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INDEPENDENCE OF DOUBLE WIENER INTEGRALS

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In this paper a necessary and sufficient condition is obtained for two double Wiener integrals to be statistically independent, first in the case of symmetric and continuous kernels. It is also shown that, for more than two double Wiener integrals, pairwise independence implies mutual independence. After that, the continuity condition on the kernels is somewhat relaxed, and it is shown that Craig's (1943, *Annals of Mathematical Statistics* 14, 195–197) theorem on the independence of quadratic forms in normal variables is a special case of the result obtained for the case of discontinuous kernels.

1. INTRODUCTION

In discussing the asymptotic theory of unit root or near unit root problems the distribution of the double Wiener integral, or a quadratic functional in a Brownian motion, plays an important role. See, for example, Tanaka (1996). The primary purpose of this paper is to find a necessary and sufficient condition for two double Wiener integrals to be statistically independent.

The double Wiener integral is a continuous analogue of the quadratic form in $NID(0,1)$ random variables. In this discrete case we have the following famous theorem.

THEOREM 1. *Let* $X_1,\ldots,X_N$ *be* $NID(0,1)$ *random variables and put* $X' = (X_1,\ldots,X_N)$. *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be* $N \times N$ *symmetric matrices. Then,* $X'AX$ *and* $X'BX$ *are statistically independent if and only if* $AB = 0$.

This theorem is known as Craig's (1943) theorem, though his proof of the "only if" part was not correct. I obtained a correct proof, and it was published in Ogawa (1949). Besides myself, Matusita (1949), Kawada (1950), and Lancaster (1954) gave a correct proof. See also Mathai and Provost (1992).

In Section 2 of this paper a necessary and sufficient condition is given for two double Wiener integrals to be independent in the case where the kernel functions are symmetric and continuous. In that section it is also shown that, for more than two double Wiener integrals, pairwise independence implies mutual independence. In Section 3 the continuity condition on the kernels is some-

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what relaxed, and it is shown that Theorem 1 can be regarded as a special case of the result obtained for the discontinuous kernels.

2. INDEPENDENCE IN THE CASE OF CONTINUOUS KERNELS

Let $K_1(r,s)$ and $K_2(r,s)$ be two symmetric and continuous functions defined on the unit square $[0,1] \times [0,1]$ and let us discuss the statistical independence of

$$Z_1 = \int_0^1 \int_0^1 K_1(r,s)\, dW(r)\, dW(s) \quad \text{and} \quad Z_2 = \int_0^1 \int_0^1 K_2(r,s)\, dW(r)\, dW(s), \tag{1}$$

where $W(r)$ denotes a standard Brownian motion defined on $[0,1]$.

First, we derive the joint moment generating function (m.g.f.) for (Z_1, Z_2). For $n = 1,2,\ldots$ define two $n \times n$ matrices

$$A_n = \begin{pmatrix} K_1(r_1,r_1) & K_1(r_1,r_2) & \ldots & K_1(r_1,r_n) \\ K_1(r_2,r_1) & K_1(r_2,r_2) & \ldots & K_1(r_2,r_n) \\ \quad \ldots\ldots & & & \\ K_1(r_n,r_1) & K_1(r_n,r_2) & \ldots & K_1(r_n,r_n) \end{pmatrix}$$

and

$$B_n = \begin{pmatrix} K_2(r_1,r_1) & K_2(r_1,r_2) & \ldots & K_2(r_1,r_n) \\ K_2(r_2,r_1) & K_2(r_2,r_2) & \ldots & K_2(r_2,r_n) \\ \quad \ldots\ldots & & & \\ K_2(r_n,r_1) & K_2(r_n,r_2) & \ldots & K_2(r_n,r_n) \end{pmatrix}.$$

Then, for any a and b, the m.g.f. for $aZ_1 + bZ_2$ is given by

$$E[\exp \theta(aZ_1 + bZ_2)] = \bar{D}(2\theta; a, b)^{-1/2},$$

where $\bar{D}(\lambda; a, b)$ is the Fredholm determinant for the kernel $aK_1(r,s) + bK_2(r,s)$ that is defined by

$$\bar{D}(\lambda; a, b) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n \int_0^1 \cdots \int_0^1 \det(aA_n + bB_n) dr_1 \ldots dr_n \tag{2}$$

(see Tanaka, 1996, p. 136). Note that the determinant for $n = 0$ in (2), also in the following, is understood to be 1.

