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# Chromatic-durable graphs†

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1. *Introduction.* One of the most studied parameters in all of graph theory is the chromatic number. Undoubtedly, its popularity as a subject for research is due to its intimate relationship with the famous Four Colour Problem.

We denote the chromatic number of a graph $G$ by $\chi(G)$. It is a basic fact that the removal of any point $v$ from $G$ results in a graph $G-v$ whose chromatic number is little different from that of $G$; in fact, $\chi(G-v) = \chi(G)$ or $\chi(G-v) = \chi(G)-1$. Those graphs $G$ for which $\chi(G-v) = \chi(G)-1$ for each point $v$ of $G$ have been studied extensively, particularly by Dirac (see (5), (6)). Such graphs have been called *chromatic-critical*. If $\chi(G) = n$ and $G$ is chromatic-critical, then $G$ is said to be *n-chromatic-critical* or *n-critical*. It is also natural to consider graphs $G$ such that $\chi(G-v) = \chi(G)$ for each point $v$ of $G$. We call such graphs *chromatic-durable*. If $\chi(G) = n$ and $G$ is chromatic-durable, then $G$ is said to be *n-chromatic-durable* or simply *n-durable*. This article studies some of the properties of $n$-chromatic-durable graphs.

We also define the line analogues of the concepts given in the above paragraph. A graph $G$ is said to be *chromatic-minimal* if $\chi(G-e) = \chi(G)-1$ for each line $e$ of $G$. If $\chi(G) = n$ and $G$ is chromatic-minimal, then we say $G$ is *n-cromatic-minimal* or *n-minimal*. Correspondingly, we say a graph $G$ is *chromatic-permanent* if $\chi(G-e) = \chi(G)$ for each line $e$ of $G$. If $\chi(G) = n$ and $G$ is chromatic-permanent, then $G$ is said to be *n-chromatic permanent* or simply *n-permanent*.

To illustrate the concepts of chromatic-durable and chromatic-permanent graphs, we present a number of examples.

(i) The complete graph $K_n$ is $n$-critical and $n$-minimal, but neither $n$-durable nor $n$-permanent.

(ii) The star $K_{1,n}$ (a complete bipartite graph with two point sets having one and $n$ points, respectively) is 2-permanent for $n > 1$, but not 2-durable.

(iii) Any tree with at least four points, other than $K_{1,n}$, is both 2-durable and 2-permanent.

(iv) The regular complete $n$-partite graph of order $2n$, denoted by $K_{n(2)}$, is both $n$-durable and $n$-permanent.

(v) The graph consisting of two disjoint copies of $K_n$ joined by a line $e$, denoted by $2K_n + e$, is both $n$-durable and $n$-permanent (see Fig. 1 for $2K_4 + e$).

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(vi) The one point union of two complete graphs $K_n$, denoted by $K_n.K_n$, is $n$-permanent, but not $n$-durable (see Fig. 1 for $K_4.K_4$).




Fig. 1

2. *Chromatic-durable graphs.* It is clear that a graph which is $n$-durable is also $n$-permanent. This contrasts with the fact that an $n$-minimal graph (having no isolated points) is also $n$-critical.

The proofs of the following two propositions are immediate.

LEMMA 1. *A graph $G$ is $n$-durable if and only if $\chi(G) = n$ and in every $n$-colouring each colour class contains more than one point.*

LEMMA 2. *For any $n$-colouring of an $n$-durable graph, each colour class contains at least two points each of which is joined by a line to at least one point of every other colour class.*

From these two lemmas, lower bounds for the number of points and lines of an $n$-durable graph are readily deduced.

THEOREM 1. *Any $n$-durable graph contains at least $2n$ points and at least $n^2-n$ lines.*

The next lemma follows directly from the definitions, and will be used to find the unique $n$-durable graph attaining both minimal values given in Theorem 1.

