Redundant axioms in teaching linear algebra

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1. *Scientific background*

What is an axiom? The meaning of the term 'axiom' varies between different fields of study. In philosophy it means a statement that is accepted without argument, while in physics the term 'postulate' is used and it means a theory that was verified in an experiment, and will be considered as true unless it is disproved by other experiments.

In mathematics the notion of 'axiom' is used in two related but distinguishable meanings: 'logical axioms' and 'non-logical axioms'. Logical axioms are certain statements that are always true, and from them all tautologies of the language can be derived. Non-logical axioms are statements that are taken to be true, to serve as a premise or starting point for further reasoning and arguments.

The meanings of the words 'axiom' and 'postulate' were separated in the beginning. Ancient Greeks used the term 'axiom' as an assumption that is common to many fields of science, such as 'The whole is greater than the part'. The term 'postulate' was an assumption specific for its field, and based on experience in this field, such as 'It is possible to draw a straight line from any point to any other point' was considered a postulate because it was specific to the field of geometry. Modern mathematics is much more abstract, so this separation does not exist any more. The notions 'axiom' and 'postulate' have the same meaning.

A common mistake among students of mathematics is that axioms are like theorems but need no proof because they are so obvious and intuitive. This is not true because axioms cannot be proved; they are the building blocks of the theory with which they are associated. Moreover, in modern mathematics there are many axioms, some of which are not intuitive at all. Almost every mathematical theory starts by stating a set of axioms. This set must be consistent, so that it is not possible for them to create a contradiction. A famous example of a set of axioms is Peano's axiomatization of the natural numbers [1]:

(1) Zero is a number;
(2) If a is a number, the successor of a is a number;
(3) Zero is not the successor of any number;
(4) Two numbers of which the successors are equal are themselves equal;
(5) (induction axiom) If a set S of numbers contains zero and also the successor of every number in S, then every number is in S.

In [2], Shianghaw developed a new set of axioms for the natural numbers, and this axiom is minimal, in the sense that no axiom can be

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derived from other axioms. However, even the author admits that this set is much more difficult to teach, meaning that sometimes redundant axioms help in the teaching process.

The focus of our paper is on redundancy within sets of axioms; we will bring some examples of redundant axioms that are common in the teaching of linear algebra, and distinguish between the following two cases:

(1) 'Pedagogical' redundant axioms: in this case the redundancy helps us in the teaching process;

(2) 'Inherited' redundant axioms: in this case the redundancy comes from a former set of axioms that we develop for our set of axioms, and in the process some axioms become redundant.

Our paper is divided as follows: Section 2 shows redundant axioms in the definition of an invertible matrix; Section 3 proves that axioms in the definition of the determinant are redundant; Section 4 focuses on the redundant axioms in the definition of a group; Section 5 focuses on redundant axioms in the definition of a ring; Section 6 shows redundant axioms in the definition of a linear transformation.

The redundancy in Sections 2 and 3 is 'pedagogical', because the redundant axioms assist in the learning process. The redundancy in Sections 4 and 5 is 'inherited', because it comes from a former set of axioms that we rely upon. It is very important to distinguish between 'pedagogical' and 'inherited' redundant axioms, as the former help teachers and students in the learning process. In general, a student who is asked to find a redundant axiom must provide a proof; and this gives the student a better insight into the material. In this paper we discuss the linear algebra domain only, but 'pedagogical' redundant axioms exist in all mathematical domains.

2. *Redundant axioms in defining 'inverse matrix'*

Our first example of a redundant axiom is taken from a basic linear algebra course. The definition of an invertible matrix requires it to be both left and right invertible, as in the following definition [3]:

Definition 1:

(a) A matrix A is a left inverse matrix of the matrix M if AM is the identity matrix.

(b) A matrix A is a right inverse matrix of the matrix M if MA is the identity matrix.

A matrix M with a two-sided inverse is an invertible matrix. That two-sided inverse is called the inverse matrix and is denoted M^{-1}.

In practice, when we find the inverse matrix of M, we start with the augmented matrix $(M \mid I)$ and make elementary row operations to get the augmented matrix $(I \mid A)$ and then $A = M^{-1}$. Notice that in this process we only found out that $MA = I$, so we need to check that $AM = I$.

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A perceptive student may ask his teacher why he needs to check $AM = I$; their conversation is described below:

Teacher: Good morning class, today we are going to learn about invertible matrices. The definition is ...

Student: OK, but how do we actually find the inverse?