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If we put $\lambda_1 = \lambda a$, $\lambda_2 = \lambda b$, and

$$D(\lambda_1,\lambda_2) = \bar{D}(1;\lambda_1,\lambda_2)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 \cdots \int_0^1 \det(\lambda_1 A_n + \lambda_2 B_n)\, dr_1 \ldots dr_n, \tag{3}$$

then the joint m.g.f. for (Z_1, Z_2) is given by

$$E[\exp(\theta_1 Z_1 + \theta_2 Z_2)] = D(2\theta_1, 2\theta_2)^{-1/2}.$$

The marginal m.g.f.'s for Z_1 and Z_2 are given by $D(2\theta_1,0)^{-1/2}$ and $D(0,2\theta_2)^{-1/2}$, respectively, where

$$D(\lambda_1,0) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \lambda_1^p \int_0^1 \cdots \int_0^1 \det(A_p)\, dr_1 \ldots dr_p \tag{4}$$

and

$$D(0,\lambda_2) = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \lambda_2^q \int_0^1 \cdots \int_0^1 \det(B_q)\, dr_1 \ldots dr_q. \tag{5}$$

Hence, a necessary and sufficient condition for the independence of Z_1 and Z_2 is that the equation

$$D(\lambda_1,\lambda_2) = D(\lambda_1,0)D(0,\lambda_2) \tag{6}$$

should hold for all λ_1 and λ_2. Now we have the following lemma, the proof of which is given in the Appendix.

LEMMA. *Equation (6) is equivalent to*

$$\int_0^1 \cdots \int_0^1 \det(C_{pq})\, dr_1 \ldots dr_{p+q}$$
$$= \int_0^1 \cdots \int_0^1 \det(A_p)\, dr_1 \ldots dr_p \times \int_0^1 \cdots \int_0^1 \det(B_q)\, dr_1 \ldots dr_q \tag{7}$$

for all $p,q = 0,1,\ldots,$ *where*

$$C_{pq} = \begin{pmatrix} K_1(r_1,r_1) & \ldots & K_1(r_1,r_p) & K_1(r_1,r_{p+1}) & \ldots & K_1(r_1,r_{p+q}) \\ & \ldots\ldots & & & \ldots\ldots & \\ K_1(r_p,r_1) & \ldots & K_1(r_p,r_p) & K_1(r_p,r_{p+1}) & \ldots & K_1(r_p,r_{p+q}) \\ K_2(r_{p+1},r_1) & \ldots & K_2(r_{p+1},r_p) & K_2(r_{p+1},r_{p+1}) & \ldots & K_2(r_{p+1},r_{p+q}) \\ & \ldots\ldots & & & \ldots\ldots & \\ K_2(r_{p+q},r_1) & \ldots & K_2(r_{p+q},r_p) & K_2(r_{p+q},r_{p+1}) & \ldots & K_2(r_{p+q},r_{p+q}) \end{pmatrix}.$$

By using this lemma, we have the following theorem.

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THEOREM 2. *Let $K_1(r,s)$ and $K_2(r,s)$ be symmetric and continuous functions defined on $[0,1] \times [0,1]$. Then Z_1 and Z_2 given by (1) are statistically independent, if and only if the equation*

$$\int_0^1 K_1(r,s)K_2(s,t)\,ds = 0 \tag{8}$$

holds for all $(r,t) \in [0,1] \times [0,1]$.

The condition that (8) holds for all $(r,t) \in [0,1] \times [0,1]$ is completely analogous to the condition $AB = 0$ given in Theorem 1 for the discrete case. The "if" part of this theorem is proved by showing that (8) for all $(r,t) \in [0,1] \times [0,1]$ implies (7) for all $p,q = 0,1,\ldots$. On the other hand, the "only if" part is proved by showing that (7) for $(p,q) = (1,1),(2,1),(1,2)$, and $(2,2)$ implies (8) for all $(r,t) \in [0,1] \times [0,1]$. The detail of the proof is given in the Appendix. The idea of the proof of the "only if" part is based on Kawada (1950) dealing with the discrete case.

