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The rotating body problem

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Abstract. It is shown that for an axisymmetric stationary system there exists, *in vacuo*, a canonical coordinate system, analogous to Weyl's canonical coordinates for the static case. Some approximate solutions of the vacuum field equations are then obtained in these canonical coordinates.

In the second part of this paper, the vacuum field equations for an isolated axisymmetric stationary system are set up and approximate solutions obtained by means of a multipole expansion. The physical components of the Riemann tensor are then examined and it is found that the 'mass term' predominates, the 'rotation term' being of the next order of magnitude.

1. *The canonical coordinate system.* It is well known that for an isolated axisymmetric system it is possible to define a canonical coordinate system *in vacuo*. In this section it is shown that the motion of a canonical coordinate system may be generalized to the case of axisymmetric metrics, which are stationary though not necessarily static. It is easily shown from symmetry arguments and coordinate transformations that such metrics may be put in the form

$$ds^2 = e^{2u}(dt + a\,d\phi)^2 - e^{-2u}(e^{2k}dr^2 + e^{2m}dz^2 + r^2 d\phi^2), \tag{1}$$

where u, a, k and m are functions of r and z only which vanish at spatial infinity, and where r, ϕ, z are to be interpreted as something like cylindrical polar coordinates.

The vacuum field equations $R_{ij} = 0$ are imposed and in particular the equation

$$g^{22}R_{22} + g^{44}R_{44} + 2g^{24}R_{24} = 0$$

yields

$$\frac{1}{r}e^{2u-2k}\left(\frac{\partial k}{\partial r} - \frac{\partial m}{\partial r}\right) = 0. \tag{2}$$

(The convention adopted is $r = x^1$, $\phi = x^2$, $z = x^3$, $t = x^4$. For a complete list of Christoffel brackets and components of the Ricci tensor see Appendix I.)

Equation (2) implies that there is no loss in generality in setting $k = m$ for this could always be ensured by a coordinate transformation of the form $z' = \int e^{f(z)}dz$.

Now the case $a = 0$ corresponds to the static situation and the coordinate system with $k = m$ to Weyl's canonical coordinates. We may thus define a canonical coordinate system to be one in which the metric (1) takes the form

$$ds^2 = e^{2u}(dt + a\,d\phi)^2 - e^{2(k-u)}(dr^2 + dz^2) - r^2 e^{-2u}d\phi^2. \tag{3}$$

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We shall henceforth adopt such a canonical coordinate system. Then the five non-trivial field equations $R_{11} = R_{13} = R_{24} = R_{33} = R_{44} = 0$ for the three functions k, a and u may be further reduced to

$$R_{44} \equiv \nabla^2 u + \frac{1}{2r^2} e^{4u} \left[\left(\frac{\partial a}{\partial r} \right)^2 + \left(\frac{\partial a}{\partial z} \right)^2 \right] = 0, \tag{4}$$

$$R_{24} - aR_{44} \equiv \nabla^2 a - \frac{2}{r} \frac{\partial a}{\partial r} + 4 \frac{\partial a}{\partial r} \frac{\partial u}{\partial r} + 4 \frac{\partial a}{\partial z} \frac{\partial u}{\partial z} = 0, \tag{5}$$

$$R_{33} - R_{11} \equiv \frac{2}{r} \frac{\partial k}{\partial r} + 2 \left(\frac{\partial u}{\partial z} \right)^2 - 2 \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{2r^2} e^{4u} \left[\left(\frac{\partial a}{\partial r} \right)^2 - \left(\frac{\partial a}{\partial z} \right)^2 \right] = 0, \tag{6}$$

$$R_{13} \equiv -\frac{1}{r} \frac{\partial k}{\partial z} + 2 \frac{\partial u}{\partial r} \frac{\partial u}{\partial z} - \frac{1}{2r^2} \frac{\partial a}{\partial r} \frac{\partial a}{\partial z} e^{4u} = 0, \tag{7}$$

where ∇^2 is the Laplacian. The equation $R_{11} = 0$ and the integrability condition $\partial^2 k / \partial r \partial z = \partial^2 k / \partial z \partial r$ are automatically satisfied as a consequence of the above four equations.

