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The drag on a flattened bubble moving across a plane substrate

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The equilibrium shape of an axisymmetric liquid drop or gas bubble in an immiscible supporting liquid held under gravity against a horizontal plane rigid surface is derived. A thin film of supporting liquid remains between the drop/bubble surface and the rigid substrate at equilibrium, of thickness h_0 determined by the balance of the disjoining pressure $\Pi(h)$ between the drop and the substrate and the internal Laplace pressure of the drop/bubble. The interface is macroscopically flat around the drop axis out to a radius a but matches smoothly into the outer shape of radius A through a boundary layer region of width $aH_0^{1/2}$ where $H_0 = 2h_0A/a^2$ is a small parameter. The outer drop shape is determined by a balance of buoyancy forces and local Laplace pressure and is roughly spherical if $A/\lambda \ll 1$, where $\lambda = \sqrt{\gamma/\Delta\rho g}$ is the capillary length in the interface with a logarithmic correction due to the action of the disjoining pressure across the flattened region. With the shape determined, we calculate the drag force on this flattened bubble to lowest order in the velocity as it moves across the rigid substrate using a lubrication approximation valid to terms of $O((h_0/a)^2)$ as an integral over the flattened bubble surface of the hydrodynamic pressure. The lubrication theory of itself is not sufficient to determine the drag due to the divergence of that integral if the outer flow field properties are neglected. By using the known exact result for the drag force on an undistorted bubble, the drag on the flattened bubble can be computed as an integral over the lubrication region alone. We derive the drag as a series expansion in the small parameter H_0 by means of a fairly intricate boundary layer analysis. The logarithmic divergence of the translational drag with film thickness h_0 for the undistorted bubble is replaced by the stronger divergence $h_0^{-1/2}$ to leading order for the flattened bubble case. We present explicit numerical results for the first few terms in the expansion for the case of an exponentially repulsive disjoining pressure, and analytic expressions for these terms in the limit of very short-range disjoining pressure forces. The results of this calculation are compared with recent work (Hodges, Jensen & Rallison, *J. Fluid Mech.*, vol. 512, 2004, p. 95) where disjoining pressure is neglected and hydrodynamic pressure is balanced against buoyancy and Laplace forces. The limits of validity of this linear drag theory are also presented.

Key words: capillary flows, drops and bubbles, thin films

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1. Introduction

The interaction of drops and bubbles with surfaces is important in many industrial processes and has been the subject of much recent study. This has ranged from high-speed drop impacts with subsequent bouncing and bursting (Yarin 2006) to very gentle probing of droplet interfaces with the atomic force microscope (Dagastine *et al.* 2006) and the surface force apparatus (Clasohm *et al.* 2005; Manica *et al.* 2007). The nature of the hydrodynamics involved varies with the size range of the drops, their velocity and viscosity ratio. In the case of bubbles, the cleanliness of the interface and surfactant transport also has a major effect (Leal 2007).

Starting with the work of Bretherton (1961) on the steady motion of a bubble through a capillary, a body of work has emerged on the tangential motion of drops and bubbles past solid surfaces. Examples include the study of bubbles confined between two plane surfaces in a Hele-Shaw cell (Maruvada & Park 1996) and bubbles confined in rectangular microchannels (Rabaud *et al.* 2011; Ajaev & Homsy 2006). The study of single bubbles is used to elucidate processes involved in emulsions and foams (Denkov *et al.* 2005; Raufaste *et al.* 2007).

In recent years, interesting experimental studies on the rolling, sliding and slipping of drops down an inclined solid surface (Aussillous & Quere 2002; Quere 2005) have provoked theoretical studies that exhibit a startling variety of dynamical behaviours, depending on only a few key parameters (Hodges, Jensen & Rallison 2004). This work relied on the assumption of fairly mild distortions from static equilibrium, where pressure forces are balanced by surface tension and buoyancy. As such it is limited to fairly low velocities, hence inclination angles. A strength of this treatment is the unifying view it gives to a wide range of behaviour, together with scaling laws for the steady velocity in each region of the 'phase diagram', largely derived from lubrication theory. This theory treats both drops of arbitrary viscosity, and bubbles, assumed clean.

On the more computational side, the tangential drag on a hemispherical bubble near a surface has been studied at a range of velocities, including the low-Reynolds-number range considered here (Legendre, Colin & Coquard 2008). Sophisticated adaptive boundary integral computations have explored the detailed shape of a drop sliding down a surface under creeping flow conditions (Griggs, Zinchenko & Davis 2008).

Much of the work cited above considers only hydrodynamic interactions and surface tension and ignores the surface forces that keep the thin liquid film separating the drop/bubble and the solid surface stable in the first place. At equilibrium, the effect of surface forces on the film profile is well-understood (Chan, Dagastine & White 2001). Recent experimental and theoretical work has begun to explore the complexities of film deformation when a bubble approaches a surface with a velocity in the normal direction (Ajaev, Tsekov & Vinogradova 2007, 2008; Pushkarova & Horn 2008). Good agreement between experiment and models (Manor *et al.* 2008) shows that these interactions are also well-understood. The situation in this paper, however, involves the bubble moving with a lateral velocity past a surface.

At sufficiently high velocities, the hydrodynamic film pressure will dominate the disjoining pressure and the purely hydrodynamic treatments discussed above will become valid. At very low velocities there must be a regime which shows the first effects of fluid flow as perturbation of the equilibrium between surface forces and surface tension. It is such a 'zeroth-order' theory of the steady tangential motion of a bubble, assumed clean, that we explore in this paper. Because of the limited velocity range we consider, we are able to give a description largely in analytical terms, as asymptotic expressions supported by limited numerical calculations.

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Since we are considering weak hydrodynamic perturbations to the equilibrium situation, we first analyse carefully the shape of a sessile bubble held against a horizontal surface by buoyancy, with a stable film generated by repulsive disjoining pressure. In § 3, we compute the flow fields for a bubble slipping past a surface, adapting the methods of Goldman, Cox & Brenner (1967) and O'Neill & Stewartson (1967). Then we extract the leading-order behaviour of the hydrodynamic drag as a function of the film thickness for an undistorted bubble. Finally we consider the sensitivity of this result to the precise nature of the disjoining pressure and its influence on the leading meniscus of the bubble. We determine the leading effects of the hydrodynamic film pressure on the film profile and hence the limits where this analysis is valid.

The analysis here has recently been used as a framework for discussing experimental studies on bubbles sliding up a quartz surface in salt solution and which show some dependence on salt concentration (Del Castillo *et al.* 2011). While those experiments use bubbles too large for the analysis here to be valid, they do point to an influence of the disjoining pressure on sliding velocity, an effect that cannot be explained by purely hydrodynamic theories.

2. Shape of the equilibrium bubble

The equation for the shape $h(r)$ of a drop/bubble held against a plane horizontal interface by gravity g and stabilized by a repulsive disjoining pressure $\Pi(h(r))$ is given by the local force balance (Deryaguin & Kussakov 1939; Teletzke, Davis & Scriven 1988; Chan *et al.* 2001)

$$\frac{h''}{(1+h'^2)^{3/2}} + \frac{1}{r}\frac{h'}{(1+h'^2)^{1/2}} + \frac{\Pi(h)}{\gamma} = \frac{2}{A} - \frac{\Delta\rho g(h-h(0))}{\gamma} \tag{2.1}$$

where $h(r)$ is the inter-surface separation at horizontal axial distance r, γ is the surface tension of the fluid interface, $\Delta\rho = \rho_m - \rho_b$ is the density difference between the suspending liquid and the drop/bubble. The Laplace pressure in the drop is $2\gamma/A$ where A can be regarded as a suitable scaling length for the drop dimension. The capillary length λ of the interface is given by

$$\lambda = \sqrt{\gamma/\Delta\rho g}. \tag{2.2}$$

When $A/\lambda \ll 1$, the length scale A is the measured radius of the spherical cap that is the outer shape of the sessile drop. We will restrict ourselves to this regime of small drops. The apparent shape of the drop in the region where the disjoining pressure stabilizes the interface is flat at a separation h_0 from the plane surface. The local force balance equation (2.1) yields a definition for this separation

$$\Pi(h_0) = \frac{2\gamma}{A}. \tag{2.3}$$

The drop is macroscopically flat over an axial distance a to be determined as part of the description of the drop shape. Since $A/\lambda \ll 1$ and $a/A \ll 1$ in this regime, it follows that $a/\lambda \ll 1$. Equation (2.1) can be rearranged as

$$h'' + \frac{1}{r}h' + \left(\frac{\Pi(h)}{\gamma} - \frac{2}{A}\right) = \left(\frac{\Pi(h)}{\gamma} - \frac{2}{A}\right)\left(1-(1+h'^2)^{3/2}\right) - \frac{h'^3}{r} - \frac{h-h(0)}{\lambda^2}(1+h'^2)^{3/2}. \tag{2.4}$$

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If we now introduce the scalings

$$\eta(R) = \frac{h(r)}{h_0}, \quad R = \frac{r}{a}, \tag{2.5}$$

the drop shape equation becomes

$$\begin{aligned}\eta'' + \frac{1}{R}\eta' + \frac{4}{H_0}\left(\frac{A\Pi(h_0\eta)}{2\gamma} - 1\right) \\ = \frac{4}{H_0}\left(\frac{A\Pi(h_0\eta)}{2\gamma} - 1\right)\left(1 - \left(1 + \left(\frac{h_0}{a}\right)^2 \eta'^2\right)^{3/2}\right) \\ - \left(\frac{h_0}{a}\right)^2 \frac{\eta'^3}{R} - \left(\frac{a}{\lambda}\right)^2 (\eta - 1)\left(1 + \left(\frac{h_0}{a}\right)^2 \eta'^2\right)^{3/2}\end{aligned} \tag{2.6}$$

where the parameter H_0 is defined by

$$H_0 = \frac{2h_0 A}{a^2}. \tag{2.7}$$

As shown below, $H_0 \ll 1$ under the small-drop conditions invoked above. Provided $h_0/a \ll 1$ the drop shape equation becomes

$$\eta'' + \frac{1}{R}\eta' + \frac{4}{H_0}\left(\frac{A\Pi(h_0\eta)}{2\gamma} - 1\right) = 0 + \cdots \tag{2.8}$$

where the omitted terms are $O((h_0/a)^2)$. We see immediately that the drop shape cannot be microscopically flat for $0 \leqslant R \sim O(1)$ otherwise the trivial solution of (2.6) or (2.8), i.e. $\eta(R) = 1$, will hold for all R correct to terms of $O((a/\lambda)^2)$. Accordingly we seek a solution

$$\eta(R) = 1 + \delta N(R) \tag{2.9}$$

where $N(R) \sim O(1)$ and δ is a small parameter. Around $R = 0$ we may expand $\Pi(h_0\eta)$ about $\eta = 1$ so that the drop shape (2.8) becomes, to leading order in δ,

$$N'' + \frac{1}{R}N' - \frac{4\alpha^2}{H_0}N = 0 + \cdots \tag{2.10}$$

where

$$\alpha^2 = \left(-\frac{h}{\Pi}\frac{\mathrm{d}\Pi}{\mathrm{d}h}\bigg|_{h_0}\right) = O(1). \tag{2.11}$$

