Historical mathematical problems suitable for classroom activities

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Introduction

The human mind, by nature, is curious and enjoys dealing, both independently and competitively, with intellectual challenges. Throughout time, mathematical tasks, riddles, and puzzles have offered such challenges. In today's modern era, marketing companies and the media even offer prizes and rewards for the successful solution of puzzles, expanding the audience exposed to the various challenges. In addition, the styles, subjects and depths of the puzzles have become diversified. Experience gained in solving puzzles aids in developing reasoning abilities and deepening thought.

The study of mathematics is a classic example of structured learning based on the solutions of problems and exercises at different levels of reasoning. Problems often require deep analysis to understand the extent of the theories and operations involved. Puzzles and mathematical games have been widely used to augment the study of 'bare' mathematics. In academic courses for pre-service mathematics teachers, in enrichment classes for outstanding students, and even as an engaging part of the regular mathematics lesson, the addition of mathematical games helps diversify and enrich the study process. The inclusion of mathematical puzzles – even to a small extent – in the mathematics classroom produces considerable interest among students. Some even join extracurricular classes and became 'addicted' to solving puzzles in mathematics and other areas.

Various studies have demonstrated the contribution that puzzles and mathematical tasks and games make to the development of mathematical thinking [1-7]. Solving puzzles encourages original thought, creativity, and intuitive reasoning that often encompasses different fields of study and the search for unconventional methods of solution. Furthermore, and no less importantly, mathematical puzzles and riddles reveal the richness, beauty and wisdom embodied in mathematics.

Puzzles and mathematics games are fantastic tools for training students to be creative mathematical thinkers. They should encourage students to do the following:

- make sense of problems and persevere in solving them;
- reason abstractly and quantitatively;
- use appropriate tools strategically;
- look for and make use of structure;
- look for regularity in repeated reasoning.

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Below, we present three puzzle tasks that were given to elementary and high school mathematics trainee teachers as part of an academic course in number theory. The tasks were subsequently presented to students aged 12 and above.

Task 1: The shepherd's bequest

A rich shepherd loved all his sons dearly.

His will stated that his sheep were to be bequeathed as follows: the oldest son would get a single sheep and 1/7 of the remainder.

The second son would get two sheep and 1/7 of the remainder.

The third son would get three sheep and 1/7 of the remainder.

And so on and so forth: each son would get the number of sheep in accordance with his order of birth (fourth, fifth, sixth, and so on) plus 1/7 of the remainder.

Since he loved all of his sons equally, his will ensured that each of them received an equal number of sheep.

How many sheep did the shepherd have? How many sons? How many sheep did each son receive?

Solution:

The first action of the students is to simply test all numbers beginning with 1 to see when the requested method is possible, i.e. the number of sheep left is always divisible by 7. This action leads to discovering that the number for which the procedure works is 36, essentially by completing Table 1. Hence they observe that every son receives the same number of sheep, in this case 6, which is the same as the number of sons.

The son's ordinal number	No. of sheep	The number of sheep he got on account of his ordinal number	1/7 of the remaining sheep	Total number of sheep he got
1	36	1	(35) 5	6
2	30	2	(28) 4	6
3	24	3	(21) 3	6
4	18	4	(14) 2	6
5	12	5	(7) 1	6
6	6	6	(0) 0	6

TABLE 1

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Generalisation of the problem

After being presented as a particular case (each son receives 1/7 of the remainder of sheep), it is interesting to generalise the problem to reach an algebraic solution.

The oldest son receives one sheep plus a part that equals $1/k$ of the rest of the flock (k is a natural number and is specified in the will). The second son then gets two sheep plus $1/k$ of the rest. Next, the third son gets three sheep plus $1/k$ of the rest, and so on.

Given that, after distributing the sheep, the first two sons have an equal number of sheep, the following questions might be posed:

(a) How many sheep were distributed in total (as a function of k)?

(b) Did all the sons receive an equal number of sheep (i.e. was there distributive justice)? How many sons are there?

The solution

(a) We denote the overall number of sheep by S.

From the data:

The first son gets $a_1 = 1 + \dfrac{S-1}{k}$ sheep.

The second son gets $a_2 = 2 + \dfrac{S-a_1-2}{k}$ sheep.

However, it is given that $a_1 = a_2$, therefore

$$1 + \frac{S-1}{k} = 2 + \frac{S-\left(1+\frac{S-1}{k}\right)-2}{k}.$$

By manipulating the equation we obtain $S = k^2 - 2k + 1 = (k-1)^2$.

However, $a_1 = 1 + \dfrac{S-1}{k} = 1 + \dfrac{(k-1)^2-1}{k} = k-1$, so it follows that $a_1 = a_2 = k-1$.

