Math. Struct. in Comp. Science (2018), *vol.* 28, *pp.* 392–411. © Cambridge University Press 2016
doi:10.1017/S096012951600030X First published online 14 October 2016

Autostability spectra for decidable structures

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Received 9 December 2015; revised 30 May 2016

We study autostability spectra relative to strong constructivizations (SC-autostability spectra). For a decidable structure $\mathcal{S}$, the SC-autostability spectrum of $\mathcal{S}$ is the set of all Turing degrees capable of computing isomorphisms among arbitrary decidable copies of $\mathcal{S}$. The degree of SC-autostability for $\mathcal{S}$ is the least degree in the spectrum (if such a degree exists).

We prove that for a computable successor ordinal α, every Turing degree c.e. in and above $\mathbf{0}^{(\alpha)}$ is the degree of SC-autostability for some decidable structure. We show that for an infinite computable ordinal β, every Turing degree c.e. in and above $\mathbf{0}^{(2\beta+1)}$ is the degree of SC-autostability for some discrete linear order. We prove that the set of all PA-degrees is an SC-autostability spectrum. We also obtain similar results for autostability spectra relative to n-constructivizations.

To the memory of Barry Cooper

1. Introduction

We study algorithmic complexity of isomorphisms between computable copies of a countable structure. A structure $\mathcal{S}$ is *computable* if the universe of $\mathcal{S}$ is a computable subset of ω, and all basic predicates and functions of $\mathcal{S}$ are uniformly computable. Equivalently, $\mathcal{S}$ is computable if its atomic diagram $D(\mathcal{S})$ is a computable set. (Here, we identify first-order formulas with their Gödel numbers.)

The key notion in the study of effective isomorphisms is that of *autostability*. The research of autostable structures goes back to the works by Fröhlich and Shepherdson (1956) and Mal'tsev (1961, 1962). Since then, the notion of autostability has been relativized to the levels of different complexity hierarchies and to arbitrary Turing degrees $\mathbf{d}$. In recent years, the focus of the research has been on *autostability spectra* of structures.

Definition 1.1. Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{A}$ is $\mathbf{d}$-*autostable* if, for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. $\mathbf{0}$-autostable structures are also called *autostable*.

The *autostability spectrum* of the structure $\mathcal{A}$ is the set

$$\mathrm{AutSpec}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-autostable}\}.$$

A Turing degree $\mathbf{d}_0$ is the *degree of autostability* of $\mathcal{A}$ if $\mathbf{d}_0$ is the least degree in $\mathrm{AutSpec}(\mathcal{A})$.

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Autostability spectra and degrees of autostability were introduced in Miller (2009) and Fokina *et al.* (2010). Note that some of the literature (see, e.g., Miller 2009, Fokina *et al.* 2010 and Csima *et al.* 2013) use the terms *categoricity spectrum* and *degree of categoricity* in place of autostability spectrum and degree of autostability, respectively. In this paper, we follow the terminology of Goncharov (2011).

Suppose that n is a natural number and α is a computable ordinal. Fokina *et al.* (2010) proved that every Turing degree $\mathbf{d}$ that is d.c.e. in and above $\mathbf{0}^{(n)}$ is the degree of autostability of a computable structure. This result was extended by Csima *et al.* (2013) to hyperarithmetical degrees. They proved that every degree that is d.c.e. in and above $\mathbf{0}^{(\alpha+1)}$ is a degree of autostability. They also showed that $\mathbf{0}^{(\alpha)}$ is a degree of autostability. Frolov (2015) proved that every degree $\mathbf{d}$ that is d.c.e. in and above $\mathbf{0}''$ is the degree of autostability for a computable linear order. Anderson and Csima (2016) investigated Turing degrees that cannot be degrees of autostability. In particular, they showed that a noncomputable hyperimmune-free degree cannot be a degree of autostability. They also constructed a Σ^0_2 set whose degree is not a degree of autostability.

Recall that a computable structure $\mathcal{S}$ is *decidable* if its complete diagram $D^c(\mathcal{S})$ is computable, i. e., given a first-order formula $\phi(\bar{x})$ and a tuple $\bar{a}$ from $\mathcal{S}$, one can effectively determine whether $\phi(\bar{a})$ is true in $\mathcal{S}$ or not. Goncharov (2011) started the investigation of autostability restricted to decidable copies of a structure. This restricted notion of autostability is a natural one, since it simply changes the word *computable* to *decidable*.

Definition 1.2. Suppose that $\mathbf{d}$ is a Turing degree. A structure $\mathcal{A}$ is $\mathbf{d}$-*autostable relative to strong constructivizations* ($\mathbf{d}$-SC-*autostable*) if $\mathcal{A}$ has a decidable copy and for any decidable structures $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism $f : \mathcal{B} \cong \mathcal{C}$. In case $\mathbf{d} = \mathbf{0}$, we say that $\mathcal{A}$ is SC-*autostable*.

The *autostability spectrum relative to strong constructivizations* (SC-*autostability spectrum*) is the set

$$\mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-SC-autostable}\}.$$

A Turing degree $\mathbf{d}_0$ is the *degree of autostability relative to strong constructivizations* (*degree of SC-autostability*) for $\mathcal{A}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{A})$.

This paper is focused on the following problem.

Problem 1.1. What sets of Turing degrees can be SC-autostability spectra? In particular, what degrees can be degrees of SC-autostability?

Goncharov (2011) proved that every c.e. Turing degree is the degree of SC-autostability for some prime model. In Bazhenov (2015a), the author announced the following result: For a computable successor ordinal α, every degree $\mathbf{d}$ that is c.e. in and above $\mathbf{0}^{(\alpha)}$ is a degree of SC-autostability.

The outline of the paper is as follows. Section 2 contains the necessary preliminaries. In Section 3, we give the complete proof of the result from Bazhenov (2015a) mentioned above. Section 4 discusses related results for n-decidable structures. In Section 5, we construct new examples of degrees of SC-autostability for linear orders. In Section 6,

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we show that the set of all PA-degrees is an SC-autostability spectrum. The last section discusses open questions related to Problem 1.1.

2. Preliminaries

We consider only computable languages and structures with universe contained in ω. For a structure $\mathcal{S}$, $D(\mathcal{S})$ denotes the atomic diagram of $\mathcal{S}$, and $\mathrm{Th}(\mathcal{S})$ denotes the first-order theory of $\mathcal{S}$.

Let n be a natural number. A structure $\mathcal{S}$ is called *n-decidable* if the Σ_n-diagram of $\mathcal{S}$ is computable, i.e., given a finitary Σ_n formula $\phi(\bar{x})$ and a tuple $\bar{a}$ from $\mathcal{S}$, one can effectively determine whether $\mathcal{S} \models \phi(\bar{a})$. Note that every decidable structure is n-decidable, and a structure is computable iff it is 0-decidable.

For a function f, $\mathrm{dom}(f)$ denotes the domain of f. We assume $\{\varphi_e\}_{e\in\omega}$ to be a standard effective enumeration of all unary partial computable functions. We also assume $\langle\cdot,\cdot\rangle$ to be a standard computable pairing function over ω. We refer the reader to the works of Ash and Knight (2000) and Ershov and Goncharov (2000) for further background on computable and decidable structures.

2.1. *Boolean algebras and coloured algebras*

Let L_{BA} be the language $\{\vee^2, \wedge^2, \mathrm{C}^1; 0, 1\}$. We treat Boolean algebras as L_{BA}-structures. For a Boolean algebra $\mathcal{B}$, there is a natural partial order: $a \leqslant_{\mathcal{B}} b$ iff $a \wedge b = a$. If b is an element of $\mathcal{B}$, then $(\widehat{b})_{\mathcal{B}}$ denotes the *relative algebra* with the universe $\{a \in \mathcal{B} : a \leqslant_{\mathcal{B}} b\}$.

For an element $a \in \mathcal{B}$, let $a^0 = \mathrm{C}(a)$ and $a^1 = a$. For a tuple $\bar{a} = a_0, \ldots, a_{n-1}$ from $\mathcal{B}$ and $\sigma \in 2^n$, $\bar{a}^\sigma$ denotes the element $a_0^{\sigma(0)} \wedge \ldots \wedge a_{n-1}^{\sigma(n-1)}$.

Let k be a non-zero natural number. A *k-partition* of a non-zero element a in a Boolean algebra $\mathcal{B}$ is a sequence $b_1, \ldots, b_k$ of pairwise disjoint non-zero elements (i.e., $b_i \wedge b_j = 0$ when $i \neq j$, and $b_i \neq 0$) such that $a = b_1 \vee \ldots \vee b_k$. The formula $(b_1, \ldots, b_k \mid a)$ denotes that $b_1, \ldots, b_k$ is a k-partition of a.

An element $a \in \mathcal{B}$ is an *atom* if a is a minimal non-zero element in $\mathcal{B}$. Let $\mathrm{At}(\mathcal{B})$ denote the set of atoms of $\mathcal{B}$. We say that $\mathcal{B}$ is *atomic* if for every non-zero element $b \in \mathcal{B}$, there is an atom $a \leqslant_{\mathcal{B}} b$. An algebra $\mathcal{B}$ is *superatomic* if every subalgebra of $\mathcal{B}$ is atomic.