The solution satisfying $\eta'(0) = 0$ is

$$\eta(R) = 1 + \delta \mathrm{I}_0(2\alpha R/H_0^{1/2}) \tag{2.12}$$

where $\mathrm{I}_0(z)$ is the modified Bessel function of zeroth-order (Abramowitz & Stegun 1965). For large argument we have

$$\mathrm{I}_0(z) = \frac{\mathrm{e}^z}{(2\pi z)^{1/2}}\left(1 + \frac{1}{8z} + O(z^{-2})\right) \tag{2.13}$$

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so that

$$\eta(R) = 1 + \frac{\Delta}{R^{1/2}} \exp\left(\frac{2\alpha(R-1)}{H_0^{1/2}}\right)\left(1 + \frac{H_0^{1/2}}{16\alpha R} + O(H_0)\right) \tag{2.14}$$

where

$$\Delta = \delta \exp\left(\frac{2\alpha}{H_0^{1/2}}\right) \Big/ \sqrt{\frac{4\pi\alpha}{H_0^{1/2}}}. \tag{2.15}$$

Since, at $R = 1$, the separation $\eta(R)$ will still be $O(1)$, we may expect $\Delta \sim O(1)$ and hence the central separation $\eta(0) = 1 + \delta$ will be astonishingly close to 1 because

$$\delta = \Delta \exp\left(-\frac{2\alpha}{H_0^{1/2}}\right) \sqrt{\frac{4\pi\alpha}{H_0^{1/2}}} \lll 1 \tag{2.16}$$

when $H_0 \ll 1$, as will be demonstrated below. Thus, even though the argument of the Bessel function will be $\gg 1$ for $R \gg H_0^{1/2}/2\alpha \ll 1$, it is only when $R \sim 1 - O(H_0^{1/2})$ that $N(R)$ is sufficiently large to counteract the incredibly small value of δ in order to produce an $O(1)$ increase in $\eta(R)$. We see, from (2.14), that there is a boundary layer of width $O(H_0^{1/2})$ around $R = 1$ in which the inter-surface shape begins to increase from its flat form $\eta(R) = 1 + O(\delta)$. In that region we can no longer expand $\Pi(h_0\eta)$ about $\eta = 1$ but must use the drop shape equation in its essentially 'exact' form, namely

$$\eta'' + \frac{1}{R}\eta' = -\frac{4}{H_0}\left(\frac{A\Pi(h_0\eta)}{2\gamma} - 1\right) + \cdots. \tag{2.17}$$

It is convenient to make a change of axial variable

$$s = R^2 - 1 \tag{2.18}$$

so that the drop axis is at $s = -1$ and the end of the flattened region $r = a$ is at $s = 0$. The region where the inter-surface separation increases to match into the macroscopic drop shape is $s > 0$. The drop shape equation (2.17) in this axial variable is

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((1+s)\frac{\mathrm{d}\eta}{\mathrm{d}s}\right) = -\frac{1}{H_0}\left(\frac{A\Pi(h_0\eta)}{2\gamma} - 1\right). \tag{2.19}$$

Once $\eta(s)$ begins to increase by $O(1)$, the disjoining pressure decreases rapidly to zero since typically its range is $O(h_0)$. Thus for $s \gg H_0^{1/2}$, the right-hand side of (2.19) becomes s independent and (2.19) can then be integrated to yield

$$\eta(s) = \frac{1}{H_0}(s - \log(1+s)) + O(1). \tag{2.20}$$

In unscaled variables the drop shape in this region is

$$h(r) = \frac{r^2 - a^2}{2A} - \frac{a^2}{A}\log(r/a) + O(h_0) \tag{2.21}$$

where the first term is the small-r form for a spherical cap truncated at $r = a$ and the second term is the distortion of the spherical cap due to the action of the disjoining pressure over the flattened region.

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To demonstrate that $H_0 \ll 1$, we consider a global force balance on the small drop. The buoyancy force is just $\Delta\rho g V_d$ where

$$V_d = \frac{4\pi}{3}A^3(1 + O((a/A)^4) + O((a/A)^2)) \tag{2.22}$$

is the drop volume. The omitted term $O((a/A)^4)$ is the correction for the truncation of the sphere and the omitted term $O((a/A)^2)$ is the correction due to the distortion of the spherical cap due to the action of the disjoining pressure over the flattened region. This latter term is of the same order as the gravitational distortion of the drop which we have already omitted in the small-drop approximation (see below). The upthrust is globally balanced by the disjoining pressure force

$$\frac{4\pi\Delta\rho g}{3}A^3 + \cdots = 2\pi\int_0^\infty \mathrm{d}r\; r\Pi(h(r)) + O((h_0/a)^2) \tag{2.23}$$

to terms consistent with the Derjaguin approximation for the action of disjoining pressure on non-parallel interfaces (White 1983). The macroscopic flattened radius a is defined consistently so that the macroscopic expression for the global disjoining pressure force matches the microscopic expression in (2.23), namely

$$\pi a^2\Pi(h_0) = 2\pi\int_0^\infty \mathrm{d}r\; r\Pi(h(r)). \tag{2.24}$$

Substituting (2.24) into (2.23), it follows from (2.3) that

$$a = \left(\frac{2}{3}\right)^{1/2}\frac{A^2}{\lambda}, \tag{2.25}$$

a result first derived by Deryaguin & Kussakov (1939). Thus we see that the distortion of the spherical cap produces a drop volume correction of order $(a/A)^2 = O((A/\lambda)^2)$ which is the order of the gravitational distortion correction. From (2.7) we have $H_0 = 3(h_0/A)\,(\lambda/A)^2$. For $A/\lambda \sim 0.1$ ($A \sim 0.3$ mm for the air/water system) and $h_0 \sim 100$ nm we have $H_0 \sim 0.1$. Slightly larger values of A or smaller values of h_0 will produce significantly smaller H_0 values. Too small values of A will cause H_0 to become $O(1)$ and the boundary layer analysis unnecessary since flattening will no longer be as pronounced. We discuss this situation in § 6.

Equation (2.24) in scaled variables reduces to

$$\int_{-1}^\infty \mathrm{d}s\left(\frac{A\Pi(h_0\eta(s))}{2\gamma} - \theta(-s)\right) = 0 \tag{2.26}$$

where $\theta(z)$ is the step function. To obtain the drop shape in the boundary layer, (2.19) is formally integrated to yield

$$(1+s)\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\frac{1}{H_0^{1/2}}\xi(s) \tag{2.27}$$

where

$$\xi(s) = \frac{1}{H_0^{1/2}}\int_{-1}^s \mathrm{d}s\left(\frac{A\Pi(h_0\eta(s))}{2\gamma} - 1\right). \tag{2.28}$$

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It follows from (2.28) that $\xi(s) \sim (1+s)O(\delta)$ around $s=-1$ and that $(\mathrm{d}\eta/\mathrm{d}s)(-1) \sim O(\delta)$ so that we may invoke the boundary condition

$$\frac{\mathrm{d}\eta}{\mathrm{d}s}(-1) = 0 + \cdots \tag{2.29}$$

correct to terms exponentially small in H_0.

We introduce a boundary layer variable z where

$$s = H_0^{1/2} z \tag{2.30}$$

so that the drop axis is at $z = -H_0^{-1/2} \approx -\infty$. We seek a series expansion of the drop shape in powers of $H_0^{1/2}$:

$$\eta(s) = \eta_0(z) + H_0^{1/2}\eta_1(z) + O(H_0). \tag{2.31}$$

Substituting into (2.28) we obtain

$$\xi(s) = \xi_0(z) + H_0^{1/2}\xi_1(z) + O(H_0) \tag{2.32}$$

where

$$\xi_0(z) = \int_{-\infty}^{z} \mathrm{d}z \left(\frac{A\Pi(h_0\eta_0(z))}{2\gamma} - 1 \right) \tag{2.33}$$

and

$$\xi_1(z) = \int_{-\infty}^{z} \mathrm{d}z \frac{A}{2\gamma} \frac{\mathrm{d}\Pi}{\mathrm{d}\eta}(h_0\eta_0(z))\eta_1(z). \tag{2.34}$$

Substituting (2.30)–(2.32) into the shape (2.27) and equating powers of $H_0^{1/2}$, we obtain the coupled equation set

$$\frac{\mathrm{d}\eta_0}{\mathrm{d}z} = -\xi_0, \quad \frac{\mathrm{d}\xi_0}{\mathrm{d}z} = \frac{A\Pi(h_0\eta_0)}{2\gamma} - 1 \tag{2.35}$$

and

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}z} = -\xi_1 + z\xi_0, \quad \frac{\mathrm{d}\xi_1}{\mathrm{d}z} = \frac{A}{2\gamma} \frac{\mathrm{d}\Pi}{\mathrm{d}\eta}(h_0\eta_0(z))\eta_1. \tag{2.36}$$

For large negative z we may expand (2.14) in powers of $H_0^{1/2}$ to obtain

$$\eta_0(z) = 1 + \Delta \mathrm{e}^{\alpha z} \tag{2.37}$$

and

$$\eta_1(z) = \frac{\Delta \mathrm{e}^{\alpha z}}{\alpha} \left(\frac{1}{16} - \frac{\alpha z}{4} - \frac{(\alpha z)^2}{4} \right) \tag{2.38}$$

from whence it follows that

$$\xi_0(z) = -\alpha \Delta \mathrm{e}^{\alpha z} \tag{2.39}$$

and

$$\xi_1(z) = -\Delta \mathrm{e}^{\alpha z} \left(-\frac{3}{16} + \frac{\alpha z}{4} - \frac{(\alpha z)^2}{4} \right). \tag{2.40}$$

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The constraint equation (2.26) when expanded in powers of $H_0^{1/2}$ becomes

$$\int_{-\infty}^{\infty} \mathrm{d}z \left(\frac{A\Pi(h_0\eta_0(z))}{2\gamma} - \theta(-z) \right) = 0 \tag{2.41}$$

and

$$\xi_1(\infty) = \int_{-\infty}^{\infty} \mathrm{d}z \frac{A}{2\gamma} \frac{\mathrm{d}\Pi}{\mathrm{d}\eta}(h_0\eta_0(z))\eta_1(z) = 0. \tag{2.42}$$

It follows from (2.33) and (2.41) that, for large positive z,

$$\xi_0(z) = -z + \cdots \tag{2.43}$$

where the omitted term is exponentially small in z.

