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Statistical electrodynamics

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Abstract. The random pure radiation field postulated in an earlier paper is set up in a relativistically invariant manner. The requirement that this field be isotropic in three dimensions makes much of the formalism identical with the theory of isotropic turbulence, as has been noted by a previous author. It is found, however, that the more stringent requirement of invariance under the Lorentz group, together with the fact that the field components satisfy Maxwell's equations, mean that, to within a multiplicative constant, the entire random process is uniquely specified.

In accordance with results obtained previously, the constant is effectively identified as Planck's constant. There then results a striking resemblance between the formalism of the present theory, and that of the quantized pure radiation field, with random variables taking the place of operators, and correlations between these random variables taking the place of commutator brackets.

The infinite field energy, which arises also in quantum field theory, where, however, it is usually considered to be purely formal, remains a central difficulty of the random theory. It can be shown to be a natural consequence of assuming Maxwell's equations, and it therefore becomes necessary to consider non-linear modifications of these.

The connexion between the present results and the inconsistencies of classical and quantum electrodynamics is discussed. Also some new information is obtained concerning the behaviour of thermodynamic quantities under a Lorentz transformation.

1. *Introduction.* In a previous paper ((7)), it was shown that, in the presence of a random electromagnetic field, certain charged classical systems, namely, the free particle and the harmonic oscillator, behave essentially like their quantum-mechanical counterparts. The energy spectrum of the field needed to produce this behaviour was obtained, in the non-relativistic limit, and was found to be the familiar Planck spectrum superposed on a certain fixed spectrum. The latter was therefore identified as a zero-temperature spectrum, and it was suggested that such a random radiation field might be capable of explaining other quantum effects in material systems.

In the present paper we investigate the requirements which special relativity places on such a random theory. Attention is confined to the pure radiation field, and the question of how non-linear material systems behave in such a field is left for a later paper.

As in quantum electrodynamics, we find that a crucial decision has to be made, namely, 'What are the quantities which are observable?' It is necessary to frame

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an extra hypothesis regarding this—the question of how our 'observables' are actually observed will naturally involve a more detailed discussion of the interaction of radiation with matter.

One pleasant feature of the theory is worth noting at this stage. We find it possible to work consistently with the field variables rather than the electromagnetic potentials. This means that the gauge difficulties which beset quantum electrodynamics ((5), page 87) do not arise at all here.

The standpoint we shall take is that the *entire* energy spectrum of the radiation field is observable. By this we shall mean not only that part of the spectrum that is furnished by the Planck formula, but also the hidden zero-temperature spectrum. We are thereby supposing that the latter is indeed responsible for quantum effects, and that measurement of h constitutes an observation of it.

We shall suppose further, that two inertial observers moving in a zero-temperature 'hohlraum' observe the same spectrum. This is an obvious requirement of special relativity, for otherwise it would presumably be possible to distinguish that unique inertial frame which was stationary with respect to the hohlraum. We would then have returned to the pre-relativistic concept of aether!

In considering such a random field, it becomes necessary to study not merely the energy, but the entire stress–energy–momentum tensor, of which the energy is a component. Indeed, we can go further and state that, in a random theory, it is natural to regard this tensor in turn as the contraction of a fourth-rank tensor. If $f_{\alpha\beta}(\mathbf{r}, t)$ is the (antisymmetric) electromagnetic field tensor, then define

$$\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \langle f_{\alpha\beta}(\mathbf{r}_1, t_1) f_{\gamma\delta}(\mathbf{r}_2, t_2)\rangle. \tag{1·1}$$

Assuming stationarity in space and time, we get immediately

$$\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2). \tag{1·2}$$

Then the (expectation) stress tensor is simply

$$S_{\alpha\beta} = \mathscr{E}_{\alpha\gamma}{}^{\gamma}{}_{\beta}(0, 0) - \tfrac{1}{4} g_{\alpha\beta} \mathscr{E}_{\gamma\delta}{}^{\delta\gamma}(0, 0). \tag{1·3}$$

