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Pruning fronts and the formation of horseshoes

ANDRÉ DE CARVALHO†

Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
(e-mail: andre@math.berkeley.edu)

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Abstract. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of the plane. We define open sets $P \subset \mathbb{R}^2$, called *pruning fronts* after the work of Cvitanović *et al* in 1988, for which it is possible to construct an isotopy $H : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$ with open support contained in $\bigcup_{n\in\mathbb{Z}} f^n(P)$ such that $H(\cdot, 0) = f(\cdot)$ and $H(\cdot, 1) = f_P(\cdot)$, where f_P is a homeomorphism under which every point of P is wandering. Applying this construction when f is Smale's horseshoe, it is possible to obtain an uncountable family of homeomorphisms, depending on infinitely many parameters, going from trivial to chaotic dynamic behavior. This family is a two-dimensional analog of a one-dimensional universal family.

0. *Introduction*

Since the universality discoveries of Feigenbaum [**Fe**] and Coullet and Tresser [**CT**] in the late 1970s, much effort has been devoted to the study of *families* of dynamical systems. In dimension one, this led to what today is a practically complete theory. In particular, the *Kneading Theory* of Milnor and Thurston [**MT**] gives a complete description of all topological possibilities for the dynamics of a family of endomorphisms of the interval which pass from simple to complicated or *chaotic* dynamical behavior.

In dimension two, no such theory exists. In fact, it is not clear what restrictions should be imposed on the families under consideration in order to develop an analogous theory.

We give a precise definition of *pruning fronts*, a concept introduced by Cvitanović *et al* [**CGP**]. We, like them, hope that pruning fronts will play, for two-dimensional dynamical systems, a role analogous to that which *kneading sequences* play for one-dimensional dynamical systems. We construct an uncountable family of two-dimensional homeomorphisms, depending on infinitely many parameters and containing the countable family of *pseudo-Anosov* maps associated with horseshoe periodic orbits. Conjecturally (see the *Pruning Front Conjecture* below) this family contains, as sub-families, the Hénon and Lozi families. We hope to be able to give a complete combinatorial picture of all ($C^{1+\epsilon}$) entropy carrying two-dimensional dynamical systems using these techniques.

† Current address: IMS, SUNY at Stony Brook, NY 11794-3660, USA. (e-mail: andre@math.sunysb.edu)

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The Hénon family $H_{a,b}(x, y) = (1-by-ax^2, x)$ [**He**] and the Lozi family $L_{a,b}(x, y) = (1 - by - a|x|, x)$ [**Lo**] just mentioned are interesting examples of families of plane homeomorphisms passing from simple to chaotic dynamic behavior. They lack, however, a defining topological characteristic analogous, for example, to the condition that a one-dimensional map be *unimodal* (i.e. piecewise monotone with exactly one turning point). Nevertheless, Cvitanović *et al* [**CGP**] introduced the concept of pruning fronts and gave interesting numerical evidence that it should be useful in the study of families of two-dimensional homeomorphisms. Their work, however, lacked a rigorous mathematical framework and this was the main motivating force for our work.

They conjectured that pruning fronts, analogously to kneading sequences, give a complete topological description of the dynamics of the Hénon or Lozi families. To be more specific, given a unimodal map $F : I \to I$, its kneading sequence $K(F)$ is the itinerary of the critical point. $K(F)$ determines a set of forbidden orbits of the 1-sided shift on two symbols and the complementary set of *allowed* orbits is exactly the set of orbits that the map F has. For $\mu > 4$, the map $F_\mu(x) = \mu x(1 - x)$ has a Cantor set of non-wandering points where the dynamics is conjugated to the 1-sided 2-shift. Therefore, the dynamics of any unimodal map F can be obtained by destroying or *pruning away* some orbits of F_μ determined by $K(F)$.

Smale's horseshoe map is a two-dimensional analog of the map F_μ. It is a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane all of whose interesting dynamics is contained in a square. The map squeezes, stretches and folds the square, placing it on top of itself in a horseshoe shape and the set Λ of non-wandering points inside the square is a Cantor set where the dynamics is conjugated to a 2-sided shift on two symbols $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$. Both the Hénon and the Lozi families present horseshoes for certain parameters values. Cvitanović *et al* [**CGP**] introduced the concept of *pruning fronts*. These are open subsets P of shift space $\{0, 1\}^{\mathbb{Z}}$ (or of $\Lambda \subset \mathbb{R}^2$, by the conjugacy mentioned above) satisfying some conditions that need not concern us here. The orbits that fall inside P under iteration by σ (or f, if we take P to be a subset of Λ) are the *forbidden* orbits. The complementary set of orbits is the set of *allowed* orbits. They stated the following conjecture.

PRUNING FRONT CONJECTURE (PFC). *Given parameter values a, b, there exists a pruning front $P = P(a, b)$ which determines a set of forbidden orbits for Smale's horseshoe map and the complementary set of allowed orbits is exactly the set of orbits found in the non-wandering set of the Hénon map $H_{a,b}$.*

In other words, PFC states that any map in the Hénon (and it could also be stated for Lozi) family can be described as a *pruned horseshoe*, that is, as a horseshoe from which some orbits are destroyed or pruned away.

We give a precise definition of pruning fronts, which for us are open subsets of the dynamical plane, and prove the following.

THEOREM. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of the plane and $P \subset \mathbb{R}^2$ a pruning front for f. Then there exists an isotopy $H : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$ with $\operatorname{supp} H \subset \bigcup_{k\in\mathbb{Z}} f^k(P)$, such that $H(\cdot, 0) = f(\cdot)$ and $H(\cdot, 1) = f_P(\cdot)$ is a homeomorphism under which every point of P is wandering.*

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In words, our theorem says that it is possible to destroy all the dynamics of f inside a pruning front by an isotopy which does not affect f outside the pruning front (and all its images and preimages).

Our theorem allows us to construct a family $\mathcal{F}$ of two-dimensional homeomorphisms, depending on infinitely many parameters, going from trivial dynamics to a complete horseshoe. This family is an analog of a *universal* family of endomorphisms of the interval. Although we do not prove PFC, it can be restated to say that the Hénon or Lozi families are two-parameter sub-families of the family $\mathcal{F}$ (see also Ishii [**I**] for a pruning theory for the Lozi family).

We show in §2 that the family $\mathcal{F}$ contains a one-parameter sub-family whose dynamics mimics that of a *full* unimodal family. In particular, $\mathcal{F}$ is uncountable. This one-parameter sub-family completes Hall's family of *no bogus transition (NBT)* orbits [**H**]. It is also equivalent to the family of inverse limits of a unimodal family (for more on inverse limits, see Barge and Martin [**BM**]).

We have recently shown [**dC2**] that $\mathcal{F}$ contains the family of all pseudo-Anosov maps associated with horseshoe periodic orbit collections. In fact, we give a constructive proof of Thurston's classification theorem for surface homeomorphisms as a consequence of a more general framework that relates pruning to generalized kneading theories for tree and graph endomorphisms. This is further evidence that $\mathcal{F}$ should be viewed as a universal family of two-dimensional homeomorphisms.

By a result of Katok's [**Ka**], all the topological entropy of any $C^{1+\epsilon}$ surface diffeomorphism is carried by a horseshoe in some iterate. Therefore, describing the formation of horseshoes can be seen as having implications on the mechanism of two-dimensional transition to chaos in a quite general context.

The techniques used in the present work are those of point set topology of the plane. In §1 we give some preliminaries and then introduce the concept of (c, e)-*disks* (a generalization of disks bounded by one arc of stable and one arc of unstable manifolds), define pruning fronts and state the main theorem. We also present an outline of the proof in the case of a pruning collection with only one (c, e)-disk. This case is rather simple and contains the main ideas used in the proof of the general case, which is quite technical and complicated. The outline is added to provide a road map one can refer to when reading the general proof.

Before proving the main theorem, we go, in §2, to the second part of the title—the formation of horseshoes. We present three examples of pruning fronts for Smale's horseshoe map. The first of them is, in fact, a family of such examples, namely the *one-dimensional-like* family we mentioned above. The second example gives rise to a *renormalizable* map, that is, a homeomorphism which interchanges two closed disks. The second iterate restricted to each one of these disks is again a full horseshoe. Finally, in the third example, we present a pruning front that gives rise to a 'lax pseudo-Anosov' homeomorphism. Together these examples should suggest different ways in which a horseshoe can be formed.

§3 contains the proof of the main theorem. It is divided into four subsections. In §3.1 we develop the plane topology tools we will use in §3.2 to construct some special collections of disks and arcs. In §3.3 we state and prove some results about isotopies of homeomorphisms

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of the plane. §3.4 puts together the work done in the previous subsections to finish the proof of the main theorem.

1. *Pruning fronts*

We will denote the two-dimensional plane π or $\mathbb{R}^2$. A *Jordan curve* J is the homeomorphic image of the circle $S^1 = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$ and a *closed arc* L is the homeomorphic image of the closed interval $[0, 1]$, the images of $\{0\}$ and $\{1\}$ being its *endpoints*. By an *open arc* we will mean the set obtained by taking the endpoints away from a closed arc. If L is a closed arc L° will denote the corresponding open arc.

Definition 1.1. Let J be a Jordan curve and I the bounded region of $\pi \setminus J$. We call I a *Jordan domain* and sometimes refer to it as the *inner domain* determined by J.

Definition 1.2. Let U be a domain in the plane and α an open (closed) arc whose endpoints lie on ∂U and all others lie in U. Such an α is called an *open (closed) cross-cut*.

We will use the following plane topology theorem. A proof can be found in [**Ne**].

THEOREM 1.3. *Let J be a Jordan curve, I its inner domain and $\alpha \subset I$ a cross-cut. Then α separates I into two Jordan domains I_1 and I_2 whose boundaries are $L_1 \cup \alpha$ and $L_2 \cup \alpha$, where L_1 and L_2 are the arcs into which the endpoints of α separate J.*

We define what we call (c, e)-disks (c for *contracting* and e for *expanding* or, rather, *contracting in the past*). Let $f : \pi \to \pi$ be a plane homeomorphism.

Definition 1.4. A closed disk D is called a (c, e)-*disk* if there are closed arcs $C, E \subset \partial D$ specified such that $\partial D = C \cup E$ and C and E only intersect at endpoints, which we call the *vertices* of D. We require moreover that C and E satisfy the following:

(i) $\lim_{n\to\infty} \operatorname{diam} f^n(C) = 0$;

(ii) $\lim_{m\to-\infty} \operatorname{diam} f^m(E) = 0$.

We now introduce a relation on pairs of (c, e)-disks which will be used presently in the definition of pruning fronts.

Definition 1.5. Let D_1, D_2 be (c, e)-disks such that $I_1 \cap I_2 \neq \emptyset$. We say D_1 is *e-longer* or simply *longer* than D_2, denoted $D_1 \succ D_2$, if (i), (ii) and (iii) hold (see Figure 1):

(i) $C_1 \cap I_2 = \emptyset$ and $E_2 \cap I_1 = \emptyset$;

(ii) if $C_1 \cap C_2 \neq \emptyset$ then $C_1 \cup C_2$ is an arc and if $E_1 \cap E_2 \neq \emptyset$ then $E_1 \cup E_2$ is an arc;

(iii) if $C_1^\circ \cap \overline{I_1 \cap I_2} \neq \emptyset$ then $C_1 \subset C_2$ and if $E_2^\circ \cap \overline{I_1 \cap I_2} \neq \emptyset$ then $E_2 \subset E_1$.

The main purpose of this article is to show how to isotop away the dynamics of f in a controlled manner. We will now define sets within which it is possible to do this, namely, to destroy all dynamics within them by an isotopy which is identically equal to f without them. We call them *pruning fronts* after the work of Cvitanović *et al* [**CGP**].

Definition 1.6. Let $\{D_i\}_{i=1}^L$ be a collection of (c, e)-disks such that (i), (ii) and (iii) hold:

(i) $\succ$ can be extended by transitivity to a partial order on $\{D_i\}_{i=1}^L$ or, equivalently, there are no 'loops' $D_{i_1} \succ D_{i_2} \succ \cdots \succ D_{i_n} \succ D_{i_1}$;

(ii) for every $n > 0$ and $i, j \in \underline{L}$, either $f^n(I_i) \cap I_j = \emptyset$ or $f^n(D_i) \succ D_j$;

(iii) for every $m < 0$ and $i, j \in \underline{L}$, either $f^m(I_i) \cap I_j = \emptyset$ or $f^m(D_i) \prec D_j$.

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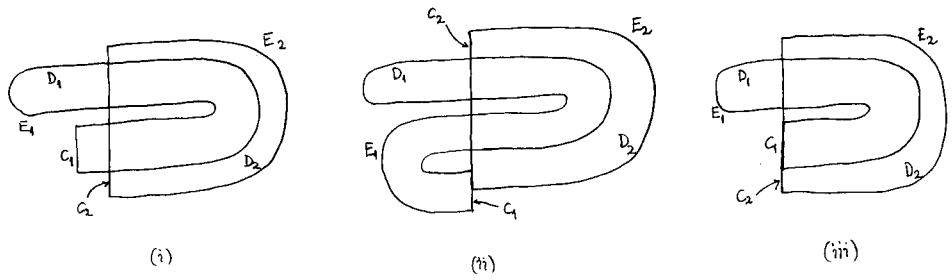


FIGURE 1. The relation $\succ$.

Such a collection will be called a *pruning collection*. Its locus $\overline{P} = \bigcup_{i=1}^{L} D_i$ (see [**CGP**] and the comments before the definition) will be called a *pruning front*.

We will use the following.

Definition 1.7. A point x is said to be *wandering* under the homeomorphism $f : X \to X$ if there exists an open set U containing x such that $f^n(U) \cap U = \emptyset$ for all $n \in \mathbb{Z}$.

We can now state the main theorem.

THEOREM 1.8. *Let $f : \pi \to \pi$ be a homeomorphism of the plane, $\{D_i\}_{i=1}^{L}$ a pruning collection and $P = \bigcup_{i=1}^{L} I_i$, where I_i is the interior of the disk D_i. Then there exists an isotopy $H : \pi \times [0, 1] \to \pi$ of the identity such that* $\operatorname{supp} H \subset \bigcup_{k\in\mathbb{Z}} f^k(P)$, *and if we set $f_P(\cdot) = f \circ H(\cdot, 1)$, every point of P is wandering under f_P.*

(For definitions of isotopy and support of an isotopy, see §3.3.)

Outline of the Proof. We now sketch the proof for the case of a pruning collection with only one (c, e)-disk D. We will divide D into two closed disks D^c and D^e by a cross cut $\alpha = \alpha(0)$ joining the vertices of D. We then construct an isotopy k at the end of which all points in the interiors of both D^c and D^e are wandering.

From the definition of pruning collection we have that for $n > 0$, either $f^n(I) \cap I = \emptyset$ or $f^n(D) \succ D$ and for $m < 0$, either $f^m(I) \cap I = \emptyset$ or $f^m(D) \prec D$ (where I is the interior of D). From (i) in the definition of $\succ$ we have that $f^n(C) \cap I = \emptyset$ for all $n > 0$ and $f^m(E) \cap I = \emptyset$ for all $m < 0$.

Assume first that D satisfies the stronger hypotheses: $f^n(C) \cap D = \emptyset$ for all $n > 0$ and $f^m(E) \cap D = \emptyset$ for all $m < 0$. First consider D and $f(D)$. Under the stronger hypotheses, we must have $f(C) \cap D = \emptyset$ and $E \cap f(D) = \emptyset$. Therefore we can find arcs $\alpha(0) \subset D$ and $\alpha(1) \subset f(D)$ joining the vertices of the respective disks wherein they lie and such that $\alpha(0) \cap f(D) = \emptyset$ and $\alpha(1) \cap D = \emptyset$ (see Figure 2).

We construct an isotopy k_0 of f with $\operatorname{supp} k_0 \subset I$ so that $k_0(\alpha(0), 1) = \alpha(1)$.

Now consider $f^2(D)$. Again from our hypotheses, $f^2(C) \cap [D \cup f(D)] = \emptyset$ and we can find an arc $\alpha(2) \subset f^2(D)$ joining vertices so that $\alpha(2) \cap [D \cup f(D)] = \emptyset$. We then construct an isotopy k_1 of f, with $\operatorname{supp} k_1 \subset f(I)$, such that $k_1(\alpha(1), 1) = \alpha(2)$. Notice that the support of k_1 has to contain a neighborhood of $f^{-1}(\alpha(2)) \cup \alpha(1)$ and that this set is disjoint from D. Therefore we can take k_1 so that $\operatorname{supp} k_1 \subset f(I) \setminus D$. Continuing like this

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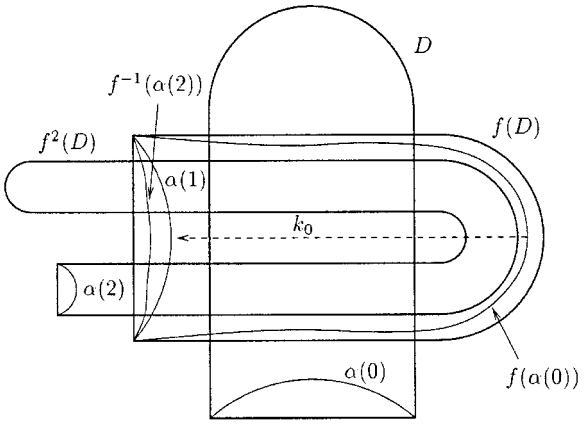


FIGURE 2. The arcs $\alpha(i)$, for $i = 0, 1, 2$, and the isotopy k_0. Notice that, since $\alpha(2) \cap f(D) = \emptyset$, $f^{-1}(\alpha(2)) \cap D = \emptyset$.

we construct a sequence of isotopies k_n of f with disjoint supports. Moreover, since diam $f^n(C) \to 0$ as $n \to +\infty$, we can choose $\alpha(n)$ so that $\operatorname{diam}(\operatorname{supp} k_n) \to 0$ as $n \to +\infty$.

If we let $k_+ = \bigcup_{n\geq 0} k_n$, then k_+ is an isotopy of f with the following property: denoting by $D^c(\alpha(n))$ the closed disk bounded by $f^n(C) \cup \alpha(n)$, then $k_+(D^c(\alpha(n)), 1) = D^c(\alpha(n+1))$. But the $\alpha(n)$'s were chosen so that $D^c(\alpha(m)) \cap D^c(\alpha(n)) = \emptyset$ for $m \neq n$ so that the interior $I^c(\alpha(0))$ of $D^c(\alpha(0))$ is a *wandering domain*† under $k_+(\cdot, 1)$.

We then proceed analogously with $D^e(\alpha(0))$ (the closed disk bounded by $E \cup \alpha(0)$) and f^{-1} to construct an isotopy $k_- = \bigcup_{n<0} k_n$ so that $I^e(\alpha(0))$, the interior of $D^e(\alpha(0))$, is wandering under $k_-(\cdot, 1)$. In this way it is possible to construct the isotopies k_n, $n \in \mathbb{Z}$, with disjoint supports so that $H = f^{-1}(k_+ \cup k_-)$ satisfies the conclusion of Theorem 1.8.

Assume now our stronger hypotheses do not hold, say, there exists a positive n such that $f^n(C) \cap D \neq \emptyset$. It turns out that the only case in which this would force us to alter the previous proof in any substantial way is when there exists an $N \geq 1$ such that $f^N(C) \subset C$ and $f^N(C) \subset \overline{f^N(I) \cap I}$. This implies that C contains a periodic point p of period N (which we take to be the least integer ≥ 1 satisfying the property above).

If this is the case, we proceed as follows. Construct arcs $\alpha(0), \alpha(1), \ldots, \alpha(N)$ as before so that $\alpha(N) \subset I^c(\alpha(0))$ (remember that $\alpha(N)$ joins the endpoints of $f^N(C) \subset C$, see Figure 3), and construct isotopies $k_0, k_1, \ldots, k_{N-1}$ as before. If we let $\Psi(\cdot) = k(\cdot, 1)$, the N-fold composition Ψ^N maps $D^c(\alpha(0))$ homeomorphically onto $D^c(\alpha(N)) \subset D^c(\alpha(0))$.

By a further isotopy if necessary, we can make the return map Ψ^N into a contraction with fixed point p. Again, the interior $I^c(\alpha(0))$ of $D^c(\alpha(0))$ consists now of wandering points.

This finishes the sketch of the proof in the case of a pruning collection with only one disk D. □

Remark. Let us remark that, for the sake of clarity, we explained how to construct the cross-cuts $\alpha(n)$ and isotopies k_n for $n \geq 0$ and said that $\alpha(m)$ and k_m for $m < 0$ were

† An open connected set $U \subset X$ is called a *wandering domain* for the homeomorphism $f : X \to X$ if $f^n(U) \cap U = \emptyset$ for all $n \in \mathbb{Z}$. Clearly all points in a wandering domain are wandering.