To calculate the drop shape in the boundary layer, we solve the differential equations (2.35) and (2.36) from a large negative z-value with an assumed value for Δ in (2.37)–(2.40) as initial conditions to a sufficiently large positive z that the integral on the left-hand side of (2.41) has converged to within a specified tolerance. The process is iterated in Δ until the integral is zero to within a specified tolerance. Finally we note that (2.43) and (2.42) imply that

$$\eta_0(z) = 1 + c + \frac{z^2}{2} + \cdots \tag{2.44}$$

and

$$\eta_1(z) = -\frac{z^3}{3} + \cdots \tag{2.45}$$

where the omitted terms are exponentially small in z and c is an $O(1)$ constant. In terms of the outer variable s the drop shape is, to this order,

$$\eta(s) = \frac{1}{H_0}\left(\frac{s^2}{2} - \frac{s^3}{3} \right) + O(1), \tag{2.46}$$

which is the first two terms in a small-s expansion of the outer drop shape (2.20). The results of this boundary layer analysis are schematically summarized in figure 1.

3. Lubrication analysis for the drag force

Unlike other calculations in this area, the present problem does not rely on a second-order-in-velocity pressure field to achieve normal force balance (see Hodges *et al.* 2004). In other words, when the correct physics (i.e. disjoining pressure) is included in the analysis, the bubble can sustain a thin uniform liquid film in the absence of motion. When transverse motion appears, the flow field produces stresses on the bubble surface linearly proportional to the bubble velocity to leading order. These stresses cause a deviation of the bubble shape and consequent correction to the flow field which are higher order in the bubble velocity. The only contributions to the linear drag force on the bubble come from the leading-order flow field acting on the zero-velocity bubble shape. The second-order corrections come from two separate contributions, namely the second-order flow field due to first-order bubble distortion acting on the zero-order bubble shape, and the leading-order (linear) flow field acting on the linear bubble distortion.

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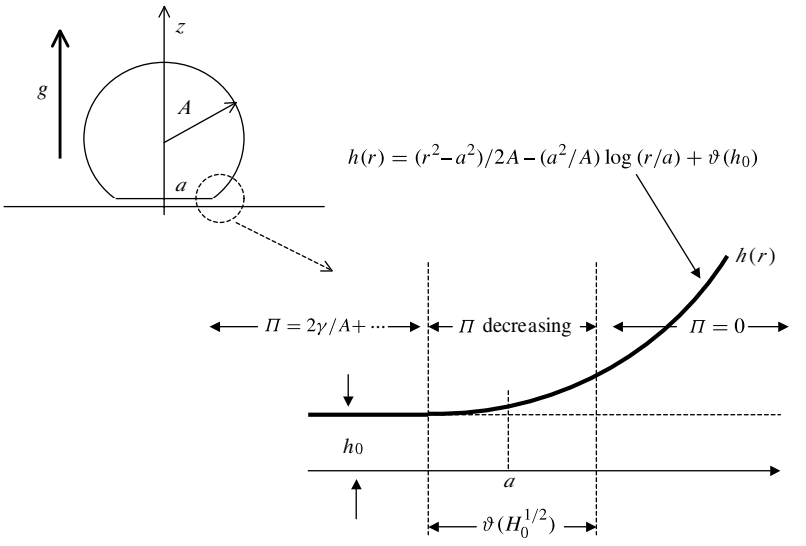


FIGURE 1. Drop shape in the flattened region.

Since we aim to calculate the drag force on a bubble moving along a plane rigid substrate to leading order in the steady drift velocity $\boldsymbol{U} = U\hat{\boldsymbol{x}}$ we can take the bubble shape to be the shape at rest. Distortion of the bubble due to flow will produce corrections to the drag force of higher order in U. We will examine the upper limit of validity of the linear drag regime in the conclusion. We seek a solution of the creeping flow equations (Batchelor 1967)

$$\mu\nabla^2\boldsymbol{v} - \boldsymbol{\nabla}p = 0, \tag{3.1a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{3.1b}$$

in the form

$$v_R(R, Z, \phi) = U\cos\phi\ (u_R(R, Z) - 1), \tag{3.2a}$$

$$v_\phi(R, Z, \phi) = U\sin\phi\ (u_\phi(R, Z) + 1), \tag{3.2b}$$

$$v_Z(R, Z, \phi) = U\cos\phi\ u_Z(R, Z), \tag{3.2c}$$

$$p(R, Z, \phi) = \mu U\cos\phi\ p(R, Z), \tag{3.2d}$$

where R is the radial coordinate scaled with a and Z is the vertical coordinate scaled with h_0 in the cylindrical polar coordinate system where the origin is located on the plane surface $Z = 0$ at the axis of symmetry of the bubble and ϕ is the azimuthal angle in the plane measured anticlockwise from the $\hat{\boldsymbol{x}}$-axis in the plane. The bubble interface is located at $Z = h(r)/h_0$, where $h(r)$ is the equilibrium bubble shape described in the previous section correct to terms of $O((h_0/a)^2)$.

The boundary conditions on the flow field are

$$\left.\begin{matrix} u_R \\ u_\phi \\ u_Z \\ p \end{matrix}\right\} \xrightarrow{R\to\infty, Z>0} 0. \tag{3.3}$$

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No-slip boundary conditions on the substrate require

$$\left.\begin{matrix} u_R \\ u_\phi \\ u_Z \end{matrix}\right\} \xrightarrow{Z\to 0, R\geqslant 0} 0. \tag{3.4}$$

On the bubble interface we require no transverse traction

$$(-\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma})_{transverse}\big|_{Z=h(r)/h_0} = 0 \tag{3.5}$$

and no normal velocity

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{v}\big|_{Z=h(r)/h_0} = 0. \tag{3.6}$$

Introducing the lubrication scalings

$$\eta(R) = \frac{h(R)}{h_0}, \quad P(R,Z) = \frac{h_0^2}{a}p(R,Z), \quad S(R,Z) = \frac{a}{h_0}u_Z(R,Z), \tag{3.7}$$

the creeping flow equations become

$$\frac{\partial^2 u_R}{\partial Z^2} - \frac{\partial P}{\partial R} = 0 + \cdots, \tag{3.8a}$$

$$\frac{\partial^2 u_\phi}{\partial Z^2} + \frac{P}{R} = 0 + \cdots, \tag{3.8b}$$

$$\frac{\partial P}{\partial Z} = 0 + \cdots, \tag{3.8c}$$

$$\frac{\partial u_R}{\partial R} + \frac{u_R + u_\phi}{R} + \frac{\partial S}{\partial Z} = 0, \tag{3.8d}$$

where the omitted terms are $O((h_0/a)^2)$. The boundary conditions are in this scaling

$$\left.\begin{matrix} u_R \\ u_\phi \\ S \end{matrix}\right|_{Z=0} = 0, \tag{3.9}$$

$$\left.\frac{\partial u_R}{\partial Z}\right|_{Z=\eta(R)} = 0 + \cdots, \tag{3.10a}$$

$$\left.\frac{\partial u_\phi}{\partial Z}\right|_{Z=\eta(R)} = 0 + \cdots, \tag{3.10b}$$

$$S - \eta'(u_R - 1)\big|_{Z=\eta(R)} = 0 + \cdots, \tag{3.10c}$$

where the omitted terms are $O((h_0/a)^2)$. The solution of these equations is

$$u_R = -P'Z(\eta - Z/2), \tag{3.11a}$$

$$u_\phi = \frac{P}{R}Z(\eta - Z/2), \tag{3.11b}$$

$$S = -\left(\frac{Z^3}{6} - \frac{Z^2\eta}{2}\right)\left(P'' + \frac{1}{R}P' - \frac{P}{R^2}\right) + \left(\frac{\eta' Z^2}{2}\right)P', \tag{3.11c}$$

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where $P(R)$ satisfies

$$P'' + P'\left(\frac{1}{R} + \frac{3\eta'}{\eta}\right) - \frac{P}{R^2} = -\frac{3\eta'}{\eta^3}. \tag{3.12}$$

To calculate the drag force on the bubble we need the component of surface traction in the direction of $\boldsymbol{U}$, which is

$$\cos\phi[\sigma_{RR}h' - \sigma_{RZ}]\,(1+h'^2)^{-1/2}\Big|_{Z=h(r)/h_0}$$

$$= \frac{\mu U\cos^2\phi}{(1+h'^2)^{1/2}}\left[-ph' + 2\frac{\partial u_R}{\partial r}h' - \frac{\partial u_Z}{\partial r}\right]_{Z=h(r)/h_0}. \tag{3.13}$$

Integrating over the bubble surface yields the drag force

$$\boldsymbol{F} = -\pi\mu\boldsymbol{U}\int_{bubble} \mathrm{d}r\, r\left[-ph' + 2\frac{\partial u_R}{\partial r}h' - \frac{\partial u_Z}{\partial r}\right]_{Z=h(r)/h_0}. \tag{3.14}$$

Inserting the lubrication scalings gives

$$\boldsymbol{F} = -2\pi\mu A\boldsymbol{U}\left[\frac{1}{H_0}\int_{bubble} \mathrm{d}R\; R[P\eta' + \cdots]\right] \tag{3.15}$$

where the omitted terms are $O((h_0/a)^2)$ in the lubrication region but produce an $O(1)$ contribution to the outer bracket for $R \sim O(A/a)$ in the macroscopic spherical cap region of the bubble surface. Thus the inner bracket is a notation for the exact surface traction to be integrated over the bubble surface, while the explicit term in the bracket serves to remind us of the leading-order behaviour of the outer solution contribution as $R/A \to 0$.