We wish, in what follows, to treat not only $S_{\alpha\beta}$, but also its spectrum as observable. This involves the consideration of a spatio-temporal correlation tensor

$$S_{\alpha\beta}(\mathbf{r}, t) = \mathscr{E}_{\alpha\gamma}{}^{\gamma}{}_{\beta}(\mathbf{r}, t) - \tfrac{1}{4} g_{\alpha\beta} \mathscr{E}_{\gamma\delta}{}^{\delta\gamma}(\mathbf{r}, t). \tag{1·4}$$

It is this tensor which we shall take as the basic observable of the radiation field. In effect, the requirement that the zero-temperature energy spectrum be left invariant under a Lorentz transformation means that this tensor must be isotropic.

It has been recognized by Bourret ((2)) that if a radiation field of this type is to be *spatially* isotropic, many of its transformation properties can be deduced by methods used in the theory of isotropic turbulence. We shall see below that, under the more stringent requirement of four-dimensional isotropy, only one form of spectrum is possible, namely

$$I(\nu) = C\nu^3. \tag{1·5}$$

This was precisely the spectrum postulated in the author's previous paper, within the constant C taking the value $4\pi h/c^3$.

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Of course, such a spectrum, implying, as it does, an infinite field energy, cannot possibly be correct for all values of ν, but it seems unlikely that any corrections will affect the low-frequency part of the spectrum. It will be shown below that this divergence is an inevitable result of assuming the linear Maxwell–Lorentz field equations. We shall also see that the difficulty is intimately linked with the divergences of both classical and quantum electrodynamics.

It is hoped that if this divergence in the pure radiation field can be removed by using a non-linear correction to the Maxwell–Lorentz equations, it will at least cast some light on the divergences of quantum electrodynamics. The prospect is reinforced by the discovery that 'randomization' of the radiation field, as envisaged in the author's previous paper, now bears a distinct formal resemblance to quantization. This resemblance will be made apparent in the following sections.

2. *Theory.* Our aim is to set up an ensemble of solutions of the Maxwell–Lorentz field equations in free space. Since we shall later wish to generalize these, it is as well to explain, that by Maxwell's equations we shall mean the set

$$\left.\begin{aligned} \operatorname{div}\mathbf{D} &= 0, \\ \operatorname{curl}\mathbf{H} &= \frac{1}{c}\dot{\mathbf{D}}, \\ \operatorname{div}\mathbf{B} &= 0, \\ \operatorname{curl}\mathbf{E} &= -\frac{1}{c}\dot{\mathbf{B}}. \end{aligned}\right\} \tag{2·1}$$

These become the Maxwell–Lorentz field equations if we add the assumption that in free space

$$\mathbf{D} = \mathbf{E} \quad \text{and} \quad \mathbf{H} = \mathbf{B}. \tag{2·2}$$

We shall require that the generalized field equations continue to be relativistically invariant, and this requirement will cause the actual Maxwell equations to remain unchanged. However the Lorentz assumption can be modified in a relativistically invariant manner, giving rise to vacuum polarization effects.

Assuming (2·2) for the time being, we find that both **E** and **B** must satisfy the wave equation. In addition, both of these quantities, considered as 3-vectors, have vanishing divergence. Using these two properties, Bourret ((2)) has shown that, in a classical hohlraum,

$$\begin{aligned} \langle E_i(\mathbf{r}_1, t_1)\, E_j(\mathbf{r}_2, t_2)\rangle &= \int J(k)\,(k^2\delta_{ij} - k_i k_j)\cos\left[\mathbf{k}.(\mathbf{r}_1 - \mathbf{r}_2) - kc(t_1 - t_2)\right] d^3\mathbf{k} \\ &= \langle B_i(\mathbf{r}_1, t_1)\, B_j(\mathbf{r}_2, t_2)\rangle. \end{aligned} \tag{2·3}$$