No exact solutions of the field equations with $a \neq 0$ have been found. However, some approximate solutions are obtained as follows.

'Spherical polar coordinates' R, θ are defined by $r = R \sin\theta$, $z = R \cos\theta$, and it is assumed that the functions a, u and k can be expanded as power series in $1/R$ whose coefficients are functions of θ only. This ensures that the boundary conditions at spatial infinity are satisfied. If only the dominant terms are retained, (5) reduces to

$$\nabla^2 a - \frac{2}{r} \frac{\partial a}{\partial r} = 0, \tag{8}$$

since the remaining terms must always be at least one order of magnitude smaller. The solutions of (8) are

$$a = \frac{K \sin\theta}{R^l} P_l^1(\cos\theta),$$

where K is a constant and $P_l^1(\cos\theta)$ are the associated Legendre functions. Of particular interest is the solution corresponding to $l = 1$, since this must give rise to the largest possible deviations from the static field. Adopting this solution in the remaining equations, and again keeping only the dominant terms we obtain

$$a = \frac{K}{R} \sin^2\theta,$$

$$u = \sum_{m=0}^{\infty} \frac{A_m}{R^{m+1}} P_m(\cos\theta) - \frac{K^2}{4R^4} \cos^2\theta,$$

$$k = -\frac{A_0^2}{2R^2} \sin^2\theta,$$

where the A_m are constants.

2. *The vacuum field equations.* 2·1. *Introduction.* A line element is derived which has properties appropriate to an axially symmetric stationary rotating system. This line element is expressed in terms of four unknown functions. The vacuum field

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equations are obtained and solutions of these are sought. Assuming the metric to be Minkowskian at infinity, and that the unknown x functions can be expressed as multipole expansions, it is found possible to obtain the first few coefficients in these expansions. It would appear that the functions can be determined to as many terms as are desired, but the calculations soon become very heavy.

Das, Florides and Synge (1) and Florides and Synge (2) have considered a similar problem. Their method which consisted of modifying a Newtonian model is quite different from the method adopted here.

2·2. *Determination of the line element.* The line element

$$d\bar{s}^2 = g_{ij}dx^i dx^j \quad (i,j = 1,2,3,4)$$

is expressed in cylindrical polar coordinates $x^1 = r, x^2 = \phi, x^3 = z, x^4 = t$ and is assumed to be stationary, i.e. $\partial g_{ij}/\partial x^4 = 0$. The metric possesses axial symmetry if it is invariant under the transformation $\bar{x}^2 = x^2 + a$, where a is a constant, the condition for this being $\partial g_{ij}/\partial x^2 = 0$. Thus the components of the metric tensor are functions of the two independent variables r and z.

The metric represents a rotating system if it is invariant under the single transformation $\bar{x}^2 = -x^2$, $\bar{x}^4 = -x^4$. Consequently the only non-zero off-diagonal elements are $g_{13} = g_{31}$ and $g_{24} = g_{42}$.

On the 2-surface $x^2 =$ constant, $x^3 =$ constant, the line element takes the form

$$ds^2 = g_{11}dr^2 + 2g_{12}dr\,dz + g_{22}dz^2.$$

Since any 2-surface is conformally flat, the line element can be expressed in diagonal form with $g_{11} = g_{22}$.

The expression

$$d\bar{s}^2 = -e^{2\alpha}(dr^2 + dz^2) - e^{2(\sigma+\rho)}\cos 2\psi\, d\phi^2 + e^{2(\sigma-\rho)}\cos 2\psi\, dt^2 + 2e^{2\sigma}\sin 2\psi\, dt\,d\phi, \quad (9)$$

where α, σ, ρ and ψ are functions of the variables r and z thus represents the line element of a stationary axially symmetric rotating system, the units being relativistic with $c = 1$.

Since the metric takes the Minkowski form when r and z approach infinity, ψ, $\sigma - \rho$ and α tend to zero and $\sigma + \rho$ tends to $\log r$.