For an ordinal α, $F_\alpha(\mathcal{B})$ denotes the αth *Fréchet ideal* of $\mathcal{B}$. Recall that $F_0(\mathcal{B}) = \{0^{\mathcal{B}}\}$,

$$\mathrm{At}_\alpha(\mathcal{B}) = \left\{a \in \mathcal{B} : a/_{F_\alpha(\mathcal{B})} \in \mathrm{At}\left(\mathcal{B}/_{F_\alpha(\mathcal{B})}\right)\right\},$$
$$F_{\alpha+1}(\mathcal{B}) = \{b \in \mathcal{B} : b \text{ is a finite sum of elements from } \mathrm{At}_\alpha(\mathcal{B})\},$$
$$F_\delta(\mathcal{B}) = \bigcup_{\beta<\delta} F_\beta(\mathcal{B}), \text{ if } \delta \text{ is limit.}$$

Suppose that $\{\mathcal{B}_n\}_{n\in\omega}$ is a sequence of Boolean algebras. The *direct sum* $\sum_{n\in\omega}^{\{0,1\}} \mathcal{B}_n$ is the subalgebra of the direct product $\prod_{n\in\omega} \mathcal{B}_n$ with the universe

$$\left\{f \in \prod_{n\in\omega} \mathcal{B}_n : (\exists c \in \{0,1\})\exists m(\forall k \geqslant m)(f(k) = c^{\mathcal{B}_k})\right\}.$$

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If $\{\mathcal{B}_n\}_{n\in\omega}$ is a computable sequence, then we identify the algebra $\sum_{n\in\omega}\{0,1\}\mathcal{B}_n$ with its natural computable copy. For $n \in \omega$, let $E^{\mathcal{B}_n}$ denote the element

$$\left(0^{\mathcal{B}_0}, 0^{\mathcal{B}_1}, \ldots, 0^{\mathcal{B}_{n-1}}, 1^{\mathcal{B}_n}, 0^{\mathcal{B}_{n+1}}, 0^{\mathcal{B}_{n+2}}, \ldots\right)$$

from the direct sum.

For a linear order $\mathcal{L}$, $B(\mathcal{L})$ denotes the corresponding *interval algebra*. For further background on countable Boolean algebras, see Goncharov (1997).

Consider the new language $L_0 = L_{\mathrm{BA}} \cup \left\{P_k^1 : k \in \omega\right\}$, where P_k^1 is a unary predicate.

Definition 2.1. Suppose that $\mathcal{B}$ is a Boolean algebra. An L_0-structure $\mathcal{B}^c = (\mathcal{B}, P_k)_{k\in\omega}$ is a *coloured algebra* if there exists a computable sequence of first-order L_{BA}-formulas $\{\Phi_k(x, \bar{y}_k)\}_{k\in\omega}$ such that for any k, there is a tuple $\bar{b}_k$ from $\mathcal{B}$ with the property

$$\mathcal{B}^c \models \forall x \left(P_k(x) \leftrightarrow \Phi_k(x, \bar{b}_k)\right). \tag{1}$$

Such a sequence $\{\Phi_k\}_{k\in\omega}$ is called a *colouring sequence* of $\mathcal{B}^c$. The Boolean algebra $\mathcal{B}$ is called the *underlying algebra* of $\mathcal{B}^c$.

Coloured algebras were introduced in Bazhenov (2015a). The informal explanation of the term 'coloured algebra' is as follows. We treat the predicates P_k as colours and assign these colours to elements of a Boolean algebra $\mathcal{B}$. Note the key difference between our colouring and the graph colouring: we do not require that an element of $\mathcal{B}$ have only one colour.

A coloured algebra $\mathcal{B}^c$ is *atomic* (*superatomic*) if its underlying algebra $\mathcal{B}$ is atomic (superatomic). We are chiefly interested in atomic coloured algebras because of the following result.

Proposition 2.1. Suppose that $\mathcal{B}^c = (\mathcal{B}, P_k)_{k\in\omega}$ is a computable atomic coloured algebra, and $\{\Phi_k(x, \bar{y}_k)\}_{k\in\omega}$ is a colouring sequence of $\mathcal{B}^c$.

i. The structure $\mathcal{B}^c$ is decidable if and only if it satisfies the following conditions:
   a. the set of atoms $\mathrm{At}(\mathcal{B})$ is computable, and
   b. there exists a computable function $g(x)$ such that for any k, the value $g(k)$ is equal to the Gödel number of some tuple $\bar{b}_k$ with the property (1).

ii. Suppose that $n \in \omega$, and every Φ_k is a Σ_n formula. Then, the structure $\mathcal{B}^c$ is decidable iff it is $(n+1)$-decidable.

Proof.

i. It is easy to see that decidability of the structure $\mathcal{B}^c$ implies (a) and (b).
Now suppose that $\mathcal{B}^c$ satisfies (a) and (b). Ershov (1964) proved that a computable atomic Boolean algebra $\mathcal{A}$ is decidable iff the set of atoms $\mathrm{At}(\mathcal{A})$ is computable. Therefore, the Boolean algebra $\mathcal{B}$ is decidable.
Suppose that $\psi(\bar{z})$ is an L_0-formula, and $\bar{c}$ is a tuple from $\mathcal{B}$. Let m be the greatest number such that the predicate P_m occurs in ψ. For $k \in \omega$, $\bar{b}_k$ denotes the tuple from $\mathcal{B}$ with the Gödel number $g(k)$. The L_{BA}-formula $\psi'(\bar{z}, \bar{y}_0, \ldots, \bar{y}_m)$ is constructed in the following way. We replace every occurrence of a formula $P_k(t)$ (where t is an L_{BA}-term

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and $k \leqslant m$) in ψ by $\Phi_k(t, \bar{y}_k)$. It is easy to show that $\mathcal{B}^c \models \psi(\bar{c})$ iff $\mathcal{B} \models \psi'(\bar{c}, \bar{b}_0, \ldots, \bar{b}_m)$. Thus, the structure $\mathcal{B}^c$ is decidable.

ii. Notice that $(n+1)$-decidability of $\mathcal{B}^c$ ensures that the conditions (a) and (b) are satisfied. □

2.2. *Relative Δ^0_α categoricity and Scott families*

Let α be a computable ordinal. A computable structure $\mathcal{A}$ is *relatively Δ^0_α categorical* if for every $\mathcal{B} \cong \mathcal{A}$, there is a $\Delta^0_\alpha(D(\mathcal{B}))$ isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Note that relative $\Delta^0_{\alpha+1}$ categoricity implies $\mathbf{0}^{(\alpha)}$-autostability. Goncharov (1977) constructed an autostable structure that is not relatively Δ^0_1 categorical. For a computable successor ordinal α, Goncharov *et al.* (2005) built a $\mathbf{0}^{(\alpha)}$-autostable structure that is not relatively $\Delta^0_{\alpha+1}$ categorical. Chisholm *et al.* (2009) extended this result to computable limit ordinals α. Downey *et al.* (2015) proved that for any computable ordinal α, there exists an autostable structure that is not relatively Δ^0_α categorical.

There are nice syntactical conditions that are equivalent to relative Δ^0_α categoricity. These conditions involve the existence of special families of infinitary formulas.

Suppose that L is a language. *Infinitary formulas* of L are formulas of the logic $L_{\omega_1\omega}$. These formulas allow disjunctions and conjunctions over countable sets of formulas, and only finite nesting of quantifiers. For a countable ordinal α, infinitary Σ_α and Π_α formulas are defined in a standard way (see, e.g., Ash and Knight 2000, Chapter 6).

We give an informal description for the class of *computable infinitary formulas* of L that was introduced in Ash (1986b). The key point of the definition is that disjunctions and conjunctions are restricted to computably enumerable (c.e.) sets of formulas. Let α be a non-zero computable ordinal.

a. Σ^c_0 and Π^c_0 formulas are quantifier-free first-order L-formulas.
b. A Σ^c_α formula is a c.e. disjunction $\bigvee_i \exists \bar{u}_i \psi_i(\bar{x}, \bar{u}_i)$, where ψ_i is a $\Pi^c_{\beta_i}$ formula for some $\beta_i < \alpha$.
c. A Π^c_α formula is a c.e. conjunction $\bigwedge_i \forall \bar{u}_i \psi_i(\bar{x}, \bar{u}_i)$, where ψ_i is a $\Sigma^c_{\beta_i}$ formula for some $\beta_i < \alpha$.

For the formal definition of computable infinitary formulas and their properties, we refer the reader to Chapter 7 in Ash and Knight (2000). Note that for $n \in \omega$, every first-order Σ_n formula is a Σ^c_n formula.

Definition 2.2. Let α be a computable ordinal, and $\mathcal{S}$ be a structure. A *formally Σ^0_α Scott family* for $\mathcal{S}$ is a c.e. set Φ of Σ^c_α formulas (with a fixed finite tuple of parameters $\bar{c}$ from $\mathcal{S}$) such that

1. every tuple from $\mathcal{S}$ satisfies some $\phi \in \Phi$, and
2. if $\bar{a}$ and $\bar{b}$ are tuples from $\mathcal{S}$ satisfying the same formula $\phi \in \Phi$, then $(\mathcal{S}, \bar{a}) \cong (\mathcal{S}, \bar{b})$.

Theorem 2.1 (Ash *et al.* 1989; Chisholm 1990). Let α be a non-zero computable ordinal. A computable structure $\mathcal{S}$ is relatively Δ^0_α categorical if and only if it has a formally Σ^0_α Scott family.