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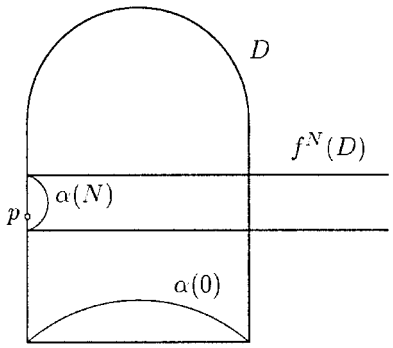


FIGURE 3. The case in which C contains a periodic point.

obtained analogously. In the course of the proof, however, it is necessary to construct the cross-cuts in pairs $\alpha(n) \cup \alpha(-n)$, $n \geq 0$, and likewise with the isotopies.

2. *Examples*

In this section we present examples of pruning collections for Smale's horseshoe map $f : \mathbb{R}^2 \to \mathbb{R}^2$. We begin by choosing a rigid model for f and describing some well known results, offered without proof. We also present some elementary concepts of kneading theory which we will need (the reader is referred to the books of Devaney [**De**] and de Melo and van Strien [**MS**] for further details on the horseshoe and on one-dimensional dynamics). We then get to the examples. In describing the dynamics of the 'pruned' maps f_P in each example, we will make several assertions and only sketch the proofs. The reason for proceeding thus is two-fold. First, this is a section to give examples of pruning collections and this aspect is presented fully. Second, the details we omit are part of a more general theory which is deserving of separate treatment, which we intend to do in forthcoming papers.

We now fix a rigid model of Smale's horseshoe map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Foliate the square

$$S = \{(x, y) : |x| \leq \tfrac{1}{2},\ |y| \leq \tfrac{1}{2}\}$$

with horizontal *unstable* leaves and vertical *stable* leaves, and begin by choosing the action of f on S as depicted in Figure 4. We require that f should stretch the unstable leaves uniformly, contract the stable leaves uniformly, and map segments of unstable (respectively, stable) leaf in $S \cap f^{-1}(S)$ onto segments of unstable (respectively, stable) leaf in S. Moreover, we choose f to map the corner of S marked with a circle on Figure 4 onto the corner of $f(S)$ marked with a circle. Extend f to the half-disks A_1 and A_2 as depicted in the diagram: let f be a strict contraction of $A_1 \cup A_2$, so that there is a fixed point x of f lying in A_1 with the property that $f^i(y) \to x$ as $i \to \infty$ for all $y \in A_1 \cup A_2$. Finally, extend f over the rest of $\mathbb{R}^2$ without introducing any new non-wandering points.

The non-wandering set $\Omega(f)$ of f consists of the fixed point $\{x\}$ and an invariant Cantor set $\Lambda \subset S$. Moreover, there exists a homeomorphism $h : \Sigma \to \Lambda$, where $\Sigma = \{0, 1\}^{\mathbb{Z}}$ is the two-sided shift on two symbols, which conjugates the shift map $\sigma : \Sigma \to \Sigma$ and

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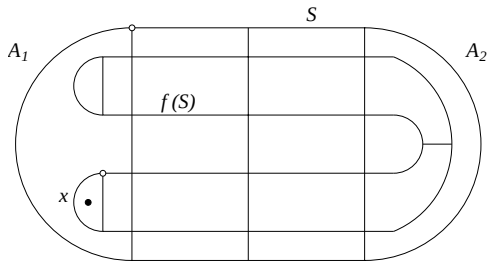


FIGURE 4. A rigid model for the horseshoe.

$f|_\Lambda : \Lambda \to \Lambda$. Λ is a hyperbolic invariant set and each point $p \in \Lambda$ has one-dimensional stable and unstable manifolds which intersect transversally. Notice that if two points p_0 and p_1 lie on the stable (unstable) manifold of some point $q \in \Lambda$, p_0 and p_1 are the endpoints of exactly one arc contained in the stable (unstable) manifold of p.

We now describe the *unimodal order* on the one-sided shift space $\Sigma_+ = \{0, 1\}^{\mathbb{N}}$ and, using it, define *kneading sequences*.

Definition 2.1. Let $s = s_0 s_1 \ldots$ and $t = t_0 t_1 \ldots$ lie in Σ_+ and suppose $s_i = t_i$ for $i < k$ *and* $s_k \neq t_k$. We set $s \lhd t$ if $\sum_{i=1}^{k} s_i$ is even. We set $s \trianglelefteq t$ if either $s = t$ or $s \lhd t$.

Definition 2.2. Let $\sigma : \Sigma_+ \to \Sigma_+$ be the shift map and $\kappa \in \Sigma_+$. We say κ is a *kneading sequence* if, for every $n \in \mathbb{N}$, $\sigma^n(\kappa) \trianglelefteq \kappa$.

The unimodal order just defined is used in the study of one-dimensional *unimodal* maps, i.e. piecewise monotone endomorphisms of the interval with exactly one critical (turning) point. In this context, kneading sequences are defined as the itinerary of the critical value. It is possible to check that kneading sequences associated with unimodal maps satisfy the definition above.

The unimodal order describes the horizontal and vertical ordering of points in Λ as follows: if $(x_1, y_1), (x_2, y_2) \in \Lambda$, with $h(x_1, y_1) = s_{-2}s_{-1} \cdot s_0 s_1 \ldots$ and $h(x_2, y_2) = t_{-2}t_{-1} \cdot t_0 t_1 \ldots$, then

$$x_1 < x_2 \iff s_0 s_1 s_2 \cdots \lhd t_0 t_1 t_2 \ldots \text{ and}$$
$$y_1 < y_2 \iff s_{-1} s_{-2} \cdots \lhd t_{-1} t_{-2} \ldots.$$

We shall often use the elements of Σ to describe points of Λ without explicitly invoking the map h. Thus, for example, we may talk about 'the fixed point $\overline{1}$', 'the periodic orbit $\overline{10011}$', or 'the point $\overline{0}.0\overline{101}$'. Here, a bar over a group of symbols stands for infinite repetition of the group. If the group is to the right (left) of the decimal point, it should be repeated infinitely to the right (left) and if there is no decimal point, the group should be repeated infinitely to both sides (so $\overline{0}.0\overline{101} = \ldots 000.0101101101 \ldots$ and $\overline{10} = \ldots 1010.1010 \ldots$). If the symbolic sequence is an element of Σ_+, a bar over a group of symbols means infinite repetition of the group to the right. Let $p \in \Lambda$ and $h(p) = \ldots s_{-2}s_{-1} \cdot s_0 s_1 \ldots$, we will sometimes refer to $\ldots s_{-2}s_{-1} \cdot s_0 s_1 \ldots$ as the *symbolic representation* of p and to $\ldots s_{-2}s_{-1}.$ and $.s_0 s_1 \ldots$ as the *symbolic vertical and horizontal coordinates of* p, respectively.

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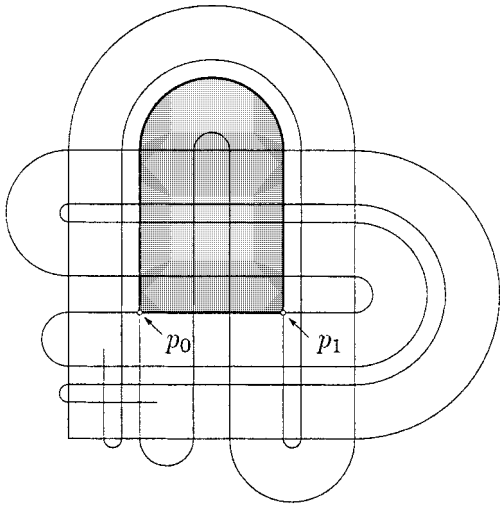


FIGURE 5. (c, e)-disk determined by $p_0 = \overline{0}10.011\overline{0}$ and $p_1 = \overline{0}10.111\overline{0}$.

If two points $p_0, p_1 \in \Lambda$ have symbolic representation $\ldots t_{-2}t_{-1} \cdot t_0t_1 \ldots$ and $\ldots s_{-2}s_{-1} \cdot s_0s_1 \ldots$, respectively, p_0 and p_1 lie on the same stable (unstable) manifold if there exists $N \in \mathbb{Z}$ such that $s_i = t_i$ for every $i \geq N$ ($i \leq N$). Consequently, p_0 and p_1 lie on the same stable *and* unstable manifolds if their symbolic representations differ in at most finitely many entries. If the symbolic representations of p_0 and p_1 differ at exactly one entry, the stable and unstable arcs of which they are the endpoints form a simple closed curve bounding a closed disk which we denote by $D(p_0, p_1)$ (see Figure 5). Because the boundary of $D(p_0, p_1)$ is the union of a stable and an unstable arcs, $D(p_0, p_1)$ is a (c, e)-disk for f as defined in §1 whose vertices are p_0 and p_1.

Notation. If $p_0, p_1 \in \Lambda$ lie on the same stable (unstable) manifold, we denote the closed arc of stable (unstable) manifold whose endpoints are p_0 and p_1 by $[p_0, p_1]_s$ ($[p_0, p_1]_u$). Let $s = s_0s_1 \cdots \in \Sigma_+$. We define the *vertical segment with horizontal coordinate* s to be $[\overline{0}.s, \overline{0}1.s]_s$ and denote it by $\mathrm{ver}(s)$. A vertical segment is thus an arc of stable manifold extending from the lowest to the highest possible symbolic vertical coordinates (notice that $000\ldots$ and $100\ldots$ are, respectively, the smallest and largest elements of Σ_+ in the unimodal order), having symbolic horizontal coordinate s. Notice also that, if we use σ to denote the shift map on Σ_+, $\sigma(\mathrm{ver}(s)) \subset \mathrm{ver}(\sigma(s))$.

We are now ready to present examples of pruning collections for f.

Remark. So that the figures below are not hopelessly complicated and unintelligible, we will represent the Cantor set Λ as a solid square. Formally, what we are depicting is the quotient of Λ under an equivalence relation which collapses the 'gaps' of the vertical and horizontal Cantor sets, the product of which is Λ. The ambiguity thus created is easily understood and the clarity gained plentifully compensates for it.

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Example 1. Let κ be a kneading sequence and

$$D = D(\overline{0}.0\kappa, \overline{0}.1\kappa)$$

be the (c, e)-disk determined by $\overline{0}.0\kappa$ and $\overline{0}.1\kappa$ (that is, the disk bounded by the union of $C = [\overline{0}.0\kappa, \overline{0}.1\kappa]_s$ and $E = [\overline{0}.0\kappa, \overline{0}.1\kappa]_u$). We claim that the collection $\{D\}$, containing D alone, is a pruning collection (see Figure 6). In order to see this we have to show that, if $f^k(I) \cap I \neq \emptyset$ (where I is the interior of D), then $f^k(D) \succ D$, if $k > 0$, and $f^k(D) \prec D$, if $k < 0$. Notice that conditions (ii) and (iii) in Definition 1.5 are automatically satisfied since C and E are arcs of stable and unstable manifolds which intersect transversally. All there is left to check is that $f^n(C) \cap I = \emptyset$ for $n > 0$ and $f^m(E) \cap I = \emptyset$ for $m < 0$.

Notice that $E \subset [\overline{0}, \overline{0}.1\overline{0}]_u$, that

$$f^{-1}([\overline{0}, \overline{0}.1\overline{0}]_u) = [\overline{0}, \overline{0}.01\overline{0}]_u \subset [\overline{0}, \overline{0}.1\overline{0}]_u$$

and that

$$[\overline{0}, \overline{0}.1\overline{0}]_u \cap I = \emptyset.$$

Thus, if $m \leq -1$, $f^m(E) \cap I = \emptyset$. On the other hand, observe that $f(C) = \mathrm{ver}(\kappa)$ and, therefore, $f^n(C) \subset \mathrm{ver}(\sigma^{n-1}(\kappa))$ for $n \geq 1$. If $f^n(C) \cap I \neq \emptyset$, then $f^n(C) \subset I$ and, in fact, $\mathrm{ver}(\sigma^{n-1}(\kappa)) \subset I$. This implies that

$$0\kappa \lhd \sigma^{n-1}(\kappa) \lhd 1\kappa$$

and, applying σ to this inequality, we get $\kappa \lhd \sigma^n(\kappa)$, which contradicts the assumption that κ is a kneading sequence.

Let f_κ denote the map obtained using Theorem 1.8 for $\overline{P} = D$. The family $\mathcal{F} = \{f_\kappa; \kappa$ is a kneading sequence$\}$ mimics in dimension two a *full family* of unimodal maps of the interval. In particular, $\mathcal{F}$ is an uncountable family of two-dimensional homeomorphisms passing from trivial dynamics to a full horseshoe as κ varies from $\overline{0}$ to $1\overline{0}$.

Example 2. Consider the two (c, e)-disks

$$D_1 = D(\overline{1}.010\overline{1},\ \overline{1}.110\overline{1}) \quad \text{and} \quad D_2 = D(\overline{1}01.0\overline{1},\ \overline{1}01.\overline{1}).$$

Because of the periodicity in the coordinates of the vertices of D_1 and D_2, it is easy to check that $\{D_1, D_2\}$ is a pruning collection (see Figure 7). Let $\alpha_i(k) \subset f^k(D_i)$, for $i = 1, 2$, $k \in \mathbb{Z}$, be closed cross-cuts as given by Proposition 3.15. Then

$$\gamma_0 = \bigcup \{\alpha_i(2k);\ i = 1, 2,\ k \in \mathbb{Z}\}$$

and

$$\gamma_1 = \bigcup \{\alpha_i(2k-1);\ i = 1, 2,\ k \in \mathbb{Z}\}$$

are Jordan curves (see Figure 8) such that $\gamma_0 \cap \gamma_1 = \{\overline{1}\}$. If f_P is the map given by Theorem 1.8 for the pruning front $\overline{P} = D_1 \cup D_2$ and U_0 and U_1 are the closed disks bounded by γ_0 and γ_1, f_P interchanges U_0 and U_1, that is, $f_P(U_0) = U_1$ and $f_P(U_1) = U_0$. Moreover, if $\Lambda_0 = \Omega(f_P) \cap U_0$ is the intersection of the non-wandering set of f_P with U_0, $f_P^2|_{\Lambda_0} : \Lambda_0 \to \Lambda_0$ is topologically conjugated to the full horseshoe $f|_\Lambda : \Lambda \to \Lambda$ restricted to the set Λ. This is an example of a 'renormalizable' or 'reducible' map.

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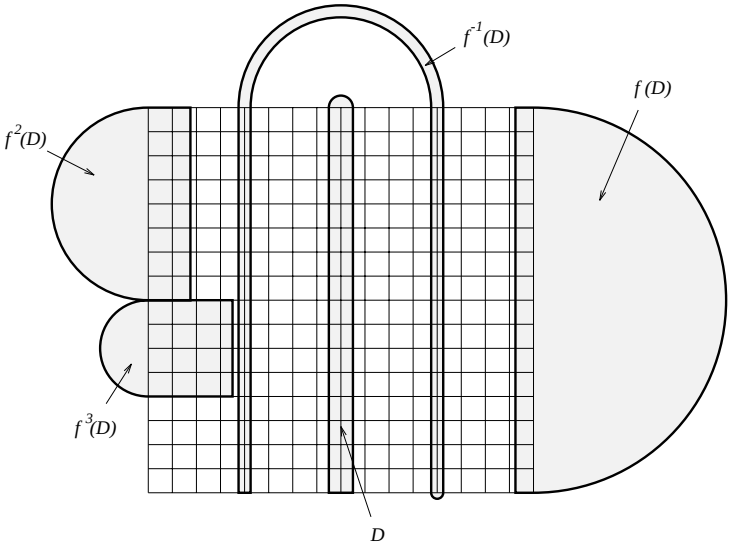


FIGURE 6. A *one-dimensional-like* pruning front for the horseshoe.

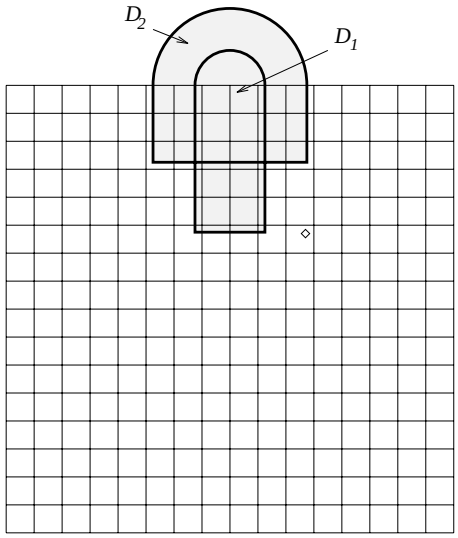


FIGURE 7. The (c, e)-disks D_1 and D_2 of Example 2. $\diamond$ is the fixed point $\overline{1}$.

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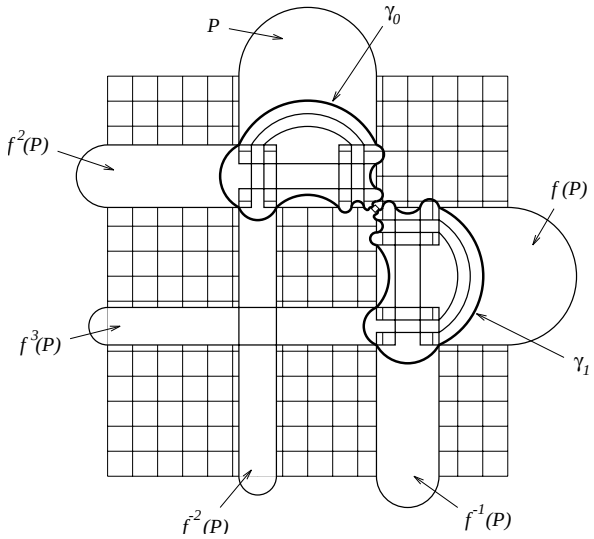


FIGURE 8. The Jordan curves γ_0 and γ_1.

Example 3. Consider the (c, e)-disks

$$D_1 = D(\overline{0}.0\overline{1000100},\ \overline{0}.1\overline{1000100}) \quad \text{and} \quad D_2 = D(\overline{0}1100010.0\overline{1},\ \overline{0}1100010.\overline{1}).$$

Notice that D_1 is of the kind $D(\overline{0}.0\kappa, \overline{0}.1\kappa)$, since $\kappa = \overline{1000100} \in \Sigma_+$ is a kneading sequence, as may easily be verified. As in the previous example, the periodicity in the coordinates of the vertices of D_1 and D_2 make it an easy computation to check that $\{D_1, D_2\}$ is a pruning collection (see Figure 9).

Let $s_7^6(0)$ denote the periodic orbit containing the point $\overline{1000100}$ (see [**HWh**] for an explanation of this name). Since none of its seven points lies in $P = I_1 \cup I_2$, none of them lies in $\bigcup_{n\in\mathbb{Z}} f^n(P)$. In Figure 9, $f^k(P)$, for $-1 \le k \le 3$, are shown. Notice that the non-wandering points of f_P lie outside the shaded region. (In fact, it is not hard to see that the points on the open e-side E_1° of D_1 are also wandering under f_P.)

We claim that the map f_P obtained using Theorem 1.8 realizes the minimum topological entropy among all maps in the isotopy class of f relative to $s_7^6(0)$. In order to see this, we construct a Markov partition for f_P as in Figure 10. The horizontal and vertical sides of the rectangles R_i are contained in $\bigcup_{n\in\mathbb{Z}} f_P^n(E_1 \cup E_2)$ and $\bigcup_{n\in\mathbb{Z}} f_P^n(C_1 \cup C_2)$, respectively. (In fact, it is enough to take the unions ranging from $n = -7$ to $n = 7$, say.) It is easy to see that, if we define $\Lambda_P = \Omega(f_P) \setminus \{\overline{0}, \overline{1}\}$, where $\Omega(f_P)$ is the non-wandering set of f_P and $\overline{0}$ and $\overline{1}$ are the fixed points of f inside S, which are also fixed points under f_P, then $\Lambda_P \subset \bigcup_{i=1}^8 R_i$. The vertices of each R_i lie outside of $\bigcup_{n\in\mathbb{Z}} f^n(P)$ and it therefore makes sense to refer to them using their symbolic representation in Σ. In Table 1 we give the symbolic horizontal and vertical coordinates of the vertices of the rectangles R_i. The columns under x_L and x_R contain the left and right horizontal coordinates, respectively, whereas those under y_L and y_U contain the lower and upper vertical coordinates, respectively.