The case $\psi = 0$ corresponds to the static Weyl and Levi-Civita metric so that ψ can be regarded as the rotation term. Putting

$$a = e^{2\rho}\tan^2\psi,$$

$$u = -\rho + \tfrac{1}{2}\log r + \tfrac{1}{2}\log\cos 2\psi,$$

$$k = \alpha - \sigma - \rho + \log r + \tfrac{1}{2}\log\cos 2\psi,$$

it can be shown that $$ds^2 = re^{-2\sigma}d\bar{s}^2$$

so that the metrics of equations (3) and (9) are conformal. Thus the metrics correspond to two different axially symmetric distributions of matter.

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2·3. *Field equations.* The components of the Ricci tensor and the Christoffel symbols arising from (8) are listed in Appendix II. The vacuum field equations $R_{ij} = 0$ then lead to the following six equations:

$$\left.\begin{aligned}(1+P)(\sigma_{11}+2\sigma_1^2) &= 0,\\ (1+P)(\rho_{11}+2\sigma_1\rho_1-4\rho_1\psi_1\tan 2\psi) &= 0,\\ (1+P)(\psi_{11}+2\sigma_1\psi_1-\rho_1^2\sin 4\psi) &= 0,\\ (1+P)(\alpha_{11}-\sigma_1^2-\psi_1^2+\rho_1^2\cos^2 2\psi) &= 0,\\ \sigma_{13}+\sigma_1\sigma_3+\rho_1\rho_3\cos^2 2\psi-\psi_1\psi_3-\sigma_1\alpha_3-\sigma_3\alpha_1 &= 0,\\ (1-P)(\sigma_{11}+\sigma_1^2+\rho_1^2\cos^2 2\psi-\psi_1^2-2\sigma_1\alpha_1) &= 0,\end{aligned}\right\} \tag{10}$$

where the suffices 1 and 3 represent partial differentiation with respect to r and z, respectively, and where the operator P has the effect of changing the suffix 1 into 3 and vice versa. These equations are not independent but are related by means of the Bianchi identities.

To solve them it is convenient, as in §1, to introduce spherical polar coordinates and to assume that ρ, ψ, α and σ can be expressed as infinite series having the required forms at infinity as follows:

$$\begin{aligned}\rho &= \tfrac{1}{2}\log r+\sum_{m=0}^{\infty} A_m R^{-(m+1)},\\ \psi &= \sum_{m=0}^{\infty} B_m R^{-(m+1)},\\ \alpha &= \sum_{m=0}^{\infty} C_m R^{-(m+1)},\\ \sigma &= \tfrac{1}{2}\log r+\sum_{m=0}^{\infty} D_m R^{-(m+1)},\end{aligned}$$

where A_m, B_m, C_m and D_m are functions of θ only. By substituting these expressions in equations (10) it is possible to derive ordinary differential equations for the coefficients. It is found that the equations for A_m make it advantageous to introduce Legendre polynomials $P_m(\cos\theta)$ and for D_m to introduce the associated Legendre polynomials $P_m^1(\cos\theta)$. The first three coefficients were found for each function.

$$\left.\begin{aligned}\rho &= \tfrac{1}{2}\log r+\frac{a_0}{R}+\left(\frac{a_1 z}{R^3}+\frac{d_1}{R^2}\right)+\left\{a_2\left(\frac{3z^2}{R^5}-\frac{1}{R^3}\right)+\frac{d_2 z}{R^4}-\frac{2a_0 d_1}{R^3}\right\}+O(R^{-4}),\\ \psi &= \frac{b_1 r}{R^3}+\left(\frac{B_2 rz}{R^5}-\frac{a_0 b_1 r}{R^4}\right)+O(R^{-4}),\\ \alpha &= \frac{a_0}{R}+\left\{\frac{a_1 z}{R^3}+d_1\left(\frac{4z^2}{R^4}-\frac{2}{R^2}\right)-\frac{a_0^2 r^2}{2R^4}\right\}\\ &\qquad+\left\{a_2\left(\frac{3z^2}{R^5}-\frac{1}{R^3}\right)+d_2\left(\frac{8z^3}{R^6}-\frac{6z}{R^4}\right)-\frac{2a_0 a_1 r^2 z}{R^6}-\frac{2a_0 d_1}{3R^3}\right\}+O(R^{-4}),\\ \sigma &= \tfrac{1}{2}\log r+\frac{d_1}{R^2}+\frac{d_2 z}{R^4}+O(R^{-4}).\end{aligned}\right\} \tag{11}$$

The terms a_m, b_m and d_m are constants, the constant a_0 corresponding to the Schwarzschild mass. The only singularity in the field is at the origin.