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A computable structure $\mathcal{A}$ is *uniformly relatively* Δ^0_α *categorical* if, given an X-computable index for a structure $\mathcal{B} \cong \mathcal{A}$, one can effectively find a $\Delta^0_\alpha(X)$-computable index for an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. We will use the following corollary of Theorem 2.1.

Corollary 2.1. Suppose that a computable structure $\mathcal{S}$ has a formally Σ^0_α Scott family with no parameters. Then, $\mathcal{S}$ is uniformly relatively Δ^0_α categorical.

Proof. See Goncharov *et al.* (2005, p. 230). □

2.3. *Back-and-forth relations and pairs of computable structures*

Suppose that L is a language, $\mathcal{A}$ and $\mathcal{B}$ are L-structures, and α is a countable ordinal. We say that $\mathcal{A} \leqslant_\alpha \mathcal{B}$ if every infinitary Π_α sentence true in $\mathcal{A}$ is true in $\mathcal{B}$. The relations $\leqslant_\alpha$ are called *standard back-and-forth relations*. If one has a sufficiently nice characterization of standard back-and-forth relations for L-structures, then some subsets of ω can be effectively coded into computable sequences of L-structures.

Let α be a non-zero computable ordinal. A finite family of computable L-structures $K = \{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ is α-*friendly* if the relations

$$B_\beta = \{(i, \bar{a}, j, \bar{b}) : \ i, j \leqslant n, \ \bar{a} \text{ is from } \mathcal{A}_i, \ \bar{b} \text{ is from } \mathcal{A}_j, \ (\mathcal{A}_i, \bar{a}) \leqslant_\beta (\mathcal{A}_j, \bar{b})\}$$

are computably enumerable uniformly in $\beta < \alpha$. A structure $\mathcal{A}$ is α-*friendly* if the family $\{\mathcal{A}\}$ is α-friendly.

Theorem 2.2 (Ash and Knight 1990, Theorem 3.1). Let α be a non-zero computable ordinal. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are computable L-structures such that $\mathcal{B} \leqslant_\alpha \mathcal{A}$ and the family $\{\mathcal{A}, \mathcal{B}\}$ is α-friendly. Then, for any Π^0_α set S, there is a uniformly computable sequence of structures $\{\mathcal{C}_n\}_{n \in \omega}$ such that

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}, & \text{if } n \in S, \\ \mathcal{B}, & \text{otherwise.} \end{cases}$$

2.4. *Back-and-forth relations for Boolean algebras*

Here, we give some useful results on standard back-and-forth relations for Boolean algebras. Recall that for a Boolean algebra $\mathcal{B}$, $\mathrm{At}(\mathcal{B})$ denotes the set of atoms of $\mathcal{B}$.

Lemma 2.1 (Ash and Knight 2000, Lemma 15.12). Suppose that α is a countable ordinal, $\mathcal{A}$ and $\mathcal{B}$ are Boolean algebras, $\bar{a} = a_0, \ldots, a_n$ and $\bar{b} = b_0, \ldots, b_n$ are tuples from $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, $(\mathcal{A}, \bar{a}) \leqslant_\alpha (\mathcal{B}, \bar{b})$ if and only if $(\widehat{\bar{a}^\sigma})_{\mathcal{A}} \leqslant_\alpha (\widehat{\bar{b}^\sigma})_{\mathcal{B}}$ for all $\sigma \in 2^n$.

Lemma 2.2 (Alaev 2004, Lemmas 3, 4, 9). Suppose that α is a countable ordinal, and $\mathcal{A}$ and $\mathcal{B}$ are Boolean algebras.

a. Let $\bar{a} = a_0, \ldots, a_n$ and $\bar{b} = b_0, \ldots, b_n$ be tuples from $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, $(\mathcal{A}, \mathrm{At}, \bar{a}) \leqslant_\alpha (\mathcal{B}, \mathrm{At}, \bar{b})$ if and only if $((\widehat{\bar{a}^\sigma})_{\mathcal{A}}, \mathrm{At}) \leqslant_\alpha ((\widehat{\bar{b}^\sigma})_{\mathcal{B}}, \mathrm{At})$ for all $\sigma \in 2^n$.

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b. $(\mathcal{A}, \mathrm{At}) \leqslant_0 (\mathcal{B}, \mathrm{At})$ iff the following holds: If one of the algebras $\mathcal{A}$, $\mathcal{B}$ has at most two elements, then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.
c. For $\alpha \geqslant 1$, $(\mathcal{A}, \mathrm{At}) \leqslant_\alpha (\mathcal{B}, \mathrm{At})$ iff $\mathcal{A} \leqslant_{\alpha+1} \mathcal{B}$.

2.5. *Prime models*

Let T be a complete first-order theory. A structure $\mathcal{M}$ is a *prime model* of T if $\mathcal{M} \models T$ and for every model $\mathcal{N} \models T$, there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. An L-structure $\mathcal{M}$ is an *atomic model* if for any tuple $\bar{a} = a_0, \ldots, a_n$ from $\mathcal{M}$, there exists a first-order L-formula $\phi(x_0, \ldots, x_n)$ such that $\mathcal{M} \models \phi(\bar{a})$, and every first-order L-formula $\psi(x_0, \ldots, x_n)$ satisfies the following condition: if $\mathcal{M} \models \psi(\bar{a})$, then $\mathcal{M} \models \forall \bar{x}(\phi(\bar{x}) \to \psi(\bar{x}))$. Such a formula ϕ is called a *complete formula* (of the theory $\mathrm{Th}(\mathcal{M})$).

Theorem 2.3 (Vaught, see Chang and Keisler 1973). Let $\mathcal{M}$ be a model of a complete theory T. $\mathcal{M}$ is a prime model of T if and only if $\mathcal{M}$ is a countable atomic model.

2.6. *PA-degrees*

We identify the set $\omega^{<\omega}$ with a tree with the standard ordering: $\sigma \preccurlyeq \tau$ iff σ is an initial segment of τ. For any $\sigma, \tau \in \omega^{<\omega}$, we use $\sigma^\frown\tau$ to denote the concatenation of σ and τ. Suppose that T is a subtree of $\omega^{<\omega}$. Let $b(\sigma; T)$ denote the *branching function* of T that is defined as follows: For $\sigma \in T$, $b(\sigma; T)$ is the cardinality of the set $\{n \in \omega : \sigma^\frown\langle n\rangle \in T\}$.

A Turing degree $\mathbf{d}$ is called a PA-degree if it computes some complete extension of Peano arithmetic. Recall the characterization of PA-degrees due to Scott (1962), Jockusch and Soare (1972) and Solovay.

Theorem 2.4 (see Simpson 1977, Theorem 6.6). Let $\mathbf{d}$ be a Turing degree. Then, the following conditions are equivalent:

1. $\mathbf{d}$ is a PA-degree;
2. $\mathbf{d}$ is the degree of a set that separates an effectively inseparable, disjoint pair of computably enumerable sets;
3. for every infinite computable finite-branching subtree T of $\omega^{<\omega}$ with a computable branching function $b(\sigma; T)$, there is a $\mathbf{d}$-computable path through T.

3. Degrees c.e. in and above $\mathbf{0}^{(\alpha+1)}$

In this section, we give the complete proof of the result below.

Theorem 3.1. Suppose that α is a computable ordinal, and $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is c.e. in and above $\mathbf{0}^{(\alpha+1)}$. There exists a decidable atomic coloured algebra $\mathcal{S}$ such that $\mathbf{d}$ is the degree of SC-autostability for $\mathcal{S}$.

The proof uses the ideas from Bazhenov (2013, 2016). The main tool of the proof is the technique of pairs of computable structures from Ash and Knight (1990). We consider two cases according to whether the ordinal α is even or odd. For even α, we construct

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a superatomic coloured algebra. For odd α, $\mathcal{S}$ is not superatomic, and the quotient $\mathcal{B}/_{F_{\alpha+1}(\mathcal{B})}$ of the underlying algebra $\mathcal{B}$ is atomless.

In our proof, we assume that α is infinite. The constructions for finite ordinals are essentially the same, except that one has to keep in mind the following fact: $\Sigma^0_1(\varnothing^{(\alpha+1)}) = \Sigma^0_{\alpha+1}$ if α is infinite, and $\Sigma^0_1(\varnothing^{(\alpha+1)}) = \Sigma^0_{\alpha+2}$ otherwise.

3.1. *Proof for* $\alpha = 2\beta$

First, we give the explicit calculations of the standard back-and-forth relations for superatomic Boolean algebras from Ash (1987).

Lemma 3.1 (Ash 1987, Lemma 5). Suppose that α, β, γ are countable ordinals, and $0 < m, n < \omega$.

1. $B(\omega^\alpha \times m) \leqslant_{2\gamma+1} B(\omega^\beta \times n)$ iff one of the following holds:
   a. $\alpha = \beta < \gamma$ and $m = n$,
   b. $\alpha = \beta = \gamma$ and $m \geqslant n$,
   c. $\alpha \geqslant \gamma + 1$ and $\beta \geqslant \gamma$.
2. Suppose that $\gamma \neq 0$. Then, $B(\omega^\alpha \times m) \leqslant_{2\gamma} B(\omega^\beta \times n)$ iff either $(\alpha = \beta < \gamma)$ & $(m = n)$, or $(\alpha \geqslant \gamma)$ & $(\beta \geqslant \gamma)$.