Teacher: Good question, let us see an example of finding the inverse ....

Student: Why do we need to check $AM = I$? Isn't finding $MA = I$ enough?

Teacher: That is an excellent question, in a few lessons we will prove that $AM = I$ is equivalent to $MA = I$.

As the perceptive student noticed, it looks as though there are two axioms, left *and* right invertible, but we only need M to be left *or* right invertible, as the following proof shows:

Theorem 1: Let M be a square matrix. If there exists a matrix A such that $MA = I$ then M is invertible and $A = M^{-1}$.

Proof:

$$\begin{aligned} MA = I &\Rightarrow \det(MA) = \det(I) \\ &\Rightarrow \det(M)\det(A) = 1 \neq 0 \\ &\Rightarrow \det(M) \neq 0 \Rightarrow M \text{ is invertible.} \end{aligned}$$

and we denote its inverse as M^{-1}.

We now prove that the inverse M^{-1} is A:

$$\begin{aligned} & MA = I \\ \Rightarrow\ & M^{-1}(MA) = M^{-1}I \\ \Rightarrow\ & \left(M^{-1}M\right)A = M^{-1} \\ \Rightarrow\ & A = M^{-1}. \end{aligned}$$

Remark: The above proof is used in most books, and requires knowledge of determinants. However, there is a proof without determinants in [4].

In most linear algebra courses determinants are studied after studying the invertible matrices, which explains why this proof is not given immediately after Definition 1. After Theorem 1 is proved, either (a) or (b) in Definition 1 is redundant, and Definition 1 can be revised to:

Definition 2: Let M be a square matrix. M is invertible if it has a left (or right) inverse matrix.

So why do most books use a definition that is not minimal?

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The drawback of this definition is that it causes some to question why M is invertible only from the left (or the right). Definition 1 looks more natural for students; only after a few lessons can it be reduced to Definition 2.

Moreover, Definition 1 allows multiplication of matrix M with its inverse from both sides, since $AM = MA = I$, while Definition 2 only allows multiplication from the left (or the right, but not both), and is therefore much more restrictive for problem solving.

3. *Redundant axioms in defining 'determinant'*

Our next example of a redundant axiom is also in basic linear algebra. The determinant function is defined in most cases as follows [5]:

Definition 3: If A is a matrix of order n

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix},$$

then the determinant of A, denoted by $\det(A)$, is defined by

$$\det(A) = \sum_{(m_1,\dots,m_n)} \operatorname{sign}\begin{pmatrix} m_1, \dots, m_n \end{pmatrix} \cdot a_{m_1,1} \cdot \dots \cdot a_{m_n,n},$$

where $(m_1, \dots, m_n)$ is a permutation of $\{1, 2, \dots, n\}$.

However, it is sometimes defined equivalently as follows [6]:

Definition 4: A determinant of order n is a function $\det : M_n \to \mathbb{R}$ such that:

(i) $\det(R_1, \dots, kR_i + R_j, \dots, R_n) = \det(R_1, \dots, R_j, \dots, R_n)$ for $i \neq j$, where $k \in \mathbb{R}$;

(ii) $\det(R_1, \dots, R_j, \dots, R_i, \dots, R_n) = -\det(R_1, \dots R_i, \dots, R_j, \dots, R_n)$ for $i \neq j$;

(iii) $\det(R_1, \dots, mR_i, \dots, R_n) = m \cdot \det(R_1, \dots R_i, \dots, R_n)$;

(iv) $\det(I) = 1$ where I is the identity matrix,

where the Rs are the rows of the matrix.

The following theorem proves that in Definition 4, axiom (ii) is redundant.

Theorem 2: Definition 4 is not minimal, because axiom (ii) is redundant.

Proof: We begin our proof with a simple example. Suppose A is the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

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We will switch row 1 with row 2, using these four row operations.

(1) $R_1 \leftarrow R_1 + R_2$. In words: replace row 1 with the sum of row 1 and row 2.

After this operation, $\det(A)$ is changed to be:

$$\det(A) = \begin{vmatrix} 5 & 7 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

(2) $R_2 \leftarrow R_2 + (-1)R_1$. In words: replace row 2 with row 2 minus row 1.

After this operation, $\det(A)$ is changed to be:

$$\det(A) = \begin{vmatrix} 5 & 7 & 9 \\ -1 & -2 & -3 \\ 7 & 8 & 9 \end{vmatrix}.$$

(3) $R_1 \leftarrow R_1 + R_2$. In words: replace row 1 with the sum of row 1 and row 2.