Example 1

If

$$K_1(r,s) = \sin \pi r + \sin \pi s \quad \text{and} \quad K_2(r,s) = \sin 2\pi r \sin 2\pi s,$$

or if

$$K_1(r,s) = \min(r,s) - \frac{8}{3} rs + r^2 s + rs^2 \quad \text{and} \quad K_2(r,s) = 1,$$

then Z_1 and Z_2 are statistically independent.

Next, consider more than two double Wiener integrals

$$Z_i = \int_0^1 \int_0^1 K_i(r,s)\,dW(r)\,dW(s) \quad (i = 1,\ldots,n) \tag{9}$$

and discuss the mutual independence of $Z_1,\ldots,Z_n$. In the discrete case, if $Z_1,\ldots,Z_n$ are quadratic forms in $NID(0,1)$ random variables $X_1,\ldots,X_N$, then the pairwise independence of $Z_1,\ldots,Z_n$ implies mutual independence, as is easily seen by using Theorem 1. A corresponding result in the continuous case is given by the following theorem, the proof of which is given in the Appendix.

THEOREM 3. *Let Z_i $(i = 1,\ldots,n)$ be defined by (9) with symmetric and continuous functions $K_i(r,s)$ defined on $[0,1] \times [0,1]$. If Z_i and Z_j are independent for every pair i and j $(i,j = 1,\ldots,n;\ i \neq j)$, then $Z_1,\ldots,Z_n$ are mutually independent.*

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3. INDEPENDENCE IN THE CASE OF DISCONTINUOUS KERNELS

As discussed by Nabeya (2001) the Fredholm method of deriving m.g.f.'s can be applied to a wider class of symmetric kernels than the class of symmetric and continuous kernels. In this section, for the sake of later convenience, $W(r)$ is assumed to be a Brownian motion defined on $[0,N]$, and the interval of integration is taken to be $[0,N]\times[0,N]$.

Let $K(r,s)$ be a bounded function defined on $[0,N]\times[0,N]$. If $K(r,s)$ is continuous in every region separated by a finite number of line segments, and if it can be extended continuously to the boundary of each region, then we call $K(r,s)$ piecewise continuous. As noted by Nabeya (2001) the Fredholm method is available in deriving the m.g.f. for the double Wiener integral with a bounded, symmetric, and piecewise continuous kernel.

Theorems 2 and 3 can be modified to cover such piecewise continuous kernels, and the extensions are given as Theorems 4 and 5, which follow. Their proofs are obtained by a small change from those of Theorems 2 and 3, respectively, and so are omitted.

THEOREM 4. *Let $K_1(r,s)$ and $K_2(r,s)$ be bounded, symmetric, and piecewise continuous functions defined on $[0,N]\times[0,N]$. Then*

$$Z_1=\int_0^N\int_0^N K_1(r,s)\,dW(r)\,dW(s)\quad\text{and}\quad Z_2=\int_0^N\int_0^N K_2(r,s)\,dW(r)\,dW(s)$$

are statistically independent, if and only if

$$\int_0^N K_1(r,s)K_2(s,t)\,ds=0 \tag{10}$$

for all $(r,t)\in[0,N]\times[0,N]$ except possibly for a finite number of line segments $r=$ constant or $t=$ constant.

Example 2

Let $W_1(r)$ and $W_2(r)$ be independent Brownian motions defined on the interval $[0,1]$ and let Z_1 and Z_2 be defined by

$$Z_1=\int_0^1 W_1(r)\,dW_1(r)+\int_0^1 W_2(r)\,dW_2(r)$$

and

$$Z_2=\int_0^1 W_1(r)\,dW_2(r)-\int_0^1 W_2(r)\,dW_1(r).$$

Then, Z_1 and Z_2 are shown subsequently not to be independent.

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In fact, if we define $W(r) = W_1(r)$ for $0 \le r \le 1$ and $W(r) = W_1(1) + W_2(r-1)$ for $1 < r \le 2$, then $W(r)$ becomes a Brownian motion defined on $[0,2]$. The Z_1 and Z_2 are expressed in the form

$$Z_1 = -1 + \int_0^2 \int_0^2 K_1(r,s)\, dW(r)\, dW(s)$$

and

$$Z_2 = \int_0^2 \int_0^2 K_2(r,s)\, dW(r)\, dW(s),$$

where

$$\begin{aligned} K_1(r,s) &= \frac{1}{2}, \quad \text{if } 0 < r < 1 \quad \text{and} \quad 0 < s < 1, \\ &\qquad \text{or if } 1 < r < 2 \quad \text{and} \quad 1 < s < 2, \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} K_2(r,s) &= \frac{1}{2}\operatorname{sgn}(s-r-1), \quad \text{if } 0 < r < 1 < s < 2, \\ &= \frac{1}{2}\operatorname{sgn}(r-s-1), \quad \text{if } 0 < s < 1 < r < 2, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