Lemma 3.2. Suppose that $\beta, \gamma_0, \ldots, \gamma_n$ are computable ordinals. There exists a β-friendly family $K = \{(\mathcal{A}_0, \mathrm{At}), \ldots, (\mathcal{A}_n, \mathrm{At})\}$ such that for $i \leqslant n$, $\mathcal{A}_i$ is isomorphic to $B(\gamma_i)$.

Proof. We use the construction from Ash and Knight (2000, Proposition 15.15) to build a β-friendly Boolean algebra $\mathcal{A}_i \cong B(\gamma_i)$, $i \leqslant n$. The algebras $\mathcal{A}_i$ have the following useful feature: For an element $a \in \mathcal{A}_i$, we can effectively determine the invariants λ and n, where $(\widehat{a})_{\mathcal{A}_i} \cong B(\omega^\lambda \times n)$. Lemmas 2.1 and 3.1 imply that $K_0 = \{\mathcal{A}_i : i \leqslant n\}$ is a β-friendly family of Boolean algebras. It is straightforward to show that the sets $\mathrm{At}(\mathcal{A}_i)$ are computable. This fact, β-friendliness of K_0, and Lemma 2.2 ensure that the family $\{(\mathcal{A}_i, \mathrm{At}) : i \leqslant n\}$ is β-friendly. □

Now, we are ready to build the coloured algebra $\mathcal{S} = \mathcal{C}^c$. We assume that $\alpha = 2\beta$, and fix a $\Pi^0_{2\beta+1}$ set $S \in \mathbf{d}$. Choose Boolean algebras $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ such that the family $\{(\mathcal{A}_i, \mathrm{At}) : i \leqslant 2\}$ is $(2\beta+1)$-friendly, and $\mathcal{A}_i \cong B(\omega^\beta \times (i+1))$. Note that $\mathcal{A}_1 \leqslant_{2\beta+1} \mathcal{A}_0$ and $\mathcal{A}_2 \leqslant_{2\beta+1} \mathcal{A}_1$. By Lemma 2.2.c and Theorem 2.2, there is a uniformly computable sequence $\{(\mathcal{C}_n, \mathrm{At})\}_{n\in\omega}$ such that

$$\mathcal{C}_{2k} \cong \begin{cases} B(\omega^\beta), & k \in S, \\ B(\omega^\beta \times 2), & k \notin S; \end{cases} \qquad \mathcal{C}_{2k+1} \cong \begin{cases} B(\omega^\beta \times 2), & k \in S, \\ B(\omega^\beta \times 3), & k \notin S. \end{cases} \tag{2}$$

Let $\mathcal{C} = \sum_{n\in\omega} \sum_{\{0,1\}} \mathcal{C}_n$. The colouring for $\mathcal{C}$ is defined as follows:

$$\mathcal{C}^c \models P_k(a) \;\Leftrightarrow\; (a = E^{\mathcal{C}_{2k}}) \vee (a = E^{\mathcal{C}_{2k+1}}).$$

Let $\phi(x, y_1, y_2)$ denote the formula $(x = y_1) \vee (x = y_2)$. It is obvious that the sequence $\{\phi_k(x, \bar{y})\}_{k\in\omega}$, where $\phi_k = \phi$ for all k, is a colouring sequence for $\mathcal{C}^c$. Furthermore, $\mathcal{C}^c$

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satisfies the conditions (a) and (b) from Proposition 2.1. Therefore, $\mathcal{C}^c$ is a decidable atomic coloured algebra. Moreover, $\mathcal{C}$ is isomorphic to $B(\omega^{\beta+1})$.

Lemma 3.3 below proves that $\mathcal{C}^c$ is a **d**-SC-autostable structure. Therefore, $\{\mathbf{c} : \mathbf{d} \leqslant \mathbf{c}\} \subseteq \mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{C}^c)$. Lemma 3.4 shows that if $\mathbf{c} \in \mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{C}^c)$, then $\mathbf{d} \leqslant \mathbf{c}$.

Lemma 3.3. Suppose that $\mathcal{D}^c$ is a decidable copy of $\mathcal{C}^c$. Then, there exists a **d**-computable isomorphism f from $\mathcal{D}^c$ onto $\mathcal{C}^c$.

Proof. For $k \in \omega$, let $P_k^{\mathcal{D}^c} = \{a_k < b_k\}$. To build a desired isomorphism f, it is sufficient to do (effectively in the oracle S and uniformly in k) the following:

1. find the elements $f(a_k)$ and $f(b_k)$ from the set $\{E^{\mathcal{C}_{2k}}, E^{\mathcal{C}_{2k+1}}\}$ such that $(\widehat{f(a_k)})_{\mathcal{C}} \cong (\widehat{a_k})_{\mathcal{D}}$ and $(\widehat{f(b_k)})_{\mathcal{C}} \cong (\widehat{b_k})_{\mathcal{D}}$, and
2. build isomorphisms $g_k : (\widehat{a_k})_{\mathcal{D}} \cong (\widehat{f(a_k)})_{\mathcal{C}}$ and $g_k' : (\widehat{b_k})_{\mathcal{D}} \cong (\widehat{f(b_k)})_{\mathcal{C}}$.

We choose the elements $f(a_k)$ and $f(b_k)$ as follows. First, we define the parameter $t \in \omega$. If $k \in S$, then set $t = 2$; otherwise, set $t = 3$. Note that $\mathcal{C}_{2k} \cong B(\omega^\beta \times (t-1))$ and $\mathcal{C}_{2k+1} \cong B(\omega^\beta \times t)$. Thus, we can find elements $c_0, \dots, c_{t-1} \in \mathcal{D}$ such that $c_0, \dots, c_{t-1} \notin F_\beta(\mathcal{D})$ and $c_0, \dots, c_{t-1}$ is a t-partition of either a_k or b_k. If $(c_0, \dots, c_{t-1} \mid a_k)$, then define $f(a_k) = E^{\mathcal{C}_{2k+1}}$ and $f(b_k) = E^{\mathcal{C}_{2k}}$. Otherwise, set $f(a_k) = E^{\mathcal{C}_{2k}}$ and $f(b_k) = E^{\mathcal{C}_{2k+1}}$.

It is well known that the βth Fréchet ideal F_β is definable by a $\Sigma^c_{2\beta}$ formula (see, e.g., Example 5 from Ash and Knight 2000, Section 6.2). Therefore, the set $F_\beta(\mathcal{D})$ is $\Sigma^0_{2\beta}$, and $F_\beta(\mathcal{D}) \leqslant_T \varnothing^{(2\beta+1)} \leqslant_T S$. Hence, the search of $f(a_k)$ and $f(b_k)$ is effective with oracle S. By (2), our choice of $f(a_k)$ and $f(b_k)$ is correct.

We describe the construction of g_k and g_k' for the case where $k \notin S$, $f(a_k) = E^{\mathcal{C}_{2k+1}}$ and $f(b_k) = E^{\mathcal{C}_{2k}}$. Find the partitions $(d_0, d_1 \mid b_k)$, $(c_0', c_1', c_2' \mid E^{\mathcal{C}_{2k+1}})$ and $(d_0', d_1' \mid E^{\mathcal{C}_{2k}})$ such that the elements d_i, d_i' and c_j' do not belong to the βth Fréchet ideal. By (2), we have $(\widehat{c_i})_{\mathcal{D}} \cong (\widehat{c_i'})_{\mathcal{C}} \cong (\widehat{d_j})_{\mathcal{D}} \cong (\widehat{d_j'})_{\mathcal{C}} \cong B(\omega^\beta)$.

We know (see Ash and Knight 2000, Theorem 17.8) that the Boolean algebra $B(\omega^\beta)$ has a formally $\Sigma^0_{2\beta}$ Scott family with no parameters. By Corollary 2.1, we can find $\Delta^0_{2\beta}$-computable indices for isomorphisms $h_i : (\widehat{c_i})_{\mathcal{D}} \cong (\widehat{c_i'})_{\mathcal{C}}$ and $h_j' : (\widehat{d_j})_{\mathcal{D}} \cong (\widehat{d_j'})_{\mathcal{C}}$, $i \leqslant 2$, $j \leqslant 1$. Therefore, we can find $\Delta^0_{2\beta}$-computable indices for isomorphisms g_k and g_k'. The described procedure is effective with oracle S. □

Lemma 3.4. There exists a decidable structure $\mathcal{D}^c \cong \mathcal{C}^c$ such that for every isomorphism f from $\mathcal{D}^c$ onto $\mathcal{C}^c$, f computes S.

Proof. The structure $\mathcal{D}^c$ is constructed using the technique of pairs of computable structures. We build a computable sequence $\{(\mathcal{D}_n, \mathrm{At})\}_{n\in\omega}$ such that

$$\mathcal{D}_{2k} \cong \begin{cases} B(\omega^\beta), & k \in S, \\ B(\omega^\beta \times 3), & k \notin S; \end{cases} \qquad \mathcal{D}_{2k+1} \cong B(\omega^\beta \times 2). \tag{3}$$

Assume that $\mathcal{D} = \sum_{n\in\omega} \mathcal{D}_n$, and $\mathcal{D}^c \models P_k(a)$ iff either $a = E^{\mathcal{D}_{2k}}$ or $a = E^{\mathcal{D}_{2k+1}}$. It is easy to see that $\mathcal{D}^c$ is a decidable coloured algebra isomorphic to $\mathcal{C}^c$.