After this operation, $\det(A)$ is changed to be:

$$\det(A) = \begin{vmatrix} 4 & 5 & 6 \\ -1 & -2 & -3 \\ 7 & 8 & 9 \end{vmatrix}.$$

(4) $R_2 \leftarrow (-1)R_2$. In words: replace row 2 with row 2 multiplied by (-1).

After this operation, $\det(A)$ is changed to be $-\det(A)$, since we multiplied a row by (-1):

$$-\det(A) = \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}.$$

This example is generalised to a switch of row i (R_i) with row j (R_j) using four row operations:

(1) $R_i \leftarrow R_i + R_j$. After this operation, only R_i is changed to be $R_i + R_j$.

(2) $R_j \leftarrow R_j + (-1)R_i$. After this operation, only R_j is changed to be $R_j + (-1)R_i$, but in this stage R_i is $R_i + R_j$, so in fact R_j is changed to be the original R_i multiplied by (-1).

(3) $R_i \leftarrow R_i + R_j$. After this operation, only R_i is changed to be $R_i + R_j$, but in this stage R_i is $R_i + R_j$, and R_j is $(-1)R_i$, so in fact R_i is changed to be the original R_j.

(4) $R_j \leftarrow (-1)R_j$. After this operation, only R_j is changed to be $(-1)R_j$, but in this stage R_j is the original R_i multiplied by (-1), so in fact R_j is changed to be the original R_i.

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Definition 4 can now be revised to a minimal form:

Definition 5: A determinant of order n is a function $\det : M_n \to \mathbb{R}$ such that:

(i) $\det(R_1, \dots, kR_i + R_j, \dots, R_n) = \det(R_1, \dots, R_j, \dots, R_n)$ for $i \neq j$;

(ii) $\det(R_1, \dots, mR_i, \dots, R_n) = m \cdot \det(R_1, \dots, R_i, \dots, R_n)$;

(iii) $\det(I) = 1$, where I is the identity matrix,

where the Rs are the rows of the matrix.

So why do we use a non-minimal definition? As in Section 2, this helps the learning process, because Definition 5 does not allow switching of rows, so instead of row switching we will be forced to make the equivalent 3 row operations, which is a waste of time.

4. *Redundant axioms in defining a 'group'*

In this section we present the definition of a 'group' from an advanced linear algebra course. Then we show that in the definition there are redundant axioms.

The common definition of a group is taken from [7]:

Definition 6: A non-empty set G, together with a binary operation in G, is called a group if

(i) for all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(ii) there exists an identity element (usually denoted by 1) $\in G$, such that

(ii.1) $a \cdot 1 = a$,

(ii.2) $1 \cdot a = a$;

(iii) every $a \in G$ has an inverse element in G, i.e. there is $b \in G$ such that

(iii.1) $a \cdot b = 1$,

(iii.2) $b \cdot a = 1$.

In this definition, the identity element must be identity from left and right, i.e. $a \cdot 1 = a = 1 \cdot a$, and the inverse element must be left and right inverse, i.e. $a \cdot b = 1 = b \cdot a$.

However, as in Section 2, this can be reduced to one side only; this means that we can define a group without using (ii.1) and (iii.1), or (ii.2) and (iii.2).

We can now prove the following theorem:

Theorem 3: Let S be a non-empty set with a binary operation in S, which has the following properties:

(i) for all $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(ii) there exists a right identity element $1 \in S$ such that $a \cdot 1 = a$;

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(iii) every $a \in S$ has a right inverse element in S, i.e. $a \cdot b = 1$.

Then there are inverse and identity elements from both sides, i.e. $a \cdot 1 = a = 1 \cdot a$ and $a \cdot b = 1 = b \cdot a$.

Proof: We denote the right inverse element of b as b', i.e. $b \cdot b' = 1$. We start by proving that the right identity element is an identity element from both sides.

$$1 \cdot a = (a \cdot b) \cdot a = a \cdot b \cdot a = (a \cdot b \cdot a) \cdot 1 = (a \cdot b \cdot a) \cdot (b \cdot b') = a \cdot b \cdot (a \cdot b) \cdot b'$$

$$= a \cdot b \cdot 1 \cdot b' = \big((a \cdot b) \cdot 1\big) \cdot b' = (a \cdot b) \cdot b' = a \cdot (b \cdot b') = a \cdot 1 = a.$$

Now we prove that the right inverse element is an inverse element from both sides.

$$b \cdot a = (b \cdot a) \cdot 1 = (b \cdot a) \cdot (b \cdot b') = b \cdot (a \cdot b) \cdot b' = b \cdot 1 \cdot b' = (b \cdot 1) \cdot b' = b \cdot b' = 1.$$

Theorem 3 gives us the following minimal definition of a group:

Definition 7: A non-empty set S together with a binary operation in S is called a group if

(i) for all $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(ii) there exists an identity element (usually denoted by 1) $\in S$, such that $a \cdot 1 = a$;

(iii) every $a \in S$ has an inverse element in S, i.e. there is $b \in S$ such that $a \cdot b = 1$.