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2·4. *The physical components of the Riemann tensor.* The physical components of the Riemann tensor are denoted by $R_{(abcd)}$ and are defined by the equation

$$R_{(abcd)} = R_{pqrs} \underset{a}{\lambda p} \underset{b}{\lambda q} \underset{c}{\lambda r} \underset{d}{\lambda s},$$

where $\underset{a}{\lambda p}$ $(a, p = 1, 2, 3, 4)$ are the components of an orthonormal tetrad of vectors. The following four vectors form such a tetrad

$$\underset{1}{\lambda p} = (-e^{-\alpha}\cos\xi,\ 0,\ -e^{-\alpha}\sin\xi, 0),$$

$$\underset{2}{\lambda p} = [0, e^{-(\rho+\sigma)}(\sin\psi\sinh\eta - \cos\psi\cosh\eta),\ 0, e^{\rho-\sigma}(\sin\psi\cosh\eta + \cos\psi\sinh\eta)],$$

$$\underset{3}{\lambda p} = (e^{-\alpha}\sin\xi, 0,\ -e^{-\alpha}\cos\xi, 0),$$

$$\underset{4}{\lambda p} = [0, e^{-(\rho+\sigma)}(\sin\psi\cosh\eta - \cos\psi\sinh\eta), 0, e^{\rho-\sigma}(\sin\psi\sinh\eta + \cos\psi\cosh\eta)]$$

for real values of ξ and η, the vector $\underset{4}{\lambda p}$ being time-like, the others space-like. The functions ξ and η are determined by assuming that the tetrad is parallelly propagated along a curve to which $(e^{-\alpha}, 0, 0, 0)$ is a tangent vector, the necessary conditions reducing to $\xi_1 = \alpha_3$, $\eta_1 + \rho_1 \sin 2\psi = 0$.

Assuming the tetrad to be the axes of an observer at a great distance from the origin, the leading terms in the multipole expansions of ξ and η were found, the arbitrary functions arising being taken as zero for convenience. Thus

$$\xi = -\frac{a_0 r}{Rz} + O(R^{-2}), \quad \eta = -\frac{b_1 r}{Rz^2} + O(R^{-3})$$

so that the leading terms in $\underset{a}{\lambda p}$ and thus $R_{(abcd)}$ were found. These results are listed in Appendix II.

It is found that at great distances from the origin, the largest physical components of the Riemann tensor are quadrupole terms which correspond precisely to the Schwarzschild mass terms. The components which depend on the rotation of the system are octupole terms. Thus the field is determined to a first approximation by the mass of the system, as in the Newtonian case. However, taking terms of higher order into account, the rotation of the system has an effect on the field. This result is at variance with Newtonian theory for an axially symmetric rotating system.

2·5. *The residual energy tensor.* From equations (9) and (11) it is possible to calculate the components of the metric tensor. The components are not exact owing to the finite number of terms in the expansions of σ, ρ, ψ and α. If these errors in g_{ij} are ignored the following components result,

$$g_{11} = g_{33} = -1 - \frac{2a_0}{R} - \frac{1}{R^2}\{2a_1\cos\theta + 4d_1\cos 2\theta + a_0^2(1+\cos^2\theta)\}$$

$$-\frac{1}{R^3}\{2a_2(3\cos^2\theta - 1) + 4d_2\cos 3\theta + 4a_0a_1\cos^3\theta + \tfrac{4}{3}a_0d_1(12\cos^2\theta - 7) + \tfrac{2}{3}a_0^3(3\cos^2\theta - 1)\},$$