The nature of the scaling of the macroscopic hydrodynamic problem determines the behaviour of $P(R)$ as $R \to \infty$ in the lubrication region. We use an argument of Goldman *et al.* (1967) and O'Neill & Stewartson (1967). First, we note that, for large R, (2.20) becomes

$$\eta(R) = \frac{1}{H_0}R^2 + \cdots \tag{3.16}$$

so that (3.12) implies that asymptotically

$$P(R) = H_0^2 G(R) \tag{3.17}$$

where $G(R)$ is an $O(1)$ decaying function of its argument. Suppose that

$$G(R) \sim R^{-\beta} \tag{3.18}$$

for large R. In unscaled variables the flow field functions derived from the lubrication analysis become

$$u_R \sim (2Aa^{\beta-3})\left[\frac{\beta\tilde{z}}{r^{\beta-1}} + O(A\tilde{z}^2/r^{\beta+1})\right], \tag{3.19a}$$

$$u_\phi \sim (2Aa^{\beta-3})\left[\frac{\tilde{z}}{r^{\beta-1}} + O(A\tilde{z}^2/r^{\beta+1})\right], \tag{3.19b}$$

$$u_Z \sim (2Aa^{\beta-3})\left[\frac{\tilde{z}^2}{r^{\beta}}(\beta^2 - \beta - 1) + O(A\tilde{z}^3/r^{\beta+2})\right], \tag{3.19c}$$

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$$p \sim 2A(2Aa^{\beta-3})\left[\frac{1}{r^{\beta}} + O(\tilde{z}/Ar^{\beta})\right], \tag{3.19d}$$

where $\tilde{z} = h_0 Z$ is the unscaled vertical coordinate. These forms must match smoothly into the leading-order macroscopic solution as $r/A \to 0$. The scaling of the macroscopic problem requires that u_R, u_ϕ, u_Z, Ap are $O(1)$ functions of the macroscopic scaled variables $r/A, \tilde{z}/A$. In order that (3.19) has this scaling to leading order we require $\beta = 3$, i.e.

$$P(R) \sim H_0^2 R^{-3} \tag{3.20}$$

for large R. No slower decay is permissible since it will not be able to match into the macroscopic solution with the correct scaling. Other more rapidly decaying functions can be added to the R^{-3} dependence since they will become negligible at large R. With these asymptotic forms for $P(R), \eta(R)$ *the lubrication part of the drag force* (3.15) *diverges logarithmically*. In the next section we show how this apparent divergence may be overcome for the case of the undistorted bubble. In § 5 we derive the drag force on the flattened bubble as a power series in H_0 correct to $O(H_0^{1/2})$.

In the axial region where $(\mathrm{d}\eta/\mathrm{d}R) \sim O(\delta)$ the lubrication pressure equation (3.12) reduces to

$$P'' + \frac{1}{R}P' - \frac{1}{R^2}P = 0 + \cdots \tag{3.21}$$

where the omitted terms are $O(\delta)$. This equation has solution $P(R) = \text{constant}\ R$. Thus it proves convenient to change axial variable to s (see (2.18)) and introduce the function $F(s)$ defined by

$$P(R) = RF(s). \tag{3.22}$$

Equation (3.12) for the lubrication pressure becomes

$$F'' + F'\left(\frac{2}{1+s} + \frac{3\eta'}{\eta}\right) + F\left(\frac{3\eta'}{2(1+s)\eta}\right) = -\frac{3\eta'}{2(1+s)\eta^3} \tag{3.23}$$

where all derivatives are with respect to s. The asymptotic constraint equation (3.20) becomes

$$F(s) \sim H_0^2 s^{-2}. \tag{3.24}$$

Outside the boundary layer region $s \gg H_0^{1/2}$ we may introduce the asymptotic bubble shape (2.20) into (3.23) and write $F(s) = H_0^2(F_0(s) + H_0 F_1(s) + O(H_0^2))$ to obtain, to leading order,

$$F_0'' + \frac{F_0'}{s}P(s) + \frac{F_0}{s^2}Q(s) = G(s) + \cdots, \tag{3.25}$$

where

$$P(s) = \frac{s}{1+s}\left(2 + \frac{3s}{s - \log(1+s)}\right), \tag{3.26a}$$

$$Q(s) = \frac{3s^3}{2\,(1+s)^2(s - \log(1+s))}, \tag{3.26b}$$

$$G(s) = -\frac{3s}{2\,(1+s)^2\,(s - \log(1+s))^3}. \tag{3.26c}$$

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The *particular solution* of (3.25) for small s is

$$F_0^{PS}(s) = \frac{2}{s^3} + \frac{1}{s^2} - \frac{1/4}{s} - \frac{17}{180}\log s + \frac{17}{360}s\log s + O(s) \tag{3.27}$$

using Frobenius' method (Bender & Orszag 1978). For $s \gg \log(1+s)$ the outer differential equation reduces to

$$F_0'' + \frac{5}{1+s}F_0' + \frac{3/2}{(1+s)^2}F_0 = -\frac{3/2}{(1+s)^2 s^2} \tag{3.28}$$

with particular solution

$$F_0^{PS}(s) = \frac{3/5}{(1+s)^2} + \frac{2}{(1+s)^3} + \cdots. \tag{3.29}$$

The homogeneous solutions of (3.28) are

$$F_0^{hom}(s) = (1+s)^{-2\pm\sqrt{5/2}}. \tag{3.30}$$

Only the lower solution is allowed in order to satisfy the asymptotic constraint (3.24). The homogeneous solutions of (3.25) for small s are, using Frobenius' method,

$$F_0^{hom}(s) = \begin{cases} 1 + \cdots \\ s^{-5} + \cdots. \end{cases} \tag{3.31}$$

The highly divergent solution for small s cannot match the boundary layer solution as $z \to \infty$ as we shall demonstrate below. Only the finite homogeneous solution is a possible allowed component of $F_0(s)$. However, numerical solution of (3.25) shows that both the asymptotic solutions (3.30) diverge as s^{-5} as $s \to 0$. Clearly the finite small-s homogeneous solution is a precise linear combination of these two divergent solutions. Thus neither of the small-s homogeneous solutions is acceptable since the only one that is suitable for matching with the boundary layer solution as $s \to H_0^{1/2}$ will not obey the large-s asymptotic constraint equation (3.24). It follows that the particular solution of (3.25) given by (3.27) for small s and (3.29) for large s is the *complete solution* to the outer problem.

In the region under the bubble where $\eta(s) = 1 + O(\delta)$ the solution of (3.23) is $F(s) =$ constant and the contribution to the drag force integral (3.15) is $O(\delta)$, i.e. negligible in this region. Only in the boundary layer region around $s = 0$ does the solution begin to increase and the local contribution to the drag become important. To calculate $F(s)$ in the boundary layer we change to the boundary layer variable z (see (2.30)) and (3.23) becomes

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \frac{\mathrm{d}F}{\mathrm{d}z}\left[\frac{2H_0^{1/2} - 3(\xi/\eta)}{1 + H_0^{1/2}z}\right] - F\left[\frac{H_0^{1/2}(3\xi/2\eta)}{(1+H_0^{1/2}z)^2}\right] = H_0^{1/2}\frac{(3\xi/2\eta^3)}{(1+H_0^{1/2}z)^2}. \tag{3.32}$$

Using the expansions for η, ξ in powers of z derived in § 2 we seek an expansion for $F(z)$ in the form

$$F(z) = H_0^{1/2}f_0(z) + H_0 f_1(z) + H_0^{3/2}f_2(z) + \cdots. \tag{3.33}$$

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Substituting these expansions in (3.32) and equating powers of H_0 we obtain an hierarchy of coupled equations for the $f_j(z)$, the first of which is

$$f_0'' - \frac{3\xi_0}{\eta_0} f_0' = \frac{3\xi_0}{2\eta_0^3}. \tag{3.34}$$

Using (2.35), we integrate (3.34) to obtain

$$f_0' = -\frac{3(\eta_0 - 1)}{2\eta_0^3}. \tag{3.35}$$

The second equation of the hierarchy is

$$f_1'' - \frac{3\xi_0}{\eta_0} f_1' = \frac{\Phi(z)}{\eta_0^3} \tag{3.36}$$

where

$$\Phi(z) = 3(\eta_0 - 1) - 3\xi_1 + \frac{3}{2} f_0 \eta_0^2 \xi_0 - \frac{9\xi_0\eta_1}{2\eta_0^2} + \frac{3}{2}\xi_0 z - \frac{9\xi_0 z}{2\eta_0} + \frac{9\xi_1}{2\eta_0}. \tag{3.37}$$

Integrating (3.36) we obtain, using (2.35),

$$f_1' = -\frac{\chi(z)}{\eta_0^3} \tag{3.38}$$

where

$$\begin{aligned}\chi(z) &= -\int_{-\infty}^{z} \mathrm{d}z \Phi(z) \\ &= \frac{1}{2}[\eta_0^3 f_0 - f_0(-\infty)] - 3\eta_1 - \frac{3}{2}(\eta_0 - 1)z + \frac{9}{2}\frac{\eta_1}{\eta_0} - \frac{3}{4}\tau(z)\end{aligned} \tag{3.39}$$

and

$$\tau(z) = \int_{-\infty}^{z} \mathrm{d}z\,[\eta_0 - 1] \tag{3.40}$$

where we have invoked the constant $F(s)$ for $s \sim -1$ in the form

$$f_0'(-\infty) = 0 \quad f_1'(-\infty) = 0. \tag{3.41}$$

Integrating (3.35) and (3.38) yields

$$f_0(z) = f_0(-\infty) - \frac{3}{2}\int_{-\infty}^{\infty} \mathrm{d}z \frac{\eta_0 - 1}{\eta_0^3} + \frac{3}{2}\int_{z}^{\infty} \mathrm{d}z \frac{\eta_0 - 1}{\eta_0^3} \tag{3.42}$$

and

$$f_1(z) = f_1(-\infty) - \int_{-\infty}^{\infty} \mathrm{d}z \frac{\chi}{\eta_0^3} + \int_{z}^{\infty} \mathrm{d}z \frac{\chi}{\eta_0^3}. \tag{3.43}$$

For large z, we may replace η_0 by its asymptotic form (2.44) in the second integral of (3.42) to produce

$$\frac{3}{2}\int_{z}^{\infty} \mathrm{d}z \frac{\eta_0 - 1}{\eta_0^3} = \frac{2}{z^3} + O(z^{-5}) \tag{3.44}$$

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and using (2.43)–(2.45) in the second integral of (3.43) produces

$$\int_z^\infty \mathrm{d}z \frac{\chi}{\eta_0^3} = \frac{1}{z^2} + O(z^{-4}). \tag{3.45}$$

We have already demonstrated that the small-s form of the outer solution is

$$F(s) = H_0^2 \left(\frac{2}{s^3} + \frac{1}{s^2} - \frac{1/4}{s} + O(\log s) \right) + O(H_0^3) \tag{3.46}$$

which in the boundary layer variable can be written as

$$F(s) = H_0^{1/2} \left(\frac{2}{z^3} \right) + H_0 \left(\frac{1}{z^2} \right) + H_0^{3/2} \left(-\frac{1/4}{z} \right) + O(H_0^2 \log H_0). \tag{3.47}$$