This minimal definition appears for example in [8].

Why do most books use the definition that has a redundancy? Again, there is a tension between using a minimal set of axioms and making the definition understandable in class.

In our opinion, the drawback of this minimal definition is that it causes students to question why there are inverse and identity elements from only one side. Definition 6 looks more natural for students and after a few lessons it can be reduced to Definition 7. Because this is studied in an advanced course, the proof can be given as an exercise for students.

5. *Redundant axioms in defining a 'ring'*

In this section we present a redundant axiom that is taken from a commutative algebra course and is related to the definition of a ring, which is one of the fundamental concepts of modern algebra. The common definition of a ring as stated in [8], is as follows:

Definition 8: A ring A is a non-empty set, together with two laws of composition called multiplication and addition respectively, and written as a product and as a sum respectively, satisfying the following conditions:

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(i) with respect to addition, A is a commutative group, i.e.
  (i.1) (commutativity) for all $a, b \in A, a + b = b + a$,
  (i.2) (associativity) for all $a, b, c \in A, (a + b) + c = a + (b + c)$.
  (i.3) (0 element) there exists the element 0 in A, such that for all $a \in A, a + 0 = a$,
  (i.4) (inverse element) for all $a \in A$, there exists an inverse element $(-a) \in A$, such that $a + (-a) = 0$;

(ii) multiplication is associative, and has an identity element, denoted by 1, i.e.
  (ii.1) (associativity) for all $a, b, c \in A, (a \cdot b) \cdot c = a \cdot (b \cdot c)$,
  (ii.2) (identity element) $a \cdot 1 = 1 \cdot a = a$;

(iii) for all $a, b, c \in A$ we have distributivity, $(a + b) \cdot c = a \cdot b + b \cdot c$ and $c \cdot (a + b) = c \cdot a + c \cdot b$.

Some examples for rings are the following: real numbers, square matrices with real numbers.

Surprisingly, the axiom of commutativity for addition suddenly becomes redundant. Many mathematicians do not believe in this result, because they are so accustomed to having commutativity.

Theorem 4: The axiom of commutativity for addition is redundant in a ring.

Proof:

Step 1: we will prove that 0 and the inverse element are commutative, i.e. $0 + a = a + 0 = a, (-a) + a = a + (-a) = 0$.

This proof follows the same outline as the proof of Theorem 3.

$$\begin{aligned} 0 + a &= \big(a + (-a)\big) + a = a + (-a) + a = a + (-a) + a + 0 \\ &= a + (-a) + a + \big((-a) + \big(-(-a)\big)\big) \\ &= a + (-a) + \big(a + (-a)\big) + \big(-(-a)\big) = a + (-a) + \big(-(-a)\big) \\ &= a + 0. \end{aligned}$$

So we have 0 from both sides, and now we prove that there is an inverse element from both sides, as follows:

$$\begin{aligned} (-a) + a &= (-a) + a + 0 = (-a) + a + \big((-a) + \big(-(-a)\big)\big) \\ &= (-a) + \big(a + (-a)\big) + \big(-(-a)\big) = (-a) + 0 + \big(-(-a)\big) \\ &= (-a) + \big(-(-a)\big) = 0. \end{aligned}$$

Step 2: $0 \cdot a = 0, (-1) \cdot a = (-a)$.

We note that in this step we do not use general commutativity, but only the results of step 1.

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$$\begin{aligned}
a &= 1\cdot a = (1+0)\cdot a = 1\cdot a + 0\cdot a = a + 0\cdot a \\
&\Rightarrow a = a + 0\cdot a \Rightarrow (-a) + a = (-a) + a + 0\cdot a \\
&\Rightarrow 0 = 0 + 0\cdot a \Rightarrow 0 = 0\cdot a \Rightarrow (-a) + 0 = (-a) + 0\cdot a \\
&\Rightarrow (-a) = (-a) + \big(1 + (-1)\big)\cdot a = (-a) + 1\cdot a + (-1)\cdot a \\
&\quad = (-a) + a + (-1)\cdot a = 0 + (-1)\cdot a \Rightarrow (-a) = (-1)\cdot a.
\end{aligned}$$

Step 3: $a + b = b + a$.