The function $J(k)$ is related to the energy spectrum, $I(\nu)$, in the hohlraum by the relation

$$k^4 J(k) = \frac{c}{4\pi} I\left(\frac{ck}{2\pi}\right). \tag{2·4}$$

Similarly, using the relations connecting **E** and **B**, one obtains the further correlations

$$\langle E_i(\mathbf{r}_1, t_1)\, B_j(\mathbf{r}_2, t_2)\rangle = -\int J(k)\,\epsilon_{ijl} k k_l \cos\left[\mathbf{k}.(\mathbf{r}_1 - \mathbf{r}_2) - kc(t_1 - t_2)\right] d^3\mathbf{k}. \tag{2·5}$$

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Equations (2·3) and (2·5) may be expressed in a single expression by observing that, in covariant notation, **E** and **B** together make an antisymmetric covariant 4-tensor of rank two

$$E_1 = f_{10}, \quad B_1 = f_{23}, \quad \text{etc.} \tag{2·6}$$

Then defining the field correlation as in (1·2), we can write in place of (2·3) and (2·5)

$$\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r},t) = \int J(k)\,(k_\alpha k_\gamma g_{\beta\delta} - k_\alpha k_\delta g_{\beta\gamma} - k_\beta k_\gamma g_{\alpha\delta} + k_\beta k_\delta g_{\alpha\gamma}) \cos(k_\lambda x^\lambda)\, d^3\mathbf{k}, \tag{2·7}$$

where x^λ is the 4-vector $(\mathbf{r}, ct)$ and k^λ is the null 4-vector $(\mathbf{k}, k)$.

The components of this tensor are already invariant under space rotation, reflexion, and time inversion. Therefore, in order to ensure that the tensor is isotropic, it remains only to investigate its behaviour under the special Lorentz transformation

$$\left.\begin{aligned} x^{1'} &= x', & x^{2'} &= x^2, \\ x^{3'} &= \gamma(x^3 - \beta x^0), & x^{0'} &= \gamma(x^0 - \beta x^3), \end{aligned}\right\} \tag{2·8}$$

where $|\beta| < 1$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. The transformed correlation tensor is

$$\mathscr{E}'_{\alpha\beta\gamma\delta}(\mathbf{r}',t') = \int J[\gamma(k' - \beta k'_3)](k'_\alpha k'_\gamma g_{\beta\delta} - k'_\alpha k'_\delta g_{\beta\gamma} - k'_\beta k'_\gamma g_{\alpha\delta} + k'_\beta k'_\delta g_{\alpha\gamma}) \cos(k'_\lambda x^{\lambda'})\, \gamma\left(1 - \frac{\beta k'_3}{k'}\right) d^3\mathbf{k}'. \tag{2·9}$$

By comparison between (2·7) and (2·9), we can infer that $\mathscr{E}_{\alpha\beta\gamma\delta}$ will be isotropic if and only if

$$J(k) = \frac{\text{constant}}{k}. \tag{2·10}$$

This implies the relation quoted in the previous section, namely

$$I(\nu) = C\nu^3. \tag{1·5}$$

As a matter of fact this is not *quite* the only spectrum which can satisfy our criteria. We have not yet specified that either $\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r},t)$ or even its contraction $S_{\alpha\beta}(\mathbf{r},t)$ is observable. If we had made either of these assumptions, then (1·5) would indeed be the only possible spectrum. If, however, we adhere to the assumption that only the energy spectrum (or time correlation tensor) is observable, this gives the less restrictive condition that $S_{\alpha\beta}(0,t)$ is isotropic. There then results the integral equation

$$J(k) = \frac{1}{2k^2}\int_{-k}^{k} \gamma(k - \beta k_3)\, J[\gamma(k - \beta k_3)]\, dk_3. \tag{2·11}$$

This has the general solution

$$J(k) = \frac{A}{k} + \frac{B}{k^3}. \tag{2·12}$$

The second term cannot be completely discounted, but corresponds to a spectrum which increases linearly with frequency, which is very unfamiliar. Also it will cease to be permissible as soon as we regard spatial as well as time correlations as observable. We shall not consider this term further in the present paper.