In this case we have

$$\begin{aligned} \int_0^2 K_1(r,s)K_2(s,t)\, ds &= \frac{t}{2} - \frac{3}{4}, \quad \text{if } 0 < r < 1 < t < 2, \\ &= -\frac{t}{2} + \frac{1}{4}, \quad \text{if } 0 < t < 1 < r < 2, \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

so Z_1 and Z_2 are not independent.[1]

Next, we show that Theorem 1 can be regarded as a special case of Theorem 4.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be symmetric $N \times N$ matrices and let $W(r)$ be a Brownian motion defined on the interval $[0, N]$. Define $K_1(r,s)$ and $K_2(r,s)$ on $[0,N] \times [0,N]$ in the following way:

$$K_1(r,s) = K_2(r,s) = 0, \quad \text{if } r \text{ or } s \text{ is an integer,}$$

$$K_1(r,s) = a_{ij} \quad \text{and} \quad K_2(r,s) = b_{ij},$$

$$\text{if } i-1 < r < i \quad \text{and} \quad j-1 < s < j \quad (i,j = 1,\ldots,N).$$

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If we define $X_i = W(i) - W(i-1)$ $(i = 1,\ldots,N)$, then $X_1,\ldots,X_N$ are $NID(0,1)$, and we have

$$\int_0^N \int_0^N K_1(r,s)\, dW(r)\, dW(s) = X'AX$$

and

$$\int_0^N \int_0^N K_2(r,s)\, dW(r)\, dW(s) = X'BX,$$

putting $X' = (X_1,\ldots,X_N)$. The left hand side of (10) is 0, if r or t is an integer, whereas it is equal to $\sum_{j=1}^N a_{ij} b_{jk}$, the (i,k) element of the matrix AB, if $i-1 < r < i$ and $k-1 < t < k$ $(i,k = 1,\ldots,N)$. Hence the condition stated in Theorem 4 reduces to $AB = 0$.

Finally, we give an extension of Theorem 3 to the case of piecewise continuous kernels.

THEOREM 5. *Let $K_i(r,s)$ $(i = 1,\ldots,n)$ be bounded, symmetric, and piecewise continuous functions defined on $[0,N] \times [0,N]$ and define double Wiener integrals*

$$Z_i = \int_0^N \int_0^N K_i(r,s)\, dW(r)\, dW(s) \quad (i = 1,\ldots,n).$$

If Z_i and Z_j are independent for every pair i and j $(i,j = 1,\ldots,n;\ i \neq j)$, then $Z_1,\ldots,Z_n$ are mutually independent.

NOTE

1. The fact that Z_1 and Z_2 are not independent can also be deduced from the joint m.g.f. for (Z_1,Z_2) given by Nabeya (2001). It was shown in the same paper that, if Z_1 and Z_2 are divided by $(\int_0^1 [W_1(r)^2 + W_2(r)^2]\, dr)^{1/2}$, then they are independent. But, it was shown by Nabeya (2000) that, if Z_1 and Z_2 are divided by $\int_0^1 [W_1(r)^2 + W_2(r)^2]\, dr$, then, they are again not independent. See also "Acknowledgment of Authorship" published in *Econometric Theory* 16, p. 796.

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APPENDIX

Proof of Lemma. The $\det(\lambda_1 A_n + \lambda_2 B_n)$ can be expressed as the sum of 2^n determinants in which the ith $(i = 1,\dots,n)$ row is given by

$$\lambda_1(K_1(r_i,r_1),K_1(r_i,r_2),\dots,K_1(r_i,r_n)) \tag{A.1}$$

or

$$\lambda_2(K_2(r_i,r_1),K_2(r_i,r_2),\dots,K_2(r_i,r_n)). \tag{A.2}$$

Let us take up a determinant in which the $i_1,\dots,i_p$th rows are of the form (A.1) and the remaining $j_1,\dots,j_q$th rows are of the form (A.2), where $q = n - p$ and $\{j_1,\dots,j_q\} = \{1,\dots,n\}\backslash\{i_1,\dots,i_p\}$.

