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SOLUTION TO PROBLEM

Solution to Problem Posed in Volume 20(3)

04.3.1. An I(2) Model for VAR(1) Processes[1]—Solution

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Let α_* be any basis of $\mathrm{col}(\Pi)$ and β_* be any basis of $\mathrm{col}(\Pi')$; we note $\mathrm{col}(\alpha_\perp) = \mathrm{col}(\alpha_{*\perp})$, $\mathrm{col}(\beta_\perp) = \mathrm{col}(\beta_{*\perp})$ so no distinction is made between $\alpha_\perp$ and $\alpha_{*\perp}$, $\beta_\perp$ and $\beta_{*\perp}$ in the following discussion.

1. Condition (b) is equivalent to $rank(\alpha_\perp'\beta_\perp) = s < p - r$; this implies that $\mathcal{A} := \mathrm{col}(\alpha_\perp) \cap \mathrm{col}(\beta)$ has dimension $p_2 := p - r - s$. Because $p_2 = \dim(\mathcal{A}) \leq \min(r, p - r)$, one has that $p_2 \leq r$.

2. Condition (b) can be written as $\alpha_\perp'\beta_\perp = \xi\eta'$, with ξ, η $(p - r) \times s$ full column rank matrices; $\alpha_\perp$ can thus be chosen as follows:

$$\alpha_\perp = (\bar{\beta}_*, \bar{\beta}_\perp)\begin{pmatrix} a & b_1 \\ 0 & b_2 \end{pmatrix} = (\bar{\beta}_*, \bar{\beta}_\perp)\underset{p\times(p-r)}{q},$$

where a is $r \times p_2$, $p_2 \leq r$, $b := (b_1' : b_2')'$ is $p \times s$, and q is of full column rank. The matrix $\alpha_\perp'\beta_\perp$ becomes

$$\alpha_\perp'\beta_\perp = \begin{pmatrix} 0 \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 \\ I_{p_2} \end{pmatrix} b_2',$$

so that $\xi = (0 : I_{p_2})'$ and $\eta = b_2$; thus $\beta_1 = \bar{\beta}_\perp b_2$, $\beta_2 = \beta_\perp b_{2\perp}$, $\alpha_1 = \bar{\alpha}_\perp(0 : I_{p_2})'$, $\alpha_2 = \alpha_\perp(I_s : 0)' = \bar{\beta}_* a$.

Using orthogonal projections, one can represent α_* as $\alpha_* = (\beta_*, \beta_\perp)c$; from $\alpha_\perp'\alpha = 0$ one deduces that $c = q_\perp$, where $c = q_\perp$ can be chosen as follows:

$$\alpha_* = (\beta_*, \beta_\perp)\begin{pmatrix} a_\perp & 0 \\ -(\beta_\perp'\beta_\perp)^{-1}b_2(b_2'(\beta_\perp'\beta_\perp)^{-1}b_2)^{-1}b_1'a_\perp & b_{2\perp} \end{pmatrix} v$$

$$= (\beta_* a_\perp + \beta_1\omega : \beta_2)v,$$

where v is a nonsingular square matrix and $\omega := -(b_2'(\beta_\perp'\beta_\perp)^{-1}b_2)^{-1}b_1'a_\perp$. Note that the second block of columns is equal to $\beta_2 = \beta_\perp\kappa$, where we have named $\kappa := b_{2\perp}$.

Now let $\alpha := (\beta_* a_\perp + \beta_1\omega : \beta_2)$ and define $\beta' := \bar{\alpha}'\Pi$; because β and β_* are two bases of $\mathrm{col}(\Pi')$, they are related by the relation $\beta u = \beta_*$ for a nonsingular matrix u. Substituting within α one finds $\alpha := (\beta\varphi + \beta_1\omega : \beta_2)$ with $\varphi := ua_\perp$

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of full column rank. Summarizing, the coefficients (α, β, Ω) are constrained as follows:

$$\alpha := (\beta\varphi + \beta_1\omega : \beta_2), \tag{1}$$

where $\beta_1 = \bar{\beta}_\perp \kappa_\perp$, $\beta_2 = \beta_\perp \kappa$. The I(2) model is thus indexed by the parameters $\tau := (\beta, \kappa, \varphi, \omega, \Omega)$, which are freely varying.