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3. *Randomization and quantization.* We identify the constant C in equation (1·5) in accordance with the results of the previous paper, and obtain

$$J(k) = \frac{\hbar c}{4\pi^2 k}. \tag{3·1}$$

Now clearly, since the integral in (2·9) is divergent, many of the operations used in deriving it are not mathematically justified. However, we now wish to develop a formal connexion with quantization, where the situation is similar. We therefore postpone considerations of convergence and take them up in the next section.

Substituting (3·1) in (2·9), we obtain

$$\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r},t) = -2\pi\hbar c(\partial_\alpha\partial_\beta g_{\beta\delta} - \partial_\alpha\partial_\delta g_{\beta\gamma} - \partial_\beta\partial_\gamma g_{\alpha\delta} + \partial_\beta\partial_\delta g_{\alpha\beta})\,\Delta_1(\mathbf{r},t), \tag{3·2}$$

where

$$\Delta_1(\mathbf{r},t) = (2\pi)^{-3}\int \cos(k_\lambda x^\lambda)\frac{d^3\mathbf{k}}{k}. \tag{3·3}$$

The latter integral is, of course, divergent, but it is a generalization, well known in quantum electrodynamics, of the Dirac δ-function ((5), page 72).

Equation (3·2) resembles closely the field commutator obtained from quantization. For if we define

$$\mathscr{Q}_{\alpha\beta\gamma\delta}(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2) = [f_{\alpha\beta}(\mathbf{r}_1,t_1), f_{\gamma\delta}(\mathbf{r}_2,t_2)], \tag{3·4}$$

where $f_{\alpha\beta}$ are now, of course, operators, then the expression obtained for $\mathscr{Q}_{\alpha\beta\gamma\delta}$ is ((5)), page 78)

$$\mathscr{Q}_{\alpha\beta\gamma\delta}(\mathbf{r},t) = 4\pi i\hbar c(\partial_\alpha\partial_\gamma g_{\beta\delta} - \partial_\alpha\partial_\delta g_{\beta\gamma} - \partial_\beta\partial_\gamma g_{\alpha\delta} + \partial_\beta\partial_\delta g_{\alpha\gamma})\,\Delta(\mathbf{r},t), \tag{3·5}$$

where

$$\Delta(\mathbf{r},t) = -(2\pi)^{-3}\int \sin(k_\lambda x^\lambda)\frac{d^3\mathbf{k}}{k}. \tag{3·6}$$

The randomization which leads to (3·2) is now seen to resemble closely the quantization which leads to (3·5). Whereas the basic elements of quantum electrodynamics are *commutators* like $\mathscr{Q}_{\alpha\beta\gamma\delta}$, we find in random electrodynamics *correlators* like $\mathscr{E}_{\alpha\beta\gamma\delta}$. The outstanding differences between (3·2) and (3·5) are all faithful reflexions of this difference between $\mathscr{Q}$ and $\mathscr{E}$. Thus we find the even function Δ_1 in (3·2) replacing the odd function Δ in (3·5), because by their nature $\mathscr{E}$ and $\mathscr{Q}$ are even and odd, respectively. Similarly the extra factor of $-2i$ in (3·5) is natural, since a commutator is an anti-hermitian operator made up of two terms.

The resemblance becomes even closer, if we calculate the vacuum expectation value of $f_{\alpha\beta}(\mathbf{r}_1,t_1)\,f_{\gamma\delta}(\mathbf{r}_2,t_2)$ according to quantum electrodynamics. We require the vacuum state vector to have the property

$$f^-_{\alpha\beta}(\mathbf{r}_1,t_1)\,\Psi_0 = 0, \tag{3·7}$$

where $f^-_{\alpha\beta}(\mathbf{r}_1,t_1)$ is the 'negative frequency part' of the field intensity operator (for definition, see (5), page 94). Then the vacuum expectation value of $f_{\alpha\beta}(\mathbf{r}_1,t_1)f_{\gamma\delta}(\mathbf{r}_2,t_2)$ is

$$\langle\Psi_0^* f_{\alpha\beta}(\mathbf{r}_1,t_1)f_{\gamma\delta}(\mathbf{r}_2,t_2)\,\Psi_0\rangle = [f^-_{\alpha\beta}(\mathbf{r}_1,t_1)f^+_{\gamma\delta}(\mathbf{r}_2,t_2)] \quad \text{(using (3·7))}. \tag{3·8}$$

