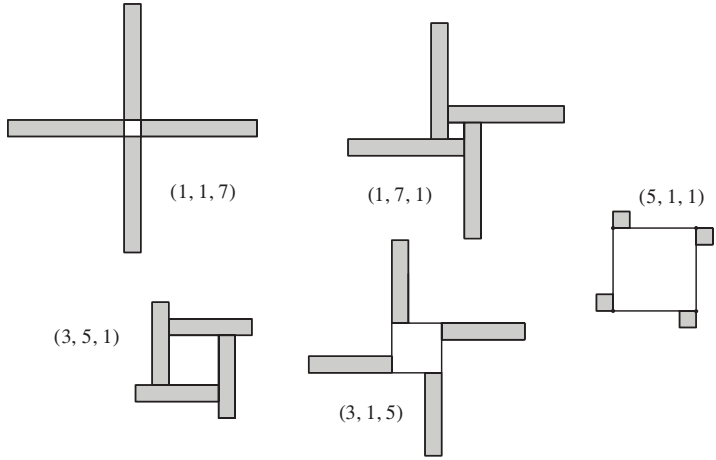


FIGURE 11

Look now at the *profile* of each windmill – the shape of the boundary, ignoring the internal divisions and the shading. You will note that (3, 5, 1) and (5, 1, 1) form a pair, since they share the same profile, and the same is true of (1, 7, 1) and (3, 1, 5). The windmill for (1, 1, 7) is not paired, so the number of diagrams is odd.

It is worth working through the cases of $p = 37$ and $p = 43$ to see that this 'almost pairing' occurs in these cases as well. This visual representation of pairing leads to the involution in the Zagier proof. What is happening is that the side of the central square is changed by 1 unit and the dimensions of the surrounding blades are altered to compensate. It is worth noting that the 'odd man out' is not the one you might expect – the triple which gives rise to the required solution of the equation. This windmill actually participates in the pairing.

Of course, one cannot conclude from these three examples that this always happens, and it requires a rigorous treatment of the pairing to show that it does. The easy step is to show that there is always one windmill which cannot be paired. For $p = 4n + 1$ this corresponds to the solution $(1, 1, n)$, where the windmill is a Greek cross of unit width. It is obvious that this cannot be paired, but the same is true for windmills of the form (n, n, m), which are thicker Greek crosses. However, it is easy to see that, unless $n = 1$, these cannot occur if p is prime. The hard step is to characterise all the ways in which the blades of the windmill can wrap around the core. It turns out that there are essentially three cases, corresponding to the three cases in Zagier's involution. It is possible, of course, to produce a rigorous geometrical argument that there are only three ways for two windmills to be paired

Here, then, is an example when a visual approach supplies a vital key to understanding how an algebraic approach is discovered and why it works.

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This cannot conceivably be regarded as a PWW, since there is a little number theory needed to link the windmills to the original problem, and a rigorous argument for existence of the pairing is necessary. However, the visual representation is such a key feature of this proof that is worthwhile celebrating it. Incidentally, the windmill approach also allows permits a proof that the two squares representation for primes is unique.

I hope that this short exploration of what makes (and what fails to make) a good PWW will stimulate some discussion amongst the readership of the journal.

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Quotations for *Nemo* (continued from page 208)

5. However great a man's fear of life, suicide remains the courageous act, the clear-headed act of a mathematician. The suicide has judged by the laws of chance—so many odds against one that to live will be more miserable than to die. His sense of mathematics is greater than his sense of survival.

6. There stands the Inspector at thy door.
Like a dog, he hunts for boys who know not two and two are four.

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