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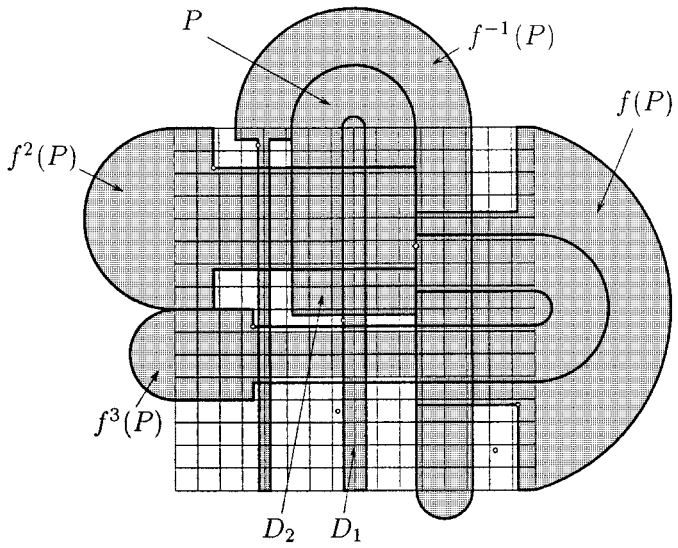


FIGURE 9. The (c, e)-disks D_1 and D_2 of Example 3 and a few images of $P = D_1 \cup D_2$ under f. The points marked by $\circ$ are the periodic orbit $s_7^6(0)$ and $\diamond$ is the fixed point $\overline{1}$.

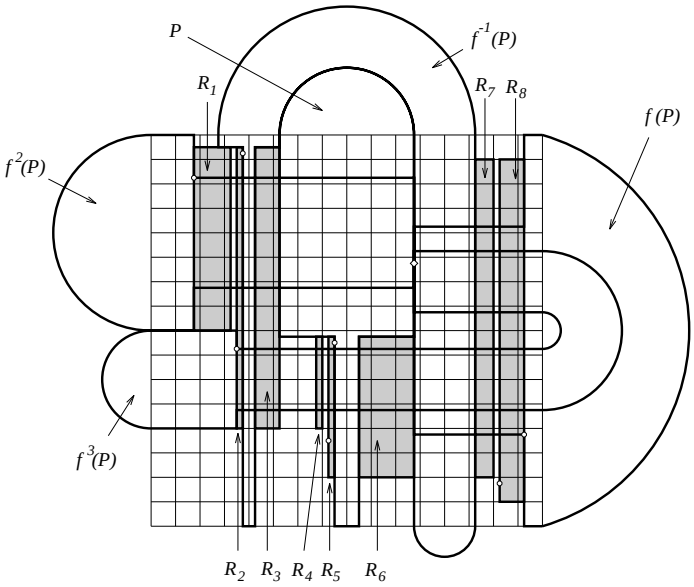


FIGURE 10. A few images of P under f and the Markov partition for f_P.

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TABLE 1. The coordinates of the vertices of the rectangles R_i.

	x_L	x_R	y_L	y_U
R_1	$.\overline{0001001}$	$.00101\overline{1000100}$	$\overline{0}11.$	$\overline{0}110001.$
R_2	$.\overline{0010010}$	$.\overline{0010001}$	$\overline{0}110.$	$\overline{0}110001.$
R_3	$.011\overline{000100}$	$.0\overline{1}$	$\overline{0}110.$	$\overline{0}110001.$
R_4	$.010\overline{1}$	$.0101\overline{000100}$	$\overline{0}110.$	$\overline{0}1100010.$
R_5	$.\overline{0100100}$	$.\overline{0100010}$	$\overline{0}1100.$	$\overline{0}1100010.$
R_6	$.1\overline{000100}$	$.\overline{1}$	$\overline{0}1100.$	$\overline{0}1100010.$
R_7	$.10\overline{1}$	$.101\overline{000100}$	$\overline{0}1100.$	$\overline{0}11001.$
R_8	$.\overline{1001000}$	$.\overline{1000100}$	$\overline{0}11000.$	$\overline{0}11001.$

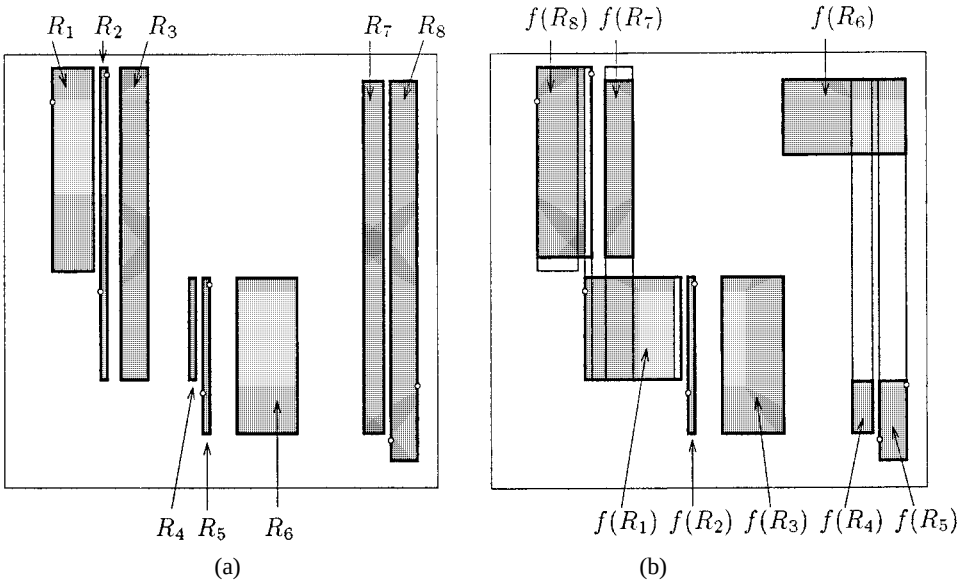


FIGURE 11. The Markov partition and its image. (a) the rectangles R_i and (b) their images under f_P.

In Figure 11 we show how the rectangles R_i are mapped under f_P. The transition matrix $M = (m_{ij})$ associated with this partition is the 8×8 matrix defined by

$$m_{ij} = \begin{cases} 1\,, & \text{if } I(f_P(R_j)) \cap R_i \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

where $I(R_i)$ stands for the interior of the rectangle R_i. Using the notation $R_j \to R_i$ for $I(f_P(R_j)) \cap R_i \neq \emptyset$, we have $R_1 \to R_2R_3R_4$, $R_2 \to R_5$, $R_3 \to R_6$, $R_4 \to R_7$,

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$R_5 \to R_8,\ R_6 \to R_8R_7,\ R_7 \to R_3,\ R_8 \to R_2R_1,$ so that

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Let

$$\Sigma_M = \{s = \ldots s_{-2}s_{-1} \cdot s_0 s_1 \ldots \in \{1, 2, \ldots, 8\}^{\mathbb{Z}};\ m_{s_i s_{i+1}} = 1\ \forall i \in \mathbb{Z}\}$$

be the subshift of finite type associated to M. It is possible to show that, if $s = \ldots s_{-2}s_{-1} \cdot s_0 s_1 \ldots \in \Sigma_M$, then $\bigcap_{n\in\mathbb{Z}} f_P^{-n}(R_{s_n})$ consists of a single point in Λ_P and that the map $k : \Sigma_M \to \Lambda_P$, given by $k(s) = \bigcap_{n\in\mathbb{Z}} f_P^{-n}(R_{s_n})$, is a topological conjugacy between the shift map $\sigma : \Sigma_M \to \Sigma_M$ and $f_P|_{\Lambda_P} : \Lambda_P \to \Lambda_P$. Under these circumstances, $h(f_P) = \log\lambda$, where λ is the spectral radius of M. Using your favorite matrix computation program, you may check that $\lambda = 1.46557$ and that $\log\lambda = 0.382244$, which agrees with Table 1 of [**H2**]. Although this is not a proof, one may be obtained using the algorithm in [**BH**] to find the pseudo-Anosov homeomorphism in the isotopy class of f rel. $s_7^6(0)$. It is known that this map realizes the minimum topological entropy in its isotopy class and it is possible to find a Markov partition for it with the same transition matrix M.

As was mentioned in the introduction, we intend to show in a forthcoming article that, as in Example 3, given a horseshoe periodic orbit collection $\mathcal{O}$, there exists a pruning front $P = P(\mathcal{O})$ such that f_P restricted to $\Omega(f_P)$ is semi-conjugated to the Thurston minimal representative $\phi_{\mathcal{O}}$ in the isotopy class of f rel. $\mathcal{O}$ and that $h(f_P) = h(\phi_{\mathcal{O}})$.

3. *The Proof of the main theorem*

This section, as its title indicates, contains a proof of Theorem 1.8. It consists essentially of two parts; one in which we construct the cross cuts $\alpha_i(n)$ and another in which we construct the isotopy. We divide the section into four subsections. In §3.1 we develop some plane topology tools which will be used in §3.2 to construct some special collections of (c, e)-disks 'cut' by cross cuts joining vertices and having the disjointness properties that will be necessary in the construction of the isotopies. In §3.3 we show that there exist individual isotopies which move cross-cuts in the necessary way. In §3.4 we put together the work done in the previous subsections to construct the various isotopies satisfying the necessary disjointness conditions and finish the proof of the main theorem. The section is rather long and technical and the reader should bear in mind the special case we described above.

3.1. *Plane topology.*

We now state the plane topology propositions that will be used later. The proofs use the Jordan Curve Theorem as well as Whyburn's Separation Theorem (Theorem 3.1 below) and are omitted. We refer the interested reader to [**dC**].

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THEOREM 3.1. (Separation theorem [**Wh**]) *Let A be compact and B closed subsets of the plane such that $A \cap B$ is totally disconnected, $a \in A \setminus (A \cap B)$, $b \in B \setminus (A \cap B)$ and ε a positive number. Then there exists a Jordan curve J which separates a and b and is such that $J \cap (A \cup B) \subset A \cap B$ and every point of J is at a distance less than ε from some point of A.*

Notation. Unless stated explicitly otherwise, we will use the following notations: J will stand for a Jordan curve, I and O for its inner and outer domains, respectively, and D for the closed disk $I \cup J$. If D is a closed disk we will sometimes use $I(D)$ to denote its inner domain. Subscripts will match in the obvious way, so that the inner domain determined by the Jordan curve J_1 is I_1 and $D_1 = I_1 \cup J_1$, etc.

If k is a positive integer $\underline{k}$ will stand for the set $\{1, 2, \ldots, k\}$.

Definition 3.2. Let $J_1, \ldots, J_n$ be Jordan curves and $L \subset J_1 \cap \cdots \cap J_n$ an arc. We say the closed disks $D_1, \ldots, D_n$ *lie on the same side of* L, denoted $D_1, \ldots, D_n|_L$, if $L \subset \overline{I_1 \cap \cdots \cap I_n}$.

PROPOSITION 3.3. *The closed disks $D_1, \ldots, D_n$ lie on the same side of a closed arc L if and only if there exists an open arc $\alpha \subset \bigcap_{i=1}^n I_i$ with the same endpoints as L. As a consequence, if U is the Jordan domain bounded by $\alpha \cup L$, $U \subset \bigcap_{i=1}^n I_i$. Moreover, α may be taken to lie in a ε-neighborhood of L, for any ε chosen in advance.*

Remark. The arc α of Proposition 3.3 is clearly a cross-cut in each of the domains I_i for each $i \in \underline{n}$.

PROPOSITION 3.4. *If $D_1, D_2|_L$, D_2, $D_3|_{L'}$ and $L'' \subset L \cap L'$ then $D_1, D_2, D_3|_{L''}$.*

PROPOSITION 3.5. *Let $J_0, J_1, \ldots, J_n$ be Jordan curves, $L \subset J_0$ an open arc and for $i \in \underline{n}$, $L \cap \overline{I_0 \cap I_i} = \emptyset$. Then given $\varepsilon > 0$ there exists an open cross-cut α in I_0 joining the endpoints of L such that $\alpha \subset V_\varepsilon(L)$ and if U is the Jordan domain bounded by $\alpha \cup L$, then $(U \cup \alpha) \cap D_i = \emptyset$ for each $i \in \underline{n}$.*

Definition 3.6. Let A be a Jordan curve or an arc and $L, L' \subset A$ closed arcs. We say that L and L' are *unlinked* if either $L \subset L'$ or $L' \subset L$ or L and L' intersect at most at endpoints.

Remark. Notice that saying that L and L' are unlinked in a Jordan curve is more than the usual definition of their endpoints being unlinked.

PROPOSITION 3.7. *Let $J_0, \ldots, J_n$ be Jordan curves, and $L_i \subset J_i \cap J_0$, $i \in \underline{n}$, closed arcs, pairwise unlinked in J_0, no two of which are identical. Assume that $D_0, D_i|_{L_i}$ for $i \in \underline{n}$. Then for each $\varepsilon > 0$ there exist disjoint open cross-cuts $\alpha_i \subset I_0$ joining the endpoints of L_i such that $\alpha_i \subset V_\varepsilon(L_i) \cap I_i$ for $i \in \underline{n}$.*

3.2. *Cut collections.* We now define *cut collections* and show special ones having disjoint/nested properties that will be used later in the construction of the isotopies. The main propositions in this subsection are Proposition 3.15, its companion, Proposition 3.18 and Proposition 3.20, where we define the sets $\mathcal{V}_i(n)$ which will contain the support of the isotopies to be constructed.

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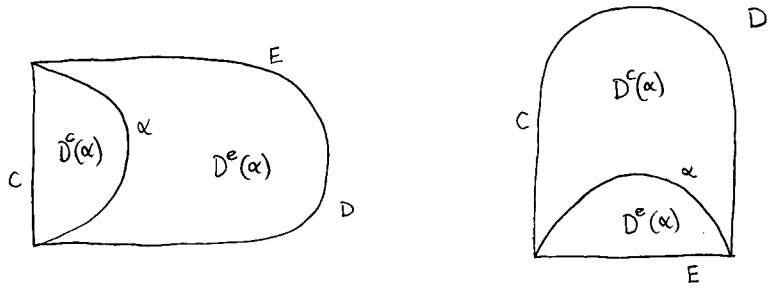


FIGURE 12. Cut (c, e)-disks.

Notation. Let D be a (c, e)-disk and α a cross-cut joining the vertices of D. We have seen that α separates the interior I of D into two Jordan domains whose boundaries are $C \cup \alpha$ and $E \cup \alpha$. We denote them by $I^c(\alpha)$ and $I^e(\alpha)$, respectively, and their closures by $D^c(\alpha)$, $D^e(\alpha)$ (see Figure 12). Moreover, when the disks are indexed and so are the cross-cuts we will only use the index inside the parentheses so that $D^c(\alpha_i)$ will denote the disk bounded by $C_i \cup \alpha_i$.

Conventions. If $D_1, \ldots, D_L$ is a collection of (c, e)-disks, we say they are *related* by $\succ$ if for any $i,\ j \in \underline{L}$ either $I_i \cap I_j = \emptyset$ or $D_i \succ D_j$ or $D_j \succ D_i$. When nothing is mentioned about a collection of (c, e)-disks it is assumed they are related by $\succ$. Cross-cuts in (c, e)-disks, when nothing is mentioned to the contrary, are assumed to be open and to join the vertices of the disk wherein they lie.

The following proposition follows easily from what we have developed so far and we omit the proof.

PROPOSITION 3.8. *Let D_1 and D_2 be (c, e)-disks and $D_1 \succ D_2$. Then the following are true:*

(i) *if $\alpha, \beta, \gamma \subset D_1$ are open cross-cuts joining vertices such that $\beta \subset I^c(\alpha)$ and $\gamma \subset I^e(\alpha)$ then $I^c(\beta) \subset I^c(\alpha) \subset I^c(\gamma)$ and $I^e(\beta) \supset I^e(\alpha) \supset I^e(\gamma)$;*

(ii) *if $C_1^\circ \cap \overline{I_1 \cap I_2} \neq \emptyset$, then $D_1, D_2|_{C_1}$ and if $E_2^\circ \cap \overline{I_1 \cap I_2} \neq \emptyset$, then $D_1, D_2|_{E_2}$;*

(iii) *if $C_1 = C_2$ or $E_1 = E_2$ then $D_1 = D_2$; and*

(iv) *assuming only that $D_1 \not\succ D_2$ and $\alpha_1 \subset D_1$ and $\alpha_2 \subset D_2$ are disjoint open cross-cuts joining vertices, then $\alpha_1 \cap I_2 \subset I^c(\alpha_2)$ and $\alpha_2 \cap I_1 \subset I^e(\alpha_1)$; moreover, $I^c(\alpha_1) \cap I^e(\alpha_2) = \emptyset$.*

Definition 3.9. A collection of pairs $\{(D_i, \beta_i)\}_{i=1}^L$, where $\{D_i\}_{i=1}^L$ is a collection of (c, e)-disks related by $\succ$ and $\{\beta_i \subset D_i\}_{i=1}^L$ is a collection of open cross-cuts joining vertices, will be called a *cut collection*.

Definition 3.10. Let $\{(D_i, \beta_i)\}_{i=1}^L$ be a cut collection, $S \subset \underline{L}$ and $\varepsilon > 0$. The collection $\{\alpha_i\}_{i \in S}$ of disjoint open cross-cuts is said to be a (ε, c)-*collection compatible with* $\{(D_i, \beta_i)\}_{i=1}^L$ (see Figure 13) if $\alpha_i \subset I^c(\beta_i) \cap V_\varepsilon(C_i)$ and for every $i \in S$ and $j \in \underline{L}$ such that $D_i \not\succ D_j$ (i.e. either $I_i \cap I_j = \emptyset$ or $D_i \succ D_j$ and $D_i \neq D_j$) either (i) or (ii) holds:

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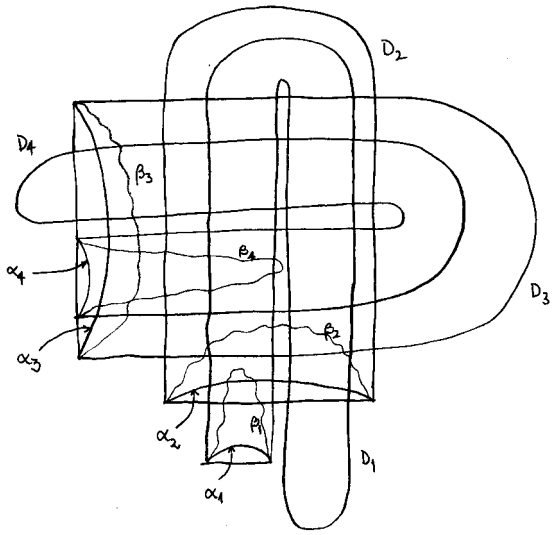


FIGURE 13. $\{\alpha_1, \alpha_2\}$ is a (ε, e)-collection and $\{\alpha_3, \alpha_4\}$ is a (ε, c)-collection, both compatible with $\{(D_i, \beta_i)\}_{i=1}^4$.

(i) if $C_i^\circ \cap \overline{I_i \cap I_j} \neq \emptyset$ then $[I^c(\alpha_i) \cup \alpha_i] \subset I^c(\beta_j)$;
(ii) otherwise $[I^c(\alpha_i) \cup \alpha_i] \cap D_j = \emptyset$.

The collection $\{\alpha_i\}_{i\in S}$ is called a (ε, e)-*collection compatible with* $\{(D_i, \beta_i)\}_{i=1}^L$ if $\alpha_i \subset I^e(\beta_i) \cap V_\varepsilon(E_i)$ and for every $i \in S$ and $j \in \underline{L}$ such that $D_i \not\succ D_j$ either (iii) or (iv) holds:
(iii) if $E_i^\circ \cap \overline{I_i \cap I_j} \neq \emptyset$ then $[I^e(\alpha_i) \cup \alpha_i] \subset I^e(\beta_j)$;
(iv) otherwise $[I^e(\alpha_i) \cup \alpha_i] \cap D_j = \emptyset$.

Remarks. Notice that if $\{\alpha_i\}_{i\in S}$ is a (ε, c)-collection compatible with $\{(D_i, \beta_i)\}$ and $\{\gamma_i\}_{i\in S}$ is a collection of open cross-cuts joining the vertices of D_i and such that $\gamma_i \subset I^c(\alpha_i)$, $\{\gamma_i\}_{i\in S}$ is also a (ε, c)-collection compatible with $\{(D_i, \beta_i)\}$. If moreover $\gamma_i \subset V_{\varepsilon'}(C_i)$ then $\{\gamma_i\}_{i\in S}$ is a (ε', c)-collection. The analogous statement holds true for (ε, e)-collections.

Warning. As the reader may have already noticed, statements about c-'things' and e-'things' are 'dual' to one another and most proofs are totally analogous in both cases. We will henceforward, whenever there is nothing essentially different between the two, present only the 'c-proof' without further comments.

We still have to show that (ε, c)- and (ε, e)-collections exist. Let us start showing that one element collections exist.