The calculation of this commutator is straightforward and its value is

$$-2\pi\hbar c(\partial_\alpha\partial_\gamma g_{\beta\delta} - \partial_\alpha\partial_\delta g_{\beta\gamma} - \partial_\beta\partial_\gamma g_{\alpha\delta} + \partial_\beta\partial_\delta g_{\alpha\gamma})\,(\Delta_1 - i\Delta)\,(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2). \tag{3·9}$$

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Δ is an odd function of $\mathbf{r}$ and t, so that we obtain finally the result that

$$\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2) = \langle \tfrac{1}{2}\{f_{\alpha\beta}(\mathbf{r}_1, t_1)f_{\gamma\delta}(\mathbf{r}_2, t_2)+f_{\gamma\delta}(\mathbf{r}_2, t_2)f_{\alpha\beta}(\mathbf{r}_1, t_1)\}\rangle_0. \qquad (3\cdot10)$$

So the correlation tensor $\mathscr{E}$ has an exact counterpart in quantum electrodynamics.

The expression on the right-hand side of equation (3·10) in turn resembles another important quantum theoretical function, the 'causal propagation function'. This may be defined

$$\mathscr{E}^c_{\alpha\beta\gamma\delta}(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2) = \langle T f_{\alpha\beta}(\mathbf{r}_1, t_1)f_{\gamma\delta}(\mathbf{r}_2, t_2)\rangle_0, \qquad (3\cdot11)$$

where T is the chronological ordering operator.

The detailed expression for $\mathscr{E}^c_{\alpha\beta\gamma\delta}$ differs from the right-hand side of (3·2), in that the function Δ_1 is replaced by

$$\Delta_2 = \begin{cases} \Delta_1 - i\Delta & (t_1 > t_2), \\ \Delta_1 + i\Delta & (t_1 < t_2). \end{cases} \qquad (3\cdot12)$$

The function Δ_2 ((2), page 75) is a causal Green's function, and plays an important role in the theory of the S-matrix. It is identical, apart from a multiplicative constant with the Feynman function D_F.

Our correlation tensor thus has the form of a 'non-causal propagation function'. (The name is as apt in random electrodynamics as in quantum electrodynamics.) The random theory appears to have a smaller element of causality than quantum theory. This view will be reinforced when we discuss, in the next section, the nature of the frequency cut-off.

4. *The frequency cut-off.* It now becomes a matter of urgency to consider ways and means of dealing with the infinite field energy. Actually the problem is directly connected with the failure of classical electrodynamics to produce a model of the electron. This failure is now known to stem from the fact that the stress tensor has a vanishing trace ((5), Chap. XVI), for one then finds that the field energy and momentum transform like the energy and momentum of a particle, if and only if the stress tensor vanishes identically.

It now becomes apparent, however, that the transformation paradox arises at a more fundamental level. For the model of the pure radiation field which we have constructed also relies on the stress tensor. Our expectation stress tensor must still have a vanishing trace, since all the fields of the ensemble have this property. At the same time, we have demanded that it be isotropic, and since its off-diagonal elements vanish, this obviously implies that the (expectation) stress tensor is a multiple of the metric tensor $g_{\alpha\beta}$. Thus the stress tensor must again vanish identically. It is therefore not surprising that our efforts to construct a non-vanishing stress tensor in a random theory should lead to divergent integrals!

