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Realizing algebraic 2-complexes by cell complexes

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Abstract

The realization theorem asserts that for a finitely presented group G, the D(2) property and the realization property are equivalent as long as G satisfies a certain finiteness condition. We show that the two properties are in fact equivalent for all finitely presented groups.

1. *Introduction*

An open question in low dimensional topology is: given a finite cell complex of cohomological dimension n (with respect to all coefficient bundles) must it be homotopy equivalent to a finite n-complex (that is a finite cell complex whose cells are all ofdimension less than or equal to n)? In 1965 Wall introduced the problem and showed that a counter example must have $n \leqslant 2$(see [**5**]). The Stallings–Swan theorem (proved a few years later (see [**3**, **4**]) implied that $n > 1$. So a counter example must be (up to homotopy equivalence) a finite 3-complex of cohomological dimension 2, which is not homotopy equivalent to any finite 2-complex. The question of whether or not such a complex exists is known as 'Wall's D(2) problem'.

This question may be parameterized by fundamental group. That is,we say a finitely presented group G satisfies the D(2) property if every connected finite cohomologically 2 dimensional 3-complex with fundamental group G is homotopy equivalent to a finite 2-complex. Wall's D(2) problem then becomes: "Does every finitely presented group G satisfy the D(2) property?".

We say a module over $\mathbb{Z}[G]$ is f.g. free if it has a finite basis and we say a module S is f.g. stably free if $S \oplus F_1 \cong F_2$, where F_1 and F_2 are f.g. free.

An algebraic 2-complex over G is an exact sequence of f.g. stably free modules over $\mathbb{Z}[G]$:

$$S_2 \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0$$

where $\operatorname{coker}(d_1) \cong \mathbb{Z}$.

Given a finite 2-complex, X, with fundamental group G we may obtain an algebraic 2-complex by taking the cellular chain complexof its universal cover. We denote this algebraic 2-complex $C_*(X)$.

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We say a finitely presented group G satisfies the realization property if every algebraic 2-complex over G is 'geometrically realizable'. That is, it is chain homotopy equivalent to $C_*(X)$, for some finite 2-complex X.

This property may be stated in purely algebraic terms. Given a connected finite 2-complex X with $\pi_1(X) \cong G$, we may quotient out a maximal tree in its 1-skeleton to attain the Cayley complex $\mathcal{K}_\mathcal{G}$ where $\mathcal{G}$ is some finite presentation of G. Then $\mathcal{K}_\mathcal{G}$ is in the same homotopy type as X.

An algebraic 2-complex is therefore geometrically realized precisely when it is chain homotopy equivalent to $C_*(\mathcal{K}_\mathcal{G})$ for some finite presentation $\mathcal{G}$ of G. Note $C_*(\mathcal{K}_\mathcal{G})$ may be computed directly from $\mathcal{G}$ using Fox's free differential calculus [**1**, section 48].

For finite groups it has been shown that the 'geometric' D(2) property is equivalent to the 'algebraic' realization property [**1**, theorem I]. Indeed in one direction this is extended to all finitely presented groups [**1**, appendix B]:

Weak realization theorem. The realization property for G implies the D(2) property for G.

In the other direction it is sufficient to show that every algebraic 2-complex $\mathcal{A}$ over G is chain homotopy equivalent to $C_*(Y)$, for some finite cell complex Y. Then the D(2) property for G would imply that Y was homotopy equivalent to a finite 2-complex X, which would geometrically realize $\mathcal{A}$.

Johnson does this in [**1**, appendix B] subject to the existence of a truncated resolution:

$$F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the F_i are f.g. free modules over $\mathbb{Z}[G]$. A group where such a truncated resolution exists is said to satisfy the FL(3) condition.

We follow Johnson's approach but use a more general "eventual stability" result so that we do not require the FL(3) condition. Thus our main result is:

THEOREM 1·1. *A finitely presented group G satisfies the D(2) property if and only if it satisfies the realization property.*

2. *Realizing algebraic 2-complexes by finite cell complexes*

Let G be a finitely presented group and let $\mathcal{A}$ denote an algebraic 2-complex over G:

$$S_2 \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0$$

THEOREM 2·1. *We may realize $\mathcal{A}$ up to chain homotopy equivalence by a finite cell complex.*

Proof. Let Y_1 denote the Cayley complex of a finite presentation of G. Then $C_*(Y_1)$ is an algebraic 2-complex:

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

and the C_i are f.g. free modules over $\mathbb{Z}[G]$.