PROPOSITION 3.11. *Let $\{(D_i, \beta_i)\}_{i=1}^L$ be a cut collection, ε a positive number and assume $D_1 \not\succ D_i$ for every $i \in \{2, 3, \ldots, L\}$. Then there exists an open cross-cut $\alpha_1 \subset D_1$ joining vertices such that $\{\alpha_1\}$ is a (ε, c)-collection compatible with $\{(D_i, \beta_i)\}_{i=1}^L$ as defined above. The statement remains true if we write $D_1 \not\succ D_i$ and (ε, e)-collection in place of $D_1 \not\succ D_i$ and (ε, c)-collection respectively.*

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Proof. Divide the disks D_i into two groups: (i) those for which $C_1 \cap \overline{I_1 \cap I_i} \neq \emptyset$ and (ii) those for which $C_1 \cap \overline{I_1 \cap I_i} = \emptyset$. If D_i is in group (i), $I_1 \cap I_i \neq \emptyset$, so that by our assumption $D_1 \succ D_i$ and, by Proposition 3.8, $D_1, D_i|_{C_1}$. Clearly $D^c(\beta_1), D_1|_{C_1}$ and $D^c(\beta_i), D_i|_{C_i}$ and, since $C_1 \subset C_i$, we see that $D^c(\beta_1), D_1|_{C_1}$, $D_1, D_i|_{C_1}$ and $D_i, D^c(\beta_i)|_{C_i}$, by Proposition 3.4, imply that $D^c(\beta_1), D^c(\beta_i)|_{C_1}$ for every D_i in group (i). It now follows from Proposition 3.3 that there exists an open cross-cut $\alpha \subset I^c(\beta_1) \cap V_\varepsilon(C_1)$ such that $[I^c(\alpha) \cup \alpha] \subset I^c(\beta_i)$.

On the other hand, for the disks D_j in group (ii), $C_1 \cap \overline{I_1 \cap I_j} = \emptyset$ and since $I^c(\alpha) \subset I_1$, it is also the case that $C_1 \cap \overline{I^c(\alpha) \cap I_j} = \emptyset$. Thus, by Proposition 3.5 there exists an open cross-cut α_1 in $I^c(\alpha)$ such that for every D_j in group (ii), $[I^c(\alpha_1) \cup \alpha_1] \cap D_j = \emptyset$. It is clear that such α_1 also satisfies $[I^c(\alpha_1) \cup \alpha_1] \subset I^c(\beta_i)$ for every D_i in group (i).

In the event that all the disks belong to one or the other of the groups, the modifications necessary in the above proof are minor and are left to the reader. □

Definition 3.12. Let $\{D_i\}$ be a collection of (c, e)-disks related by $\succ$. We say D_i and D_j are c-equivalent, and write $D_i \sim_c D_j$, if there exists D_k in the collection such that $C_i, C_j \subset C_k$ and $D_i, D_k|_{C_i}$ and $D_j, D_k|_{C_j}$. We define e-equivalence analogously by changing c-sides to e-sides above, and denote it by $\sim_e$.

Remark. Notice that by this definition, $D_i \sim_c D_j$ if $C_i \subset C_j$ and $D_i, D_j|_{C_j}$ or *vice versa* and analogously for $\sim_e$.

PROPOSITION 3.13. *The relations $\sim_c$ and $\sim_e$ defined above are equivalence relations. If the collection $\{D_i\}_{i=1}^{L}$ is finite, each equivalence class for $\sim_c$ ($\sim_e$) has a distinguished representative whose c-(e-)side contains the c-(e-)sides of all other disks in its c-(e-)equivalence class. Moreover, in each c-(e-)equivalence class the c-(e-)sides are unlinked in the c-side of its distinguished representative.*

Proof. The proof is easy and we omit it. □

Standing convention. If the lower index in an indexed union or collection is larger than the upper one we will take the union or collection to be empty, so that $\bigcup_{-n+1}^{n-1} f^k(P) = \emptyset$ when $n = 0$. Also, recall that a bar under a positive integer denotes the set of all positive integers smaller than or equal to it: $\underline{L} = \{1, 2, \ldots, L\}$. If $L = 0$ we take $\underline{L}$ to be the empty set as well.

We now go on to prove the existence of (ε, c)- and (ε, e)-collections (see Figure 14).

PROPOSITION 3.14. *Let $\{(D_i(k), \beta_i(k));\ k = -1, 0, 1 \text{ and } i \in \underline{L(k)}\}$ (where $L(k)$ is a non-negative integer for each $k = -1, 0, 1$) be a cut collection such that if $k < l$ then $D_i(k) \not\succ D_j(l)$ for $i \in \underline{L(k)}$ and $j \in \underline{L(l)}$. Then given $\varepsilon, \delta > 0$ there exist a (δ, e)-collection $\{\alpha_i(-1) \subset D_i(-1)\}_{i=1}^{L(-1)}$ and a (ε, c)-collection $\{\alpha_j(1) \subset D_j(1)\}_{j=1}^{L(1)}$ both compatible with $\{(D_i(k), \beta_i(k));\ k = -1, 0, 1 \text{ and } i \in \underline{L(k)}\}$.*

Proof. (See remark before the statement.) We may assume, without loss of generality, that the distinguished representatives in the c-equivalence classes among $\{D_i(1);\ i \in \underline{L(1)}\}$ are the first n disks $D_1(1), \ldots, D_n(1)$. For each $i \in \underline{n}$ consider the cut collection

$$\{(D_j(k), \beta_j(k));\ k = -1, 0, 1,\ j \in \underline{L(k)}, D_j(k) \not\succ D_i(1)\} \cup \{(D_i(1), \beta_i(1))\}.$$

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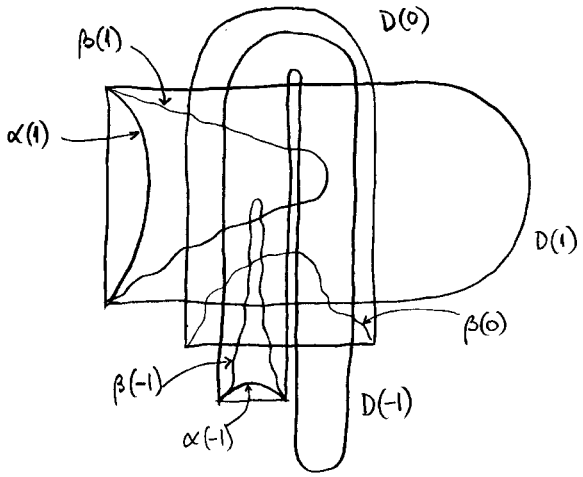


FIGURE 14. $\{\alpha(1)\}$ is the (ε, c)-collection and $\{\alpha(-1)\}$ is the (ε, e)-collection, both compatible with $\{(D(k), \beta(k); k = -1, 0, 1\}$.

By Proposition 3.11 there exists an open cross-cut $\alpha_i(1) \subset I^c(\beta_i(1)) \cap V_\varepsilon(C_i(1))$ satisfying (i) and (ii) of that proposition (with $\alpha_i(1)$ in place of α_1). We do the same for every $i \in \underline{n}$ obtaining $\{\alpha_i(1)\}_{i=1}^n$. These cross-cuts clearly satisfy (i) and (ii) in the definition of (ε, c)-collections and $\alpha_i(1) \subset I^c(\beta_i(1)) \cap V_\varepsilon(C_i(1))$ by construction. In order to see they are disjoint, let $i, j \in \underline{n}$. If $I_i(1) \cap I_j(1) = \emptyset$, $\alpha_i(1) \cap \alpha_j(1) = \emptyset$ since $\alpha_i(1) \subset I_i(1)$ and $\alpha_j(1) \subset I_j(1)$. If $I_i(1) \cap I_j(1) \neq \emptyset$, then either $D_i(1) \succ D_j(1)$ or $D_j(1) \succ D_i(1)$, say, $D_i(1) \succ D_j(1)$. It follows that $C_i^\circ(1) \cap \overline{I_i(1) \cap I_j(1)} = \emptyset$ for otherwise $C_i(1) \subset C_j(1)$ and $D_i(1), D_j(1)|_{C_i(1)}$, which goes against our assumption that $C_i(1)$ was the distinguished representative in its c-equivalence class. From this we can conclude that $[I^c(\alpha_i(1)) \cup \alpha_i(1)] \cap D_j = \emptyset$ and thus that $\alpha_i(1) \cap \alpha_j(1) = \emptyset$. Indeed we have shown more, namely that

$$[I^c(\alpha_i(1)) \cup \alpha_i(1)] \cap [I^c(\alpha_j(1)) \cup \alpha_j(1)] = \emptyset$$

for any $i, j \in \underline{n}$.

We now look at the disks in one c-equivalence class. By Proposition 3.13 the c-sides of the elements in the class are unlinked in the c-side of its distinguished representative, $D_i(1)$ say. By Proposition 3.7 it is possible to find disjoint open cross-cuts $\alpha_j(1) \subset I^c(\alpha_i(1))$ joining the endpoints of $C_j(1)$ such that $\alpha_j(1) \subset I^c(\beta_j(1)) \cap V_\varepsilon(C_j(1))$ for every j such that $D_j \sim_c D_i$. Doing this for each c-equivalence class we find a collection of disjoint open cross-cuts $\{\alpha_i(1)\}_{i=1}^{L(1)}$ satisfying the conditions in the definition of a (ε, c)-collection compatible with $\{(D_i(k), \beta_i(k))\}$. □

Before we proceed, let us say a word about finite partially ordered sets. If $(X, \geq)$ is one such we define the set of *initial elements* of X to be

$$I(X) = \{x \in X;\ \forall y \in X,\ y \leq x \Longrightarrow y = x\}.$$

It is easy to see that if X is finite and non-empty, $I(X)$ is non-empty and that no two distinct elements in $I(X)$ are related by $\geq$. Now let $X_1 = I(X)$ and inductively set

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$X_n = I(X \setminus \bigcup_{i=1}^{n-1} X_i)$. From what we have said, X_n is non-empty if $X \setminus \bigcup_{i=1}^{n-1} X_i$ is non-empty. Since X is finite, there exists $n \geq 1$ such that $X_1, X_2, \ldots, X_n$ are all non-empty and for $m > n$, $X_m = \emptyset$. Clearly $X_1, \ldots, X_n$ is a partition of X and if X_i has s_i elements we can list the elements of $X = \{x_1, x_2, x_3, \ldots, x_L\}$ so that the first s_1 elements are those in X_1, the next s_2 elements are those in X_2 and so on. In this way the subscripts reflect the partial order in the sense that if $i < j$ then $x_i \not\succeq x_j$. Having said this we adopt the following notation.

Notation. We will use $\geq$ to denote the extension of $\succ$ to a partial order and keep $\succ$ to denote the binary relation as we defined previously.

Convention. Henceforth it will be assumed that the subscripts in a pruning collection reflect the partial order $\geq$ in the sense that if $i < j$ then $D_i \not\geq D_j$. Notice that, in particular, if $i < j$ then $D_i \not\succ D_j$.

We can now state a proposition containing one of the main ingredients in the proof of the main theorem (see Figure 2).

PROPOSITION 3.15. *Let $\{D_i\}_{i=1}^{L}$ be a pruning collection and $\{\varepsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers converging to zero. Then there exists a collection $\{\alpha_i(n) \subset f^n(D_i);\ i \in \underline{L},\ n \in \mathbb{Z}\}$ of disjoint open cross-cuts joining the vertices of $f^n(D_i)$ such that* (i) *and* (ii) *below hold:*

(i) *for each $n \geq 1$, $\{\alpha_i(n);\ i \in \underline{L}\}$ is a (ε_n, c)-collection compatible with*

$$\{(f^k(D_j),\ \alpha_j(k));\ j \in \underline{L},\ -n+1 \leq k \leq n-1\}$$
$$\cup \{(f^n(D_j),\ f(\alpha_j(n-1)));\ j \in \underline{L}\};$$

(ii) *for each $m \leq 0$, $\{\alpha_i(m);\ i \in \underline{L}\}$ is a $(\varepsilon_{|m|}, e)$-collection compatible with*

$$\{(f^k(D_j), \alpha_j(k));\ j \in L, m+1 \leq k \leq -m+1\}$$
$$\cup \{(f^m(D_j),\ f^{-1}(\alpha_j(m+1)));\ j \in \underline{L}\}.$$

Proof. We will use the following easy lemma.

LEMMA 3.16. *If $\{\alpha_i\}_{i\in S}$ is a (ε, c)-collection compatible with the cut collection $\{(D_i, \beta_i)\}_{i=1}^{L}$ and $\{\beta_i' \subset D_i\}_{i=1}^{L}$ is a collection of cross-cuts such that $\beta_i' \subset D^e(\beta_i)$ for each $i \in \underline{L}$ then $\{\alpha_i\}_{i\in S}$ is also compatible with $\{(D_i, \beta_i')\}_{i=1}^{L}$. If above we change (ε, c)-to (ε, e)- and $D^e(\beta_i)$ to $D^c(\beta_i)$ the resulting statement is also true.*

We will let $m = -n + 1$ and use induction on n. In order to prove the proposition for $n = 1$, choose any collection $\{\beta_i \subset D_i\}_{i=1}^{L}$ of open cross-cuts joining vertices and apply Proposition 3.14 with $L(0) = 0$ (so that $\underline{L(0)} = \emptyset$ and $\{(D_i(0), \beta_i(0))\} = \emptyset$) to the cut collection

$$\mathcal{D} = \{(D_i, \beta_i);\ i \in \underline{L}\} \cup \{(f(D_i),\ f(\beta_i));\ i \in \underline{L}\}$$

where $\{(D_i, \beta_i)\}$ and $\{(f(D_i), f(\beta_i))\}$ play the roles of $\{(D_i(-1), \beta_i(-1))\}$ and $\{(D_i(1),\ \alpha_i(1))\}$ respectively in the statement of that proposition, whereas $\varepsilon = \varepsilon_1$ and $\delta = \varepsilon_0$. By the definition of pruning collection, $f(D_i) \not\succ D_j$ for any $i, j \in \underline{L}$ so that $\mathcal{D}$ satisfies the hypotheses and we can conclude there exist $\{\alpha_i(1)\}_{i=1}^{L}$ and $\{\alpha_i\}_{i=1}^{L}$ a (ε_1, c)-

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and a (ε_0, e)-collection respectively, both compatible with $\mathcal{D}$. Since $\alpha_i \subset I^e(\beta_i)$ and therefore $f(\alpha_i) \subset I^e(f(\beta_i))$, by Lemma 3.16, $\{\alpha_i(1)\}_{i=1}^L$ is a (ε_1, c)-collection compatible with

$$\{(D_i, \alpha_i);\ i \in \underline{L}\} \cup \{(f(D_i), f(\alpha_i));\ i \in \underline{L}\}.$$

By the same token $\{\alpha_i\}_{i=1}^L$ is a (ε_0, e)-collection compatible with

$$\{(D_i, f^{-1}(\alpha_i(1)));\ i \in \underline{L}\} \cup \{(f(D_i), a_i(1)); i \in \underline{L}\}.$$

That $\alpha_i(1) \cap \alpha_j = \emptyset$ for $i, j \in \underline{L}$ is a consequence of Lemma 3.16. This proves the proposition for $n = 1,\ m = 0$.

Assume we have constructed a collection $\{\alpha_i(k);\ i \in \underline{L},\ -n+2 \le k \le n-1\}$ of disjoint open cross-cuts satisfying the conclusions of the proposition. Consider the cut collection

$$\begin{aligned}\mathcal{D} = &\{(f^n(D_i),\ f(\alpha_i(n-1)));\ i \in \underline{L}\}\\ &\cup \{(f^k(D_i), \alpha_i(k));\ i \in \underline{L},\ -n+2 \le k \le n-1\}\\ &\cup \{(f^{-n+1}(D_i), f^{-1}(\alpha_i(-n+2)));\ i \in \underline{L}\}\end{aligned}$$

and apply Proposition 3.14 with $\{(D_i(1), \beta_i(1))\}$, $\{(D_i(0), \alpha_i(0))\}$ and $\{(D_i(-1), \alpha_i(-1))\}$ equal to the first, second and third collections respectively, in the above union, letting $\varepsilon = \varepsilon_n$ and $\delta = \varepsilon_{|-n+1|}$. From the definition of pruning collection, $f^n(D_i) \not\prec f^k(D_j)$ for any $k < n$ and any $i, j \in \underline{L}$ and $f^{-n+1}(D_i) \not\succ f^k(D_j)$ for any $k > -n+1$ and any $i, j \in \underline{L}$, so that the hypotheses of the proposition are satisfied. We may then conclude there exist $\{\alpha_i(n) \subset f^n(D_i)\}_{i=1}^L$ and $\{\alpha_i(-n+1) \subset f^{-n+1}(D_i)\}_{i=1}^L$ a (ε_n, c)- and a $(\varepsilon_{|-n+1|}, e)$-collection respectively, both compatible with $\mathcal{D}$. From Lemma 3.16, $\{\alpha_i(n)\}_{i=1}^L$ is compatible with

$$\begin{aligned}&\{(f^k(D_i), \alpha_i(k));\ i \in \underline{L},\ -n+1 \le k \le n-1\}\\ &\quad \cup \{(f^n(D_i),\ f(\alpha_i(n-1)));\ i \in \underline{L}\}\end{aligned}$$

and $\{\alpha_i(-n+1)\}_{i=1}^L$ is compatible with

$$\begin{aligned}&\{(f^k(D_i), \alpha_i(k));\ i \in \underline{L},\ -n+2 \le k \le n\}\\ &\quad \cup \{(f^{-n+1}(D_i),\ f^{-1}(\alpha_i(-n+2)));\ i \in \underline{L}\}.\end{aligned}$$

That $\alpha_i(n) \cap \alpha_j(k) = \emptyset$ for $-n+1 \le k \le n-1$ and $\alpha_i(-n+1) \cap \alpha_j(k) = \emptyset$ for $-n+2 \le k \le n$ is a consequence of Lemma 3.16. This finishes the induction step and proves the proposition. □

COROLLARY 3.17. *With the notation of Proposition 3.15, for every* $n \in \mathbb{Z}$, $\alpha_i(n) \subset I^c(f(\alpha_i(n-1)))$ *and* $\alpha_i(n) \subset I^e(f^{-1}(\alpha_i(n+1)))$.

The next proposition is but a 'fattened' version of Proposition 3.15 (see Figure 15). We could have proven it together with Proposition 3.15 had we stated the 'fattened' versions of the propositions we proved before. Although feasible, this would have been rather cumbersome. It is also possible to give a direct proof using the techniques we have used so far. We leave it to the interested reader.

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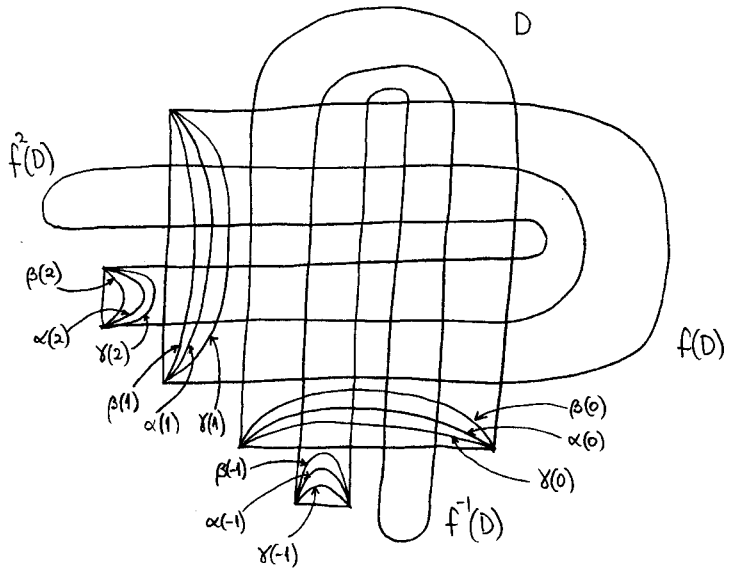


FIGURE 15. The first few $\gamma(n)$'s and $\beta(n)$'s for a pruning collection containing only one (c, e)-disk D.