LEMMA 3. *For $n \geqslant 2$, if an $n$-durable graph is coloured in any way with $n$ colours, and the points of any one colour class are deleted, then the resulting subgraph is $(n-1)$-durable.*

THEOREM 2. *For $n \geqslant 1$, if $G$ is an $n$-durable graph containing exactly $2n$ points and $n^2-n$ lines, then $G = 2K_n$.*

*Proof.* We use induction on $n$. The result is trivial for $n = 1$. Now assume the result holds for natural numbers less than $n$, and let $G$ be $n$-durable with $2n$ points and $n^2-n$ lines, where $n \geqslant 2$. Consider any $n$-colouring of $G$, and remove the points of any one colour class. Designate the resulting subgraph by $G^*$. Clearly $G^*$ has exactly $2(n-1)$ points; since $G$ had exactly $2n$ points and was $n$-durable, each colour class of $G$ had precisely two points. Also, Lemma 2 implies that each point of $G$ had degree at least $n-1$. But since $G$ had exactly $n^2-n$ lines, each point of $G$ had degree precisely $n-1$. Hence $G^*$ has exactly $(n^2-n)-2(n-1) = (n-1)^2-(n-1)$ lines. By Lemma 3, $G^*$ is $(n-1)$-durable. By the induction hypothesis, $G^* = 2K_{n-1}$. We claim that $G = 2K_n$. Let $u$ and $v$ be the two points deleted from $G$ to obtain $G^*$. Then both $u$ and $v$ have degree $n-1$ in $G$. Suppose $u$ is adjacent, in $G$, to at least one point in each copy of $K_{n-1}$ making up $G^*$. By interchanging the colours for one copy of $K_{n-1}$ (if necessary), two of the points adjacent to $u$ in $G$ may now be assigned the same colour. Then

$$\chi(G-v) = n-1,$$

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contradicting the $n$-durability of $G$. Hence $u$ is contained in a subgraph $K_n$ of $G$. Similarly, $v$ is contained in a second subgraph $K_n$ of $G$. Since $G$ is regular of degree $n-1$, $G = 2K_n$.

Similar considerations establish the following:

THEOREM 3. *Let $G$ be a connected $n$-durable graph, for $n \geqslant 2$. Then $G$ has at least $2n$ points and $n^2-n+1$ lines. Moreover, if $G$ has exactly $n^2-n+1$ lines, then $G = 2K_n+e$.*

THEOREM 4. *Let $\chi(G) = n$. Then $\chi(G-v) = n-1$ if and only if the point $v$ is contained in each $n$-chromatic subgraph of $G$.*

*Proof.* If $\chi(G-v) < n$, then every subgraph of $G-v$ has chromatic number less than $n$, so that $v$ must be contained in every $n$-chromatic subgraph of $G$. Conversely, if $\chi(G-v) = n$, then $v$ is not contained in the $n$-chromatic subgraph $G-v$.

This theorem relates to $n$-durable graphs by the following corollary:

COROLLARY 4*a*. *A graph $G$ is $n$-durable if and only if the intersection of all $n$-chromatic subgraphs of $G$ is empty.*

Similarly, Theorem 4 may be restated for $n$-critical graphs:

COROLLARY 4*b*. *A graph $G$ is $n$-critical if and only if there are no proper (pointwise) $n$-chromatic subgraphs of $G$.*

We have already used the observation that any $n$-chromatic graph contains an $n$-critical (induced) subgraph. It is also clear that any $n$-chromatic graph $G$ is an induced subgraph of an $n$-durable graph: form the disjoint union of two copies of $G$. (If the graph $G$ is connected, then adding one line between these two copies of $G$ produces a connected $n$-durable supergraph of $G$.)

3. *Chromatic-permanent graphs.* In this section we present results for chromatic-permanency analogous to those for chromatic-durability given above. We have already observed that if $G$ is $n$-durable, then $G$ is also $n$-permanent. We now show that if $G(p, q)$ is $n$-permanent, then $p \geqslant 2n-1$ and $q \geqslant n^2-n$.

THEOREM 5. *An $n$-permanent graph has at least $2n-1$ points and $n^2-n$ lines.*

*Proof.* Let $G$ be $n$-permanent with at most $2n-2$ points. Then in any $n$-colouring of $G$ there are at least two colour classes with only one point. These two colour classes must be adjacent, since $\chi(G) = n$. But by removing the edge $e$ between these two points, we find $\chi(G-e) = n-1$, contradicting the $n$-permanency of $G$. Thus $p \geqslant 2n-1$.

Now let $G$ be $n$-permanent with a proper $n$-colouring. Each colour class must be joined to each other colour class by at least two lines, for otherwise the removal of the single line would reduce the number of colour classes. Thus each colour class has at least $2(n-1)$ lines incident with it. Therefore $q \geqslant n(n-1)$.