Suppose that f is an isomorphism from $\mathcal{D}^c$ onto $\mathcal{C}^c$. By (2) and (3), $k \in S$ if and only if $f(E^{\mathcal{D}_{2k}}) = E^{\mathcal{C}_{2k}}$. Therefore, $S \leqslant_T f$. □

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3.2. *Proof for* $\alpha = 2\beta + 1$

Let η denote the linear order of rationals.

Lemma 3.5 (Bazhenov 2014, Lemma 5). Suppose that α, β, γ are countable ordinals, and m is a non-zero natural number.

1. $B(\omega^\alpha \times m) \leqslant_{2\gamma+1} B(\omega^\beta \times (1+\eta))$ iff $\alpha \geqslant \gamma + 1$ and $\beta \geqslant \gamma$.
2. $B(\omega^\beta \times (1+\eta)) \leqslant_{2\gamma+1} B(\omega^\alpha \times m)$ iff $\alpha \geqslant \gamma$ and $\beta \geqslant \gamma$.
3. For $\gamma \neq 0$, $B(\omega^\alpha \times m) \leqslant_{2\gamma} B(\omega^\beta \times (1+\eta))$ iff $\alpha \geqslant \gamma$ and $\beta + 1 \geqslant \gamma$.
4. For $\gamma \neq 0$, $B(\omega^\beta \times (1+\eta)) \leqslant_{2\gamma} B(\omega^\alpha \times m)$ iff $\alpha \geqslant \gamma$ and $\beta \geqslant \gamma$.

Lemma 3.6. Suppose that $\alpha, \beta, \gamma_0, \ldots, \gamma_n$ are computable ordinals. There exists an α-friendly family $K = \{(\mathcal{A}_0, \mathrm{At}), \ldots, (\mathcal{A}_n, \mathrm{At}), (\mathcal{B}, \mathrm{At})\}$ such that $\mathcal{B}$ is isomorphic to $B(\omega^\beta \times (1+\eta))$ and for $i \leqslant n$, $\mathcal{A}_i \cong B(\gamma_i)$.

Proof. We form the family $K_0 = \{(\mathcal{A}_i, \mathrm{At}) : i \leqslant n\}$ as in Lemma 3.2. As in Ash and Knight (2000, Proposition 15.15), we take a computable order $\mathcal{L}$ of type ω^β in which we can effectively calculate the Cantor normal form for intervals. The desired structure $\mathcal{B}$ is the natural computable copy of the interval algebra $B(\mathcal{L} \times (1+\eta))$. α-friendliness of K_0, Lemmas 2.2 and 3.5 implies that the constructed family K is α-friendly. □

We assume that $\alpha = 2\beta + 1$ and fix a $\Pi^0_{2\beta+2}$ set S from $\mathbf{d}$. We choose Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong B(\omega^\beta \times (1+\eta))$, $\mathcal{B} \cong B(\omega^{\beta+1})$, and the family $\{(\mathcal{A}, \mathrm{At}), (\mathcal{B}, \mathrm{At})\}$ is $(2\beta+2)$-friendly. Notice that $\mathcal{B} \leqslant_{2\beta+2} \mathcal{A}$. Thus, by Theorem 2.2, there exists a computable sequence $\{(\mathcal{C}_n, \mathrm{At})\}_{n\in\omega}$ such that

$$\mathcal{C}_{4k} \cong B(\omega^\beta \times (1+\eta)), \qquad \mathcal{C}_{4k+3} \cong B(\omega^{\beta+1}),$$
$$\mathcal{C}_{4k+1} \cong \mathcal{C}_{4k+2} \cong \begin{cases} B(\omega^\beta \times (1+\eta)), & k \in S, \\ B(\omega^{\beta+1}), & k \notin S. \end{cases}$$

Let $\mathcal{C} = \sum_{n\in\omega} \sum_{\{0,1\}} \mathcal{C}_n$, and set

$$\mathcal{C}^c \models P_{2k}(a) \text{ iff } a = (E^{\mathcal{C}_{4k}} \vee E^{\mathcal{C}_{4k+1}}) \text{ or } a = (E^{\mathcal{C}_{4k+2}} \vee E^{\mathcal{C}_{4k+3}}),$$
$$\mathcal{C}^c \models P_{2k+1}(b) \text{ iff } b = E^{\mathcal{C}_{4k}} \text{ or } b = E^{\mathcal{C}_{4k+2}}.$$

It is easy to show that $\mathcal{C}^c$ is a decidable atomic coloured algebra with a colouring sequence consisting of quantifier-free formulas.

Lemma 3.7. Let $\mathcal{D}^c$ be a decidable copy of $\mathcal{C}^c$. There exists a $\mathbf{d}$-computable isomorphism f from $\mathcal{D}^c$ onto $\mathcal{C}^c$.

Proof. Assume that $P^{\mathcal{D}^c}_{2k} = \{a_k < b_k\}$, $P^{\mathcal{D}^c}_{2k+1} = \{c_k, d_k\}$, $c_k <_{\mathcal{D}} a_k$, and $d_k <_{\mathcal{D}} b_k$. Choose u_k and v_k such that $(c_k, u_k \mid a_k)$ and $(d_k, v_k \mid b_k)$. We will do the following:

1. for each $x \in \{c_k, d_k, u_k, v_k\}$, find a value $f(x)$ such that the resulting finite function $f \upharpoonright \{c_k, d_k, u_k, v_k\}$ extends to an isomorphism of coloured algebras, and
2. for each $x \in \{c_k, d_k, u_k, v_k\}$, build an isomorphism $g_x : (\widehat{x})_{\mathcal{D}} \cong (\widehat{f(x)})_{\mathcal{C}}$.

https://doi.org/10.1017/S096012951600030X Published online by Cambridge University Press

We know that the set $\mathrm{At}_\beta(B(\omega^\beta \times (1+\eta)))$ is empty, and $\mathrm{At}_\beta(B(\omega^{\beta+1}))$ is an infinite set. Thus, if $k \in S$, then we can find an element z such that $z \in \mathrm{At}_\beta(\mathcal{D})$, and $(z <_{\mathcal{D}} u_k) \vee (z <_{\mathcal{D}} v_k)$. If $z <_{\mathcal{D}} u_k$, then set

$$f(c_k) = E^{\mathcal{C}_{4k+2}},\ f(u_k) = E^{\mathcal{C}_{4k+3}},\ f(d_k) = E^{\mathcal{C}_{4k}},\ f(v_k) = E^{\mathcal{C}_{4k+1}}. \tag{4}$$

Otherwise, define

$$f(c_k) = E^{\mathcal{C}_{4k}},\ f(u_k) = E^{\mathcal{C}_{4k+1}},\ f(d_k) = E^{\mathcal{C}_{4k+2}},\ f(v_k) = E^{\mathcal{C}_{4k+3}}. \tag{5}$$

If $k \notin S$, then find an element z with the property: $z \in \mathrm{At}_\beta(\mathcal{D})$ and $(z <_{\mathcal{D}} c_k) \vee (z <_{\mathcal{D}} d_k)$. If $z <_{\mathcal{D}} c_k$, then define $f(x)$ as in (4); otherwise, use (5).

It is straightforward to show that the constructed finite function extends to an isomorphism from $\mathcal{D}^c$ onto $\mathcal{C}^c$. Since $\mathrm{At}_\beta(\mathcal{D})$ is definable by a $\Pi^c_{2\beta+1}$ formula, we have $\mathrm{At}_\beta(\mathcal{D}) \leqslant_T \varnothing^{(2\beta+2)} \leqslant_T S$.

In Bazhenov (2014, Lemma 4), it was shown that the Boolean algebra $B(\omega^\beta \times (1+\eta))$ has a formally $\Sigma^0_{2\beta+1}$ Scott family with no parameters. As in the proof of Lemma 3.3, we find $\Delta^0_{2\beta+2}$-computable indices for isomorphisms g_x, $x \in \{c_k, d_k, u_k, v_k\}$. Note that the described construction is uniform in k and effective in the oracle S. Therefore, we can build a **d**-computable isomorphism f from $\mathcal{D}^c$ onto $\mathcal{C}^c$. □

Lemma 3.8. There exists a decidable copy $\mathcal{D}^c$ of $\mathcal{C}^c$ such that for every isomorphism f from $\mathcal{D}^c$ onto $\mathcal{C}^c$, f computes S.

Proof. We build a computable sequence $\{(\mathcal{D}_n, \mathrm{At})\}_{n\in\omega}$ such that

$$\mathcal{D}_{4k+2} \cong B(\omega^\beta \times (1+\eta)), \qquad \mathcal{D}_{4k+3} \cong B(\omega^{\beta+1}),$$
$$\mathcal{D}_{4k} \cong \mathcal{D}_{4k+1} \cong \begin{cases} B(\omega^\beta \times (1+\eta)), & k \in S, \\ B(\omega^{\beta+1}), & k \notin S. \end{cases}$$

Set $\mathcal{D} = \sum_{n\in\omega} \mathcal{D}_n$, and

$$\mathcal{D}^c \models P_{2k}(a) \text{ iff } a = (E^{\mathcal{D}_{4k}} \vee E^{\mathcal{D}_{4k+1}}) \text{ or } a = (E^{\mathcal{D}_{4k+2}} \vee E^{\mathcal{D}_{4k+3}}),$$
$$\mathcal{D}^c \models P_{2k+1}(b) \text{ iff } b = E^{\mathcal{D}_{4k}} \text{ or } b = E^{\mathcal{D}_{4k+2}}.$$

Notice that $\mathcal{D}^c$ is a decidable copy of $\mathcal{C}^c$.