In order that this outer solution match smoothly with the boundary layer solution (3.33) it is necessary to choose

$$f_0(-\infty) = \frac{3}{2} \int_{-\infty}^\infty \mathrm{d}z \frac{\eta_0 - 1}{\eta_0^3} \tag{3.48}$$

and

$$f_1(-\infty) = \int_{-\infty}^\infty \mathrm{d}z \frac{\chi}{\eta_0^3}. \tag{3.49}$$

Thus

$$f_0(z) = \frac{3}{2} \int_z^\infty \mathrm{d}z \frac{\eta_0 - 1}{\eta_0^3} \sim \frac{2}{z^3} \tag{3.50}$$

and

$$f_1(z) = \int_z^\infty \mathrm{d}z \frac{\chi}{\eta_0^3} \sim \frac{1}{z^2} \tag{3.51}$$

for large positive z. For large negative z (3.35) becomes, using (2.37),

$$f_0' = -\frac{3}{2} \Delta \mathrm{e}^{\alpha z} + O(\mathrm{e}^{2\alpha z}) \tag{3.52}$$

and hence

$$f_0(z) = f_0(-\infty) - \frac{3}{2} \frac{\Delta}{\alpha} \mathrm{e}^{\alpha z} + \cdots. \tag{3.53}$$

Similarly we have

$$\tau(z) = \frac{\Delta}{\alpha} \mathrm{e}^{\alpha z} + \cdots \tag{3.54}$$

and

$$\chi(z) = \frac{3\Delta}{2\alpha} \mathrm{e}^{\alpha z} \left[\alpha f_0(-\infty) - \frac{15}{16} - \frac{5}{4}(\alpha z) - \frac{1}{4}(\alpha z)^2 \right] + \cdots \tag{3.55}$$

and hence

$$f_1(z) = f_1(-\infty) + \frac{3\Delta}{2\alpha^2} \mathrm{e}^{\alpha z} \left[\frac{3}{16} + \frac{3}{4}(\alpha z) + \frac{1}{4}(\alpha z)^2 - \alpha f_0(-\infty) \right] + \cdots. \tag{3.56}$$

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Numerically we first calculate $\xi_0(z)$, $\eta_0(z)$ as described in § 2 and the integral (3.48) which gives $f_0(-\infty)$. We then solve the coupled first-order differential equations for $\eta_0(z)$, $\eta_1(z)$, $\xi_0(z)$, $\xi_1(z)$, $f_0(z)$, $f_1(z) - f_1(-\infty)$, $\tau(z)$ from a large negative z value using the asymptotic forms derived above to a suitably large positive z value where the integral (3.49) has converged. This determines $f_1(-\infty)$ which is not known initially. In § 5 we demonstrate how to use these results to calculate the drag force on the sessile bubble as a series in the small parameter.

4. The undistorted bubble

It is useful before examining the case of the flattened bubble to treat the undistorted bubble, i.e. a spherical bubble of radius A moving along a plane substrate at separation h_0 with velocity $U\hat{\boldsymbol{x}}$. In the region $R - 1 \ll A/a$ the undistorted bubble has the shape

$$h^{(0)}(R) = h_0 + \frac{a^2}{2A}R^2 + \cdots \tag{4.1}$$

where the omitted term is $O(a^4/A^3)$. It should be noted that the radial scaling used is somewhat artificial since a does not appear in the undistorted problem and the drag force in this case must depend on h_0, A only. This unnatural scaling of the undistorted problem is useful in calculating the drag on the flattened bubble as we will demonstrate in § 5.

In terms of our scaled variables the shape may be written as

$$\eta^{(0)}(s) = 1 + \frac{1}{H_0}(1+s) + \cdots \tag{4.2}$$

where the omitted term is $O((a/A)^2)/H_0$. The undistorted pressure function $F^{(0)}(s)$ obeys

$$F^{(0)\prime\prime} + F^{(0)\prime}\left(\frac{2}{1+s} + \frac{3}{H_0+1+s}\right) + F^{(0)}\left(\frac{3}{2(1+s)(H_0+1+s)}\right) = -\frac{3H_0^2}{(1+s)\,(H_0+1+s)^3}. \tag{4.3}$$

The particular solution is precisely

$$F^{(0)}_{PS}(s) = \frac{\frac{3}{5}H_0^2}{(1+H_0+s)^2}. \tag{4.4}$$

The homogeneous differential equation can be written as

$$F^{(0)\prime\prime} + F^{(0)\prime}\left(\frac{2+5t}{t(1+t)}\right) + F^{(0)}\left(\frac{3}{2t(1+t)}\right) = 0 \tag{4.5}$$

with a change of independent variable

$$1 + s = H_0 t. \tag{4.6}$$

For large t the homogeneous solutions are

$$F^{(0)}_{hom}(t) \sim t^{-2\pm\sqrt{5/2}}. \tag{4.7}$$

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Only the lower solution is permitted by the asymptotic constraint (3.24). For $t \sim 0$ ($s \sim -1$) the solutions of (4.5) are of the form

$$F^{(0)}_{hom}(t) = \begin{cases} 1 + \cdots \\ t^{-1} + \cdots . \end{cases} \tag{4.8}$$

The divergent solution is not permitted. Numerical solution of (4.5) shows that both large-t asymptotic solutions (4.7) diverge like t^{-1} for $t \sim 0$. Clearly the finite small-t homogeneous solution is a precise linear combination of the two divergent solutions. Thus, the finite small-t homogeneous solution must be eliminated also since, for large t, it becomes a prescribed linear combination of both the asymptotic solutions (4.7) and hence decays too slowly to obey the asymptotic constraint (3.24). Hence (4.4) is *the complete lubrication solution* of the pressure function $F^{(0)}(s)$ for the undistorted bubble.

The drag force on the undistorted bubble is, from (3.15),

$$\boldsymbol{F}^{(0)} = -2\pi\mu A\boldsymbol{U}\left[\frac{3}{5}\int_{-1} \mathrm{d}s \left[\frac{1+s}{(1+H_0+s)^2} + \cdots\right]\right] \tag{4.9}$$

where the omitted terms are $O((h_0/a)^2)$ in the lubrication region but produce an $O(1)$ contribution to the outer bracket for $R \sim O(A/a)$ in the macroscopic spherical portion of the bubble surface.

Following Goldman *et al.* (1967), we introduce an arbitrary length ζ such that $H_0 \ll \zeta \ll (A/a)^2$ so that ζ lies in the matching region connecting the lubrication region to the macroscopic flow region, and write

$$\begin{aligned} &\int_{-1} \mathrm{d}s \left[\frac{1+s}{(1+H_0+s)^2} + \cdots\right] \\ &\quad = \int_{-1}^{-1+\zeta} \mathrm{d}s \left[\frac{1+s}{(1+H_0+s)^2}\right] + \int_{-1+\zeta} \mathrm{d}s \left[\frac{1+s}{(1+H_0+s)^2} + \cdots\right] \\ &\quad = -\log H_0 + \log\zeta - 1 + \frac{2H_0}{\zeta} + O((H_0/\zeta)^2) \\ &\qquad + \int_{-1+\zeta} \mathrm{d}s \left\{\frac{1}{(1+s)} - \frac{2H_0}{(1+s)^2} + \cdots\right\} \end{aligned} \tag{4.10}$$

where the explicit terms in the brackets indicate the leading terms in the outer solution contribution to the integral for small s. Thus

$$\begin{aligned} &\int_{-1+\zeta} \mathrm{d}s \left\{\frac{1}{(1+s)} - \frac{2H_0}{(1+s)^2} + \cdots\right\} \\ &\quad = -\log\zeta - \frac{2H_0}{\zeta} + O((H_0/\zeta)^2) + \frac{5C^{(0)}}{3} - \log((a/A)^2/2) + 1 + \cdots \end{aligned} \tag{4.11}$$

where the implicit terms produce a constant to leading order which we have written for convenience as $5C^{(0)}/3 - \log((a/A)^2/2) + 1$, and small terms of $O(h_0/A)$. The drag force on the undistorted bubble is then

$$\boldsymbol{F}^{(0)} = -2\pi\mu A\boldsymbol{U}\left[-\tfrac{3}{5}\log(h_0/A) + C^{(0)} + O\left((h_0/A)\log(h_0/A)\right)\right], \tag{4.12}$$

all terms involving the arbitrary length ζ cancelling as they must. The particular choice of the constant in (4.11) is necessary given the somewhat unnatural scaling

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discussed above in order to produce a drag force which only depends on h_0/A and is independent of the flattened radius a of the gravitating bubble. Meyappan & Subramanian (1987) have explicitly calculated the drag force on a spherical bubble migrating along a plane substrate using bispherical coordinates and obtained

$$\boldsymbol{F}^{(0)} = -2\pi\mu A\boldsymbol{U}\,\Lambda^{(0)}(h_0/A). \tag{4.13}$$

By numerically computing the limiting behaviour of $\Lambda^{(0)}(h_0/A)$ we obtain the value of the constant $C^{(0)}$ as

$$\begin{aligned} C^{(0)} &= \lim_{h_0\to 0}\left[\Lambda^{(0)}(h_0/A) + \tfrac{3}{5}\log(h_0/A)\right] \\ &= 1.71316. \end{aligned} \tag{4.14}$$

For comparison, the equivalent result for a rigid sphere (O'Neill & Stewartson 1967; Chaoui & Feuillebois 2003) is

$$\boldsymbol{F}^{(0)} = -2\pi\mu A\boldsymbol{U}\left[-\tfrac{8}{5}\log(h_0/A) + 2.8629 + O\left((h_0/A)\log(h_0/A)\right)\right] \tag{4.15}$$

obtained by a similar lubrication argument.