This final step shows that commutativity is redundant, because it can be proved from other axioms.

$$\begin{aligned}
a + b &= (a + b) + 0 = (a + b) + \big(-(b + a) + (b + a)\big) \\
&= \big((a + b) + \big(-(b + a)\big)\big) + (b + a) \\
&= \big((a + b) + (-1)\cdot(b + a)\big) + (b + a) \\
&= \big((a + b) + (-1)\cdot b + (-1)\cdot a\big) + (b + a) \\
&= \big((a + b) + (-b) + (-a)\big) + (b + a) \\
&= \big(a + \big(b + (-b)\big)\big) + (-a) + (b + a) \\
&= \big(a + 0 + (-a)\big) + (b + a) \\
&= \big(a + (-a)\big) + (b + a) = 0 + (b + a) = b + a.
\end{aligned}$$

In this section we proved that the axiom of commutativity is redundant in a ring. Other papers showed different cases, for example: in [9] it was proved that the axiom of commutativity for addition is redundant in a field, and in [10] it was proved for the field of real numbers that when distribution from both sides is allowed, the axiom of commutativity for multiplication is also redundant.

6. *Redundant axioms in defining linear transformations*

A possible definition of a linear transformation is as shown in [5]:

Definition 9: Let V and W be vector spaces over a field $\mathbb{F}$. A linear map from V to W is a function $T : V \to W$ with the following properties:

(i) *additivity*: for all $u, v \in V$: $T(u + v) = T(u) + T(v)$;

(ii) *homogeneity*: for all $a \in \mathbb{F}$, for all $v \in V$: $T(a\cdot v) = a\cdot T(v)$.

An attentive student asked her teacher whether the homogeneity axiom is redundant. Their conversation is described in the following lines:

Teacher: Good morning class, today we are going to learn about linear maps. The definition of a linear map is ...

Student: Why do we need homogeneity? We can always use additivity

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instead. For example $T(2v) = T(v + v) = 2{\cdot}T(v)$.

Teacher: You are taking the constant to be in $\mathbb{F} = \mathbb{Q}$, but usually it is in $\mathbb{F} = \mathbb{R}$ and in that case homogeneity is needed.

Student: OK.... so please show me an example where additivity is not enough, when $\mathbb{F} = \mathbb{R}$.

Teacher: (after a few minutes of trying) ..... This is a very good question, I will discuss this case next week ....

This example is interesting because the answer to the student's question depends on the field, $\mathbb{F}$. When $\mathbb{F} = \mathbb{Z}_p$ (p is prime) or $\mathbb{F} = \mathbb{Q}$, homogeneity is indeed redundant, but when $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, it is not redundant. Furthermore, when $\mathbb{F} = \mathbb{R}$ no actual counterexample can be shown, and this is why the teacher failed to find a counterexample.

Proposition 1: If $\mathbb{F} = \mathbb{Z}_p$ then the homogeneity axiom is redundant.

Proof: For each $k \in \mathbb{Z}_p$: $T(k{\cdot}v) = T\left(\sum_{i=1}^{k} v\right) = \sum_{i=1}^{k} T(v) = k{\cdot}T(v)$.

Proposition 2: If $\mathbb{F} = \mathbb{Q}$ then the homogeneity axiom is redundant.

Proof:

(1) For each $q \in \mathbb{N}$: $T(q{\cdot}v) = T\left(\sum_{i=1}^{q} v\right) = \sum_{i=1}^{q} T(v) = q{\cdot}T(v)$.

(2) For $q = 0$: $T(0{\cdot}v) = T(0) = 0 = 0{\cdot}T(v) \Rightarrow T(q{\cdot}v) = q{\cdot}T(v)$.

(3) For $q = -1$: $T(v) + T(-v) = T(v - v) = T(0) = 0 \Rightarrow T(-v) = -T(v) \Rightarrow T\left((-1){\cdot}v\right) = (-1){\cdot}T(v) \Rightarrow T(q{\cdot}v) \Rightarrow q{\cdot}T(v)$.