The problem exists, in a formal sense, in quantum electrodynamics also, but in that case it can be transformed away by a suitable redefinition of the stress tensor. The definition depends on a reordering of the operators arising from the classical expression ((6), page 47). This artifice is not available to us, for our correlation tensor $\mathscr{E}$ is unchanged by such reordering processes. However, we would hardly wish to transform away the zero-point energy in the random theory, since it is of the essence of this theory that *the field exists in the classical sense.*

As we have seen, the problem cannot possibly be resolved within the framework

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of the Maxwell–Lorentz theory. The assumption (1·2) will have to be modified, and this will lead us to a study of a non-linear wave equation. Such modifications have been attempted before, for example, by Born in ((1)), but subsequent attempts to quantize the non-linear equations met with little success. It seems worthwhile, however, to see whether the more classical and less formal process of randomization is more successful.

The problem will not be attempted in the present paper. However, we can see already that the assumption made in section 1 concerning the observables will also need modification. It seems natural, since the vacuum is to be a polarizable medium, to define a new correlation tensor

$$\mathscr{E}_{\alpha\beta\gamma\delta}(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2) = \langle f_{\alpha\beta}(\mathbf{r}_1, t_1)\, h_{\gamma\delta}(\mathbf{r}_2, t_2)\rangle, \tag{4·1}$$

where $h_{\gamma\delta}$ is the 'displacement tensor' whose components are **D** and **H**. Contraction of this will give us a new $S_{\alpha\beta}(\mathbf{r}, t)$, and $S_{\alpha\beta}(0, 0)$ will be the well-known asymmetric stress tensor ((8)). We would then hope that the isotropic requirement on $S_{\alpha\beta}(\mathbf{r}, t)$ would lead to a non-singular result, that is to a convergent spectrum.

If such a programme could be realized, it would again have a parallel in quantum field theory. For if the correlation tensor should turn out to be a non-singular function, this would inevitably mean that points with a space-like separation would have field variables with a non-vanishing correlation. In a random theory it is inevitable that this should be so, for although no signal could have travelled from one point to the other, this does not preclude a signal having arrived at both points from some third point. However, in quantum field theory, the commutators are given their physical meaning in terms of simultaneous observability. It is therefore normally assumed that the commutator of two field variables at points with space-like separation vanishes. This means that it is assumed that the photon field is *local*. From the standpoint of the present paper, the property of locality is one which must ultimately break down. Its status in random theory is that of a *white noise approximation*, for that is what δ-function correlations imply.

It could, of course, be objected that such arguments take too seriously the analogy we established in the previous section between randomization and quantization. It is certainly true, that at present they are nothing more than conjectures. However, if the random theory is to give a satisfactory explanation of other quantum phenomena, as is hoped, we must expect to find even more strong resemblances with quantum electrodynamics than we have found so far, in view of the success of the latter. At the same time, any light, however conjectural, which can be cast on the divergent integrals of quantum electrodynamics is well worth further investigation.

5. *Temperature in the random theory.* We have characterized the zero-temperature hohlraum by the requirements of Lorentz invariance of $S_{\alpha\beta}$. By considering, instead of (1·5), the spectrum appropriate for a hohlraum at non-zero temperature, we obtain information about the transformation properties of temperature and other thermodynamic quantities. We therefore consider the energy spectrum in an inertial frame N_0 to be

$$J(k) = \frac{\hbar c}{4\pi^2 k} \cot\frac{k}{2\theta}, \tag{5·1}$$

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where θ in normal thermodynamic notation, is given by

$$\theta = \frac{\kappa T}{\hbar c}. \tag{5·2}$$

For the purposes of this section we shall refer to θ as 'temperature'. It differs from the normal absolute temperature T only by a multiplicative constant.

We have seen that the spectrum (5·1) has to be modified for high frequencies, but for low (that is infra-red) frequencies, we may write approximately

$$J(k) = \frac{\hbar c\,\theta}{2\pi^2 k^2}. \tag{5·3}$$

According to section 2, the spectrum in a frame N defined by the Lorentz transformation (2·8) will be

$$\begin{aligned} J'(k) &= \frac{\hbar c\theta}{4\pi^2 k^2}\int_{-k}^{k} [\gamma(k-\beta k_3)]^{-1}\,dk_3 \\ &= \frac{\hbar c\theta}{4\pi^2 k^2}\frac{1}{\gamma\beta}\log\frac{1+\beta}{1-\beta}. \end{aligned} \tag{5·4}$$