5. The flattened bubble

Expressing (3.15) in terms of the axial variable s we obtain

$$\boldsymbol{F} = -2\pi\mu A\boldsymbol{U}\left[\frac{1}{H_0}\int_{bubble} \mathrm{d}s\,\left[(1+s)F\eta' + \cdots\right]\right]. \tag{5.1}$$

By adding and subtracting the drag force on an undistorted bubble (4.9) we obtain

$$\boldsymbol{F} = \boldsymbol{F}^{(0)} - 2\pi\mu A\boldsymbol{U}\left[\frac{1}{H_0}\int_{-1}^{\infty}\mathrm{d}s\,\left\{(1+s)F\eta' - \frac{\frac{3}{5}H_0(1+s)}{(1+H_0+s)^2}\right\} + O((a/A)^2)\right] \tag{5.2}$$

where we may now omit the implicit terms and integrate over the lubrication layer alone since, for large s, the integrand is now of $O(s^{-2})$ and therefore integrable. The macroscopic flow fields in both problems will differ by $O(a^2/A)$ terms (both the vertical displacement of the two spherical regions from one another and the distortion due to the disjoining pressure over the flattened region) and therefore the omitted terms will contribute an $O((a/A)^2)$ term to the outer bracket, which we neglect. Evaluating the integral up to $O(H_0^{1/2})$ contributions will then give us the first few terms in a series expansion for the drag in powers of H_0. We first consider the integral from $s=-1$ to $s=0$. Inserting the boundary layer expressions for $\eta(s), F(s)$ in the integrand, we have

$$\begin{aligned} &\frac{1}{H_0}\int_{-1}^{0}\mathrm{d}s\,\left\{(1+s)F\eta' - \frac{35H_0(1+s)}{(1+H_0+s)^2}\right\} \\ &\quad = \frac{3}{5}(1+\log H_0) - \frac{1}{H_0^{1/2}}\int_{-\infty}^{0}\mathrm{d}z(f_0\xi_0 + H_0^{1/2}(f_1\xi_0 + f_0\xi_1)) + \cdots \end{aligned} \tag{5.3}$$

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where the omitted terms are $O(H_0^{1/2})$. The drag force may be written as

$$\boldsymbol{F} = -2\pi\mu A\boldsymbol{U}\left[\begin{array}{l} -\dfrac{1}{H_0^{1/2}}\displaystyle\int_{-\infty}^{0} \mathrm{d}z f_0\xi_0 + C^{(0)} + \frac{3}{5} - \frac{3}{5}\log(a^2/2A^2) \\ \quad - \displaystyle\int_{-\infty}^{0} \mathrm{d}z(f_1\xi_0 + f_0\xi_1) \\ \quad + \dfrac{1}{H_0}\displaystyle\int_0^{\infty} \mathrm{d}s\left\{(1+s)F\eta' - \frac{35H_0(1+s)}{(1+H_0+s)^2}\right\} + \cdots \end{array}\right] \tag{5.4}$$

where we use the undistorted bubble drag result (4.12). We now consider the integral from $s = 0$ to $s = \infty$ and write

$$\frac{1}{H_0}\int_0^{\infty} \mathrm{d}s\left\{(1+s)F\eta' - \frac{\frac{3}{5}H_0(1+s)}{(1+H_0+s)^2}\right\} = \frac{1}{H_0}\int_0^{\zeta} \mathrm{d}s\{\cdots\} + \frac{1}{H_0}\int_{\zeta}^{\infty} \mathrm{d}s\{\cdots\} \tag{5.5}$$

where $H_0^{1/2} \ll \zeta \ll 1$ in the matching region between the boundary layer and the outer lubrication solution. Thus we write

$$\frac{1}{H_0}\int_0^{\zeta} \mathrm{d}s(1+s)F\eta' = -\frac{1}{H_0^{1/2}}\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z\, f_0\xi_0 - \int_0^{\zeta H_0^{-1/2}} \mathrm{d}z(f_1\xi_0 + f_0\xi_1) + \cdots \tag{5.6}$$

where the omitted terms are $O(H_0^{1/2})$. The first integral on the right-hand side can be written as

$$\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z\, f_0\xi_0 = \int_0^{\infty} \mathrm{d}z\, f_0\xi_0 - \int_{\zeta H_0^{-1/2}}^{\infty} \mathrm{d}z\left(\frac{-2}{z^2}\right) \tag{5.7}$$

where the large-z asymptotic forms for f_0 (3.50) and ξ_0 (2.43) have been used in the second integral since $\zeta H_0^{-1/2} \gg 1$. Thus

$$\frac{1}{H_0^{1/2}}\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z\, f_0\xi_0 = \frac{1}{H_0^{1/2}}\int_0^{\infty} \mathrm{d}z\, f_0\xi_0 + \int_{\zeta}^{\infty} \mathrm{d}s\left(\frac{2}{s^2}\right). \tag{5.8}$$

The second integral in (5.6) cannot be treated in quite the same way since, from (2.43) and (3.51),

$$f_1\xi_0 + f_0\xi_1 \sim -\frac{1}{z} \tag{5.9}$$

so the integral upper limit cannot be extended to infinity. The problem is circumvented by adding and subtracting an appropriate integral, namely

$$\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z(f_1\xi_0 + f_0\xi_1) = -\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z\; g(z) + \int_0^{\zeta H_0^{-1/2}} \mathrm{d}z(f_1\xi_0 + f_0\xi_1 + g(z)) \tag{5.10}$$

where the function $g(z)$ is arbitrary apart from the constraints that it must be integrable around zero and $1/z$ for large z . Thus we have

$$\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z(f_1\xi_0 + f_0\xi_1) = -\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z\; g(z) - \int_{\zeta H_0^{-1/2}}^{\infty} \mathrm{d}z\left[-\frac{1}{z} + g(z)\right] + \int_0^{\infty} \mathrm{d}z(f_1\xi_0 + f_0\xi_1 + g(z)). \tag{5.11}$$

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The first integral is dominated by its upper limit where $g(z) \sim 1/z$ and the second integral is only weakly dependent on $\zeta/H_0^{1/2}$. Thus

$$-\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z\; g(z) - \int_{\zeta H_0^{-1/2}}^{\infty} \mathrm{d}z\; \left[-\frac{1}{z} + g(z)\right] = -\log(\zeta/H_0^{1/2}) + \epsilon. \tag{5.12}$$

It is straightforward to show that the residual term ϵ is a constant dependent on the particular choice of $g(z)$ and that there are an infinity of functions, which we label $g_0(z)$, for which the residual $\epsilon = 0$. Among these are

$$g_0(z) = \frac{z^b}{1+z^{b+1}} \quad (b \geqslant 0). \tag{5.13}$$

Thus

$$\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z(f_1\xi_0 + f_0\xi_1) = -\log(\zeta/H_0^{1/2}) + \epsilon + \int_0^{\infty} \mathrm{d}z(f_1\xi_0 + f_0\xi_1 + g(z)). \tag{5.14}$$

The constant $\epsilon + \int_0^{\infty} \mathrm{d}z(f_1\xi_0 + f_0\xi_1 + g(z))$ is invariant to the choice of $g(z)$ possessing the particular properties listed above and so it is convenient to choose $g(z)$ to be any one of the zero residual functions. Thus we write

$$\int_0^{\zeta H_0^{-1/2}} \mathrm{d}z(f_1\xi_0 + f_0\xi_1) = -\log(\zeta/H_0^{1/2}) + \int_0^{\infty} \mathrm{d}z(f_1\xi_0 + f_0\xi_1 + g_0(z)). \tag{5.15}$$

The integral in (5.15) is, of course, invariant to the particular choice of the $g_0(z)$ and we have chosen the simplest, namely $b = 0$. The drag force is then

$$\boldsymbol{F} = -2\pi\mu A\boldsymbol{U}\left[\begin{array}{l} -\dfrac{1}{H_0^{1/2}}\displaystyle\int_{-\infty}^{\infty} \mathrm{d}z f_0\xi_0 + C^{(0)} + \frac{3}{5}(1 - \log(a^2/2A^2)) \\ -\displaystyle\int_{-\infty}^{\infty} \mathrm{d}z \left(f_1\xi_0 + f_0\xi_1 + \dfrac{\theta(z)}{1+z}\right) \\ \quad + \displaystyle\int_{\zeta}^{\infty} \mathrm{d}s \left\{sF_0(s) - \frac{2}{s^2} - \frac{3/5}{(1+s)}\right\} \\ \quad - \displaystyle\int_0^{\zeta} \mathrm{d}s \frac{3/5}{1+s} + \log\zeta - \frac{1}{2}\log H_0 + \cdots \end{array}\right] \tag{5.16}$$

where we have inserted the outer forms for $\eta(s), F(s)$ in the outer integral of (5.5) correct to terms of $O(H_0)$.

We note that the outer integral in (5.16) diverges as $\zeta \to 0$ since

$$\left\{sF_0(s) - \frac{2}{s^2}\right\} \sim \frac{1}{s} - \frac{1}{4} + O(s\log s). \tag{5.17}$$

In order to extend the lower limit to zero we must add and subtract a suitable function to remove the non-integrable singularity at $s = 0$:

$$\begin{aligned} &\int_{\zeta}^{\infty} \mathrm{d}s \left\{sF_0(s) - \frac{2}{s^2} - \frac{3/5}{(1+s)}\right\} \\ &\quad = \int_{\zeta}^{\infty} \mathrm{d}s \left\{sF_0(s) - \frac{2}{s^2} - G(s) - \frac{3/5}{(1+s)}\right\} + \int_{\zeta}^{\infty} \mathrm{d}s G(s) \end{aligned} \tag{5.18}$$

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where the arbitrary function $G(s)$ is integrable at infinity and $\sim 1/s$ for small s. Thus we may write

$$\int_\zeta^\infty \mathrm{d}s \left\{ sF_0(s) - \frac{2}{s^2} - \frac{3/5}{(1+s)} \right\} = \int_\zeta^\infty \mathrm{d}s\; G(s) - \int_0^\zeta \mathrm{d}s \left\{ \frac{1}{s} - G(s) \right\} + \int_0^\zeta \mathrm{d}s \frac{3/5}{(1+s)} + \int_0^\infty \mathrm{d}s \left\{ sF_0(s) - \frac{2}{s^2} - G(s) - \frac{3/5}{(1+s)} \right\} \tag{5.19}$$

where in the second integral on the right-hand side we have replaced $sF_0(s) - 2/s^2$ by its leading-order small-s form. Since the first integral is dominated by its lower limit where the integrand $\sim 1/s$ and the second integral is weakly dependent on its upper limit, we have

$$\int_\zeta^\infty \mathrm{d}s\; G(s) - \int_0^\zeta \mathrm{d}s \left\{ \frac{1}{s} - G(s) \right\} = -\log\zeta + \epsilon. \tag{5.20}$$

Again the residual is a constant dependent on the choice of $G(s)$ and there is a class of suitable functions with zero residual, e.g.