PROPOSITION 3.18. *Let $\{\alpha_i(n);\ i \in \underline{L},\ n \in \mathbb{Z}\}$ be as in Proposition 3.15. Then there exist collections of disjoint open cross-cuts $\{\beta_i(n) \subset f^n(D_i);\ i \in \underline{L},\ n \in \mathbb{Z}\}$ and $\{\gamma_i(n) \subset f^n(D_i);\ i \in \underline{L},\ n \in \mathbb{Z}\}$ joining vertices such that:*

(i) *$\beta_i(n) \subset I^c(\alpha_i(n))$ and $\gamma_i(n) \subset I^e(\alpha_i(n))$;*

(ii) *for $n \geq 1$, $\{\gamma_i(n);\ i \in \underline{L}\}$ is a (ε_n, c)-collection compatible with*

$$\{(f^k(D_i),\ \beta_i(k));\ i \in \underline{L},\ -n+1 \leq k \leq n-1\}$$
$$\cup\, \{(f^n(D_i),\ f(\beta_i(n-1)));\ i \in \underline{L}\};$$

(iii) *for $m \leq 0$, $\{\beta_i(m);\ i \in L\}$ is a $(\varepsilon_{|m|}, e)$-collection compatible with*

$$\{(f^k(D_i), \gamma_i(k));\ i \in \underline{L},\ m+1 \leq k \leq -m+1\}$$
$$\cup\, \{(f^m(D_i),\ f^{-1}(\gamma_i(m+1)));\ i \in \underline{L}\}.$$

COROLLARY 3.19. *With the notation of Proposition 3.18, for every $n \in \mathbb{Z}$, $\gamma_i(n) \subset I^c(f(\beta_i(n-1)))$ and $\beta_i(n) \subset I^e(f^{-1}(\gamma_i(n+1)))$.*

The next proposition creates the sets in whose union will lie the support of the isotopy we will construct to prove the main theorem.

PROPOSITION 3.20. *Let $\{\alpha_i(n)\}$, $\{\beta_i(n)\}$ and $\{\gamma_i(n)\}$ be as in Propositions 3.15 and 3.18. Then for every $n \in \mathbb{Z}$ and $i \in \underline{L}$, $f^{-1}(\beta_i(n+1)) \cup \gamma_i(n)$ is a Jordan curve bounding a Jordan domain $\mathcal{V}_i(n)$ such that $\mathcal{V}_i(n) \supset f^{-1}(\alpha_i(n+1)) \cup \alpha_i(n)$. Moreover, $\mathcal{V}_i(n) = I^c(\gamma_i(n)) \cap I^e(f^{-1}(\beta_i(n+1)))$.*

Proof. The proof is an easy exercise using (i) of Proposition 3.18, Corollary 3.19 and Proposition 3.8 (see Figure 16). □

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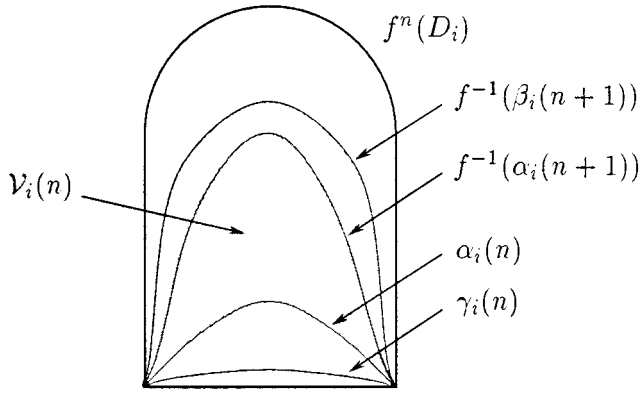


FIGURE 16. $\overline{f^{-1}(\beta_i(n+1)) \cup \gamma_i(n)}$ is a Jordan curve bounding the domain $\mathcal{V}_i(n)$.

PROPOSITION 3.21. *Let $i, j \in \underline{L}$ and $n, k \in \mathbb{Z}$.*
(i) *If $f^k(D_j) \not\asymp f^n(D_i)$, then $f^{-1}(\alpha_j(k+1)) \cap \overline{\mathcal{V}_i(n)} = \emptyset$.*
(ii) *If $f^k(D_j) \not\asymp f^n(D_i)$ then $\alpha_j(k) \cap \overline{\mathcal{V}_i(n)} = \emptyset$.*

Proof. From Proposition 3.18(i) and Proposition 3.8 it follows that $\alpha_i(n) \subset I^e(\beta_i(n)) \cap I^c(\gamma_i(n))$. From Proposition 3.18 it also follows that $\beta_j(k) \cap \gamma_i(n) = \emptyset$ for any $i, j \in \underline{L}$, $k, n \in \mathbb{Z}$. Assume $f^k(D_j) \not\asymp f^n(D_i)$. By Proposition 3.8, we see that $I^e(\beta_j(k)) \cap I^c(\gamma_i(n)) = \emptyset$. Since $\mathcal{V}_i(n) \subset I^c(\gamma_i(n))$, $\alpha_j(k) \subset I^e(\beta_j(k))$ and $I^e(\beta_j(k))$ is open we can conclude that $\overline{\mathcal{V}_i(n)} \cap \alpha_j(k) = \emptyset$. This proves (ii). In order to prove (i) assume $f^k(D_j) \not\asymp f^n(D_i)$. Then $f^{k+1}(D_j) \not\asymp f^{n+1}(D_i)$ and, as above, we can conclude that $I^e(\beta_i(n+1)) \cap I^c(\gamma_j(k+1)) = \emptyset$. It follows that

$$I^e(f^{-1}(\beta_i(n+1))) \cap I^c(f^{-1}(\gamma_j(k+1))) = \emptyset$$

and since $\mathcal{V}_i(n) \subset I^e(f^{-1}(\beta_i(n+1)))$, then

$$f^{-1}(\alpha_j(k+1)) \subset I^c(f^{-1}(\gamma_j(k+1))).$$

The latter being an open set, we see that $f^{-1}(\alpha_j(k+1)) \cap \overline{\mathcal{V}_i(n)} = \emptyset$. This completes the proof. □

PROPOSITION 3.22. *For $k \geq 1$, $f^k(C_i)$ and $f^{-k}(E_i)$ are disjoint from $\mathcal{V}_j(n)$ for every $i, j \in \underline{L}$ and every $n \in \mathbb{Z}$.*

Proof. The proof is not hard and is omitted. □

3.3. *Isotopies.* In what follows we will use a slightly unusual definition of *support* in that our supports will be *open* sets.

Definition 3.23. Let X, Y be topological spaces. By an *isotopy* we mean a continuous map $H : X \times [0,1] \to Y$ such that the 'slice' map $H_t : X \to Y$, $H_t(x) = H(x,t)$ is a homeomorphism for each $t \in [0,1]$. The *support* of an isotopy $H : X \times [0,1] \to Y$ is by definition the set $\operatorname{supp} H = \mathcal{C}\{x \in X;\ H(x,t) = H(x,0)\ \forall t \in [0,1]\}$ where, as usual, $\mathcal{C}$

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stands for complement. If $\xi : X \to X$ is a homeomorphism we define the support of ξ as $\operatorname{supp}\xi = \mathcal{C}\{x \in X;\ \xi(x) = x\}$.

Definition 3.24. Let G be a collection of subsets of a metric space. We call G a *null collection* if for every $\varepsilon > 0$ only finitely many elements of G have diameter greater than ε.

The proposition below is true in greater generality than we state and is part of the folklore of hyperbolic geometry, geodesic laminations, etc. For a proof, see [**dC**].

PROPOSITION 3.25. *Let J be a Jordan curve and $\{\alpha_n\}$ and $\{\beta_n\}$ two countable null collections of interior disjoint cross-cuts in D, the closed disk bounded by J. Assume that no two elements of each collection share both endpoints and that the endpoints of α_i and β_i coincide. Then there exists a homeomorphism $\zeta : D \to D$ such that $\zeta|_J$ is the identity, $\zeta(\alpha_n) = \beta_n$ and ζ is isotopic to the identity through an isotopy with support in I, the interior of D.*

COROLLARY 3.26. *Let J be a Jordan curve and α, β cross-cuts in D having the same endpoints. Then there exists an isotopy of the identity taking α to β with support in I.*

Notation. Let D_1, D_2 be closed disks, $D_1 \subset D_2$ and $D_1, D_2|_L$, where $L \subset \partial D_1 \cap \partial D_2$ is an arc. If $D_1 \setminus L \subset I_2$, the interior of D_2, we will write $D_1 \subset D_2|_L$.

PROPOSITION 3.27. *Let $\psi : D \to D$ be a homeomorphism onto its image so that $\psi(D) \subset D|_{\psi(L)}$, where L is a closed arc, $\psi(L) \subset L$ and $p \in L$ is a fixed point such that $\psi^n(x) \to p$ for every $x \in L$. Then there exists an isotopy $h : D \times [0,1] \to D$ of the identity such that $h|_{\partial D} = id$ and if $\zeta(\cdot) = h(\cdot, 1)$, then $(\psi \circ \zeta)^n(x) \to p$ for every $x \in D$.*

Proof. We will construct a null collection $\{\alpha_n\}_{n=1}^{\infty}$ of disjoint open cross-cuts in I with the following properties:

(i) α_n has the same endpoints as $\psi^n(L)$;

(ii) if I_{n-1} is the Jordan domain bounded by $\alpha_{n-1} \cup \psi^{n-1}(L)$, α_n is a cross-cut in $I_{n-1} \cap \psi(I_{n-1})$, for $n \geq 2$;

(iii) $\alpha_n \subset V_{1/n}(\psi^n(L))$.

Set $\alpha_1 = \psi(\partial D \setminus L)$ and $D_1 = \psi(D)$. Notice that $D_1 \subset D|_{\psi(L)}$ implies $\psi(D_1) \subset D_1|_{\psi^2(L)}$. By Proposition 3.7, it is possible to find $\alpha_2 \subset \psi(I_1) \cap I_1 = \psi(I_1)$, an open cross-cut joining the endpoints of $\psi^2(L)$ such that $\alpha_2 \subset V_{1/2}(\psi^2(L))$.

Assume we have constructed $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying (i), (ii) and (iii) above. Since $\alpha_n \subset \psi(I_{n-1}) \cap I_{n-1}$, and α_n has the same endpoints as $\psi^n(L)$, $D_n \subset \psi(D_{n-1})|_{\psi^n(L)}$ and $D_n \subset D_{n-1}|_{\psi^n(L)}$. The latter implies that $\psi(D_n) \subset \psi(D_{n-1})|_{\psi^{n+1}(L)}$ and since $\psi^{n+1}(L) \subset \psi^n(L)$, by Proposition 3.4, it follows that $D_n, \psi(D_n)|_{\psi^{n+1}(L)}$. By Proposition 3.7, there exists $\alpha_{n+1} \subset I_n \cap \psi(I_n)$ an open cross-cut with the same endpoints as $\psi^{n+1}(L)$ such that $\alpha_{n+1} \subset V_{1/(n+1)}(\psi^{n+1}(L))$. By induction, we construct the collection $\{\alpha_n\}_{n=1}^{\infty}$. That $\{\alpha_n\}_{n=1}^{\infty}$ is a null collection follows from the fact that $\alpha_n \subset V_{1/n}(\psi^n(L))$ and diam $\psi^n(L) \to 0$. That the α_n's are disjoint is clear since $\alpha_n \subset I_{n-1}$ for every $n \geq 1$. Notice also that no two α_n's share both endpoints. This is so because the endpoints of α_n are the same as those of $\psi^n(L)$ and if $\psi^n(L)$ and $\psi^m(L)$ shared both endpoints, L would contain more than one fixed point.

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Let $\beta_n = \psi^{-1}(\alpha_{n+1})$. The collection $\{\beta_n\}_{n=1}^{\infty}$ is clearly a null collection of disjoint open cross-cuts no two of which share both endpoints. Also, for each $n \geq 1$, α_n and β_n have the same endpoints. By Proposition 3.25 there exists an isotopy of the identity $h : D \times [0,1] \to D$ such that if $\zeta(\cdot) = h(\cdot, 1)$, $\zeta(\alpha_n) = \beta_n$. Then $\psi \circ \zeta(\alpha_n) = \psi(\beta_n) = \alpha_{n+1}$ and since $\psi(\psi^n(L)) = \psi^{n+1}(L)$ we see that $\psi \circ \zeta(D_n) = D_{n+1}$. But diam $D_n \to 0$ as $n \to \infty$ and therefore it follows that $(\psi \circ \zeta)^n(x) \to p$, $\forall x \in D$ as $n \to \infty$ as we wanted. □

COROLLARY 3.28. *For $i = 1, \ldots, n$ let D_i be closed disks with disjoint interiors and $L_i \subset \partial D_i$ a closed arc. Let $\psi : \pi \to \pi$ be a homeomorphism of the plane such that $\psi(L_i) \subset L_{i+1}$ and $\psi(D_i) \subset D_{i+1}|_{\psi(L_i)}$, where we let the indices 'wrap around', i.e. we set $n+1$ to be 1. Assume $\psi^n|_{D_1} : (D_1, L_1) \to (D_1, L_1)$ satisfies the hypotheses of Proposition 3.27. Then there exists an isotopy $h : \pi \times [0,1] \to \pi$ of the identity such that* $\operatorname{supp} h \subset D_1$ *and if $\zeta(\cdot) = h(\cdot, 1)$, $(\psi \circ \zeta)^{kn}(x) \to p$ as $k \to \infty$ for every $x \in D_1$ where $p \in L_1$ is the fixed point of $\psi^n|_{L_1}$.*

3.4. *The Proof of the main theorem.* In what follows $f : \pi \to \pi$ will be a uniformly continuous homeomorphism of the plane and $\{D_i\}_{i=1}^{L}$ a pruning collection for f. As we pointed out before, we may and will assume that the subscripts reflect the partial order $\geq$ in $\{D_i\}_{i=1}^{L}$, in the sense that, if $i > j$ then $D_i \not\leq D_j$. In particular, if $i > j$ then $D_i \not\prec D_j$.

Definition 3.29. For each $i \in L$ we define four numbers $n(i)$, $N(i)$, $m(i)$, $M(i) \in \mathbb{Z} \cup \{\pm\infty\}$ as follows:

(i) $n(i)$ is the smallest integer ≥ 1 such that $f^{n(i)}(D_i)$, $f(D_j)|_{f^{n(i)}(C_i)}$ for some $j \in \underline{L}$ or $n(i) = \infty$ if $f^k(D_i)$, $f(D_j) \not| f^k(C_i)$ for every $k \geq 1$ and $j \in \underline{L}$;

(ii) $N(i) = \lceil n(i)/2 \rceil$, i.e. the smallest integer greater than or equal to $n(i)/2$, if $n(i) < \infty$ or $N(i) = \infty$ if $n(i) = \infty$;

(iii) $m(i)$ is the largest integer less than or equal to zero such that $f^{m(i)}(D_i)$, $D_j|_{f^{m(i)}(E_i)}$ for some $j \in \underline{L}$ or $m(i) = -\infty$ if $f^k(D_i)$, $D_j \not| f^k(E_i)$ for every $k \leq 0$ and $j \in \underline{L}$;

(iv) $M(i) = \lceil m(i)/2 \rceil$ if $m(i) > -\infty$ or $M(i) = -\infty$ if $m(i) = -\infty$.

The following proposition is a straightforward consequence of the definitions and we omit the proof.

PROPOSITION 3.30. *If $n(i)$, $N(i)$, $m(i)$ and $M(i)$ are finite the following hold for each $i \in \underline{L}$:*

(i) *$n(i) = 2N(i) - \delta$ and $m(i) = 2M(i) - \delta'$, where $\delta, \delta' = 0$ or 1;*

(ii) *$f^{N(i)}(D_i)$, $f^{-N(i)+\delta+1}(D_j)|_{f^{N(i)}(C_i)}$ for some $j \in \underline{L}$ but for $-N(i) + \delta + 2 \leq k \leq N(i) - 1$, $f^{N(i)}(D_i)$, $f^k(D_j) \not| f^{N(i)}(C_i)$ for any $j \in \underline{L}$;*

(iii) *for $1 \leq n < N(i)$, $-n + 1 \leq k \leq n - 1$, $f^n(D_i)$, $f^k(D_j) \not| f^n(C_i)$ for any $j \in \underline{L}$;*

(iv) *$f^{M(i)}(D_i)$, $f^{-M(i)+\delta'}(D_j)|_{f^{m(i)}(E_i)}$ for some $j \in \underline{L}$ but for $M(i) + 1 \leq k \leq -M(i) + \delta' + 1$, $f^{M(i)}(D_i)$, $f^k(D_j) \not| f^{M(i)}(E_i)$ for any $j \in \underline{L}$;*

(v) *for $M(i) < m \leq 0$ and $m + 1 \leq k \leq -m + 1$, $f^m(D_i)$, $f^k(D_j) \not| f^m(E_i)$ for any $j \in \underline{L}$.*

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Moreover, for each $i \in \underline{L}$, $n(i) > 1$ *if and only if* D_i *is the distinguished representative in its c-equivalence class in* $\{D_i\}_{i=1}^{L}$. *Likewise,* $m(i) < 0$ *if and only if* D_i *is the distinguished representative in its e-equivalence class in* $\{D_i\}_{i=1}^{L}$.

We now start the construction of the isotopy for the proof of the main theorem. If the pruning collection contains only one (c, e)-disk D_1 and $N(1) = \infty$ and $M(1) = -\infty$, most of what is presented from here to the end of this subsection is very much simplified. We suggest that the reader concentrate on this case upon a first reading.

Recall (Proposition 3.20) that $\mathcal{V}_i(n) \subset f^n(I_i)$ is a Jordan domain containing $\alpha_i(n)$ and $f^{-1}(\alpha_i(n+1))$ as cross-cuts with the same endpoints. Using Corollary 3.26 construct, for each $i \in \underline{L}$ and $M(i) \le n < N(i)$, an isotopy $k_{i,n} : \pi \times [0,1] \to \pi$ of the identity such that $\operatorname{supp} k_{i,n} \subset \mathcal{V}_i(n)$ and $k_{i,n}(\alpha_i(n), 1) = f^{-1}(\alpha_i(n+1))$. If $n < M(i)$ or $n \ge N(i)$ we let $k_{i,n} \equiv$ identity. Set $\zeta_{i,n}(\cdot) = k_{i,n}(\cdot, 1)$. For $n \in \mathbb{Z}$ define

$$k_n(x,t) = \begin{cases} k_{1,n}(x, Lt), & t \in \left[0, \frac{1}{L}\right] \\ \zeta_{1,n}(k_{2,n}(x, Lt-1)), & t \in \left[\frac{1}{L}, \frac{2}{L}\right] \\ \zeta_{1,n} \circ \zeta_{2,n}(k_{3,n}(x, Lt-2)), & t \in \left[\frac{2}{L}, \frac{3}{L}\right] \\ \vdots & \vdots \\ \zeta_{1,n} \circ \zeta_{2,n} \circ \cdots \circ \zeta_{L-1,n}(k_{L,n}(x, Lt-L+1)), & t \in \left[\frac{L-1}{L}, 1\right] \end{cases}$$

and let $\zeta_n(\cdot) = k_n(\cdot, 1)$. Now let $r_0 = k_0$ and for $n \ge 1$

$$r_n(x,t) = \begin{cases} k_{-n}(x, 2t), & t \in [0, \frac{1}{2}] \\ \zeta_{-n}(k_n(x, 2t-1)), & t \in [\frac{1}{2}, 1] \end{cases}$$

and set $\rho_n(\cdot) = r_n(\cdot, 1)$ for $n \ge 0$.

Recall that the locus $\overline{P} = \bigcup_{i=1}^{L} D_i$ of a pruning collection $\{D_i\}_{i=1}^{L}$ was called a pruning front. We will denote the union of the interiors $\bigcup_{i=1}^{L} I_i$ by P.

PROPOSITION 3.31. *The isotopies* r_n *just defined have the following properties:*

(i) $\operatorname{supp} r_n \subset [f^n(P) \cup f^{-n}(P)] \setminus \bigcup_{-n+1}^{n-1} f^k(\overline{P})$ *for every* $n \ge 0$ *so that if* $n \ne m$, $\operatorname{supp} r_n \cap \operatorname{supp} r_m = \emptyset$;

(ii) *since* f *is uniformly continuous, the diameters of the connected components of* $\operatorname{supp} r_n$ *converge to zero as* $n \to \infty$;

(iii) *for each* $i \in \underline{L}$, *if* $n < N(i)$, $\rho_n(\alpha_i(n)) = f^{-1}(\alpha_i(n+1))$ *and if* $-n \ge M(i)$, $\rho_n(\alpha_i(-n)) = f^{-1}(\alpha_i(-n+1))$.