It is unlikely that the non-linear modifications envisaged in the previous section will affect the infra-red part of the spectrum. We can therefore state that infra-red observations will lead to the following transformation law for temperature:

$$\begin{aligned} \theta' &= \frac{\theta}{2\gamma\beta}\log\frac{1+\beta}{1-\beta} \\ &= \theta\left[1-\frac{\beta^2}{6}+o(\beta^4)\right]. \end{aligned} \tag{5·5}$$

The transformation properties of temperature have been discussed previously from a macroscopic standpoint ((3), (9), (10)) but the resulting transformation law is different from (5·5). The reason for the discrepancy is fundamental, and becomes evident as soon as we attempt to transform the more exact expression (5·1) instead of the approximate expression (5·3). We then get

$$J'(k) = \frac{\hbar c}{8\pi^2 k^2}\int_{-k}^{k} \coth\frac{\gamma(k-\beta k_3)}{2\theta}\,dk_3. \tag{5·6}$$

This cannot possibly be cast in the form of (5·1). It therefore seems that an observer moving inside the hohlraum can measure his velocity relative to the walls of the enclosure by *purely thermal measurements*. The concept of temperature, and even the concept of thermal equilibrium, are appropriate only in the privileged frame N_0, while the transformation law (5·5) will be valid only if observations are confined to frequencies which lie within the range of validity of the Rayleigh–Jeans distribution (5·3).

This last conclusion has important cosmological consequences. It means, in effect, that in 'free space' the temperature must be zero. On the other hand, if we are in the neighbourhood of a material body which is in a state of (approximate) thermal

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equilibrium, we can measure our velocity relative to that body by purely thermal measurements. This is analogous to the situation in general relativity, where gravitational effects also make possible such a locally privileged inertial frame.

The concept of zero-temperature is, of course, an idealization in the same way as the concept of flat space-time. They both depend on 'isolation' from the influence of all material bodies. In the thermal case, however, we specifically said that the zero-temperature radiation was an unavoidable result of the interaction of all systems, 'isolated' or not, with the external world. The resolution of the paradox must lie in a sort of thermal Mach Principle, whereby the zero-temperature field is an overall average, and local variations of temperature exist in the neighbourhood of all material bodies in the same way as local variations of gravitational field.

6. *Discussion.* We are now in a better position to evaluate the results of the previous paper ((7)). The random field there postulated has here been defined in a relativistically invariant manner. We found that the field variable correlation tensor $\mathscr{E}$ has an exact analogue in quantum electrodynamics. This formal resemblance is so close that we can justifiably claim that randomization is actually a form of quantization. The field propagation function differs significantly from the Feynman function, in that it is not causal. Also the correlation function itself is taken as the basis of the theory, whereas in quantum electrodynamics it has to be derived from a set of commutation relations.

These differences apart, however, the standpoint of the previous paper seems to have been 'second quantization without the first'. We have found a way to randomize—or quantize—the radiation field, in such a way that, at any rate some material systems can be correctly treated without needing to quantize them at all. We cannot yet say whether the same treatment will be equally successful for non-linear systems, but one thing we can reasonably expect—that any supplementary quantizations that do become necessary should be expressible directly in terms of the joint position-and-momentum distribution function. In quantum-mechanical terms, this is equivalent to stating all the assumptions in terms of expectation values of observables.

In view of the considerations of section 4, there is potentially much to be gained from basing one's assumptions directly on the energy spectrum, as we have done. Any non-linearity in the field equations is then reflected in the correlation tensor. In quantum electrodynamics also, there is a breakdown in linearity, resulting from virtual pair creation in photon–photon collisions ((4)). The analogous procedure there would then be to use this non-linearity to modify the 'basic' commutators, thereby making the field non-localizable. The formal difficulties of such a procedure are known to be very formidable ((11)) in the case of orthodox quantization, but look like being much less so for the more classical randomization.

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