Change the order of rows and columns of the determinant so that the original $i_1,\dots,i_p,j_1,\dots,j_q$th rows come to the $1,2,\dots,n$th rows, respectively, and the original $i_1,\dots,i_p,j_1,\dots,j_q$th columns come to the $1,2,\dots,n$th columns, respectively. By this change the value of the determinant is not changed; thus we can see that the integral of the determinant over $(r_1,\dots,r_n) \in [0,1]^n$ has the same value

$$\lambda_1^p \lambda_2^q \int_0^1 \cdots \int_0^1 \det(C_{pq})\, dr_1 \dots dr_{p+q}, \tag{A.3}$$

for any choice of $\{i_1,\dots,i_p\}$ from $\{1,2,\dots,n\}$, as far as p is fixed.

For a fixed p, the set $\{i_1,\dots,i_p\}$ can be selected in $\binom{n}{p}$ ways, so the integral of the sum of these determinants is given by $\binom{n}{p}$ times (A.3). Summing up the product for $p = 0,\dots,n$ we get the integral of $\det(\lambda_1 A_n + \lambda_2 B_n)$.

By using the preceding result, the $D(\lambda_1,\lambda_2)$ in (3) can be written as

$$D(\lambda_1,\lambda_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q}}{p!\,q!} \lambda_1^p \lambda_2^q \int_0^1 \cdots \int_0^1 \det(C_{pq})\, dr_1 \dots dr_{p+q}. \tag{A.4}$$

If we compare the coefficient of $\lambda_1^p \lambda_2^q$ on the right hand side of (A.4) with that in $D(\lambda_1,0)D(0,\lambda_2)$, taking (4) and (5) into account, then we can see that equation (6) is equivalent to (7) for all $p, q = 0,1,\dots$. ■

Proof of the "If Part" in Theorem 2. Assuming that (8) holds for all $(r,t) \in [0,1] \times [0,1]$, we show subsequently that (7) holds for all $p,q = 0,1,\dots$. The equality (7) is trivial if p or q is equal to 0, so we can restrict ourselves to the case $p > 0$ and $q > 0$.

To evaluate $\det(C_{pq})$ apply Laplace's expansion along the first p rows. Let us denote by $\Delta(i_1,\dots,i_p)$ the minor of $\det(C_{pq})$ consisting of the first p rows and the $i_1,\dots,i_p$th

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columns and denote by $\tilde{\Delta}(j_1,\ldots,j_q)$ the minor of $\det(C_{pq})$ consisting of the last q rows and the $j_1,\ldots,j_q$th columns. Then, $\det(C_{pq})$ is equal to the sum of $\binom{p+q}{p}$ terms of the form $\pm\Delta(i_1,\ldots,i_p)\tilde{\Delta}(j_1,\ldots,j_q)$ where $\{j_1,\ldots,j_q\} = \{1,2,\ldots,p+q\}\backslash\{i_1,\ldots,i_p\}$. The term corresponding to $\{i_1,\ldots,i_p\} = \{1,\ldots,p\}$ is $\Delta(1,\ldots,p)\tilde{\Delta}(p+1,\ldots,p+q)$, and its integral over $(r_1,\ldots,r_{p+q}) \in [0,1]^{p+q}$ is equal to the right-hand side of (7). Hence it is sufficient to show that the integral of other terms in the Laplace's expansion is equal to 0.

If $\{i_1,\ldots,i_p\} \neq \{1,\ldots,p\}$, then at least one of $i_1,\ldots,i_p$ is greater than p. Assume that $1 \leq l \leq p$ and $i_l = k > p$. If we expand $\Delta(i_1,\ldots,i_p)$ along the lth column, then we have

$$\Delta(i_1,\ldots,i_p) = \sum_{i=1}^{p} K_1(r_i,r_k)L_i,$$

where $L_i (i = 1,\ldots,p)$ does not depend on r_k. On the other hand, if we expand $\tilde{\Delta}(j_1,\ldots,j_q)$ along the $(k-p)$th row, then we have