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$$g_{22} = -R^2\sin^2\theta\left[1+\frac{2a_0}{R}+\frac{2}{R^2}(a_1\cos\theta+2d_1+a_0^2)\right.$$
$$\left.+\frac{1}{R^3}\{a_2(6\cos^2\theta-2)+4d_2\cos\theta+\tfrac{20}{3}a_0d_1+4a_0a_1\cos\theta+\tfrac{4}{3}a_0^3\}\right],$$

$$g_{44} = 1-\frac{2a_0}{R}-\frac{2}{R^2}(a_1\cos\theta-a_0^2)+\frac{1}{R^3}\{a_2(2-6\cos^2\theta)+4a_0a_1\cos\theta+\tfrac{4}{3}(a_0d_1-a_0^3)\},$$

$$g_{24} = \frac{2b_1\sin^2\theta}{R}+\frac{2\sin^2\theta}{R^2}(b_2\cos\theta-a_0b_1)$$

referred to cylindrical polar coordinates. It is found that in terms of this metric, the components of the energy tensor are as follows. T_{22} is of order R^{-4}, T_{24} is of order R^{-5} and T_{11}, T_{33} and T_{44} are of order R^{-6}.

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APPENDIX I

The non-zero components of the Christoffel brackets for the metric (1) are

$$\left\{\begin{matrix}1\\11\end{matrix}\right\} = k_1-u_1, \qquad \left\{\begin{matrix}3\\11\end{matrix}\right\} = e^{2(k-m)}(u_3-k_3),$$
$$\left\{\begin{matrix}1\\22\end{matrix}\right\} = e^{-2k}[e^{4u}(aa_1+a^2u_1)+r^2u_1-r], \qquad \left\{\begin{matrix}3\\22\end{matrix}\right\} = e^{-2m}[e^{4u}(aa_3+a^2u_3)+r^2u_3],$$
$$\left\{\begin{matrix}1\\33\end{matrix}\right\} = e^{2(m-k)}(u_1-m_1), \qquad \left\{\begin{matrix}3\\33\end{matrix}\right\} = m_3-u_3,$$
$$\left\{\begin{matrix}1\\44\end{matrix}\right\} = e^{4u-2k}u_1, \qquad \left\{\begin{matrix}3\\44\end{matrix}\right\} = e^{4u-2m}u_3,$$
$$\left\{\begin{matrix}1\\13\end{matrix}\right\} = k_3-u_3, \qquad \left\{\begin{matrix}3\\13\end{matrix}\right\} = m_1-u_1,$$
$$\left\{\begin{matrix}1\\24\end{matrix}\right\} = e^{4u-2k}(\tfrac{1}{2}a_1+au_1), \qquad \left\{\begin{matrix}3\\24\end{matrix}\right\} = e^{4u-2m}(\tfrac{1}{2}a_3+au_3),$$
$$\left\{\begin{matrix}2\\12\end{matrix}\right\} = -(e^{4u}aa_1+2r^2u_1-2r)/2r^2, \qquad \left\{\begin{matrix}4\\12\end{matrix}\right\} = (e^{4u}a^2a_1+4r^2au_1+r^2a_1-2ar)/2r^2,$$
$$\left\{\begin{matrix}2\\14\end{matrix}\right\} = -e^{4u}a_1/2r^2, \qquad \left\{\begin{matrix}4\\14\end{matrix}\right\} = (aa_1e^{4u}+2r^2u_1)/2r^2,$$
$$\left\{\begin{matrix}2\\23\end{matrix}\right\} = -(aa_3e^{4u}+2r^2u_3)/2r^2, \qquad \left\{\begin{matrix}4\\23\end{matrix}\right\} = (a^2a_3e^{4u}+4ar^2u_3+a_3r^2)/2r^2,$$
$$\left\{\begin{matrix}2\\34\end{matrix}\right\} = -e^{4u}a_3/2r^2, \qquad \left\{\begin{matrix}4\\34\end{matrix}\right\} = (aa_3e^{4u}+2r^2u_3)/2r^2,$$

where suffices 1, 3 denote partial differentiation with respect to r and z, respectively.