We observe that the one point union of two complete graphs on $n$ points, denoted by $K_n.K_n$, is $n$-permanent with the minimum number of points and lines. The following theorem shows that this graph is unique in this respect.

THEOREM 6. *The graph $G$ is an $n$-permanent $(2n-1, n^2-n)$ graph if and only if*

$$G = K_n.K_n.$$



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*Proof.* Let $G$ be $n$-permanent with $2n-1$ points and $n^2-n$ lines. Then in any $n$-colouring of $G$, some colour class has exactly one point, say $v$. Moreover, each other colour class contains exactly two points, for if $G$ contained two colour classes of one point each, then the graph formed by removing the line joining these two points could be $(n-1)$-coloured. It now follows that $v$ must be adjacent to every other point of $G$; i.e. $v$ has degree $2n-2$. We now claim that any other point $u$ in $G$ must have degree at least $n-1$. Suppose to the contrary that some point $u$ has degree less than $n-1$, and let $w$ be the other point in the same colour class with $u$. Then in the graph formed by removing the line between $v$ and $w$ from $G$, $u$ may be re-coloured with a colour it is not adjacent to, and $w$ may be re-coloured with the colour of $v$, giving an $(n-1)$-colouring of $G-vw$; which is a contradiction. It now follows that $G$ contains one point of degree $2n-2$ and $2n-2$ points of degree $n-1$, since $q = n^2-n$.

Since $\chi(G) = n$, $G$ contains an $n$-minimal subgraph $H_1$. We may assume that $H_1$ contains no isolated points, and then $H_1$ must be $n$-critical also. Hence the minimum degree of $H_1$ satisfies $\delta(H_1) \geqslant n-1$, so that $\delta(H_1) = n-1$. Let $e$ be an edge of $H_1$, and consider the graph $G-e$. Repeat the argument above to find an $n$-critical subgraph $H_2$ of $G$, distinct from $H_1$, with $\delta(H_2) = n-1$. Let $H_2$ have $k$ points not in $H_1$, $0 \leqslant k \leqslant n-1$. We claim first that $k > 0$. Assume to the contrary that $V(H_2) \subseteq V(H_1)$. We must have $E(H_2) \nsubseteq E(H_1)$, since $H_1$ is $n$-minimal. Let $x = uw \in E(H_2) - E(H_1)$. Without loss of generality, assume that $u \neq v$. Then $u$ has degree $n-1$ in $H_1$, and hence degree at least $n$ in $G$; which is a contradiction. Thus $k > 0$. Now each of those $k$ points is adjacent to $v$ but to no other point in $H_1$, since these other points each have degree $n-1$, both in $H_1$ and in $G$. But each of these $k$ points has degree $n-1$, so that we must have $k = n-1$. Hence $H_1$ has exactly $n$ points and must be $K_n$. Similarly, $H_2 = K_n$, with $V(H_1) \cap V(H_2) = \{v\}$. Hence $G = K_n . K_n$.

The following theorem is analogous to Theorem 4; its proof will be omitted.

THEOREM 7. *Let $\chi(G) = n$. Then $\chi(G-e) = n-1$ if and only if the line $e$ is contained in each $n$-chromatic subgraph of $G$.*

COROLLARY 7*a*. *A graph $G$ is $n$-permanent if and only if the intersection of the line sets of all $n$-chromatic subgraphs of $G$ is empty.*

COROLLARY 7*b*. *A graph $G$ is $n$-minimal if and only if there are no proper (line-wise) $n$-chromatic subgraphs of $G$.*

4. *The point-partition numbers.* In (1) Chartrand, Geller, and Hedetniemi defined graphs having property $P_m$. A graph $G$ is said to have property $P_m$ if it contains no subgraph which is homeomorphic from the complete graph $K_{m+1}$ or the complete bipartite graph $K([\frac{1}{2}(m+2)], \{\frac{1}{2}(m+2)\})$. The graphs having property $P_m$, $1 \leqslant m \leqslant 4$, have been studied under other names. A graph has property $P_1$ if and only if it is *totally disconnected*. The graphs with property $P_2$ are the *acyclic graphs* or *forests*, while an *outerplanar graph* is one with property $P_3$ (see (2)). Finally, the Kuratowski characterization of *planar graphs* (8) shows that the graphs with property $P_4$ are planar graphs. No special names have been given to graphs having property $P_m$, $m \geqslant 5$.