Proof. From the definition of k_n it is clear that for $n \in \mathbb{Z}$,

$$\operatorname{supp} k_n \subset \bigcup \{\mathcal{V}_i(n);\ M(i) \le n < N(i)\}$$

so that for $n \ge 0$,

$$\operatorname{supp} r_n \subset \bigcup \{\mathcal{V}_i(n);\ 1 \le n < N(i)\} \cup \bigcup \{\mathcal{V}_i(-n);\ M(i) \le -n \le 0\}$$

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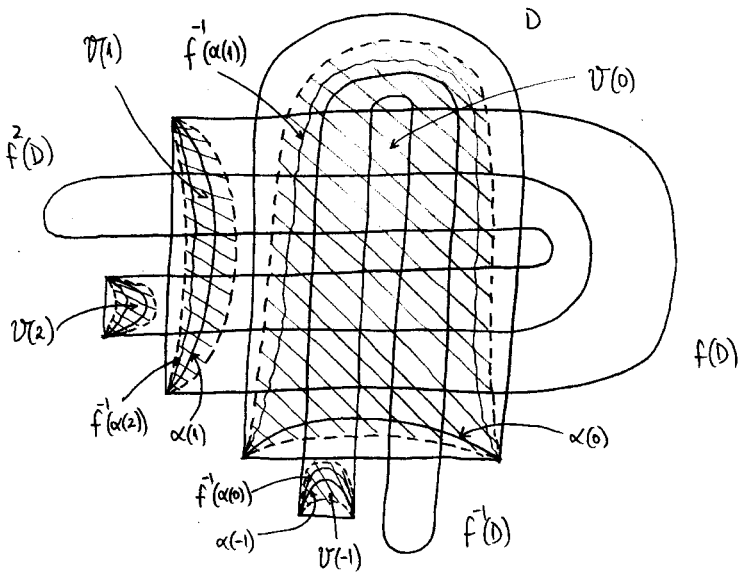


FIGURE 17. The first few $\mathcal{V}(n)$'s for a pruning collection with only one (c, e)-disk D.

and since $\mathcal{V}_i(n) \subset f^n(I_i)$, it is clear that

$$\operatorname{supp} r_n \subset f^n(P) \cup f^{-n}(P) = \bigcup_{i=1}^{L} f^n(I_i) \cup \bigcup_{i=1}^{L} f^{-n}(I_i).$$

There is nothing more to prove for $n = 0$ (recall that $\bigcup_1^{-1} f^k(P) = \emptyset$, by our convention) and we may assume that $n \geq 1$ (see Figure 17).

If $1 \leq n < N(i)$, it follows from Proposition 3.30 that $f^n(D_i),\ f^k(D_j) \not\pitchfork_{f^n(C_i)}$ for any $j \in \underline{L}$ and $-n+1 \leq k \leq n-1$. Since $\{\gamma_i(n)\}_{i=1}^{L}$ is a (ε_n, c)-collection compatible with

$$\{(f^k(D_j), \beta_j(k)) :\ j \in \underline{L},\ -n+1 \leq k \leq n-1\},$$

by Proposition 3.18, we must have $[I^c(\gamma_i(n)) \cup \gamma_j(n)] \cap f^k(D_j) = \emptyset$ for every $j \in \underline{L}$ and $-n+1 \leq k \leq n-1$. But $\mathcal{V}_i(n) \subset I^c(\gamma_i(n))$ by Proposition 3.20 and taking the union over $j \in \underline{L}$ and $-n+1 \leq k \leq n-1$ we see that $\mathcal{V}_i(n) \cap \bigcup_{-n+1}^{n-1} f^k(\overline{P}) = \emptyset$ from which it follows that

$$\bigcup \{\mathcal{V}_i(n);\ 1 \leq n < N(i)\} \cap \bigcup_{-n+1}^{n-1} f^k(\overline{P}) = \emptyset.$$

If $M(i) < m \leq 0$, it follows from Proposition 3.30 that $f^m(D_i),\ f^k(D_j) \not\pitchfork_{f^m(E_i)}$ for any $j \in \underline{L}$ and $m+1 \leq k \leq -m+1$. Again by Proposition 3.18 $\{\beta_i(m)\}_{i=1}^{L}$ is a $(\varepsilon_{|m|}, e)$-collection compatible with

$$\{(f^k(D_j), \beta_j(k));\ j \in \underline{L},\ m+1 \leq k \leq -m+1\},$$

which implies that $[I^e(\beta_i(m)) \cup \beta_i(m)] \cap f^k(D_j) = \emptyset$ for every $j \in \underline{L}$ and $m+1 \leq k \leq -m+1$, and thus that $[I^e(f^{-1}(\beta_i(m))) \cup f^{-1}(\beta_i(m))] \cap f^k(D_j) = \emptyset$ for every $j \in \underline{L}$ and

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$m \le k \le -m$. Letting $m = -n+1$ and noticing that $\mathcal{V}_i(-n) \subset I^e(f^{-1}(\beta_i(-n+1)))$, by Proposition 3.20, what we have just seen implies that for $M(i) \le -n < 0$, $\mathcal{V}_i(-n) \cap \bigcup_{-n+1}^{n-1} f^k(\overline{P}) = \emptyset$ from which it follows that

$$\bigcup \{\mathcal{V}_i(-n);\ M(i) \le -n < 0\} \cap \bigcup_{-m+1}^{n-1} f^k(\overline{P}) = \emptyset.$$

This finishes the proof of (i).

In order to prove (ii) notice that $\operatorname{supp} r_n \subset \bigcup_{i=1}^{L} \{\mathcal{V}_i(n) \cup \mathcal{V}_i(-n)\}$ and from Propositions 3.18 and 3.20, for $n \ge 1$, $\mathcal{V}_i(n) \subset I^c(\gamma_i(n)) \subset V_{\epsilon_n}(f^n(C_i))$ and

$$\mathcal{V}_i(-n) \subset I^e(f^{-1}(\beta_i(-n+1))) = f^{-1}(I^e(\beta_i(-n+1))) \subset f^{-1}(V_{\epsilon_{n-1}}(f^{-n+1}(E_i))).$$

From the (c,e)-dynamic assumption, $\operatorname{diam} f^n(C_i) \to 0$ as $n \to \infty$ and $\operatorname{diam} f^m(E_i) \to 0$ as $m \to -\infty$. Since $\epsilon_n \to 0$, it is clear that $\operatorname{diam} \mathcal{V}_i(n) \to 0$, as $n \to \infty$ and from the uniform continuity of f we can also conclude that $\operatorname{diam} \mathcal{V}_i(-n) \to 0$ as $n \to \infty$. It is now easy to see that the connected components of $\bigcup_{i=1}^{L} \{\mathcal{V}_i(n) \cup \mathcal{V}_i(-n)\}$ have diameters converging to zero as $n \to \infty$. This proves (ii).

Let us now look at (iii). From the way we indexed the pruning collection, if $i > j$, $D_i \not< D_j$ which implies $f^n(D_i) \not< f^n(D_j)$ for any $n \in \mathbb{Z}$. From Proposition 3.21 it follows that $f^{-1}(\alpha_i(n+1)) \cap \mathcal{V}_j(n) = \emptyset$. Similarly, if $l > i$ the same proposition implies that $\alpha_i(n) \cap \mathcal{V}_l(n) = \emptyset$. Since each $k_{i,n}$ is an isotopy of the identity with support contained in $\mathcal{V}_i(n)$, and $\zeta_{i,n}(\cdot) = k_{i,n}(\cdot, 1)$, we have $\operatorname{supp} \zeta_{i,n} \subset \mathcal{V}_i(n)$ and, from what we said above, we see that if $j < i$, $\zeta_{j,n}(f^{-1}(\alpha_i(n+1))) = f^{-1}(\alpha_i(n+1))$ and that if $l > i$, $\zeta_{l,n}(\alpha_i(n)) = \alpha_i(n)$. Thus, for any $M(i) \le n < N(i)$

$$\begin{aligned}
\zeta_n(\alpha_i(n)) &= \zeta_{1,n} \circ \cdots \circ \zeta_{i,n} \circ \cdots \circ \zeta_{L,n}(\alpha_i(n)) \\
&= \zeta_{1,n} \circ \cdots \circ \zeta_{i,n}(\alpha_i(n)) \\
&= \zeta_{1,n} \circ \cdots \circ \zeta_{i-1,n}(f^{-1}(\alpha_i(n+1))) \\
&= f^{-1}(\alpha_i(n+1)).
\end{aligned}$$

From the definition of pruning collections, $f^{-n}(D_i) \not< f^n(D_j)$ for any $n \ge 1$ and any $i, j \in \underline{L}$ and, by Proposition 3.21, it follows that $f^{-1}(\alpha_i(n+1)) \cap \mathcal{V}_j(-n) = \emptyset$ and $\mathcal{V}_i(n) \cap \alpha_j(-n) = \emptyset$. Thus we can conclude that for any $i \in \underline{L}$, $f^{-1}(\alpha_i(n+1)) \cap \operatorname{supp} \zeta_{-n} = \emptyset$ and that $\alpha_i(-n) \cap \operatorname{supp} \zeta_n = \emptyset$, for $n \ge 1$. Therefore if $1 \le n < N(i)$,

$$\rho_n(\alpha_i(n)) = \zeta_{-n} \circ \zeta_n(\alpha_i(n)) = \zeta_{-n}(f^{-1}(\alpha_i(n+1))) = f^{-1}(\alpha_i(n+1))$$

and if $M(i) \le -n \le -1$,

$$\rho_n(\alpha_i(-n)) = \zeta_{-n} \circ \zeta_n(\alpha_i(-n)) = \zeta_{-n}(\alpha_i(-n)) = f^{-1}(\alpha_i(-n+1)).$$

This completes the proof since, for $n = 0$, $\rho_0 = \zeta_0$ and this case had already been taken care of. □

COROLLARY 3.32. *The sequence $R_n = \bigcup_{i=0}^{n} r_i$ is a Cauchy sequence in the uniform topology and converges to an isotopy $R : \pi \times [0,1] \to \pi$. If we set $\rho(\cdot) = R(\cdot, 1)$, for each $i \in \underline{L}$ and $M(i) \le n < N(i)$, $\rho(\alpha_i(n)) = f^{-1}(\alpha_i(n+1))$. Moreover,* $\operatorname{supp} R \subset \bigcup \{\mathcal{V}_i(n);\ i \in \underline{L},\ M(i) \le n < N(i)\}$ *and for each $i \in \underline{L}$ we have:*

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(i) $\rho(D^c(\alpha_i(n))) = D^c(f^{-1}(\alpha_i(n+1)))$ *for* $1 \le n < N(i)$*; and*
(ii) $\rho(D^e(\alpha_i(m))) = D^e(f^{-1}(\alpha_i(m+1)))$ *for* $M(i) \le m < 0$.

Proof. Given $\varepsilon > 0$, by Proposition 3.31, there exists K large enough so that all the connected components of $\operatorname{supp} r_m$ have diameter smaller than ε if $m \ge K$. Let $n > m \ge K$. We then have

$$\begin{aligned} d(R_m, R_n) &= \sup_{(x,t)} d(R_m(x,t), R_n(x,t)) \\ &= \sup_{(x,t)} d\left(R_m(x,t), \left[R_m \cup \bigcup_{m+1}^{n} r_i\right](x,t)\right) \\ &= \sup_{(x,t)} d\left(x, \bigcup_{m+1}^{n} r_i(x,t)\right) \\ &< \varepsilon \end{aligned}$$

where the last inequality is a consequence of the observation that if $H : X \times [0,1] \to X$ is an isotopy of the identity and $x \in \operatorname{supp} H$, then $H(x,t)$ and x belong to the same path component of $\operatorname{supp} H$. This shows that R_n is a Cauchy sequence. We leave it for the reader to check that $\rho(\alpha_i(n)) = f^{-1}(\alpha_i(n+1))$ and that $\operatorname{supp} R \subset \bigcup\{\mathcal{V}_i(n);\ i \in \underline{L},\ M(i) \le n < N(i)\}$.

In order to prove (i) in the statement, notice that, since $\operatorname{supp} R \subset \bigcup\{\mathcal{V}_i(n);\ i \in \underline{L},\ M(i) \le n < N(i)\}$, by Corollary 3.22, for $n \ge 1$, $f^n(C_i) \cap \operatorname{supp} R = \emptyset$. Also, if $1 \le n < N(i)$, $\rho(\alpha_i(n)) = f^{-1}(\alpha_i(n+1))$. Therefore

$$\rho(f^n(C_i) \cup \alpha_i(n)) = \rho(f^n(C_i)) \cup \rho(\alpha_i(n)) = f^n(C_i) \cup f^{-1}(\alpha_i(n+1)).$$

But $f^n(C_i) \cup \alpha_i(n) = \partial D^c(\alpha_i(n))$ and

$$f^n(C_i) \cup f^{-1}(\alpha_i(n+1)) = \partial D^c(f^{-1}(\alpha_i(n+1))).$$

This completes the proof of (i). (ii) is proven analogously. □

Definition 3.33. For each $n \ge 0$, let $\psi_n = f \circ \rho_n$, $\Psi_n = \bigcup_{i=0}^{n} \psi_i$ and $\Psi = f \circ \rho$, i.e. $\psi_n(\cdot) = f \circ r_n(\cdot, 1)$, $\Psi_n(\cdot) = f \circ R_n(\cdot, 1)$ and $\Psi(\cdot) = f \circ R(\cdot, 1)$.

We are now going to define new closed disks A_i, $i \in \underline{L}$, whose union is still the closed pruning front $\overline{P}$. We will see that the cross-cut $\alpha_i(0) \subset D_i$ is also a cross-cut in A_i and divides it into two disks A_i^c and A_i^e (see Figure 18). These have disjoint/nested properties made precise in Proposition 3.50 and will be used in the proof of the main theorem.

Definition 3.34. Let $A_L = A_L(0) = D_L$ and, for $1 \le i < L$, set $A_i = A_i(0) = \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1,0}^{-1}(D_i)$. Then define inductively for $n \ge 1$, $A_i(n) = \Psi(A_i(n-1))$ and $A_i(-n) = \Psi^{-1}(A_i(-n+1))$.

We can now state a proposition and a corollary which are among the main ingredients in the proof of the main theorem. We will use the following.

Definition 3.35. Let $P(0) = P$ and define inductively $P(n) = \Psi_{n-1}(P(n-1))$ and $P(-n) = \Psi_n^{-1}(P(-n+1))$, for every $n \ge 1$.

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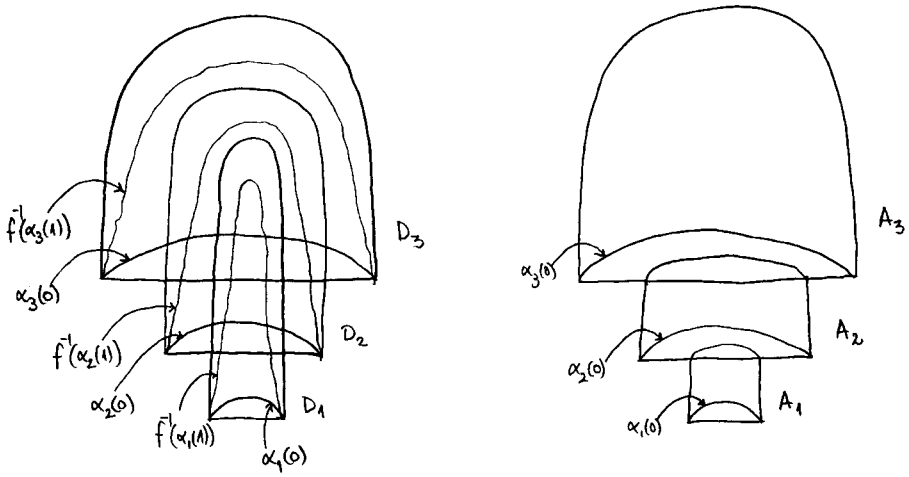


FIGURE 18. The D_is and the A_is.

PROPOSITION 3.36. *For $n \geq 1$ and $i \in \underline{L}$ we have:*

(i) $\bigcup_{j\leq i} f^n(D_j) \cup \bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{j\leq i} A_j(n) \cup \bigcup_{-n+2}^{n-1} P(k)$;

(ii) $\bigcup_{j\geq i} f^{-n+1}(D_j) \cup \bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{j\geq i} A_j(-n+1) \cup \bigcup_{-n+2}^{n-1} P(k)$.

COROLLARY 3.37. *For $n \geq 1$ and $i \in \underline{L}$ we have:*

(i) *$A_i(n) = f^n(D_i)$ in the complement of*

$$\bigcup_{j<i} f^n(D_j) \cup \bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{j<i} A_j(n) \cup \bigcup_{-n+2}^{n-1} P(k);$$

(ii) *$A_i(-n+1) = f^{-n+1}(D_i)$ in the complement of*

$$\bigcup_{j>i} f^{-n+1}(D_j) \cup \bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{j>i} A_j(-n+1) \cup \bigcup_{-n+2}^{n-1} P(k).$$

Proof. The proof of the proposition is rather long and we divide it into several lemmas.

Recall that if $\xi : X \to X$ is a homeomorphism we defined $\operatorname{supp}\xi = \mathcal{C}\{x \in X; \xi(x) = x\}$.

LEMMA 3.38. *Let $\xi, \eta : X \to X$ be homeomorphisms so that $\operatorname{supp}\xi \subset A$ and $\operatorname{supp}\eta \subset B$. Then $A \cup B = A \cup \xi \circ \eta(B)$.*

Proof. First notice that if $\operatorname{supp}\xi \subset A$ then $\xi(A) = A$ since $\xi(\mathcal{C}A) = \mathcal{C}A$ and ξ is a homeomorphism. Therefore, since $\operatorname{supp}\xi \circ \eta \subset A \cup B$ we have $A \cup B = \xi \circ \eta(A \cup B) = \xi(A) \cup (B) = A \cup \xi(B) = A \cup \xi \circ \eta(B)$. □

LEMMA 3.39. *For $n \geq 1$:*

(i) $f^{n+1}(P) \cup f^{-n+1}(P) = \psi_n(f^n(P)) \cup f^{-n+1}(P)$;

(ii) $f^n(P) \cup f^{-n}(P) = f^n(P) \cup \psi_n^{-1}(f^{-n+1}(P))$.

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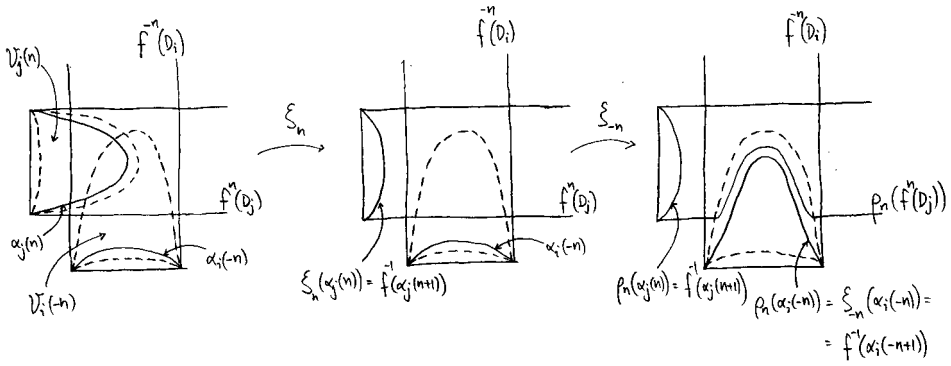


FIGURE 19. The homeomorphism ρ_n.

Proof. Let us first show that $f^n(P) \cup f^{-n}(P) = \rho_n(f^n(P)) \cup f^{-n}(P)$ and that $f^n(P) \cup f^{-n}(P) = f^n(P) \cup \rho_n^{-1}(f^{-n}(P))$. For $n = 0$, $\operatorname{supp}\rho_0 \subset P$ and the result follows. For $n \geq 1$, $\rho_n = \zeta_{-n} \circ \zeta_n$ and $\operatorname{supp}\zeta_{-n} \subset f^{-n}(P)$ and $\operatorname{supp}\zeta_n \subset f^n(P)$. The results now follow as easy applications of Lemma 3.38 (see Figure 19). To get (i) and (ii), just apply ψ_n to the equalities just obtained. □

The lemma that follows contains the main ideas and techniques to be used in the proof of Proposition 3.36.

LEMMA 3.40. *With the notation above,* $P(1) = f(P)$ *and:*
(i) *for* $n \geq 1$, $\bigcup_{-n+1}^{n-1} f^k(P) = \bigcup_{-n+1}^{n-1} P(k)$;
(ii) *for* $n \geq 2$, $\bigcup_{-n+2}^{n} f^k(P) = \bigcup_{-n+2}^{n} P(k)$.

Proof. We will use induction on n. For $n = 1$, (i) states that $P = P(0)$, which is just the definition, whereas $P(1) = \Psi_0(P) = \psi_0(P) = f \circ \rho_0(P)$ and, since $\operatorname{supp}\rho_0 \subset P$, $\rho_0(P) = P$, this shows that $P(1) = f(P)$ (see Figure 20).