$$\tilde{\Delta}(j_1,\ldots,j_q) = \sum_{m=1}^{q} K_2(r_k,r_{j_m})L_m^*,$$

where $L_m^* (m = 1,\ldots,q)$ again does not depend on r_k. By (8) we have

$$\int_0^1 K_1(r_i,r_k)K_2(r_k,r_{j_m})\,dr_k = 0$$

for all $(r_i,r_{j_m}) \in [0,1]\times[0,1]$ $(i = 1,\ldots,p;\ m = 1,\ldots,q)$, as $k = i_l \neq i$ and $k = i_l \neq j_m$. Hence we have

$$\int_0^1 \cdots \int_0^1 \Delta(i_1,\ldots,i_p)\tilde{\Delta}(j_1,\ldots,j_q)\,dr_1\ldots dr_{p+q} = 0,$$

as far as $\{i_1,\ldots,i_p\} \neq \{1,\ldots,p\}$.

Thus, (7) is established.

Proof of the "Only If" part in Theorem 2. Assuming that (7) holds for $(p,q) = (1,1),(2,1),(1,2)$, and $(2,2)$, we show subsequently that (8) holds for all $(r,t) \in [0,1]\times[0,1]$.

From equation (7) with $(p,q) = (1,1)$ we have

$$\int_0^1\int_0^1 \begin{vmatrix} K_1(r_1,r_1) & K_1(r_1,r_2) \\ K_2(r_2,r_1) & K_2(r_2,r_2) \end{vmatrix} dr_1\,dr_2$$

$$= \int_0^1 K_1(r_1,r_1)dr_1 \times \int_0^1 K_2(r_2,r_2)\,dr_2. \qquad \textbf{(A.5)}$$

By taking the difference between both sides of (A.5) we have

$$\int_0^1\int_0^1 K_1(r,s)K_2(s,r)\,drds = 0. \qquad \textbf{(A.6)}$$

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From equation (7) with $(p,q) = (2,1)$ we have

$$\int_0^1\int_0^1\int_0^1 \begin{vmatrix} K_1(r_1,r_1) & K_1(r_1,r_2) & K_1(r_1,r_3) \\ K_1(r_2,r_1) & K_1(r_2,r_2) & K_1(r_2,r_3) \\ K_2(r_3,r_1) & K_2(r_3,r_2) & K_2(r_3,r_3) \end{vmatrix} dr_1\,dr_2\,dr_3$$

$$= \int_0^1\int_0^1 \begin{vmatrix} K_1(r_1,r_1) & K_1(r_1,r_2) \\ K_1(r_2,r_1) & K_1(r_2,r_2) \end{vmatrix} dr_1\,dr_2 \times \int_0^1 K_2(r_3,r_3)\,dr_3. \tag{A.7}$$

If we expand the determinant on the left-hand side of (A.7), then the integral of the sum of two terms involving $K_2(r_3,r_3)$ is equal to the right-hand side of (A.7). Hence, the integral of the sum of the remaining four terms is equal to 0. Thus we have

$$\int_0^1\int_0^1\int_0^1 [-K_1(r_1,r_1)K_1(r_2,r_3)K_2(r_3,r_2) + K_1(r_1,r_2)K_1(r_2,r_3)K_2(r_3,r_1)$$

$$+ K_1(r_1,r_3)K_1(r_2,r_1)K_2(r_3,r_2) - K_1(r_1,r_3)K_1(r_2,r_2)K_2(r_3,r_1)]\,dr_1\,dr_2\,dr_3$$

$$= 0. \tag{A.8}$$

From (A.6) we can see that the first and the fourth terms in the square brackets of (A.8) have the value of the integral equal to 0. The remaining two terms have the same value of the integral. Thus we have

$$\int_0^1\int_0^1\int_0^1 K_1(r,s)K_1(s,t)K_2(t,r)\,drdsdt = 0. \tag{A.9}$$

In the same way, from equation (7) with $(p,q) = (1,2)$ we have

$$\int_0^1\int_0^1\int_0^1 K_1(r,s)K_2(s,t)K_2(t,r)\,drdsdt = 0. \tag{A.10}$$

Finally, from equation (7) with $(p,q) = (2,2)$ we have

$$\int_0^1\int_0^1\int_0^1\int_0^1 \begin{vmatrix} K_1(r_1,r_1) & K_1(r_1,r_2) & K_1(r_1,r_3) & K_1(r_1,r_4) \\ K_1(r_2,r_1) & K_1(r_2,r_2) & K_1(r_2,r_3) & K_1(r_2,r_4) \\ K_2(r_3,r_1) & K_2(r_3,r_2) & K_2(r_3,r_3) & K_2(r_3,r_4) \\ K_2(r_4,r_1) & K_2(r_4,r_2) & K_2(r_4,r_3) & K_2(r_4,r_4) \end{vmatrix} dr_1\,dr_2\,dr_3\,dr_4$$