Let f be an isomorphism from $\mathcal{D}^c$ onto $\mathcal{C}^c$. It is easy to see that $k \in S$ iff $f(E^{\mathcal{D}_{4k}}) = E^{\mathcal{C}_{4k}}$. □

This concludes the proof of Theorem 3.1.

4. Autostability spectra for n-decidable structures

Fokina *et al.* (2015) started the investigation of computable structures that are *autostable relative to n-constructivizations*. They calculated, in a precise way, the complexity of the index sets for m-decidable structures autostable relative to n-constructivizations, where $m, n \in \omega$. The following definition is motivated by the results of Fokina *et al.* (2015).

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Definition 4.1. Suppose that $n \in \omega$, and $\mathbf{d}$ is a Turing degree. A structure $\mathcal{A}$ is $\mathbf{d}$-*autostable relative to n-constructivizations* ($\mathbf{d}$-*categorical relative to n-decidable presentations*) if $\mathcal{A}$ has an n-decidable copy and for any n-decidable structures $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{B}$ onto $\mathcal{C}$.

The *autostability spectrum relative to n-constructivizations* is the set

$$\mathrm{AutSpec}_n(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-autostable relative to } n\text{-constructivizations}\}.$$

A Turing degree $\mathbf{d}_0$ is the *degree of autostability relative to n-constructivizations* for $\mathcal{A}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_n(\mathcal{A})$.

Note that for an n-decidable structure $\mathcal{A}$,

$$\mathrm{AutSpec}(\mathcal{A}) = \mathrm{AutSpec}_0(\mathcal{A}) \subseteq \mathrm{AutSpec}_1(\mathcal{A}) \subseteq \mathrm{AutSpec}_2(\mathcal{A}) \subseteq \ldots \subseteq \mathrm{AutSpec}_n(\mathcal{A}).$$

Moreover, if $\mathcal{A}$ is decidable, then $\mathrm{AutSpec}_n(\mathcal{A}) \subseteq \mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{A})$.

The following result was announced in Bazhenov (2015a, Theorem 1).

Theorem 4.1 (Bazhenov 2016, Theorems 2.1, 3.2). Let α be a computable ordinal. There exists a decidable atomic Boolean algebra $\mathcal{B}$ with the degree of SC-autostability $\mathbf{0}^{(\alpha)}$.

The proof of Theorem 4.1 involves the exact calculations of degrees of SC-autostability for interval algebras $B(\omega^\beta \times m)$ and $B(\omega^\beta \times (1+\eta))$, where β is a computable ordinal and $m \in \omega$.

The result of Ershov (1964) implies that an atomic Boolean algebra is decidable if and only if it is 1-decidable. Therefore, we obtain:

Corollary 4.1. Suppose that $0 < n < \omega$, and α is a computable ordinal. There exists an n-decidable Boolean algebra $\mathcal{B}$ such that $\mathbf{0}^{(\alpha)}$ is the degree of autostability relative to n-constructivizations for $\mathcal{B}$.

Note that the coloured algebra $\mathcal{C}^c$ from the proof of Theorem 3.1 has the following property: The colouring sequence of $\mathcal{C}^c$ consists of quantifier-free finitary formulas. Hence, by Proposition 2.1, a copy of $\mathcal{C}^c$ is decidable iff it is 1-decidable. Therefore, Theorem 3.1 implies the following result.

Corollary 4.2. Suppose that $0 < n < \omega$, α is a computable ordinal, and $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is c.e. in and above $\mathbf{0}^{(\alpha+1)}$. There exists an n-decidable coloured algebra $\mathcal{S}$ such that $\mathbf{d}$ is the degree of autostability relative to n-constructivizations for $\mathcal{S}$.

5. Degrees of SC-autostability for linear orders

In this section, we show that for an infinite computable ordinal α, every Turing degree c.e. in and above $\mathbf{0}^{(2\alpha+1)}$ is the degree of SC-autostability for some linear order. We refer the reader to Downey (1998) and Rosenstein (1982) for background on linear orders.

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Let $\mathcal{L}$ be a linear order. The *successor relation* $S(\mathcal{L})$ is the set $\{(x, y) : (x <_{\mathcal{L}} y) \& \neg \exists z\ (x <_{\mathcal{L}} z <_{\mathcal{L}} y)\}$. The *block relation* $B(\mathcal{L})$ is the set

$$\{(x, y) : \bigvee_{k \in \omega} \exists z_0 \dots \exists z_{k+1}((x = z_0) \& S(z_0, z_1) \& S(z_1, z_2) \& \dots \& S(z_k, z_{k+1}) \& (z_{k+1} = y))\}.$$

We say that $(x \sim_B y)$ if $x = y$, or $B(x, y)$, or $B(y, x)$. It is easy to see that $\sim_B$ is a congruence relation on $\mathcal{L}$.

ζ denotes the linear order of integers. A linear order $\mathcal{L}$ is *discrete* if $\mathcal{L}$ is isomorphic to a linear order $\zeta \times \mathcal{L}_0$ for some order $\mathcal{L}_0$. We give some useful results on discrete linear orders.

Theorem 5.1 (Langford 1926; see also Downey 1998, Theorem 2.12). A discrete linear order $\mathcal{L}$ is decidable if and only if it is 1-decidable.

Theorem 5.2 (Moses 1984, p. 257). A computable linear order $\mathcal{L}$ is 1-decidable if and only if the successor relation $S(\mathcal{L})$ is computable.

Theorem 5.3 (Frolov 2012, Theorem 10). Let $\mathcal{L}$ be a countable linear order. Then, the following are equivalent:

1. $\mathcal{L}$ has a $\mathbf{0}'$-computable copy,
2. the order $\zeta \times \mathcal{L}$ has a computable copy $\mathcal{L}'$ with computable successor relation $S(\mathcal{L}')$.

Corollary 5.1. Let $\mathcal{L}$ be a $\mathbf{0}'$-computable linear order. Then, the order $\zeta \times L$ has a decidable copy.

For a computable linear order $\mathcal{L}$, we define the *standard decidable copy* $\mathrm{St}(\zeta \times \mathcal{L})$ of the order $\zeta \times L$ in the following way. The universe of $\mathrm{St}(\zeta \times \mathcal{L})$ is equal to $\{\langle z, a\rangle : z \in \zeta,\ a \in \mathcal{L}\}$, and $(z, a) \leqslant_{\mathrm{St}(\zeta \times \mathcal{L})} (y, b)$ iff $a <_{\mathcal{L}} b$ or $(a = b) \& (z \leqslant_{\zeta} y)$. By Theorems 5.1 and 5.2, we may assume that $\mathrm{St}(\zeta \times \mathcal{L})$ is a decidable structure.

Theorem 5.4. Suppose that $\mathbf{d}$ is a Turing degree, and $\mathcal{L}$ is a computable linear order with the following properties:

a. if $\mathcal{L}'$ is a $\mathbf{0}'$-computable copy of $\mathcal{L}$, then there is a $\mathbf{d}$-computable isomorphism from $\mathcal{L}$ onto $\mathcal{L}'$;
b. there exists a computable copy $\mathcal{L}_0$ of $\mathcal{L}$ such that every isomorphism $f : \mathcal{L} \cong \mathcal{L}_0$ computes $\mathbf{d}$.

Then, $\mathbf{d}$ is the degree of SC-autostability for the order $\zeta \times \mathcal{L}$.

Proof. Assume that $\mathcal{A} = \mathrm{St}(\zeta \times \mathcal{L})$, and $\mathcal{B}$ is a decidable copy of $\mathcal{A}$. Let $\mathcal{L}_{\mathcal{B}}$ be a substructure of $\mathcal{B}$ such that the universe of $\mathcal{L}_{\mathcal{B}}$ is the set of all elements that are first in their $\sim_B$-classes (where by first we mean under the standard ordering of ω). It is easy to show that $\mathcal{L}_{\mathcal{B}}$ is a $\mathbf{0}'$-computable copy of $\mathcal{L}$.

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For non-zero $k \in \omega$ and an element $c \in \mathcal{L}_{\mathcal{B}}$, we find elements $b(c,k)$ and $b(c,-k)$ from $\mathcal{B}$ such that

$$\mathcal{B} \models \exists z_0 \ldots \exists z_k((c = z_0)\&S(z_0,z_1)\&S(z_1,z_2)\& \ldots \&S(z_{k-1},z_k)\&(z_k = b(c,k))),$$
$$\mathcal{B} \models \exists z_0 \ldots \exists z_k((b(c,-k) = z_0)\&S(z_0,z_1)\&S(z_1,z_2)\& \ldots \&S(z_{k-1},z_k)\&(z_k = c)).$$

Fix a **d**-computable isomorphism f from $\mathcal{L}$ onto $\mathcal{L}_{\mathcal{B}}$. For $a \in \mathcal{L}$, set $g(\langle 0,a\rangle) = f(a)$, and $g(\langle z,a\rangle) = b(f(a),z)$, where $z \in \zeta$ and $z \neq 0$. It is straightforward to prove that g is a **d**-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Therefore, $\mathcal{A}$ is a **d**-SC-autostable structure.