Let $C \cong S_2 \oplus C_1 \oplus S_0$ and let $S \cong C_2 \oplus S_1 \oplus C_0$.

We may pick a f.g. free module Q such that $C \oplus Q$and $S \oplus Q$ are both f.g. free. Let Y_2 denote the wedge of Y_1 with $\mathrm{rk}_{\mathbb{Z}[G]}(C \oplus Q)$ 2-spheres. We may identify $C_*(Y_2)$ with:

$$C_2 \oplus (C \oplus Q) \xrightarrow{\partial_2 \oplus 0} C_1 \xrightarrow{\partial_1} C_0$$

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Let:

$$\mathcal{A}' = S \oplus Q \xrightarrow{\iota} S_2 \oplus (S \oplus Q) \xrightarrow{d_2 \oplus 0} S_1 \xrightarrow{d_1} S_0$$

where ι denotes the natural inclusion into the second summand. The natural inclusion $\mathcal{A} \hookrightarrow \mathcal{A}'$ is a chain homotopy equivalence.

By [**2**, theorem 1.1] we have a chain homotopy equivalence:

$$\{S_2 \oplus S \xrightarrow{d_2 \oplus 0} S_1 \xrightarrow{d_1} S_0\} \tilde{\longrightarrow} \{C_2 \oplus C \xrightarrow{\partial_2 \oplus 0} C_1 \xrightarrow{\partial_1} C_0\}.$$

By extending this to the identity on Q we get a chain homotopy equivalence:

$$\begin{array}{ccccc} S_2 \oplus (S \oplus Q) & \xrightarrow{d_2 \oplus 0} & S_1 & \xrightarrow{d_1} & S_0 \\ \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_0 \\ C_2 \oplus (C \oplus Q) & \xrightarrow{\partial_2 \oplus 0} & C_1 & \xrightarrow{\partial_1} & C_0 \end{array}$$

Let $\vec{f}_1, \ldots, \vec{f}_m$ be a basis for $S \oplus Q$. Then $\phi_2 \iota \vec{f}_i \in H_2(\tilde{Y}_2; \mathbb{Z})$ for each $i \in \{1, \ldots, m\}$. Let $h\colon H_2(\tilde{Y}_2; \mathbb{Z}) \tilde{\to} \pi_2(\tilde{Y}_2)$ denote the inverse of the Hurewicz isomorphism. The covering map, $\tilde{Y}_2 \to Y_2$, induces an isomorphism $p\colon \pi_2(\tilde{Y}_2) \tilde{\to} \pi_2(Y_2)$.

For each $i \in \{1, \ldots, m\}$, let E_i be a 3-ball and let $\psi_i\colon \partial E_i \to Y_2$ denote a map representing the homotopy element $ph\phi_2 \iota \vec{f}_i$. Let Y denote the 3-complex constructed by attaching each 3-cell E_i to Y_2 via the attaching map ψ_i, for $i \in \{1, \ldots, m\}$. Then $C_*(Y)$ is given by:

$$S \oplus Q \xrightarrow{\phi_2 \iota} C_2 \oplus (C \oplus Q) \xrightarrow{\partial_2 \oplus 0} C_1 \xrightarrow{\partial_1} C_0.$$

We now have a chain homotopy equivalence $\mathcal{A}' \tilde{\to} C_*(Y)$:

$$\begin{array}{ccccccc} S \oplus Q & \xrightarrow{\iota} & S_2 \oplus (S \oplus Q) & \xrightarrow{d_2 \oplus 0} & S_1 & \xrightarrow{d_1} & S_0 \\ \downarrow 1 & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_0 \\ S \oplus Q & \xrightarrow{\phi_2 \iota} & C_2 \oplus (C \oplus Q) & \xrightarrow{\partial_2 \oplus 0} & C_1 & \xrightarrow{\partial_1} & C_0 \end{array}$$

Thus $\mathcal{A}$ is geometrically realized by Y.

Note that Y is a finite cohomologically 2-dimensional cell–complex (as cohomology is an invariant of chain homotopy type) with $\pi_1(Y) \cong G$.

In particular, if the D(2) property holds for G then we may find a finite 2-complex X in the homotopy type of Y. Thus $\mathcal{A}$ is geometrically realized by the 2-complex X.

Hence the D(2) property for G implies the realization property for G. From the Weak realization theorem [**1**, appendix B] we know that the realization property for G implies the D(2) property for G. Our proof of Theorem 1.1 is therefore complete.

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