(4) From (1)-(3) it follows that for each negative $q \in \mathbb{Z}$:
$T(q{\cdot}v) = T\left((-1){\cdot}|q|{\cdot}v\right) = (-1){\cdot}T\left(|q|{\cdot}v\right) = (-1){\cdot}|q|{\cdot}T(v) = q{\cdot}T(v)$.

(5) From (1)-(4) it follows that for each $q \in \mathbb{Z}$: $T(q{\cdot}v) = q{\cdot}T(v)$.

(6) For each $q = \frac{a}{b}$ such that $a, b \in \mathbb{Z}$ and $a, b > 0$:

(6.1) $T(q{\cdot}v) = T\left(\frac{a}{b}{\cdot}v\right) = T\left(a{\cdot}\frac{1}{b}{\cdot}c\right) = a{\cdot}T\left(\frac{1}{b}{\cdot}v\right) \Rightarrow T(q{\cdot}v) = a{\cdot}T\left(\frac{1}{b}{\cdot}v\right)$.

(6.2) $T(v) = T\left(b{\cdot}\frac{1}{b}{\cdot}v\right) = b{\cdot}T\left(\frac{1}{b}{\cdot}v\right) \Rightarrow T\left(\frac{1}{b}{\cdot}v\right) = T\left(\frac{1}{b}{\cdot}v\right) = \frac{1}{b}{\cdot}T(v)$.

From (6.1) and (6.2) we get $T(q{\cdot}v) = a{\cdot}T\left(\frac{1}{b}{\cdot}v\right) = a{\cdot}\frac{1}{b}{\cdot}T(v) = \frac{a}{b}{\cdot}T(v) = q{\cdot}T(v)$.

(7) From (3) and (6) it follows that for each $q \in \mathbb{Q}$, i.e. $q = \frac{a}{b}$ such that $a, b \in \mathbb{Z}$ and $b \neq 0$: $T(q{\cdot}v) = q{\cdot}T(v)$.

Proposition 3: If $\mathbb{F} = \mathbb{C}$ then the homogeneity axiom is not redundant.

Proof: Look at the map $T(a + bi) = a - bi$. This map is additive, as the following line proves:

$$T(u + v) = T\left((a + bi) + (c + di)\right) = T\left((a + c) + (b + d)\,i\right)$$
$$= (a + c) - (b + d)\,i = (a - bi) + (c - di)$$

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$$= T(u) + T(v).$$

But, if we assume homogeneity, we get a contradiction

$$-1 = T(-1) = T(i \cdot i) = i \cdot T(i) = i \cdot (-i) = 1.$$

Proposition 4: If $\mathbb{F} = \mathbb{R}$ then the homogeneity axiom is not redundant.

Proof: Consider the vector space V which is $\mathbb{R}^{\mathbb{Q}}$ (the vector space of $\mathbb{R}$ over $\mathbb{Q}$). According to the Axiom of Choice [11] there is a basis B in V (a Hamel basis), i.e. $B = \{b_i | i \in I\}$, such that for all $v \in \mathbb{R}^{\mathbb{Q}}$, $v = \sum_{i \in I_v} a_i^v \cdot b_i$, where I_v is a finite subset of I.

Let $b_1 \in B$. We define the map $T: \mathbb{R} \to \mathbb{R}$ such that $T(v) = a_1^v$. Now we prove that it is additive but not homogeneous (note that $\mathrm{Im}(T) = \mathbb{Q}$, and not $\mathbb{R}$):

for all $v, u \in \mathbb{R}$: $v = \sum_{i \in I_v} a_i^v \cdot b_i$ and

$$u = \sum_{i \in I_u} a_i^u \cdot b_i \Rightarrow v + u = \sum_{i \in I_v \cup I_u} \left(a_i^v + a_i^u\right) \cdot b_i$$

$$\Rightarrow T(v + u) = a_i^v + a_i^u = T(u) + T(v).$$

This proves T is additive. But, if we assume homogeneity: for all $r \in \mathbb{R}$: $T(r \cdot b_1) = r \cdot T(b_1) = r \cdot 1 = r$, and if we choose $r = \sqrt{2}$, we get $T(\sqrt{2} \cdot b_1) = \sqrt{2} \notin \mathbb{Q}$, which is a contradiction to $\mathrm{Im}(T) = \mathbb{Q}$.

We note that a similar proof appears in [12].

Notice that this proof is outside the scope of linear algebra, because $\mathbb{R} = \mathbb{Q}$ is an infinite vector space. Also, this proof does not provide us with a counterexample, because we cannot actually construct the Hamel basis.

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