(b) Now we examine how many sheep the third son gets: $a_3 = 3 + \dfrac{S-a_1-a_2-3}{k}$.

By substituting the values for S, a_1 and a_2, we indeed obtain $a_3 = k-1$.

At this stage, the answer is encouraging and suggests that the answer to question (b) is probably affirmative. If from the data, $a_1 = a_2 = k-1$, and also $a_3 = k-1$, then, by induction, we shall assume that every brother, including the mth brother, receives $k-1$ sheep.

Let us now prove that the $(m+1)$th brother also receives $k-1$ sheep:

$$a_{m+1} = m + 1 + \frac{S - m(k-1) - (m+1)}{k}$$

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$$= m + 1 + \frac{(k-1)^2 - m(k-1) - (m+1)}{k}$$

$$= k - 1.$$

Using the process of induction, we have proved that all the brothers receive an equal number of sheep.

Conclusion

In any case of this puzzle, the father distributes $(k-1)^2$ sheep between $(k-1)$ sons, and each son receives $(k-1)$ sheep, where k indicates the fraction of the remaining flock that each son receives after he has taken 'his' number.

Task 2: The camel inheritance

A rich sheikh with many camels left a strange will. 'Divide my 17 camels between my three sons as follows: the oldest will get 1/2 of the camels, the second son will get 1/3 of the camels, and the youngest son will get 1/9 of the camels.' Alas, everyone was upset. The herdsman was obviously not very learned in the topic of fractions, and made a grievous error in his will, which confounded his sons. 'How can we divide 17 camels by two?' asked the eldest son. 'Will I get $8\frac{1}{2}$ camels?' The other two sons had similar concerns.

In spite of the mistake, the problem is solved. As legend has it, a wise man rode by on his camel. He heard the sons' problem and solved it as follows: He added his camel to the 17 camels that the sons inherited. Hence the oldest son got $9 = \frac{1}{2} \times 18$ camels, the second son got $6 = \frac{1}{3} \times 18$ camels and the youngest son got $2 = \frac{1}{9} \times 18$ camels. Altogether the sons received $9 + 6 + 2 = 17$ camels, leaving one camel, which was returned to the wise man. Everyone was extremely satisfied, as each received a (whole) number of camels that was larger than the original share according to the will: nine camels instead of $8\frac{1}{2}$, six camels instead of $5\frac{2}{3}$, and two camels instead of $1\frac{8}{9}$.

The solution of the mystery is clear: $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18} < 1$.

The following is the mathematical argument of the riddle. We are seeking three fractions $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ $(a < b < c)$ such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{N-1}{N}, \tag{1}$$

where the natural number N is divisible by each of the numbers a, b, c.

Specific case 1 (original question):

In the original question, $a = 2$, $b = 3$, $c = 9$ and $N = 18$. If we substitute all these values into the equation, the equality is satisfied.

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Can we find other solutions of (1) with $a = 2$ and $b = 3$?

If we substitute $N = 42$, we obtain $c = 7$, and indeed $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$.

We can use $N = 12$, and get $c = 12$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{12} = \frac{11}{12}$.

If we substitute $N = 24$, we get $c = 8$, and indeed $\frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{23}{24}$.

Specific case 2:

What about other values for a and b?

For example, let $a = 2, b = 4$. What are possible values for c and N?

The equation in this case is:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{c} = \frac{N-1}{N} \Rightarrow c = \frac{4N}{N-4} = 4 + \frac{16}{N-4}.$$

Clearly we are solving a Diophantine equation, as we must find values for N such that c will be an integer.

If $N = 20$ then $c = 5$ and we get: $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20}$.

If $N = 12$ then $c = 6$ and we get: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$.

If $N = 8$ then $c = 8$ and we get: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$.

We can see that the riddle of the camel inheritance can be extended to any case where the values of c and N satisfy (1) for some constants a and b.

The problem can be extended further to division between other numbers of sons. For example with four sons, we might state: $a = 2, b = 3, c = 8$, and one has to find the values of d and N to satisfy the equation:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{d} = \frac{N-1}{N}.$$

From this equation we obtain:

$$N = 24 + \frac{576}{d-24} \quad \text{and} \quad d = 24 + \frac{576}{N-24}.$$

Of course, we can obtain many other solutions, the number of which, in this case, is the number of divisors of 576, which is 21.

If $d = 25$ then $N = 600$ and we get: $\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{25} = \frac{599}{600}$.

If $d = 26$ then $N = 312$ and we get: $\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{26} = \frac{311}{312}$.