$$R_{11} = m_{11}-u_{11}-k_1(m_1-u_1)-(k_1/r)+e^{2(k-m)}(u_3-k_3)(m_3-k_3)-(u_{33}-k_{33})e^{2(k-m)}$$
$$+2u_1^2-(u_1/r)+m_1(m_1-u_1)-(a_1^2e^{4u}/2r^2),$$

$$R_{22} = (1+Q)e^{4u-2k}[ak_1(a_1+au_1)-4aa_1u_1-am_1(a_1+au_1)-\tfrac{1}{2}a_1^2-aa_{11}-a^2u_{11}]$$
$$-r^2(1+Q)e^{-2k}(u_1m_1-k_1u_1+u_{11})+r(m_1-k_1-u_1)e^{-2k}+e^{4u-2k}a(a_1-au_1)/r$$
$$-(a^2/2r^2)(1+Q)a_1^2e^{8u-2k},$$

$$R_{33} = k_{33}-u_{33}-e^{2(m-k)}(u_1-m_1)/r+(k_3-u_3)(k_3-m_3)-e^{2(m-k)}(u_{11}-m_{11})$$
$$-e^{2(m-k)}m_1(u_1-m_1)+e^{2(m-k)}k_1(u_1-m_1)+2u_3^2-e^{4u}a_3^2/2r^2,$$

$$R_{44} = -e^{4u-2k}u_1/r-(1+Q)e^{4u-2k}(u_1m_1+u_{11}-u_1k_1)-(1/2r^2)(1+Q)a_1^2e^{8u-2k},$$

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$$R_{13} = 2u_1u_3 - (k_3/r) - a_1a_3e^{4u}/2r^2,$$

$$R_{24} = (1+Q)\,e^{4u-2k}(-\tfrac{1}{2}a_1m_1 - au_1m_1 - \tfrac{1}{2}a_{11} - au_{11} + \tfrac{1}{2}k_1a_1 + ak_1u_1 - 2u_1a_1)$$
$$+ e^{4u-2k}(a_1 - 2au_1)/2r - (a/2r^2)\,(1+Q)\,a_1^2e^{8u-2k},$$

where Q is an operator which interchanges 1 and 3 *and* k and m, e.g. $m_1 \to k_3$, $u_1 \to u_3$, etc.

APPENDIX II

Christoffel symbols for Metric (9)

$$\left\{{1 \atop 11}\right\} = \left\{{3 \atop 13}\right\} = -\left\{{1 \atop 33}\right\} = \alpha_r, \quad \left\{{1 \atop 13}\right\} = \left\{{3 \atop 33}\right\} = -\left\{{3 \atop 11}\right\} = \alpha_z,$$

$$\left\{{1 \atop 22}\right\} = -e^{2(\sigma+\rho-\alpha)}[(\sigma_r+\rho_r)\cos 2\psi - \psi_r \sin 2\psi],$$

$$\left\{{1 \atop 24}\right\} = e^{2(\sigma-\alpha)}(\sigma_r \sin 2\psi + \psi_r \cos 2\psi),$$

$$\left\{{1 \atop 44}\right\} = e^{2(\sigma-\rho-\alpha)}\,[(\sigma_r-\rho_r)\cos 2\psi - \psi_r \sin 2\psi],$$

$$\left\{{3 \atop 22}\right\} = e^{2(\sigma+\rho-\alpha)}\,[\psi_z \sin 2\psi - (\sigma_z+\rho_z)\cos 2\psi],$$

$$\left\{{3 \atop 24}\right\} = e^{2(\sigma-\alpha)}\,(\sigma_z \sin 2\psi + \psi_z \cos 2\psi),$$

$$\left\{{3 \atop 44}\right\} = e^{2(\sigma-\rho-\alpha)}\,[(\sigma_z-\rho_z)\cos 2\psi - \psi_z \sin 2\psi],$$

$$\left\{{2 \atop 12}\right\} = \sigma_r + \rho_r \cos^2 2\psi, \qquad \left\{{2 \atop 14}\right\} = -e^{-2\rho}(\psi_r + \rho_r \sin 2\psi \cos 2\psi),$$

$$\left\{{2 \atop 23}\right\} = \sigma_z + \rho_z \cos^2 2\psi, \qquad \left\{{2 \atop 34}\right\} = -e^{-2\rho}(\psi_z + \rho_z \sin 2\psi \cos 2\psi),$$

$$\left\{{4 \atop 12}\right\} = e^{2\rho}(\psi_r - \rho_r \sin 2\psi \cos 2\psi), \qquad \left\{{4 \atop 14}\right\} = \sigma_r - \rho_r \cos^2 2\psi,$$

$$\left\{{4 \atop 23}\right\} = e^{2\rho}(\psi_z - \rho_z \sin 2\psi \cos 2\psi), \qquad \left\{{4 \atop 34}\right\} = \sigma_z - \rho_z \cos^2 2\psi.$$