Assume that $\mathcal{A}_0 = \mathrm{St}(\zeta \times \mathcal{L}_0)$. Let f be an isomorphism from $\mathcal{A}$ to $\mathcal{A}_0$. For an element $a \in \mathcal{L}$, if $f(\langle 0,a\rangle) = \langle z,b\rangle$, then set $g(a) = b$. It is not difficult to check that g is an isomorphism from $\mathcal{L}$ to $\mathcal{L}_0$. Therefore, we have $\mathbf{d} \leqslant \deg_T(g) \leqslant \deg_T(f)$. Hence, every isomorphism from $\mathcal{A}$ to $\mathcal{A}_0$ computes **d**, and **d** is the degree of SC-autostability for $\zeta \times \mathcal{L}$. □

Corollary 5.2. Suppose that **d** and $\mathcal{L}$ are as in Theorem 5.4, and $0 < n < \omega$. Then, **d** is the degree of autostability relative to n-constructivizations for the order $\zeta \times \mathcal{L}$.

Theorem 5.4 and constructions from Bazhenov (to appear) produce new examples of degrees of SC-autostability for discrete linear orders.

Theorem 5.5 (Bazhenov to appear, Theorem 2.1). Let β be an infinite computable ordinal, and $\omega^\beta \leqslant \alpha < \omega^{\beta+1}$. Then, α has the degree of autostability $\mathbf{0}^{(2\beta)}$.

Corollary 5.3. Suppose that β is an infinite computable ordinal, and $\omega^\beta \leqslant \alpha < \omega^{\beta+1}$. Then, the order $\zeta \times \alpha$ has the degree of SC-autostability $\mathbf{0}^{(2\beta)}$.

Proof. We need to check the conditions (a) and (b) from Theorem 5.4. The first condition follows from the result of Ash (1986a): He showed that the ordinal α is relatively $\Delta^0_{2\beta}$ categorical (see also Ash and Knight 2000, Theorem 17.5). The second condition is explicit in the construction from Theorem 5.5: We build two computable copies of α such that every isomorphism between them computes $\mathbf{0}^{(2\beta)}$. □

Theorem 5.6 (Bazhenov to appear, Theorem 3.1). Suppose that α is a computable infinite ordinal, and **d** is a Turing degree such that **d** is c.e. in and above $\mathbf{0}^{(2\alpha+1)}$. There exists a computable linear order $\mathcal{L}$ with the degree of autostability **d**.

Proof. For the sake of completeness, here, we give a brief sketch of the proof. Fix a $\Pi^0_{2\alpha+1}$ set S from **d**. We use the results from Ash and Knight (2000, Section 15.3.3) and apply Theorem 2.2 to obtain a computable sequence $\{\mathcal{A}_n\}_{n\in\omega}$ such that

$$\mathcal{A}_{2k} \cong \begin{cases} \omega^\alpha, & k \in S, \\ \omega^\alpha \times 2, & k \notin S; \end{cases} \qquad \mathcal{A}_{2k+1} \cong \begin{cases} \omega^\alpha \times 2, & k \in S, \\ \omega^\alpha \times 3, & k \notin S. \end{cases}$$

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For an integer z, let $p(z)$ denote the parity of z, i.e., $p(2y) = 0$ and $p(2y+1) = 1$. Recall that η denotes the linear order of rationals. We define

$$\mathcal{L} = \sum_{k\in\omega} \left(k + 2 + \eta + 1 + \sum_{z\in\zeta} \left(1 + \eta + \mathcal{A}_{2k+p(z)}\right) \right).$$

One can prove that $\mathcal{L}$ is a **d**-autostable structure.

We also build a computable sequence $\{\mathcal{B}_n\}_{n\in\omega}$ such that

$$\mathcal{B}_{2k} \cong \begin{cases} \omega^\alpha, & k \in S, \\ \omega^\alpha \times 3, & k \notin S; \end{cases} \qquad \mathcal{B}_{2k+1} \cong \omega^\alpha \times 2.$$

The linear order

$$\mathcal{L}_0 = \sum_{k\in\omega} \left(k + 2 + \eta + 1 + \sum_{z\in\zeta} \left(1 + \eta + \mathcal{B}_{2k+p(z)}\right) \right).$$

is a computable copy of $\mathcal{L}$ with the following property: Every isomorphism $f : \mathcal{L} \cong \mathcal{L}_0$ computes S. Therefore, **d** is the degree of autostability for $\mathcal{L}$. □

Corollary 5.4. Suppose that α is a computable infinite ordinal, and **d** is a Turing degree such that **d** is c.e. in and above $\mathbf{0}^{(2\alpha+1)}$. There exists a decidable discrete linear order $\mathcal{L}'$ with the degree of SC-autostability **d**.

Proof. Let $\mathcal{L}$ be the order from Theorem 5.6, and $\mathcal{L}' = \zeta \times \mathcal{L}$. The analysis of the proof for Theorem 5.6 shows that $\mathcal{L}$ satisfies the conditions of Theorem 5.4. □

6. PA-degrees

Miller (2009) constructed the first example of a computable structure with no degree of autostability. Csima *et al.* (2013) proved the following: If a computable structure $\mathcal{S}$ is not hyperarithmetically categorical, then $\mathcal{S}$ has no degree of autostability. The natural examples of such structures $\mathcal{S}$ are the Harrison linear order $\omega_1^{\mathrm{CK}} \times (1+\eta)$ and the Harrison Boolean algebra $B(\omega_1^{\mathrm{CK}} \times (1+\eta))$. Fokina *et al.* (2016) gave examples of rigid computable structures with no degree of autostability.

In Bazhenov (2015b), it was shown that there exists a decidable prime model with no degree of SC-autostability. Miller and Shlapentokh (2015) proved that there exists a computable algebraic field whose autostability spectrum contains precisely the PA-degrees. In this section, we use the ideas from Miller and Shlapentokh (2015) to prove the following refinement of the result from Bazhenov (2015b).

Theorem 6.1. There exists a decidable structure $\mathcal{M}$ such that $\mathcal{M}$ is a prime model of the theory $\mathrm{Th}(\mathcal{M})$, and the SC-autostability spectrum of $\mathcal{M}$ contains precisely the PA-degrees.

Proof. Let $L_0 = L_{\mathrm{BA}} \cup \{P_k^1, Q_k^1 : k \in \omega\}$. We will build two isomorphic coloured algebras $\mathcal{M}_L$ and $\mathcal{M}_R$ for the language L_0. Fix a computable Boolean algebra $\mathcal{B} \cong B(\omega)$ with a computable set of atoms. The algebra $\mathcal{B}$ will be an underlying algebra for both $\mathcal{M}_L$ and $\mathcal{M}_R$.

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For $V \in \{L, R\}$, we partition the set $\mathrm{At}(\mathcal{B})$ computably into $\{a_n^V : n \in \omega\} \cup \{b_n^V : n \in \omega\}$. For $k, s \in \omega$, set $P_k^{\mathcal{M}_V} = \{a_k^V, b_k^V\}$, and

$$\begin{aligned} Q_{\langle k,s\rangle}^{\mathcal{M}_L} &= \{a_k^L, b_k^L : \varphi_{k,s+1}(k)\uparrow \text{ or } \varphi_{k,s+1}(k)\downarrow \notin \{0,1\}\} \cup \\ &\quad \{a_k^L : \varphi_{k,s+1}(k)\downarrow \in \{0,1\} \text{ and } \varphi_{k,s}(k)\uparrow\}, \\ Q_{\langle k,s\rangle}^{\mathcal{M}_R} &= \{a_k^R, b_k^R : \varphi_{k,s+1}(k)\uparrow \text{ or } \varphi_{k,s+1}(k)\downarrow \notin \{0,1\}\} \cup \\ &\quad \{a_k^R : \varphi_{k,s+1}(k)\downarrow = 0 \text{ and } \varphi_{k,s}(k)\uparrow\} \cup \{b_k^R : \varphi_{k,s+1}(k)\downarrow = 1 \text{ and } \varphi_{k,s}(k)\uparrow\}. \end{aligned}$$

It is easy to see that $\mathcal{M}_L$ and $\mathcal{M}_R$ are isomorphic L_0-structures. Similarly to the proof of Bazhenov (2015b, Lemma 2), one can show that $\mathcal{M}_L$ and $\mathcal{M}_R$ are decidable coloured algebras with a colouring sequence consisting of quantifier-free formulas.

Lemma 6.1. $\mathcal{M}_L$ is a prime model of the theory $\mathrm{Th}(\mathcal{M}_L)$.