An alternative version of the question

Based on the above explanation, one can construct a 'reverse' question:

A shepherd has 599 sheep and his will states: 'The oldest son gets 1/2, the second son gets 1/3, the third son gets 1/8 and the fourth gets 1/25 of the

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sheep. How many sheep does each son get?' (This task was presented at the 'Mini Mathematical Olympiad' for ten-year-olds which took place at the Weizmann Institute of Science in Israel, in 2015.)

Fractions of the form $1/n$, where n is a natural number, are called *unit fractions* or sometimes *Egyptian fractions*. The Egyptians used only unit fractions, the only exception being $\frac{2}{3}$. In 1880, the mathematician Sylvester proved that any fraction between 0 and 1 can be represented as a sum of distinct unit fractions [8].

Task 3: Dividing a rope

Given a rope $\frac{2}{3}$ metres long, and without a ruler or tape measure, how can one obtain a rope that is exactly $\frac{1}{2}$ metre?

Solution

Stage A: Fold the $\frac{2}{3}$ metre rope into two equal parts to obtain $\frac{1}{3}$ metres each.

Stage B: Divide one of the $\frac{1}{3}$ metre parts into two equal parts, to obtain a section that is $\frac{1}{6}$ metre, as shown in Figure 1.

Stage C: Cut off one $\frac{1}{6}$ metre section from the original $\frac{2}{3}$ metres to obtain $\frac{1}{2}$ metres, as required.

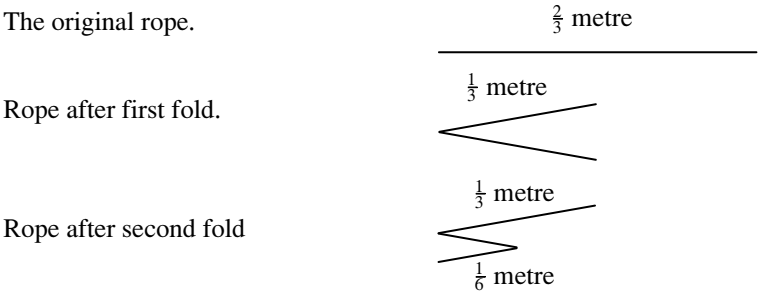


FIGURE 1

Here is a similar problem: Beginning with a length of rope that is $\frac{4}{7}$ metres, how do we cut it to obtain $\frac{1}{2}$ metre?

Solution

$$\tfrac{4}{7}\text{m} + \text{fold} \quad \rightarrow \quad \tfrac{2}{7}\text{m} + \text{fold} \quad \rightarrow \quad \tfrac{1}{7}\text{m} + \text{fold} \quad \rightarrow \quad \tfrac{1}{14}\text{m}.$$

Remove the $\frac{1}{14}$ m: $\frac{4}{7} - \frac{1}{14} = \frac{1}{2}$ metre.

Generalisation

Given any rope $\dfrac{2^{k-1}}{2^k - 1}$ metres long (k is a natural number) and a pair of scissors, how can we end up with a rope which is $\frac{1}{2}$ metre long?

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Solution

Our general expression is $\dfrac{2^{k-1}}{2^k - 1} - \dfrac{1}{2} = \dfrac{1}{2(2^k - 1)}$.

We fold the original $\dfrac{2^{k-1}}{2^k - 1}$ metre rope which results in:

$$\frac{2^{k-1}}{2^k - 1} + \text{fold} \quad \rightarrow \quad \frac{2^{k-2}}{2^k - 1} + \text{fold} \quad \rightarrow \quad \frac{2^{k-3}}{2^k - 1} + \text{fold} \ldots .$$

After k folds, we obtain a part whose length is $\dfrac{1}{2(2^k - 1)}$ metres.

By cutting this part from the original rope, we obtain:

$$\frac{2^{k-1}}{2^k - 1} - \frac{1}{2(2^k - 1)} = \frac{1}{2} \text{ metre.}$$

We can extend this example. For example, we might begin with rope $\frac{64}{127}$ metres long, and we need a rope $\frac{1}{2}$ metres or $\frac{1}{4}$ metres or $\frac{1}{2^n}$ metres long (n is a natural number). The answer can be obtained in a similar manner.

Conclusion

In didactic terms, this paper addresses the multifaceted contribution that mathematical puzzles and riddles can make to learners in the education system:

- dealing with word problems;
- becoming acquainted with methods and strategies for solving word problems, including Diophantine equations;
- understanding the concepts of Egyptian fractions and mathematical induction;
- coping with real value problems from daily life;
- developing the ability to compose tangible problems.

We believe that judiciously integrating puzzles and riddle-like tasks into mathematics lessons will boost achievement and improve the learning atmosphere.

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