$$G_0(s) = \frac{1}{s(1+s^{b+1})} \quad (b \geqslant 0). \tag{5.21}$$

Choosing the simplest of these ($b = 0$), we may write

$$\int_\zeta^\infty \mathrm{d}s \left\{ sF_0(s) - \frac{2}{s^2} - \frac{3/5}{(1+s)} \right\} = -\log\zeta + \int_0^\zeta \mathrm{d}s \frac{3/5}{(1+s)} + \int_0^\infty \mathrm{d}s \left\{ sF_0(s) - \frac{2}{s^2} - \frac{1}{s(1+s)} - \frac{3/5}{(1+s)} \right\}. \tag{5.22}$$

Thus the drag force becomes

$$\boldsymbol{F} = -2\pi\mu A \boldsymbol{U} \left[\frac{B}{H_0^{1/2}} - \frac{1}{2}\log H_0 + C + O(H_0) \right], \tag{5.23}$$

all terms in ζ cancelling as they must, and where

$$B = -\int_{-\infty}^\infty \mathrm{d}z f_0 \xi_0 \quad (>0) \tag{5.24}$$

$$C = C^{(0)} + C^{(1)} + C^{(2)} + \tfrac{3}{5} - \tfrac{3}{5}\log(a^2/2A^2) \tag{5.25}$$

where

$$C^{(1)} = -\int_{-\infty}^\infty \mathrm{d}z \left(f_0\xi_1 + f_1\xi_0 + \frac{\theta(z)}{1+z} \right) \tag{5.26}$$

and

$$C^{(2)} = \int_0^\infty \mathrm{d}s \left\{ sF_0(s) - \frac{2}{s^2} - \frac{1}{s(1+s)} - \frac{3/5}{(1+s)} \right\}. \tag{5.27}$$

The residual existence of the $-(1/2)\log H_0$ term is a curious universal result as noted by Hinch (1991) in the asymptotic evaluation of integrals of this boundary layer type.

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It is convenient to use (2.35) to rewrite (5.24) as

$$B = \frac{3}{2}\int_{-\infty}^{\infty} \mathrm{d}z \frac{(\eta_0 - 1)^2}{\eta_0^3}. \tag{5.28}$$

In a similar way $C^{(1)}$ may be written as

$$C^{(1)} = \int_{-\infty}^{\infty} \mathrm{d}z \left[-\xi_1 f_0 + \frac{(\eta_0 - 1)}{\eta_0^3}\chi - \theta(z)\frac{z}{(1+z)^2} \right] \tag{5.29}$$

$$= 1 - \frac{1}{3}f_0\,(-\infty)^2 + \int_{-\infty}^{\infty} \mathrm{d}z \left[\left\{ \frac{1}{2}(\eta_0 - 1) - \xi_1 \right\} f_0 - \frac{\theta(z)}{1+z} - \frac{(\eta_0 - 1)}{\eta_0^3}\left\{ 3\eta_1 + \frac{3}{2}(\eta_0 - 1)z - \frac{9\eta_1}{2\eta_0} + \frac{3\tau}{4} \right\} \right] \tag{5.30}$$

so that $f_1(z)$ need not be explicitly calculated.

6. Discussion and conclusions

We present below results for one functional form of the disjoining pressure, namely

$$\Pi(h) = \Pi_0 \mathrm{e}^{-\kappa h}, \tag{6.1}$$

a standard form for electrostatically stabilized bubble/rigid surface systems where κ^{-1} in this case would represent the Debye screening length in the solution containing the bubble. The value of the thin film thickness h_0 of the flattened bubble is given by

$$\frac{2\gamma}{A} = \Pi_0 \mathrm{e}^{-\kappa h_0} \tag{6.2}$$

so that we may write

$$\frac{A\Pi}{2\gamma} = \mathrm{e}^{-\Lambda(\eta - 1)} \tag{6.3}$$

where

$$\Lambda = \kappa h_0 \tag{6.4}$$

is a largish but $O(1)$ number in typical practical systems. For this assumed form for the disjoining pressure, the asymptotic constant α, given by (2.11), is

$$\alpha = \Lambda^{1/2}. \tag{6.5}$$

As promised in § 2, we first display in figure 2 the central separation deviation $\delta = h(0)/h_0 - 1$ as a function of the parameter Λ. The magnitude of δ is astonishingly small for any typical value of Λ.

The constant $C^{(2)}$ is independent of the choice of the form of the disjoining pressure as it arises from the outer form of the lubrication solution. Solving the outer differential equation for $F_0(s)$ and performing the numerical integration indicated in (5.27) we have

$$C^{(2)} = 0.498 \pm 0.005, \tag{6.6}$$

the uncertainty being due to the numerical instability of the solution for $F_0(s)$ at large s.

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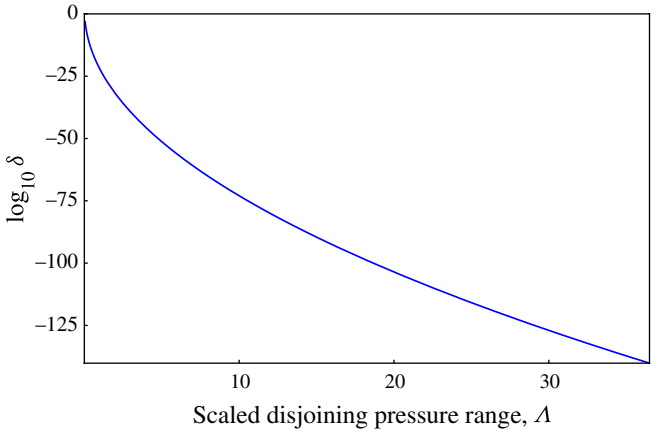


FIGURE 2. (Colour online available at journals.cambridge.org/flm) Central separation deviation δ as a function of Λ .

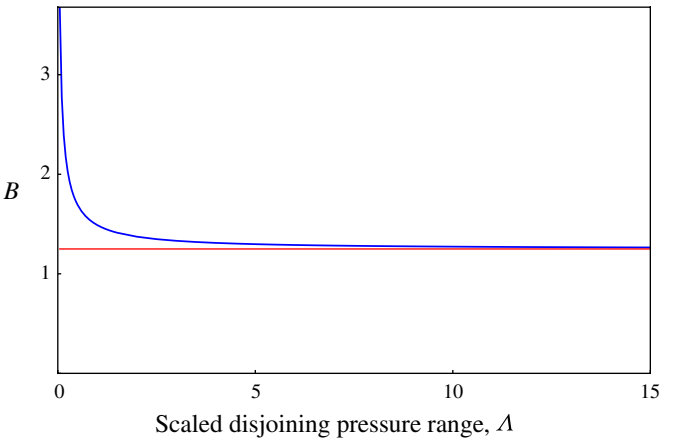


FIGURE 3. (Colour online) The constant B as a function of Λ.

In figures 3 and 4 the constants $B(\Lambda), C^{(1)}(\Lambda)$ are displayed as functions of Λ and show that these disjoining pressure-dependent asymptotic constants tend to limiting values as Λ increases.

The limit $\Lambda \to \infty$ represents the physical situation

$$\frac{A\Pi}{2\gamma} = \theta(-s) \tag{6.7}$$

for which

$$\eta(s) = \begin{cases} 1 & (s < 0) \\ 1 + \dfrac{1}{H_0}(s - \log(1+s)) & (s \geqslant 0). \end{cases} \tag{6.8}$$

Hopefully, if this limit is approximated for real systems, the drag force on the flattened bubble can be effectively independent of the particulars of the disjoining pressure form.

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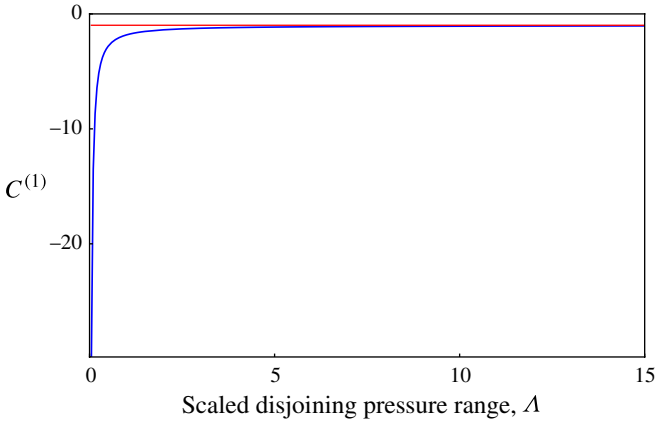


FIGURE 4. (Colour online) The constant $C^{(1)}$ as a function of Λ.

In this case

$$\eta_0(z) = \begin{cases} 1 & (z < 0) \\ 1 + \dfrac{z^2}{2} & (z \geqslant 0), \end{cases} \quad \xi_0(z) = \begin{cases} 0 & (z < 0) \\ -z & (z \geqslant 0), \end{cases}$$
$$\xi_1(z) = 0 \quad \eta_1(z) = \begin{cases} 0 & (z < 0) \\ -\dfrac{z^3}{3} & (z \geqslant 0), \end{cases} \tag{6.9}$$

so that

$$f_0(z) = \begin{cases} \dfrac{3\pi}{16\sqrt{2}} & (z < 0) \\ \dfrac{3z}{8\,(1 + z^2/2)^2} - \dfrac{3z}{16(1 + z^2/2)} + \dfrac{3\sqrt{2}}{16}\left[\dfrac{\pi}{2} - \arctan\left(\dfrac{z}{\sqrt{2}}\right)\right] & (z \geqslant 0). \end{cases} \tag{6.10}$$

From (5.28)

$$B(\infty) = \frac{9\pi}{16\sqrt{2}}. \tag{6.11}$$

The constant $C^{(1)}(\infty)$ is, from (5.30),

$$C^{(1)}(\infty) = 1 - \frac{1}{3}\left(\frac{3\pi}{16\sqrt{2}}\right)^2 + \int_0^\infty \mathrm{d}z \left[\frac{z^2 f_0}{4} + \frac{z^5}{2\,(1 + z^2/2)^3}\left\{\frac{1}{8} - \frac{3}{2(1 + z^2/2)}\right\} - \frac{1}{1 + z}\right]. \tag{6.12}$$

Some tedious but straightforward integration yields

$$C^{(1)}(\infty) = -\left[\frac{7}{12} + \frac{1}{2}\log 2 + \frac{1}{3}\left(\frac{3\pi}{16\sqrt{2}}\right)^2\right]. \tag{6.13}$$

We see from figures 3 and 4 that this limiting situation is reached for $\Lambda \geqslant 5$ in practice, at least for the exponential repulsion used here.