Proof. By Theorem 2.3, it is sufficient to show that $\mathcal{M}_L$ is an atomic model. For an atom $c \in \mathcal{B}$, we define the quantifier-free formula $\psi^c(x)$. Assume that $c \in \{a_k^L, b_k^L\}$.

a. If $\varphi_k(k)\downarrow \notin \{0,1\}$ or $\varphi_k(k)$ is undefined, then $\psi^c(x) = P_k(x)$.

b. Suppose that $\varphi_k(k) \in \{0,1\}$. Choose the smallest $s \in \omega$ such that $\varphi_{k,s+1}(k)$ is defined. If $c = a_k^L$, then $\psi^c(x) = P_k(x) \& Q_{\langle k,s\rangle}(x)$; otherwise, $\psi^c(x) = P_k(x) \& \neg Q_{\langle k,s\rangle}(x)$.

It is easy to show that $\mathcal{M}_L \models \psi^c(c)$. Moreover, if $\mathcal{M}_L \models \psi^c(a) \& \psi^c(b)$, then $a \in \mathrm{At}(\mathcal{B})$ and there is an automorphism of $\mathcal{M}_L$ taking a to b. Hence, ψ^c is a complete formula.

If d is an element from $\mathcal{M}_L$, then either $d \in F_1(\mathcal{B})$, or $\mathrm{C}(d) \in F_1(\mathcal{B})$. Therefore, using arguments similar to those in Bazhenov (2015b, Lemma 5), one can prove that for a tuple $\bar{d}$ from $\mathcal{M}_L$, there is a complete formula $\Phi(\bar{x})$ such that $\mathcal{M}_L \models \Phi(\bar{d})$. This implies that $\mathcal{M}_L$ is a prime model of $\mathrm{Th}(\mathcal{M}_L)$. □

For $k \in \omega$, set $v(2k) = a_k^L$, and $v(2k+1) = b_k^L$. The following definition is based on Miller (2009, Definition 5.1) and Bazhenov (2015b, Definition 6).

Definition 6.1. Let $\mathcal{N}$ be a decidable copy of $\mathcal{M}_L$ with universe N. The universe of the *isomorphism tree* $T_{\mathcal{N}}$ is the set of all functions f with the following properties.

a. $f = \varnothing$, or $\mathrm{dom}(f) = \{v(0), v(1), \ldots, v(s)\}$ for some $s \in \omega$.

b. Assume that L_f is the language $\{P_k : k \in \omega\} \cup \{Q_t : v(t) \in \mathrm{dom}(f)\}$. If $f \neq \varnothing$, then f is an isomorphic embedding from the L_f-structure $(\mathrm{dom}(f), L_f^{\mathcal{M}_L})$ into $(N, L_f^{\mathcal{N}})$.

The ordering of the tree $T_{\mathcal{N}}$ is standard, i.e., $f \preccurlyeq g$ iff $f \subseteq g$. We identify the tree $T_{\mathcal{N}}$ with a computable subtree of $\omega^{<\omega}$. We may assume that $T_{\mathcal{N}}$ is built effectively in the atomic diagram $D(\mathcal{N})$.

The choice of the term 'isomorphism tree' is justified by the following lemma.

Lemma 6.2. Suppose that $\mathcal{N}$ is a decidable copy of $\mathcal{M}_L$. Let I be the set of all isomorphisms from $\mathcal{M}_L$ onto $\mathcal{N}$, and P be the set of all paths through the tree $T_{\mathcal{N}}$. There exists a bijection Ψ from P onto I such that for any $\pi \in P$, π is Turing equivalent to $\Psi(\pi)$.

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Proof. Let $L_1 = L_0 \setminus L_{\mathrm{BA}}$. Suppose that π is a path through $T_{\mathcal{N}}$. Set $g_\pi = \bigcup_{f \prec \pi} f$. It is straightforward to prove that g_π is an isomorphism from the L_1-structure $(\mathrm{At}(\mathcal{M}_L), L_1^{\mathcal{M}_L})$ onto $(\mathrm{At}(\mathcal{N}), L_1^{\mathcal{N}})$. Since the underlying algebra of $\mathcal{N}$ is an isomorphic copy of $B(\omega)$, there is a unique isomorphism h_π from $\mathcal{M}_L$ onto $\mathcal{N}$ such that $h_\pi \supseteq g_\pi$. Furthermore, g_π computes h_π. Set $\Psi(\pi) = h_\pi$. It is not difficult to show that Ψ is a bijection, and $\Psi(\pi) \equiv_T \pi$. □

It is easy to verify that $T_{\mathcal{N}}$ is a finite-branching tree with a computable branching function $b(\sigma; T_{\mathcal{N}})$. Moreover, for $\sigma \in T_{\mathcal{N}}$, we have $b(\sigma; T_{\mathcal{N}}) \leqslant 2$.

Lemma 6.3. The SC-autostability spectrum of $\mathcal{M}_L$ contains precisely the PA-degrees.

Proof. Suppose that $\mathcal{N}$ is a decidable copy of $\mathcal{M}_L$, and $\mathbf{d}$ is a PA-degree. By Theorem 2.4, there is a $\mathbf{d}$-computable path through the isomorphism tree $T_{\mathcal{N}}$. Hence, by Lemma 6.2, there is a $\mathbf{d}$-computable isomorphism from $\mathcal{M}_L$ to $\mathcal{N}$.

Let $A = \{k : \varphi_k(k) = 0\}$ and $B = \{k : \varphi_k(k) = 1\}$. By Theorem 2.4, a degree $\mathbf{d}$ is a PA-degree iff $\mathbf{d}$ computes a separation of A and B. Now assume that f is an isomorphism from $\mathcal{M}_L$ onto $\mathcal{M}_R$. Note the following: If $k \in A$, then $f(a_k^L) = a_k^R$; and if $k \in B$, then $f(a_k^L) = b_k^R$. Hence, the set $\{k : f(a_k^L) = a_k^R\}$ separates A and B, and the Turing degree of f is a PA-degree. Therefore, every degree from $\mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{M}_L)$ is a PA-degree. □

This concludes the proof of Theorem 6.1. □

Recall that the colouring sequence of $\mathcal{M}_L$ consists of quantifier-free formulas. Thus, Proposition 2.1 implies the following.

Corollary 6.1. Suppose that $0 < n < \omega$. There exists a decidable structure $\mathcal{M}$ such that the autostability spectrum relative to n-constructivizations for $\mathcal{M}$ contains precisely the PA-degrees.

7. Conclusion

In conclusion, we formulate some open questions related to Problem 1.1. The results of Goncharov (2011) and Theorem 3.1 show that for a non-limit computable ordinal α, every degree c.e. in and above $\mathbf{0}^{(\alpha)}$ is a degree of SC-autostability. It is natural to ask whether one can extend this result for d.c.e. (n-c.e., β-c.e.) degrees.

Question 7.1. Suppose that α is a computable ordinal, and $2 \leqslant \beta < \omega_1^{\mathrm{CK}}$. Is every degree that is β-c.e. in and above $\mathbf{0}^{(\alpha)}$ a degree of SC-autostability?

Note that it is still unknown whether every 3-c.e. degree is a degree of autostability.

The next question is motivated by the following observation. For a decidable structure $\mathcal{S}$, let $\mathcal{S}^*$ be its morleyzation. It is easy to show that $\mathrm{AutSpec}(\mathcal{S}^*) = \mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{S})$. Therefore, every SC-autostability spectrum is an autostability spectrum.

Question 7.2. Is every autostability spectrum the SC-autostability spectrum for some decidable structure? In particular, is every degree of autostability a degree of SC-autostability?

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Hirschfeldt *et al.* (2002) showed that for a computable structure $\mathcal{S}$, one can find a computable structure $\mathcal{A}_\mathcal{S}$ such that $\mathrm{AutSpec}(\mathcal{A}_\mathcal{S}) = \mathrm{AutSpec}(\mathcal{S})$, and $\mathcal{A}_\mathcal{S}$ belongs to the familiar class of algebraic structures (directed graphs, symmetric irreflexive graphs, partial orders, lattices, rings, integral domains, two-step nilpotent groups, etc.). Miller *et al.* (to appear) proved that the similar result holds for the class of fields.

Question 7.3. Let K be one of the familiar algebraic classes (e.g., directed graphs, partial orders, lattices, groups, fields, etc.). Suppose that $\mathcal{S}$ is a decidable structure (for an arbitrary computable language). Does there always exist a decidable structure $\mathcal{A}_S$ from K such that $\mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{A}_\mathcal{S}) = \mathrm{AutSpec}_{\mathrm{SC}}(\mathcal{S})$?

Note that the structures built in Theorems 3.1 and 6.1 essentially use the infinite language. It is still unknown whether one can build similar examples using structures of a finite language.

The last question is related to the complexity of index sets. Downey *et al.* (2015) proved that the index set of autostable structures is m-complete Π^1_1. Goncharov and Marchuk (2015a, 2015b) proved that the index set of SC-autostable structures is m-complete $\Sigma^0_{\omega+2}$. Furthermore, in Goncharov *et al.* (2015b), it was shown that the index set for SC-autostable Boolean algebras is m-complete $\Sigma^0_{\omega+2}$. We refer the reader to Goncharov *et al.* (2015a) for more results on index sets of SC-autostable structures.

Question 7.4. Let α be a non-zero computable ordinal. What is the exact complexity of the index set for computable structures with the degree of SC-autostability $\mathbf{0}^{(\alpha)}$?

The author is grateful to Pavel Alaev, Sergey Goncharov, Margarita Marchuk, and Russell Miller for fruitful discussions on the subject. The reported study was funded by RFBR, according to the research projects No. 16-31-60058 mol_a_dk, and No. 14-01-00376.

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