$$= \int_0^1\int_0^1 \begin{vmatrix} K_1(r_1,r_1) & K_1(r_1,r_2) \\ K_1(r_2,r_1) & K_1(r_2,r_2) \end{vmatrix} dr_1\,dr_2$$

$$\times \int_0^1\int_0^1 \begin{vmatrix} K_2(r_3,r_3) & K_2(r_3,r_4) \\ K_2(r_4,r_3) & K_2(r_4,r_4) \end{vmatrix} dr_3\,dr_4. \tag{A.11}$$

If we expand the determinant on the left-hand side of (A.11), then we have 24 terms. Four terms come from the product of the upper left 2×2 minor and the lower right 2×2 minor. The integral of the product is equal to the right-hand side of (A.11).

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Among the remaining 20 terms 12 terms involve one or two factors of the diagonal elements, and we can see that the integral of these terms is equal to 0, because of (A.6), (A.9), or (A.10). Thus, the integral of the sum of the eight terms

$$
\begin{aligned}
&-K_1(r_1,r_2)K_1(r_2,r_3)K_2(r_3,r_4)K_2(r_4,r_1)\\
&\quad -K_1(r_1,r_2)K_1(r_2,r_4)K_2(r_3,r_1)K_2(r_4,r_3)\\
&\quad -K_1(r_1,r_3)K_1(r_2,r_1)K_2(r_3,r_4)K_2(r_4,r_2)\\
&\quad +K_1(r_1,r_3)K_1(r_2,r_4)K_2(r_3,r_1)K_2(r_4,r_2)\\
&\quad -K_1(r_1,r_3)K_1(r_2,r_4)K_2(r_3,r_2)K_2(r_4,r_1)\\
&\quad -K_1(r_1,r_4)K_1(r_2,r_1)K_2(r_3,r_2)K_2(r_4,r_3)\\
&\quad -K_1(r_1,r_4)K_1(r_2,r_3)K_2(r_3,r_1)K_2(r_4,r_2)\\
&\quad +K_1(r_1,r_4)K_1(r_2,r_3)K_2(r_3,r_2)K_2(r_4,r_1) \qquad \textbf{(A.12)}
\end{aligned}
$$

is equal to 0. Furthermore, the integrals of the fourth and eighth terms in (A.12) are equal to 0, by (A.6).

If we put

$$M(r,t)=\int_0^1 K_1(r,s)K_2(s,t)\,ds,$$

then the integrals of the first, second, third, and sixth terms in (A.12) are equal to

$$-\int_0^1\int_0^1 M(r,t)^2\,dr\,dt,$$

whereas the integrals of the fifth and seventh terms are equal to

$$-\int_0^1\int_0^1 M(r,t)M(t,r)\,dr\,dt.$$

Hence we have

$$
\begin{aligned}
&-\int_0^1\int_0^1 [4M(r,t)^2+2M(r,t)M(t,r)]\,dr\,dt\\
&\qquad =-2\int_0^1\int_0^1 [M(r,t)^2+M(r,t)M(t,r)+M(t,r)^2]\,dr\,dt=0. \qquad \textbf{(A.13)}
\end{aligned}
$$

In general we have $x^2+xy+y^2\geq 0$ for any real numbers x and y; and $x^2+xy+y^2=0$ only if $x=y=0$. The integrand in the second line of (A.13) has this style. Moreover, as $M(r,t)$ is a continuous function of (r,t), we can conclude from (A.13) that $M(r,t)=0$ for all $(r,t)\in[0,1]\times[0,1]$, which is the required result. ■

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Proof of Theorem 3. To save the space the proof is given for the case $n = 3$ only.