Ricci tensor

$$R_{11} = 2\sigma_{rr} + \alpha_{rr} + \alpha_{zz} - 2(\alpha_r\sigma_r - \alpha_z\sigma_z) + 2(\sigma_r^2 - \psi_r^2 + \rho_r^2 \cos^2 2\psi),$$

$$R_{22} = e^{2(\sigma+\rho-\alpha)}\,(1+P)\,[(\sigma_{rr}+\rho_{rr})\cos 2\psi - \psi_{rr}\sin 2\psi + 2\sigma_r^2 \cos 2\psi + 2\sigma_r\rho_r \cos 2\psi$$
$$+ 2\rho_r^2 \sin^2 2\psi \cos 2\psi - 4\rho_r\psi_r \sin 2\psi - 2\psi_r\sigma_r \sin 2\psi],$$

$$R_{13} = 2(\sigma_{rz} + \sigma_r\sigma_z - \alpha_r\sigma_z - \alpha_z\sigma_r - \psi_r\psi_z + \rho_r\rho_z \cos^2 2\psi),$$

$$R_{33} = 2\sigma_{zz} + \alpha_{rr} + \alpha_{zz} + 2(\alpha_r\sigma_r - \alpha_z\sigma_z) + 2(\sigma_z^2 - \psi_z^2 + \rho_z^2 \cos^2 2\psi),$$

$$R_{24} = e^{2(\sigma-\alpha)}\,(1+P)\,[-\sigma_{rr}\sin 2\psi - \psi_{rr}\cos 2\psi - 2\sigma_r^2 \sin 2\psi + 2\rho_r^2 \sin 2\psi \cos^2 2\psi$$
$$- 2\psi_r\sigma_r \cos 2\psi],$$

$$R_{44} = e^{2(\sigma-\rho-\alpha)}\,(1+P)\,[(\rho_{rr}-\sigma_{rr})\cos 2\psi + \psi_{rr}\sin 2\psi - 2\sigma_r^2 \cos 2\psi + 2\sigma_r\rho_r \cos 2\psi$$
$$- 2\rho_r \sin^2 2\psi \cos 2\psi - 4\rho_r\psi_r \sin 2\psi + 2\psi_r\sigma_r \sin 2\psi].$$

Riemann Tensor—Physical Components

$$R_{(1212)} = a_0(3r^2 - 2R^2)/R^5 + O(R^{-4}), \qquad R_{(1423)} = -b_1(15r^2z - 3zR^2)/R^7 + O(R^{-5}),$$

$$R_{(1313)} = a_0/R^3 + O(R^{-4}), \qquad R_{(1434)} = 3a_0rz/R^5 + O(R^{-4}),$$

$$R_{(1214)} = b_1(15r^3 - 12rR^2)/R^7 + O(R^{-5}), \qquad R_{(2323)} = a_0(3z^2 - 2R^2)/R^5 + O(R^{-4}),$$

$$R_{(1223)} = -3a_0rz/R^5 + O(R^{-4}), \qquad R_{(2334)} = -b_1(15rz^2 - 3rR^2)/R^7 + O(R^{-5}),$$

$$R_{(1234)} = b_1(15r^2z - 6zR^2)/R^7 + O(R^{-5}), \qquad R_{(2424)} = -a_0/R^3 + O(R^{-4}),$$

$$R_{(1324)} = -3b_1z/R^5 + O(R^{-5}), \qquad R_{(3434)} = a_0(3z^2 - R^2)/R^5 + O(R^{-4}),$$

$$R_{(1414)} = a_0(3r^2 - R^2)/R^5 + O(R^{-4}).$$

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