We now show that $\bigcup_0^2 f^k(P) = \bigcup_0^2 P(k)$, but before we start, let us point out that, from the definitions of ψ_n and Ψ_n, the following is clear, for each $n \geq 0$:

(a) $\Psi_n = f$ in the complement of $\operatorname{supp} R_n \subset \bigcup_{-n}^{n} f^k(P)$;
(b) $\psi_n = f$ in the complement of $\operatorname{supp}\rho_n \subset [f^n(P) \cup f^{-n}(P)] \setminus \bigcup_{-n+1}^{n-1} f^k(\overline{P})$;
(c) $\Psi_n = \begin{cases} \Psi_{n-1} & \text{within } \bigcup_{-n+1}^{n-1} f^k(P) \\ \psi_n & \text{without } \bigcup_{-n+1}^{n-1} f^k(P); \end{cases}$
(d) $\Psi_n^{-1} = \begin{cases} \Psi_{n-1}^{-1} & \text{within } \bigcup_{-n+2}^{n} f^k(P) = \Psi_{n-1}\big(\bigcup_{-n+1}^{n-1} f^k(P)\big) \\ \psi_n^{-1} & \text{without } \bigcup_{-n+2}^{n} f^k(P) = \Psi_{n-1}\big(\bigcup_{-n+1}^{n-1} f^k(P)\big). \end{cases}$

Having said this, let us go back to the proof of $\bigcup_0^2 f^k(P) = \bigcup_0^2 P(k)$. Notice that from (c) above we have

$$P(2) = \Psi_1(P(1)) = \begin{cases} \Psi_0(P(1)) & \text{within } P \\ \psi_1(P(1)) & \text{without } P \end{cases}$$

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and, since we have seen that $P(0) = P$ and $P(1) = f(P)$,

$$\begin{aligned}\Psi_1(P(1)) &= \Psi_1(P(1) \cap P(0)) \cup \Psi_1(P(1) \setminus P(0)) \\ &= [\Psi_1(P(1)) \cap \Psi_0(P(0))] \cup [\Psi_1(f(P) \setminus P)] \\ &= [P(2) \cap P(1)] \cup [\psi_1(f(P) \setminus P)] \\ &= [P(2) \cap f(P)] \cup [\psi_1(f(P)) \setminus f(P)]\end{aligned}$$

where the last equality is a consequence of (b) above. Thus

$$\begin{aligned}\bigcup_0^2 P(k) &= \Psi_1(P(1)) \cup \bigcup_0^1 P(k) \\ &= \Psi_1(P(1)) \cup \bigcup_0^1 f^k(P) \\ &= \psi_1(f(P)) \cup \bigcup_0^1 f^k(P) \\ &= \bigcup_0^2 f^k(P)\end{aligned}$$

where the last equality is a consequence of Lemma 3.39 (i), with $n = 1$.

We now show that $\bigcup_{-1}^1 f^k(P) = \bigcup_{-1}^1 P(k)$. From (d) above we have

$$P(-1) = \Psi_1^{-1}(P(0)) = \begin{cases} \Psi_0^{-1}(P(0)) & \text{within } f(P) = P(1) \\ \psi_1^{-1}(P(0)) & \text{without } f(P) = P(1) \end{cases}$$

so that

$$\begin{aligned}\Psi_1^{-1}(P(0)) &= \Psi_1^{-1}(P(0) \cap P(1)) \cup \Psi_1^{-1}(P(0) \setminus P(1)) \\ &= [\Psi_1^{-1}(P(0)) \cap \Psi_0^{-1}(P(1))] \cup \Psi_1^{-1}(P \setminus f(P)) \\ &= [P(-1) \cap P(0)] \cup [\psi_1^{-1}(P \setminus f(P))] \\ &= [P(-1) \cap P] \cup [\psi_1^{-1}(P) \setminus P]\end{aligned}$$

where the last equality is a consequence of (b). From this we see that

$$\begin{aligned}\bigcup_{-1}^1 P(k) &= \bigcup_0^1 P(k) \cup \Psi_1^{-1}(P(0)) \\ &= \bigcup_0^1 f^k(P) \cup \Psi_1^{-1}(P(0)) \\ &= \bigcup_0^1 f^k(P) \cup \psi_1^{-1}(P) \\ &= \bigcup_{-1}^1 f^k(P)\end{aligned}$$

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where the last equality is again a consequence of Lemma 3.39 (ii), with $n = 1$. This completes the proof of (i) and (ii) for $n = 2$. Suppose we have proven that (i) and (ii) hold for $2 \leq n \leq N$. From this assumption the assertions below follow.

(1) $\bigcup_{-n+1}^{n} f^k(P) = \bigcup_{-n+1}^{n} P(k)$, for $2 \leq n \leq N$, by just taking the union of (i) and (ii).

(2) $f^n(P) = P(n)$ and $f^{-n+1}(P) = P(-n+1)$ in the complement of $\bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{-n+2}^{n-1} P(k)$, for $0 \leq n \leq N$. This can be seen as follows: by (i), $\bigcup_{-n+2}^{n} f^k(P) = \bigcup_{-n+2}^{n} P(k)$ and by (1), $\bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{-n+2}^{n-1} P(k)$. Then

$$f^n(P) \cup \bigcup_{-n+2}^{n-1} f^k(P) = P(n) \cup \bigcup_{-n+2}^{n-1} P(k).$$

It then follows that $f^n(P) = P(n)$ in the complement of $\bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{-n+2}^{n-1} P(k)$. The other part is proven similarly.

(3) $\Psi_n(P(j)) = P(j+1)$ for any $-n \leq j \leq n$, $0 \leq n \leq N$. For notice that $\Psi_n = \Psi_{|j|}$ in $\bigcup_{-|j|}^{|j|} f^k(P) = \bigcup_{-|j|}^{|j|} P(k) \supset P(j)$. Thus $\Psi_n(P(j)) = \Psi_{|j|}(P(j)) = P(j+1)$ from the definition of $P(j)$. This reasoning is valid for $-n \leq j \leq n$, $0 \leq n < N$. For $n = N$ what remains to be shown is that $\Psi_N(P(N)) = P(N+1)$ and $\Psi_N(P(-N)) = P(N+1)$ or equivalently $P(-N) = \Psi_N^{-1}(P(-N+1))$. But these are just the definitions again.

We now proceed to prove (i) and (ii) for $N+1$. We start with (ii) $\bigcup_{-N+1}^{N+1} P(k) = \bigcup_{-N+1}^{N+1} f^k(P)$.

From (c) in the beginning of the proof

$$P(N+1) = \Psi_N(P(N)) = \begin{cases} \Psi_{N-1}(P(N)) & \text{within } \bigcup_{-N+1}^{N-1} f^k(P) = \bigcup_{-N+1}^{N-1} P(k) \\ \psi_N(P(N)) & \text{without } \bigcup_{-N+1}^{N-1} f^k(P) = \bigcup_{-N+1}^{N-1} P(k). \end{cases}$$

Thus,

$$\begin{aligned} \Psi_N(P(N)) &= \Psi_N\Big(P(N) \cap \bigcup_{-N+1}^{N-1} P(k)\Big) \cup \Psi_N\Big(P(N) \setminus \bigcup_{-N+1}^{N-1} P(k)\Big) \\ &= \Big[\Psi_N(P(N)) \cap \Psi_{N-1}\Big(\bigcup_{-N+1}^{N-1} P(k)\Big)\Big] \\ &\quad \cup \Psi_N\Big(f^N(P) \setminus \bigcup_{-N+1}^{N-1} f^k(P)\Big) \\ &= \Big[P(N+1) \cap \bigcup_{-N+2}^{N} P(k)\Big] \cup \psi_N\Big(f^N(P) \setminus \bigcup_{-N+1}^{N-1} f^k(P)\Big) \\ &= \Big[P(N+1) \cap \bigcup_{-N+2}^{N} f^k(P)\Big] \cup \Big[\psi_N(f^N(P)) \setminus \bigcup_{-N+2}^{N} f^k(P)\Big], \end{aligned}$$

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where we used (2) in the second equality, (3) in the third and (b) from the beginning in the fourth, not to mention the induction hypothesis here and there. From this it follows that

$$\begin{aligned}\bigcup_{-N+1}^{N+1} P(k) &= \Psi_N(P(N)) \cup \bigcup_{-N+1}^{N} P(k)\\ &= \Psi_N(P(N)) \cup \bigcup_{-N+1}^{N} f^k(P)\\ &= \psi_N(f^N(P)) \cup \bigcup_{-N+1}^{N} f^k(P)\\ &= \bigcup_{-N+1}^{N+1} f^k(P),\end{aligned}$$

where the last equality comes from Lemma 3.39(i) with $n = N$.

We now prove (i) $\bigcup_{-N}^{N} P(k) = \bigcup_{-N}^{N} f^k(P)$. From (d) we have

$$P(-N) = \Psi_N^{-1}(P(-N+1)) = \begin{cases} \Psi_{N-1}^{-1}(P(-N+1)) & \text{within } \bigcup_{-N+2}^{N} f^k(P) = \bigcup_{-N+2}^{N} P(k) \\ \psi_N^{-1}(P(-N+1)) & \text{without } \bigcup_{-N+2}^{N} f^k(P) = \bigcup_{-N+2}^{N} P(k). \end{cases}$$

Thus

$$\begin{aligned}\Psi_N^{-1}(P(-N+1)) &= \Psi_N^{-1}\Big(P(-N+1) \cap \bigcup_{-N+2}^{N} P(k)\Big)\\ &\quad \cup \Psi_N^{-1}\Big(P(-N+1) \setminus \bigcup_{N+2}^{N} P(k)\Big)\\ &= \Big[\Psi_N^{-1}(P(-N+1)) \cap \Psi_{N-1}^{-1}\Big(\bigcup_{-N+2}^{N} P(k)\Big)\Big]\\ &\quad \cup \Psi_N^{-1}\Big(f^{-N+1}(P) \setminus \bigcup_{-N+2}^{N} f^k(P)\Big)\\ &= \Big[P(-N) \cap \bigcup_{-N+1}^{N-1} P(k)\Big]\\ &\quad \cup \psi_N^{-1}\Big(f^{-N+1}(P) \setminus \bigcup_{-N+2}^{N} f^k(P)\Big)\\ &= \Big[P(-N) \cap \bigcup_{-N+1}^{N-1} f^k(P)\Big]\\ &\quad \cup \Big[\psi_N^{-1}(f^{-N+1}(P)) \setminus \bigcup_{-N+1}^{N-1} f^k(P)\Big],\end{aligned}$$

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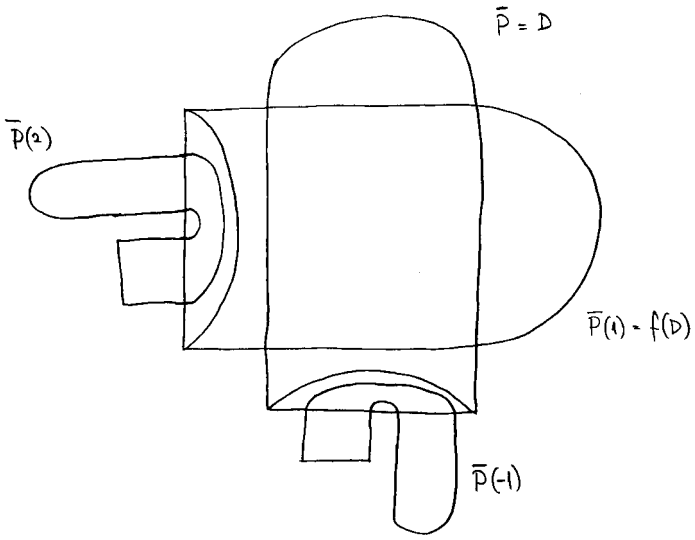


FIGURE 20. $\overline{P}(k)$, $k = -1, 0, 1, 2$, for a pruning collection containing only one (c, e)-disk D.

where we have used (2) in the second equality, (3) in the third, (b) in the fourth and the induction hypothesis.

Therefore

$$\begin{aligned}\bigcup_{-N}^{N} P(k) &= \bigcup_{-N+1}^{N} P(k) \cup \Psi_N^{-1}(P(-N+1)) \\ &= \bigcup_{-N+1}^{N} f^k(P) \cup \Psi_N^{-1}(P(-N+1)) \\ &= \bigcup_{-N+1}^{N} f^k(P) \cup \psi_N^{-1}(P(-N+1)) \\ &= \bigcup_{-N}^{N} f^k(P)\end{aligned}$$

where the last equality comes from Lemma 3.39 (ii) with $n = N$. This completes the proof. □

COROLLARY 3.41. *For $n \geq 1$:*

(i) $\bigcup_{n+1}^{n} f^k(P) = \bigcup_{-n+1}^{n} P(k)$;

(ii) $f^n(P) = P(n)$ *and* $f^{-n+1}(P) = P(-n+1)$ *in the complement of* $\bigcup_{-n+2}^{n-1} f^k(P) = \bigcup_{-n+2}^{n-1} P(k)$.

Proof. The proof is the same as that given for (1) and (2) in the proof of Lemma 3.40. □

COROLLARY 3.42. *If Ψ is as we defined above, $P(k) = \Psi(P(k-1))$ for every $k \in \mathbb{Z}$, that is $\{P(k);\ k \in \mathbb{Z}\}$ is an orbit under Ψ.*

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Proof. Just notice that $\Psi = \Psi_n$ in $\bigcup_{-n}^{n} f^k(P)$ and argue like in the proof of (3) in Lemma 3.40. □

LEMMA 3.43. *For $l \in \underline{L}$, $\bigcup_{i=l}^{L} A_i = \bigcup_{i=l}^{L} D_i$. In particular $\bigcup_{i=1}^{L} A_i = \overline{P}$.*

Proof. By definition $A_L = D_L$. Assume we have shown that $\bigcup_{i=l+1}^{L} A_i = \bigcup_{i=l+1}^{L} D_i$. Then

$$\bigcup_{i=l}^{L} A_i = \bigcup_{i=l+1}^{L} A_i \cup A_l = \bigcup_{i=l+1}^{L} D_i \cup \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{l+1,0}^{-1}(D_l) = \bigcup_{i=l}^{L} D_i$$

where the last equality holds because $\operatorname{supp} \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{l+1,0}^{-1} \subset \bigcup_{i=l+1}^{L} D_i$. □

Proof of proposition. The proof is by induction on n and goes along the same lines as that of Lemma 3.40. We omit the details and refer the interested reader to [**dC**]. Just notice that for $n = 1$, (ii) in the statement is Lemma 3.43. □

PROPOSITION 3.44. *For each $i \in \underline{L}$, $\alpha_i(0)$ is a cross-cut in A_i and divides A_i into two closed disks A_i^c and A_i^e bounded by $\rho_0^{-1}(C_i) \cup \alpha_i(0)$ and $\alpha_i(0) \cup E_i$, respectively. As consequences, we have that, for each $i \in \underline{L}$,* (i) *and* (ii) *hold:*

(i) *for $n \geq 1$, $A_i(n)$ is bounded by the Jordan curve $f^n(C_i) \cup \Psi^n(E_i)$;*

(ii) *for $m \leq 0$, $A_i(m)$ is bounded by the Jordan curve $\Psi^m(\rho_0^{-1}(C_i)) \cup f^m(E_i) = \Psi^{m-1}(f(C_i)) \cup f^m(E_i)$.*

Proof. In the proof of Proposition 3.31(iii), we have shown that for each $i \in \underline{L}$, $\rho_0(\alpha_i(0)) = \zeta_0(\alpha_i(0)) = \zeta_{1,0} \circ \cdots \circ \zeta_{i,0}(\alpha_i(0))$. Thus we see that

$$\begin{aligned} \alpha_i(0) &= \rho_0^{-1}(\zeta_{1,0} \circ \cdots \circ \zeta_{i,0}(\alpha_i(0))) \\ &= \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{1,0}^{-1} \circ \zeta_{1,0} \circ \cdots \circ \zeta_{i,0}(\alpha_i(0)) \\ &= \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1}^{-1}(\alpha_i(0)) \end{aligned}$$

so that $\alpha_i(0)$ is left fixed by $\zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1,0}^{-1}$. Since $\alpha_i(0)$ is a cross-cut in D_i and $A_i = \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1,0}^{-1}(D_i)$, $\alpha_i(0)$ is also a cross-cut in A_i.

We now show that A_i is bounded by $\rho_0^{-1}(C_i) \cup E_i$ which will complete the proof of the proposition. Notice that if $j \leq i$, $C_i \cap I_j = \emptyset$ so that $\rho_0^{-1}(C_i) = \zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1,0}^{-1}(C_i)$. On the other hand, for $j \geq i$, $I_j \cap E_i = \emptyset$ so that $\zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1,0}^{-1}(E_i) = E_i$. This shows that $\zeta_{L,0}^{-1} \circ \cdots \circ \zeta_{i+1,0}^{-1}(C_i \cup E_i) = \rho_0^{-1}(C_i) \cup E_i$, as we wanted.

We just saw that $A_i(0)$ is bounded by $\rho_0^{-1}(C_i) \cup E_i = \psi_0^{-1}(f(C_i)) \cup E_i = \Psi^{-1}(f(C_i)) \cup E_i$. Therefore $A_i(1) = \Psi(A_i(0))$ is bounded by $\Psi(\Psi^{-1}(f(C_i)) \cup E_i) = f(C_i) \cup \Psi(E_i)$, which proves (i) and (ii) for $n = 1$ and $m = 0$ respectively. The general result is now proved by induction using Corollary 3.22 to guarantee that $\Psi^n(f(C_i)) = f^{n+1}(C_i)$ for $n \geq 0$ and that $\Psi^m(E_i) = f^m(E_i)$ for $m \leq 0$. □

Definition 3.45. Let $A_i^c(0) = A_i^c$ and $A_i^e(0) = A_i^e$ as in Proposition 3.44 and define inductively for $n \geq 1$, $A_i^{c(e)}(n) = \Psi(A_i^{c(e)}(n-1))$ and $A_i^{c(e)}(-n) = \Psi^{-1}(A_i^{c(e)}(-n+1))$ (see Figure 21).

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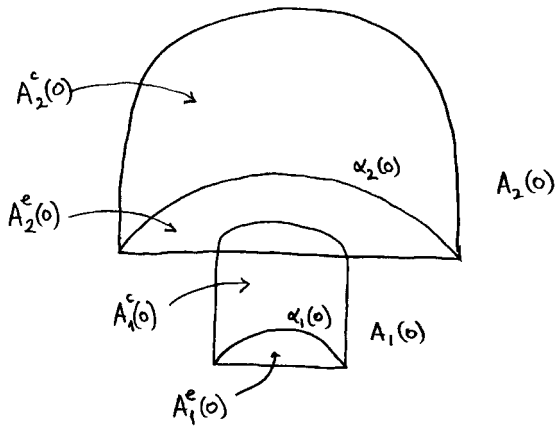


FIGURE 21. $A_i^c(0)$ and $A_i^e(0)$ for $i = 1, 2$.

PROPOSITION 3.46. *With the notation just introduced we have:*

(i) $A_i^c(n) = D^c(\alpha_i(n))$ *for* $1 \le n \le N(i)$;

(ii) $A_i^e(m) = D^e(\alpha_i(m))$ *for* $M(i) \le m \le 0$.

Let $n(i), N(i), m(i)$ and $M(i)$ be as we defined them in the beginning of this subsection. By Proposition 3.30 if $n(i), m(i)$ are finite then $n(i) = 2N(i) - \delta, m(i) = 2M(i) - \delta'$, where $\delta, \delta' = 0$ or 1. Moreover,

$$f^{N(i)}(D_i),\ \ f^{-N(i)+\delta+1}(D_j)|_{f^{N(i)}(C_i)}$$

and

$$f^{M(i)}(D_i),\ \ f^{-M(i)+\delta'}(D_l)|_{f^{M(i)}(E_i)}$$

for some $j, l \in \underline{L}$. Recall also that if D_1, $D_2|_L$ and $D_1 \setminus L \subset I_2$ we write $D_1 \subset D_2|_L$. We can now state the following.

PROPOSITION 3.47. *With the same notation as above, for every* $i \in \underline{L}$, *the following hold:*

(i) *if* $n(i) < \infty$, *then for every* $j \in \underline{L}$ *such that* $f^{N(i)}(D_i), f^{-N(i)+\delta+1}(D_j)|_{f^{N(i)}(C_i)}$ *we have*

$$A_i^c(N(i)) \subset A_j^c(-N(i) + \delta + 1)|_{f^{N(i)}(C_i)};$$

(ii) *if* $m(i) > -\infty$, *then for every* $j \in \underline{L}$ *such that* $f^{M(i)}(D_i)\ f^{-M(i)+\delta'}(D_j)|_{f^{M(i)}(E_i)}$ *we have*

$$A_i^e(M(i)) \subset A_j^e(-M(i) + \delta')|_{f^{M(i)}(E_i)}.$$

Proof. We will do the proof in steps. The first two are contained in the lemmas below.