The *point-partition number* $\chi_m(G)$, $m \geqslant 1$, of a graph $G$ is the smallest integer $n$ for which there exists a partition $V_1, V_2, \ldots, V_n$ of the point set of $G$ such that each induced

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subgraph $\langle V_i \rangle$, $i = 1, 2, \ldots, n$, has property $P_m$. The parameter $\chi_1(G)$ gives the *chromatic number* and has been discussed above. Each $\chi_m(G)$ is essentially a colouring number, since it gives the minimum number of colours in any colouring of the points of $G$ so that each colour class induces a subgraph having property $P_m$. The parameter $\chi_2(G)$ is referred to as the *point-arboricity* of the graph $G$ and has been studied in (3), (4), and (7). Virtually no research has been done on the parameters $\chi_m(G)$ for $m \geqslant 3$.

If $\chi_m(G) = n$ and $\chi_m(G-v) < n$ for each point $v$ of $G$ ($\chi_m(G-e) < n$ for each line $e$ of $G$), then we say that $G$ is $n$-critical ($n$-minimal) *with respect to* $\chi_m$. Correspondingly, if $\chi_m(G) = n$ and $\chi_m(G-v) = n$ for each point $v$ of $G$ ($\chi_m(G-e) = n$ for each line $e$ of $G$), then we say that $G$ is *$n$-durable* (*$n$-permanent*) *with respect to* $\chi_m$.

It is well known that if $G$ is an $n$-critical graph with respect to $\chi_1$ (chromatic number), then the minimum degree of $G$ satisfies $\delta(G) \geqslant n-1$. Chartrand and Kronk (3) proved that if a graph $G$ is $n$-critical with respect to $\chi_2$ (point-arboricity), then $\delta(G) \geqslant 2(n-1)$. We prove the following result for graphs $G$ which are $n$-critical with respect to $\chi_m$, $m \geqslant 3$.

THEOREM 8. *If $G$ is $n$-critical with respect to $\chi_m$, $m \geqslant 3$, then $\delta(G) \geqslant 2(n-1)$.*

*Proof.* There are no 1-critical graphs and so we assume $n \geqslant 2$. Suppose there is a point $v$ of degree $2n-3$ or less. Since $G$ is $n$-critical with respect to $\chi_m$, $\chi_m(G-v) = n-1$, and there is a partition $V_1, V_2, \ldots, V_{n-1}$ of the point set of $G-v$ such that each induced subgraph $\langle V_i \rangle$, $i = 1, 2, \ldots, n-1$, has property $P_m$. Because $\deg v \leqslant 2n-3$, at least one of these subsets, say $V_j$, contains at most one point adjacent to $v$. The set $V_j \cup \{v\}$ necessarily induces a subgraph with property $P_m$. Hence, $V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{n-1}$ is a partition of the point set of $G$ into $n-1$ subsets, each of which induces a subgraph with property $P_m$. This implies that $\chi_m(G) < n$, which is a contradiction; thus $\delta(G) \geqslant 2(n-1)$.

THEOREM 9. *Let $\chi_m(G) = n$.*

(i) *If $G$ is $n$-minimal with respect to $\chi_m$ and $G$ has no isolated points, then $G$ is $n$-critical with respect to $\chi_m$.*

(ii) *If $G$ is $n$-durable with respect to $\chi_m$, then $G$ is $n$-permanent with respect to $\chi_m$.*

The proof is straightforward and will be omitted.

The observation that any graph with at most $m$ points has property $P_m$ leads to the result that if $\chi_m(G) = n$, then $G$ has at least $m(n-1)+1$ points. Hence any graph $G$ which is $n$-critical ($n$-minimal) with respect to $\chi_m$ has at least $m(n-1)+1$ points. In fact, the following theorem shows that there is a unique graph $G$ which has $\chi_m(G) = n$ and has $m(n-1)+1$ points, namely $G = K_{m(n-1)+1}$.