3. From the I(2) representation theorem of Johansen (1992), one finds that $\delta := \bar{\alpha}'\beta_2$. By (1) one sees that $\delta = (0, I_{p_2})'$, which is full rank p_2. Let $\beta^1 := \beta(I_{r-p_2} : 0)'$ and $\beta^2 := \beta(0, I_{p_2})'$; one sees that $\beta^{1\prime}X_t$ are the CI(2,2) relations and $\beta^{2\prime}X_t + (\beta_2'\beta_2)^{-1}\beta_2'\Delta X_t$ are the multicointegrating relations.

4. By the I(2) representation theorem of Johansen (1992), $C_2 = \beta_2(\alpha_2'\theta\beta_2)^{-1}\alpha_2'$, where $\theta := (I + \bar{\beta}\bar{\alpha}')$. The choice β is by construction conformable with $\alpha := (\beta\varphi + \beta_1\omega : \beta_2)$. Going back to the representation of α_2 one finds $\bar{\beta}_* = \bar{\beta}(u')^{-1}$ and thus $\alpha_2 = \bar{\beta}_* a = \bar{\beta}\varphi_\perp$, where we note that $(u')^{-1}a = \varphi_\perp$. Thus $\alpha_2'\theta\beta_2 = \varphi_\perp'\bar{\beta}'(I + \bar{\beta}\bar{\alpha}')\beta_2 = \varphi_\perp'\bar{\beta}'\bar{\beta}\bar{\alpha}'\beta_2 = \varphi_\perp'\bar{\beta}'\bar{\beta}\delta$, given the structure (1) of α, and hence

$$C_2 = \beta_\perp\kappa\left(\varphi_\perp'(\beta'\beta)^{-1}\begin{pmatrix}0\\ I_{p_2}\end{pmatrix}\right)^{-1}\varphi_\perp'\bar{\beta}'. \tag{2}$$

Note that C_2 depends on β, κ, φ and it does not depend on ω, Ω.

5. The likelihood function depends on τ through the product $\alpha\beta'$ and Ω. Specifically $\alpha\beta' = (\beta\varphi + \beta_1\omega)\beta^{1\prime} + \beta_\perp\kappa\beta^{2\prime}$, where $\beta = (\beta^1 : \beta^2)$ is partitioned conformably with α. It is simple to see that τ is not identified because one can insert two square invertible matrices ϕ_i, $i = 1,2$, and their inverses within the two factors in $\alpha\beta'$; i.e., $\tau := (\beta, \kappa, \varphi, \omega, \Omega)$ and $\tau^* := (\beta\phi', \kappa\phi_2^{-1}, (\phi')^{-1}\varphi\phi_1^{-1}, \omega\phi_1^{-1}, \Omega)$ give the same likelihood, where $\phi := \mathrm{diag}(\phi_1, \phi_2)$. Moreover because $\beta_\perp$, β_1 are any bases of $\mathrm{col}(\beta_\perp)$ and $\mathrm{col}(\beta_1)$, one also needs to identify them.

For any matrix γ, let a be of the same dimensions such that $a'\gamma$ is nonsingular; define $\gamma_a := \gamma(a'\gamma)^{-1}$ (see, e.g., Johansen, 1996, ch. 5). To achieve identification one can identify β^1, β^2, $\beta_\perp$, and β_1 by defining $\beta^1_{a_1}$, $\beta^2_{a_2}$, $\beta_{\perp a_3}$, and β_{1a_4} for appropriate matrices a_i, $i = 1,2,3,4$. This solves the identification problem for τ.

NOTE

1. An excellent partial solution based on a state space representation has been independently proposed by Dietmar Bauer, Vienna University of Technology (Austria).

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