LEMMA 3.48. *For each* $i \in \underline{L}$, (i) *and* (ii) *hold:*

(i) *if* $n(i) < \infty$ *and* $j \in \underline{L}$ *is largest such that* $f^{N(i)}(D_i), f^{-N(i)+\delta+1}(D_j)|_{f^{N(i)}(C_i)}$, *then*

$$A_i^c(N(i)) \subset A_j(-N(i) + \delta + 1)|_{f^{N(i)}(C_i)};$$

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(ii) *if* $m(i) > -\infty$ *and* $j \in \underline{L}$ *is smallest such that* $f^{M(i)}(D_i),\ f^{-M(i)+\delta'}(D_j)|_{f^{M(i)}(E_i)}$, *then*

$$A_i^e(M(i)) \subset A_j(-M(i)+\delta')|_{f^{M(i)}(E_i)}.$$

Proof. By Proposition 3.46 we know that $A_i^c(N(i)) = D^c(\alpha_i(N(i)))$. We have to show that $f^{N(i)}(C_i) \subset \partial A_j(-N(i)+\delta+1)$ and that

$$\begin{aligned} A_i^c(N(i)) \setminus f^{N(i)}(C_i) &= D^c(\alpha_i(N(i))) \setminus f^{N(i)}(C_i) \\ &= I^c(\alpha_i(N(i))) \cup \alpha_i(N(i)) \\ &\subset I(A_j(-N(i)+\delta+1)) \end{aligned}$$

where the last set is the interior of $A_j(-N(i)+\delta+1)$ and the second equality is just the definition and only the last inclusion needs proof (see Figure 22).

Let us first show that $f^{N(i)}(C_i) \subset \partial A_j(-N(i)+\delta+1)$. By assumption $f^{N(i)}(D_i),\ f^{-N(i)+\delta+1}(D_j)|_{f^{N(i)}(C_i)}$ which implies that $f^{N(i)}(C_i) \subset f^{-N(i)+\delta+1}(C_j)$. If $n(i) = 1$, then $N(i) = 1$ and $\delta = 1$, so that by Proposition 3.44 and above we have $f(C_i) \subset f(C_j) \subset \partial A_j(1)$. If $n(i) > 1$, applying $f^{N(i)-\delta-1}$ to the inclusion $f^{N(i)}(C_i) \subset f^{-N(i)+\delta+1}(C_j)$ we get $f^{n(i)-1}(C_i) \subset C_j$. From Corollary 3.22 we know that $f^n(C_i) \cap \operatorname{supp} R = \emptyset$ for $n \geq 1$. In particular, $f^{n(i)-1}(C_i) \cap \operatorname{supp} \rho_0 = \emptyset$, and it follows that $f^{n(i)-1}(C_i) \subset \rho_0^{-1}(C_j) \subset \partial A_j(0)$, this last inclusion coming from Proposition 3.44. Also, if $k < n(i)$ then $\Psi^{-k}(f^{n(i)}(C_i)) = f^{-k}(f^{n(i)}(C_i)) = f^{n(i)-k}(C_i)$, so that

$$\begin{aligned} \Psi^{-N(i)+\delta+1}(f^{n(i)-1}(C_i)) &= f^{N(i)}(C_i) \\ &\subset \Psi^{-N(i)+\delta+1}(\rho_0^{-1}(C_j)) \\ &\subset \partial A_j(-N(i)+\delta+1) \end{aligned}$$

as we wanted.

In order to see that $I^c(\alpha_i(N(i))) \cup \alpha_i(N(i)) \subset I(A_j(-N(i)+\delta+1))$ first notice that, since $\{\alpha_l(N(i))\}_{l=1}^L$ is a $(\varepsilon_{N(i)}, c)$-collection compatible with

$$\{(f^k(D_j), \alpha_j(k));\ j \in \underline{L},\ -N(i)+1 \leq k \leq N(i)-1\}$$

and that $f^{N(i)}(D_i),\ f^{-N(i)+\delta+1}(D_j)|_{f^{N(i)}(C_i)}$ then $I^c(\alpha_i(N(i))) \cup \alpha_i(N(i)) \subset f^{-N(i)+\delta+1}(I_j)$.

We will now show that

$$[I^c(\alpha_i(N(i))) \cup \alpha_i(N(i))] \cap \left[\bigcup_{l>j} f^{-N(i)+\delta+1}(D_l) \cup \bigcup_{-N(i)+\delta+2}^{N(i)-\delta-1} f^k(\overline{P})\right] = \emptyset.$$

This is so because by assumption $f^{N(i)}(D_i),\ f^{-N(i)+\delta+1}(D_l) \not\pitchfork_{f^{N(i)}(C_i)}$ for $l > j$ and from Proposition 3.30(ii), $f^{N(i)}(D_i),\ f^k(D_j) \not\pitchfork_{f^{N(i)}(C_i)}$ for any $j \in \underline{L}$ and $-N(i)+\delta+2 \leq k \leq N(i)-1$. This together with the aforementioned compatibility of $\{\alpha_i(N(i))\}_{l=1}^L$ are exactly what we need in order to verify the equation above. By Corollary 3.37 (ii), $A_j(-N(i)+\delta+1) = f^{-N(i)+\delta+1}(D_j)$ in the complement of

$$\bigcup_{l>j} f^{-N(i)+\delta+1}(D_l) \cup \bigcup_{-N(i)+\delta+2}^{N(i)-\delta-1} f^k(P)$$

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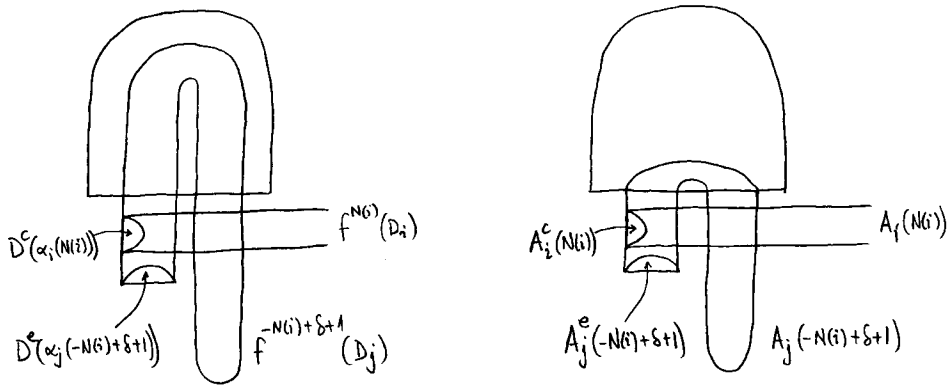


FIGURE 22. A possible configuration for $f^{N(i)}(D_i)$, $f^{-N(i)+\delta+1}(D_j)$ and $A_i(N(i))$, $A_j(-N(i)+\delta+1)$ and $A_i^c(N(i))$.

which shows that $I^c(\alpha_i(N(i)))\cup\alpha_i(N(i)) \subset A_j(-N(i)+\delta+1)$. We leave it for the reader to show that it is possible to put $I(A_j(-N(i)+\delta+1))$ in place of $A_j(-N(i)+\delta+1)$ in the inclusion above. □

LEMMA 3.49. *Under the hypotheses of Lemma 3.48*(i) *and* (ii) *respectively,* (i′) *and* (ii′) *below hold:*

(i′) $A_i^c(N(i)) \subset A_j^c(-N(i)+\delta+1)|_{f^{N(i)}(C_i)}$;

(ii′) $A_i^e(M(i)) \subset A_j^e(-M(i)+\delta')|_{f^{M(i)}(E_i)}$.

Proof. By Lemma 3.48 (i), $A_i^c(N(i)) \subset A_j(-N(i)+\delta+1)|_{f^{N(i)}(C_i)}$. Therefore all we need to prove is that

$$[I^c(\alpha_i(N(i))) \cup \alpha_i(N(i))] \cap A_j^e(-N(i)+\delta+1) = \emptyset.$$

There are two cases to be considered: $M(j) \le -N(i)+\delta+1$ and $M(j) > -N(i)+\delta+1$. If $M(j) \le -N(i)+\delta+1$, by Proposition 3.46,

$$A_j^e(-N(i)+\delta+1) = D^e(\alpha_j(-N(i)+\delta+1))$$

and, since $\{\alpha_l(N(i))\}_{l=1}^{\underline{L}}$ is a $(\varepsilon_{N(i)}, c)$-collection compatible with

$$\{(f^k(D_j), \alpha_j(k));\ j \in \underline{L},\ -N(i)+1 \le k \le N(i)-1\},$$

then

$$[I^c(\alpha_i(N(i))) \cup \alpha_i(N(i))] \subset I^c(\alpha_j(-N(i)+\delta+1))$$

so that

$$[I^c(\alpha_i(N(i))) \cup \alpha_i(N(i))] \cap D^e(-N(i)+\delta+1) = \emptyset,$$

as we wanted.

If $M_j > -N(i)+\delta+1$, there exists $l \in \underline{L}$ such that $f^{M(j)}(D_j)$, $f^{-M(j)+\delta'}(D_l)|_{f^{M(j)}(E_j)}$ where $m(j) = 2M(j)+\delta'$, and, assuming l is the smallest such, by Lemma 3.48 (ii), we

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can conclude that $A_j^e(M(j)) \subset A_l(-M(j)+\delta')$. Therefore

$$\begin{aligned} A_j^e(-N(i)+\delta+1) &= \Psi^{-M(j)-N(i)+\delta+1}(A_j^e(M(j))) \\ &\subset \Psi^{-M(j)-N(i)+\delta+1}(A_l(-M(j)+\delta')) \\ &= A_l(-m(j)-N(i)+\delta+1). \end{aligned}$$

From $M(j) \geq -N(i)+\delta+2$ we have

$$\begin{aligned} -m(j)-N(i)+\delta+1 &= -2M(j)+2\delta'-N(i)+\delta+1 \\ &\leq 2N(i)-2\delta-4+2\delta'-N(i)+\delta+1 \\ &= N(i)-\delta-3-2\delta' \\ &\leq N(i)-\delta-1. \end{aligned}$$

If $m(j) > 0$, then $-m(j)-N(i)+\delta+1 \geq -N(i)+\delta+2$, so that

$$A_l(-m(j)-N(i)+\delta+1) \subset \bigcup_{-N(i)+\delta+2}^{N(i)-\delta-1} \overline{P}(k).$$

If $m(j) = 0$, then D_j, $D_l|_{E_j}$, which implies that $D_l \succ D_j$ and therefore that $l > j$. With this we have shown that

$$\begin{aligned} A_l(-m(j)-N(i)+\delta+1) &\subset \bigcup_{l>j} A_l(-N(i)+\delta+1) \cup \bigcup_{-N(i)+\delta+2}^{N(i)-\delta-1} \overline{P}(k) \\ &= \bigcup_{l>j} f^{-N(i)+\delta+1}(D_l) \cup \bigcup_{-N(i)+\delta+2}^{N(i)-\delta-1} f^k(\overline{P}) \end{aligned}$$

where this last equality is a consequence of Proposition 3.36 (ii). But, from the proof of Lemma 3.48, we have seen that $I^c(\alpha_i(N(i))) \cup \alpha_i(N(i))$ does not intersect the set after the equal sign just above. This finishes the proof. □

Proof of Proposition 3.47. We have shown that if $j \in \underline{L}$ is largest such that $f^{n(i)}(D_i)$, $f(D_j)|_{f^{N(i)}(C_i)}$ (which is equivalent to the condition in (i) above) then the desired inclusion holds. Let $l \in \underline{L}$ be such that $f^{n(i)}(D_i)$, $f(D_l)|_{f^{n(i)}(C_i)}$, $l \neq j$ (and thus $l < j$). By Proposition 3.4, $f(D_j)$, $f(D_l)|_{f^{n(i)}(C_i)}$ and since $l < j$, we must have $f(D_l) \prec f(D_j)$. Since $\{\alpha_i(1)\}_{i=1}^L$ is a (ε, c)-collection, it follows that $[I^c(\alpha_j(1)) \cup \alpha_j(1)] \subset I^c(\alpha_l(1))$ and by Proposition 3.46 this is equivalent to $A_j^c(1) \subset A_l^c(1)|_{f(C_j)}$ (see Figure 23). Taking the $\Psi^{-N(i)+\delta}$-image of this latter inclusion, we get

$$A_j^c(-N(i)+\delta+1) \subset A_l^c(-N(i)+\delta+1)|_{\Psi^{-N(i)+\delta}(f(C_j))}.$$

Notice that $\Psi^{-N(i)+\delta}(f(C_j)) = \Psi^{-N(i)+\delta+1}(\rho_0^{-1}(C_j))$ and that in the proof of Lemma 3.48 we showed that $f^{N(i)}(C_i) \subset \Psi^{-N(i)+\delta+1}(\rho_0^{-1}(C_j))$. Therefore

$$A_i^c(N(i)) \subset A_j^c(-N(i)+\delta+1)|_{f^{N(i)}(C_i)}$$

and

$$A_j^c(-N(i)+\delta+1) \subset A_l^c(-N(i)+\delta+1)|_{\Psi^{-N(i)+\delta+1}(\rho_0^{-1}(C_i))}$$

imply that

$$A_i^c(N(i)) \subset A_l^c(-N(i)+\delta+1)|_{f^{N(i)}(C_i)}$$

as we wanted. □

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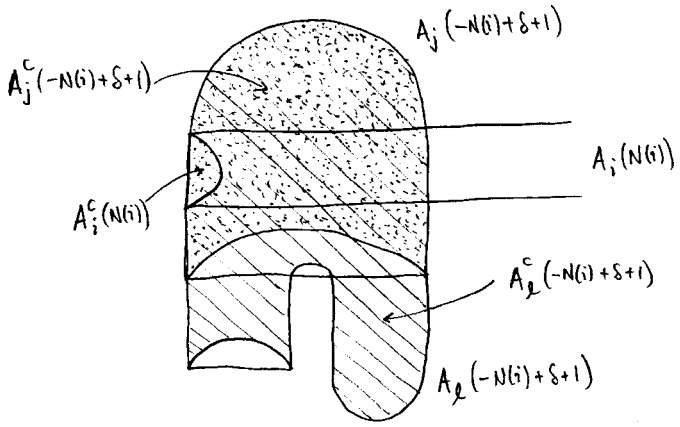


FIGURE 23. An example of $A_i(N(i))$, $A_j(-N(i)+\delta+1)$ and $A_l(-N(i)+\delta+1)$.

To sum up the important results proved so far, we state the following.

PROPOSITION 3.50. *Ψ has the following properties:*

(i) *if $n(i)=\infty$, then $\Psi^n(A_i^c)$ has interior disjoint from $\overline{P}$ for every $n>0$;*

(ii) *if $n(i)<\infty$, then for every $j\in \underline{L}$ such that $f^{n(i)}(D_i)$, $f(D_j)|_{f^{n(i)}(C_i)}$, $\Psi^{n(i)}(A_i^c)\subset A_j^c(1)|_{f^{n(i)}(C_i)}$ and $f^{n(i)}(C_i)\subset f(C_j)$;*

(iii) *if $m(i)=-\infty$, then $\Psi^m(A_i^e)$ has interior disjoint from $\overline{P}$ for every $m<0$;*

(iv) *if $m(i)>-\infty$, then for every $j\in \underline{L}$ such that $f^{m(i)}(D_i)$, $D_j|_{f^{m(i)}(E_i)}$, $\Psi^{m(i)}(A_i^e)\subset A_j^e|_{f^{m(i)}(E_i)}$ and $f^{m(i)}(E_i)\subset E_j$.*

Proof. If $n(i)=\infty$, then $N(i)=\infty$ and by Proposition 3.46 we see that $\Psi^n(A_i^c)=A_i^c(n)=D^c(\alpha_i(n))$ for every $n\geq 1$. Since $f^n(D_i)$, $f^k(D_j)\not\pitchfork_{f^n(C_i)}$ for $-n+1\leq k\leq n-1$,

$$[I^c(\alpha_i(n))\cup\alpha_i(n)]\cap\bigcup_{-n+1}^{n-1}f^k(\overline{P})=\emptyset.$$

This being true for every $n\geq 1$, we see that $[I^c(\alpha_i(n))\cup\alpha_i(n)]\cap\overline{P}=\emptyset$ for every $n\geq 1$, which proves (i). (ii) is immediate from Proposition 3.47. (iii) and (iv) are analogous and we omit the proofs. □

We can now prove the main theorem.

Proof of main theorem. Construct a directed graph G_c as follows: its vertices are the integers $\{i\in\underline{L};\ n(i)>1\}$ and there is a directed vertex from i to j if $n(i)<\infty$ and $f^{n(i)}(D_i)$, $f(D_j)|_{f^{n(i)}(C_i)}$. Since we have taken only $i\in\underline{L}$ for which $n(i)>1$, it is easy to see that from each vertex there is at most one outgoing edge (or none, if $n(i)=\infty$). A *loop* in the directed graph consists of an ordered set of distinct vertices $\{i_1<i_2<\cdots<i_l\}$ such that there is a directed edge from i_r to i_{r+1}, for $1\leq r\leq l$ where we let the indices 'wrap around', i.e. $l+1$ '$=$' 1. Since there is at most one edge emanating from each vertex, and the vertices in a loop are ordered and distinct, it follows that two loops are either equal

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or disjoint. Let $\mathcal{L} = \{i_1, \ldots, i_l\}$ be a loop in G_c, which for now we will represent by just its subscripts $\{1, \ldots, l\}$ so that the notation is not too awkward. By definition, we have

$$f^{n(r)}(D_r), \;\; f(D_{r+1})|_{f^{n(r)}(C_r)} \quad \text{for } 1 \le r \le l$$

and by Proposition 3.50

$$\Psi^{n(r)}(A_i^c) \subset A_{r+1}(1)|_{f^{n(r)}(C_r)} \quad \text{for } 1 \le r \le l$$

from which it follows that

$$\Psi^{\sum_{r=1}^{l} n(r)-(l-2)}(A_1^c(1)) \subset A_1^c(1)|_{f^{\sum_{r=1}^{l} n(r)-(l-1)}(C_i)}.$$

For a loop $\mathcal{L} = \{i_1, \ldots, i_l\}$ let $n(\mathcal{L}) = \sum_{r=1}^{l} n(i_r) - (l-2)$. By Proposition 3.27 and Corollary 3.28, there exists an isotopy $h_{\mathcal{L}}$ of the identity with $\operatorname{supp} h_{\mathcal{L}} \subset I(A_{i_1}^c(1))$ such that, if $\zeta_{\mathcal{L}}(\cdot) = h_{\mathcal{L}}(\cdot, 1)$, then $(\Psi \circ \zeta_{\mathcal{L}})^{kn(\mathcal{L})}(x) \to p$ for every $x \in A_i^c(1)$, where p is the fixed point of $\Psi^{n(\mathcal{L})}$ in $f^{n(\mathcal{L})+1}(C_i)$. We then construct isotopies $h_{\mathcal{L}}$ for each loop $\mathcal{L}$ in G_c. Since the vertices of G_c were integers $i \in \underline{L}$ for which $n(i) > 1$, the supports of isotopies associated with different loops are disjoint. Let h_c be the union of all these isotopies. By construction

$$\operatorname{supp} h_c \subset \bigcup_{i=1}^{l} I(A_i^c(1)) = \bigcup_{i=1}^{L} I^c(\alpha_i(1)).$$

In an analogous manner, we construct a directed graph G_e whose vertices are $\{i \in \underline{L}; m(i) < 0\}$ and for each loop $\mathcal{L}$ in G_e, we construct an isotopy $h_{\mathcal{L}}$ of the identity, with support in $A_{i_1}^e(0)$, playing the analogous role for Ψ^{-1} as the above ones played for Ψ. Let k_e denote the union of these isotopies, for it is again easy to check that they have disjoint supports, and define $h_e = \Psi^{-1} \circ k_e^{-1} \circ \Psi$, i.e. for each fixed t, $h_e(x,t) = \Psi^{-1}(k_e^{-1}(\Psi(x), t))$. h_e is also an isotopy of the identity and since $\operatorname{supp} k_e \subset \bigcup_{i=1}^{L} I(A_i^e(0))$,

$$\operatorname{supp} h_e \subset \bigcup_{i=1}^{L} I(A_i^e(0)) = \bigcup_{i=1}^{L} I^e(\alpha_i(0)).$$

From this it follows that $\operatorname{supp} h_e \cap \operatorname{supp} h_c = \emptyset$ and we let $h = h_c \cup h_e$ and $\zeta(\cdot) = h(\cdot, 1)$. Finally, set

$$H(x,t) = \begin{cases} h(x, 2t) & t \in [0, \frac{1}{2}] \\ R(\zeta(x), 2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is now not hard to check that H has the desired properties. □

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