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The Stokes drag analysis given in this paper contains an internal check on the region of validity of the linearity assumption wherein the bubble is undistorted by the flow field. In the flattened region under the bubble, the pressure field is

$$p(R, Z, \phi) = \frac{a\mu U}{h_0^2} F(s) R\cos\phi = \frac{a\mu U}{h_0^2}\left[H_0^{1/2} f_0(-\infty) + \cdots\right] R\cos\phi. \tag{6.14}$$

When $p \sim \Pi(h_0)$ the lubrication pressure is large enough to change locally the inter-surface separation. Thus the uniform flattened region of this analysis requires

$$\frac{Ap}{2\gamma} \ll 1 \tag{6.15}$$

or, in terms of the capillary number $Ca = \mu U/\gamma$,

$$Ca \ll \frac{\sqrt{2}}{f_0(-\infty)}\left[\frac{h_0}{A}\right]^{3/2}. \tag{6.16}$$

For $h_0 \sim 100$ nm and $A \sim 0.1$ mm, this is $Ca \ll 2 \times 10^{-4}$ using (6.10) for $f_0(-\infty)$. For comparison, in water with a velocity $U \sim 1$ mm s^{-1} the capillary number is 1.4×10^{-5} . However although the bubble shape is likely to be significantly disturbed at this capillary number, the effect on the drag force may not be quite so sensitive. Preliminary analysis of the bubble distortion in the flow field shows that the next correction to the drag force is at $O(U^3)$. At $O(U^2)$ only an extra lift force contributes to the hydrodynamic force on the flattened bubble.

As an application of the results here, we consider the steady translation of a bubble in water on the underside of a surface inclined at an angle ϕ. Balancing the drag force derived here against the buoyancy force $\Delta\rho g V_d \sin\phi$, we obtain

$$U = \frac{\frac{2}{3}\Delta\rho g A^2 \sin\phi}{\mu\left[B(\Lambda)H_0^{-1/2} - \frac{1}{2}\log H_0 + C(\Lambda)\right]}. \tag{6.17}$$

It is convenient to express this result in terms of the capillary number as a function of the scaled radius $A/\lambda = Bo^{1/2}$, where Bo is the Bond number, and the dimensionless film thickness $\varepsilon = h_0/\lambda$:

$$Ca = \frac{\frac{2}{3}Bo\sin\phi}{\left[B(\Lambda)Bo^{3/4}(3\varepsilon)^{-1/2} - \frac{1}{2}\log(\varepsilon) + \frac{3}{20}\log(Bo) + \bar{C}(\Lambda)\right]} \tag{6.18}$$

where we have written H_0 as

$$H_0 = \frac{3}{Bo}\frac{h_0}{A} = 3\varepsilon Bo^{-3/2} \tag{6.19}$$

and where

$$\bar{C}(\Lambda) = C^{(0)} + C^{(1)}(\Lambda) + C^{(2)} + \tfrac{3}{5} + \tfrac{1}{10}\log 3 \approx 2.921 + C^{(1)}(\Lambda). \tag{6.20}$$

In the limit $\Lambda \to \infty$, we can evaluate all constants using (6.11) and (6.13) to get

$$Ca \approx \frac{\frac{2}{3}Bo\sin\phi}{\left[1.250Bo^{3/4}(3\varepsilon)^{-1/2} - \frac{1}{2}\log(\varepsilon) + \frac{3}{20}\log(Bo) + 1.818\right]}. \tag{6.21}$$

In figure 5 we show the steady velocity in this form for two choices of ε, where the film thickness h_0 is assumed fixed for simplicity. We have chosen $\phi = 2^\circ \approx 0.035$

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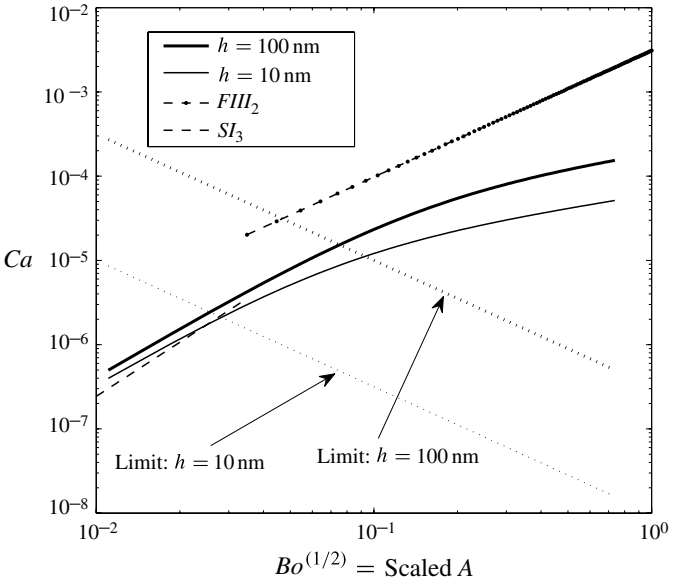


FIGURE 5. Various theoretical results for the steady velocity of bubbles sliding along an inclined surface, $\phi = 2°$. The range of Bond numbers corresponds to bubbles in water of radius 30 μm–2 mm.

in this plot, to limit ourselves to low velocities. We also show the results for two cases studied in Hodges *et al.* (2004), which neglects disjoining pressure effects, but includes effects of higher order in U than those treated here. We also show the limits of applicability, according to (6.16).

The cases from Hodges *et al.* (2004) are for bubbles (viscosity ratio with liquid $\ll 1$) with no flattened region (Case SI_3: S for spherical) and a flattened bubble (Case $FIII_2$: F for flattened). The notation marks different regions in the 'phase diagram' with axes being viscosity ratio and Bond number. Different regions exhibit different dynamical behaviour, such as slipping, sliding or rolling. Only the slipping cases are relevant here.

The theory here appears to link smoothly with the case of a spherical bubble covered in Hodges *et al.* (2004) for small bubbles, as might be expected. That case is claimed to be applicable for $Bo < \phi$. There are detectable deviations from the hydrodynamic theory within the limits of applicability for each value of h_0, but these would be difficult to measure at such low speeds.

The case $FIII_2$ gives much faster speeds than the theory here, because the film thickness is determined by a normal force balance on the whole bubble and is typically a few microns, much larger than the equilibrium thicknesses considered here.

These theories can be compared with recent experimental results in Del Castillo *et al.* (2011). The experiments use bubbles with radius 0.3–1.4 mm, which are outside the range of applicability of the present theory. The measured velocities are higher than would be predicted here for film thicknesses in the range 10–100 nm, but much slower than the case $FIII_2$ of Hodges *et al.* (2004). In turn, they are consistent with earlier experiments (Aussillous & Quere 2002). Of most interest, they show a weak dependence on salt concentration, suggesting an influence of disjoining pressure of the kind explored here.

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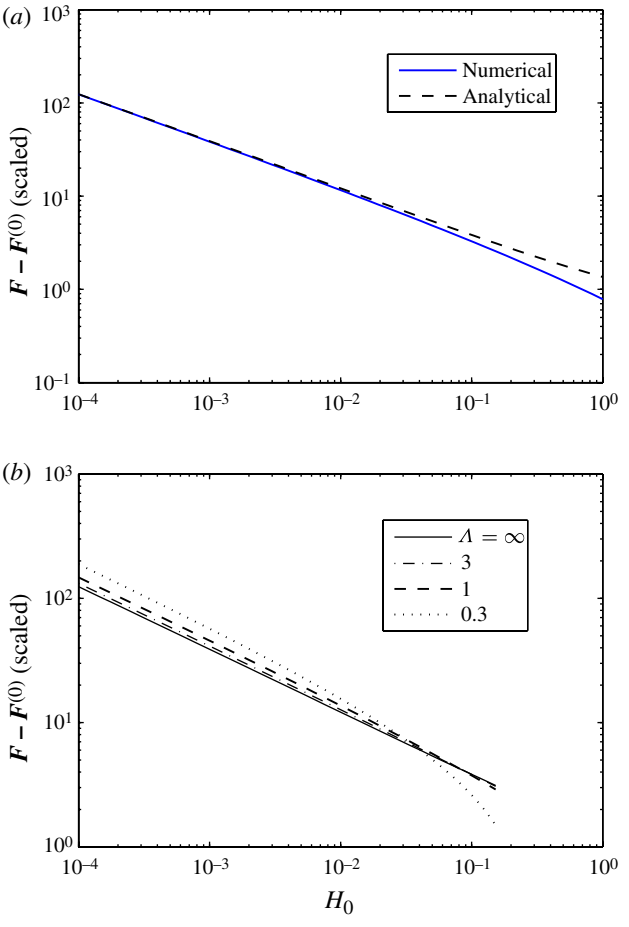


FIGURE 6. (Colour online) The drag increment due to flattening, scaled by $2\pi\mu AU$, as a function of H_0. (*a*) Numerical solution of (5.2) and (3.23) compared to asymptotic results obtained from (5.23), for the case $\Lambda \to \infty$. (*b*) The effect of Λ on the drag given by (5.23), analytical approximation.

The analysis in this paper is based on two assumptions, namely that $H_0 \ll 1$ and that $h_0/a \ll 1$. It is possible, for A/λ sufficiently small, that $H_0 \sim 1$ and the boundary layer analysis above is not appropriate. This does not prevent the calculation of the Stokes drag force to order $(a/A)^2 \sim (A/\lambda)^2$. Equation (2.8) for equilibrium bubble shape and (5.2) for the drag force are still valid. Only the boundary layer analysis of these equations is negated. By solving (2.8) and (3.23) from $s = -1$ through $s = 0$ to large positive s the integral (5.2) may be evaluated for the drag force. It will still be a function of the parameter H_0 but its evaluation as an expansion in H_0 will be lost. In this case the result is likely to be sensitive to the details of the disjoining pressure functionality however.

As an illustration, we show in figure 6(*a*) the drag increment due to flattening $\boldsymbol{F} - \boldsymbol{F}^{(0)}$, scaled by $2\pi\mu AU$, as a function of H_0 for the case $\Lambda \to \infty$. The solid curve is obtained from the integral (5.2), where $\eta(s)$ is given by (6.8), so that only the second term in the integrand contributes for $s < 0$ and the integral for $s > 0$ is done numerically, after solving (3.23) numerically. The dashed curve is obtained from (5.23)

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but removing the terms originating from the drag on an undistorted bubble $\boldsymbol{F}^{(0)}$. This verifies that our asymptotic expression is accurate for $H_0 \leqslant 0.1$.

As can be seen from figure 6(*b*), the effect of disjoining pressure, for the exponentially repulsive form studied here, seems quite modest for common values of Λ, i.e. $\Lambda \geqslant 1$, which suggests that the analytical results here may be employed as a guide in the absence of any knowledge of the disjoining pressure functionality.

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