Define $(p_1 + p_2 + p_3) \times (p_1 + p_2 + p_3)$ matrix $C_{p_1 p_2 p_3}$ $(p_1, p_2, p_3 = 0,1,2,\ldots)$ in the following way. When we make a partition

$$C_{p_1 p_2 p_3} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

with $A_{ij}(i, j = 1,2,3)$ a $p_i \times p_j$ matrix, then the (k, l) element of A_{ij} is defined to be $K_i(r_{ik}, r_{jl})$, where $r_{1k} = r_k$ $(k = 1,\ldots,p_1)$, $r_{2k} = r_{p_1+k}$ $(k = 1,\ldots,p_2)$, and $r_{3k} = r_{p_1+p_2+k}$ $(k = 1,\ldots,p_3)$.

The joint m.g.f. for (Z_1, Z_2, Z_3) is given by

$$E[\exp(\theta_1 Z_1 + \theta_2 Z_2 + \theta_3 Z_3)] = D(2\theta_1, 2\theta_2, 2\theta_3)^{-1/2},$$

where

$$D(\lambda_1, \lambda_2, \lambda_3) = \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \frac{(-1)^{p_1+p_2+p_3}}{p_1! p_2! p_3!} \lambda_1^{p_1} \lambda_2^{p_2} \lambda_3^{p_3} \times \int_0^1 \cdots \int_0^1 \det(C_{p_1 p_2 p_3})\, dr_1 \ldots dr_{p_1+p_2+p_3},$$

similarly to (A.4) in the case of (Z_1, Z_2). A necessary and sufficient condition for the mutual independence of Z_1, Z_2, Z_3 is given by the equation

$$D(\lambda_1, \lambda_2, \lambda_3) = D(\lambda_1, 0, 0) D(0, \lambda_2, 0) D(0, 0, \lambda_3), \tag{A.14}$$

which should hold for all λ_1, λ_2, and λ_3.

In a similar way as the lemma in the text we can see that (A.14) is equivalent to

$$\int_0^1 \cdots \int_0^1 \det(C_{p_1 p_2 p_3})\, dr_1 \ldots dr_{p_1+p_2+p_3} = \int_0^1 \cdots \int_0^1 \det(A_{11})\, dr_1 \ldots dr_{p_1} \times \int_0^1 \cdots \int_0^1 \det(A_{22})\, dr_{p_1+1} \ldots dr_{p_1+p_2} \times \int_0^1 \cdots \int_0^1 \det(A_{33})\, dr_{p_1+p_2+1} \ldots dr_{p_1+p_2+p_3} \tag{A.15}$$

for all $p_1, p_2, p_3 = 0,1,2,\ldots$.

By the assumption of the theorem, Z_1, Z_2, Z_3 are pairwise independent. Hence by Theorem 2 we have

$$\int_0^1 K_i(r, s) K_j(s, t)\, ds = 0 \tag{A.16}$$

for $i, j = 1,2,3$, $i \neq j$ and for all $(r, t) \in [0,1] \times [0,1]$.

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If we apply Laplace's expansion to $\det(C_{p_1p_2p_3})$ along the first p_1 rows, then, by using (A.16) with $(i,j) = (1,2)$ and with $(i,j) = (1,3)$, we can show as in the proof of the "if" part of Theorem 2 that

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \det(C_{p_1p_2p_3})\, dr_1 \ldots dr_{p_1+p_2+p_3} \\
&\quad = \int_0^1 \cdots \int_0^1 \det(A_{11})\, dr_1 \ldots dr_{p_1} \\
&\qquad \times \int_0^1 \cdots \int_0^1 \det\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} dr_{p_1+1} \ldots dr_{p_1+p_2+p_3}. \qquad \textbf{(A.17)}
\end{aligned}$$

If we use (A.16) with $(i,j) = (2,3)$, then we have

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \det\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} dr_{p_1+1} \ldots dr_{p_1+p_2+p_3} \\
&\quad = \int_0^1 \cdots \int_0^1 \det(A_{22})\, dr_{p_1+1} \ldots dr_{p_1+p_2} \\
&\qquad \times \int_0^1 \cdots \int_0^1 \det(A_{33})\, dr_{p_1+p_2+1} \ldots dr_{p_1+p_2+p_3}. \qquad \textbf{(A.18)}
\end{aligned}$$

From (A.17) and (A.18) we have (A.15); thus the mutual independence of Z_1, Z_2, Z_3 is established. ■

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