THEOREM 10. *If $\chi_m(G) = n$, $n \geqslant 2$, and $G$ has $m(n-1)+1$ points, then $G = K_{m(n-1)+1}$.*

*Proof.* Assume that $\chi_m(G) = n$ and $G$ has $m(n-1)+1$ points. Let $u$ and $v$ be any pair of points of $G$. If $u$ and $v$ are not adjacent in $G$, then we may partition the point set of $G$ into $n-2$ sets, each containing $m$ points, and one set containing $u$ and $v$ as well as $m-1$ other points. Since any graph with $m$ points has property $P_m$ and any subgraph of $K_{m+1}-e$ has property $P_m$, we have partitioned the point set of $G$ into $n-1$

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sets such that each set induces a subgraph with property $P_m$. Thus $\chi_m(G) < n$; which is a contradiction. Therefore each pair of points of $G$ is adjacent and $G = K_{m(n-1)+1}$.

Since the smallest (with respect to the number of points) graph $G$ with $\chi_m(G) = n$ is $G = K_{m(n-1)+1}$, the smallest $n$-critical ($n$-minimal) graph with respect to $\chi_m$ is also $K_{m(n-1)+1}$. It follows that any $n$-durable ($n$-permanent) graph with respect to $\chi_m$, $m \geqslant 2$, has at least $m(n-1)+2$ points. This leads to the following theorems.

THEOREM 11. *If $G$ is an $n$-durable ($n$-permanent) graph, $n \geqslant 3$, with respect to $\chi_m$, $m \geqslant 2$, having $m(n-1)+2$ points, then $G = K_{m(n-1)+2}$.*

*Proof.* Let $G$ be any $n$-permanent graph with respect to $\chi_m$, $m \geqslant 2$, $n \geqslant 3$ having $m(n-1)+2$ points (then $G$ has at least 6 points). Let $u$ and $v$ be any two points of $G$. If $u$ and $v$ are not adjacent, let $e = wz$ be any line of $G$ not incident with $u$ and $v$. Since $G$ is $n$-permanent with respect to $\chi_m$, $\chi_m(G-e) = n$. Then the point set of $G$ can be partitioned into $n-3$ sets each containing $m$ points, one set containing $m+1$ points of which two are $u$ and $v$, and one set containing $m+1$ points of which two are $w$ and $z$. Thus the graph $G-e$ has its point set partitioned into $n-1$ sets each inducing a subgraph with property $P_m$. Thus $\chi_m(G-e) < n$, which is a contradiction. Therefore each pair of points is adjacent and $G = K_{m(n-1)+2}$.

Since any $n$-durable graph with respect to $\chi_m$ is $n$-permanent with respect to $\chi_m$ (see Theorem 9), we have completed the proof.

THEOREM 12. *If $G$ is a 2-durable graph with respect to $\chi_m$, $m \geqslant 2$, having $m+2$ points, then $G = K_{m+2}$.*

*Proof.* Let $G$ be any 2-durable graph with respect to $\chi_m$, $m \geqslant 2$, having $m+2$ points. Let $u$ and $v$ be any two points of $G$. If $u$ and $v$ are not adjacent in $G$, let $w$ be any point of $G$ other than $u$ or $v$. Then $G-w$ is a subgraph of $K_{m+1}-e$ and so $\chi_m(G-w) = 1$. This is a contradiction, since $G$ was assumed to be 2-durable. Thus $u$ and $v$ are adjacent and $G = K_{m+2}$.

Since the graph $K_{m+2}$ with a line removed is 2-permanent with respect to $\chi_m$, Theorem 11 cannot be extended to include 2-permanency with respect to $\chi_m$.

Since the minimum degree of an $n$-critical graph ($n$-minimal graph with no isolated points) with respect to $\chi_2$ is at least $2(n-1)$, the smallest (with respect to the number of lines) such a graph is also $K_{2(n-1)+1} = K_{2n-1}$. Likewise the smallest $n$-durable ($n$-permanent), $n \geqslant 3$, graph with respect to $\chi_2$ is $K_{2(n-1)+2} = K_{2n}$. This argument does not generalize for cases where $m > 2$, since, for example, $\delta$ may be less than $3(n-1)$ for an $n$-critical graph with respect to $\chi_3$. The complete bipartite graph $K_{2,3}$ is 2-critical with respect to $\chi_3$, but the minimum degree of $K_{2,3}$ is 2.

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