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WAS ULAM RIGHT? III: INDECOMPOSABLE IDEALS

TANMAY INAMDAR[1] AND ASSAF RINOT[2]

[1] *Department of Mathematics, Ben-Gurion University of the Negev, Israel*
(tci.math@protonmail.com)

[2] *Department of Mathematics, Bar-Ilan University, Israel*
http://www.assafrinot.com
(rinotas@math.biu.ac.il)

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This paper is dedicated to Saharon Shelah on the occasion of his 80th birthday

Abstract We continue our study of Ulam's measure problem. In contrast to our previous works, we shift our focus from measures stratified by their additivity, to measures stratified by their indecomposability. The breakthrough here is obtained by replacing the classical 'least' function associated with ideals by a two-dimensional 'last' function associated with walks on ordinals. Consequently, we obtain conditions under which a measure admits not just infinite pairwise disjoint families of positive sets, but in fact families of maximum possible size. As an application we solve a problem left open in Shelah's *Cardinal Arithmetic* book, proving that for every weakly inaccessible cardinal κ, if there exists a stationary subset of κ that does not reflect at regulars, then the strong Ramsey relation $\kappa \nrightarrow [\kappa]^2_\kappa$ holds.

1. Introduction

Ulam's measure problem [Ula30] studies when there is a total two-valued measure on the powerset of a set. To identify some of the obstructions, Ulam introduced the *Ulam matrices*, which in particular he used to show that there is no total countably-additive two-valued measure on the powerset of the first uncountable cardinal, $\aleph_1$. This is done by inspecting saturation properties of the corresponding ideal of null sets. As it turns out, roughly around that time Sierpiński [Sie34] in his work on the Continuum Hypothesis

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(CH, the assertion that $\mathbb{R}$ has size $\aleph_1$) had formulated some colouring principles which are in effect generalizations of Ulam's matrices. Having made this connection, and anticipating applications in infinite combinatorics, in [IR23, IR24], we initiated the study of Sierpiński-type colourings in some generality.

The simplest of all, the principle $\mathsf{onto}(J,\theta)$ for an ideal J over a cardinal κ asserts the existence of an edge-colouring of the complete graph on κ, $c:[\kappa]^2\to\theta$, satisfying that for every J-positive set B, there exists a vertex $\eta<\kappa$ such that the bipartite subgraph $\{(\eta,\beta)\mid\beta\in B \text{ above } \eta\}$ attains all colours, that is, $c[\{\eta\}\circledast B]=\theta$. This elegant principle as well as some of its weaker variants are all sufficient to imply that every J-positive set may be partitioned into θ many J-positive sets (see [IR24, §4] for exact statements). It also allows to characterize classical large cardinals, for instance, a regular uncountable cardinal κ is weakly compact iff $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],3)$ fails, and is ineffable iff $\mathsf{onto}(\mathrm{NS}_\kappa,2)$ fails.[1] Deeper applications to infinite combinatorics have quickly emerged [LHR23, CIR26, KRS26], for instance, Shalev [Sha24] using our results constructed a partial order previously obtained by Todorčević and Kuzeljević [KT23] under an additional axiomatic assumption.

As for this paper, being the third part of the series, the various technical results of [IR23, IR24], and in particular [IR24, Theorem A] clarify the picture considerably, leaving us with a more focused goal in this paper, to take care of the remaining extreme cases, and to consider an important class of ideals that was previously out of our reach. The classes of ideals we studied earlier were characterized mainly by their additivity (or *completeness degree* – closure under arbitrary unions of a given size), and here the classes of ideals are determined by their indecomposability spectrum (closure under increasing unions of a given length). As will be seen momentarily, this subtle difference will enable to solve a problem left open in Shelah's monograph [She94a].

As an advertisement, we note that this paper can be read without a familiarity with the previous parts of this series, taking results from there as a black box to draw corollaries. Indeed, this paper can be read by anyone with background in set theory; the last two sections also assume familiarity with walks on ordinals [Tod87].

Our first result is the following improvement of the characterization of large cardinals mentioned earlier.

Theorem A. *Suppose that $\theta<\kappa$ is a pair of infinite regular cardinals.*

- *If there exists a κ-Aronszajn tree, then $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\theta)$ holds;*
- *If the ineffable tree property fails at κ, then $\mathsf{onto}(\mathrm{NS}_\kappa,\theta)$ holds.*

Our second result is an instance of $\mathsf{unbounded}^+(\ldots)$ with a provably optimal number of colours. In fact, all the results here are sharp as will be explained soon.

Theorem B. *Suppose that κ is a regular uncountable cardinal that is not strongly inaccessible, and let θ denote the least cardinal to satisfy $2^\theta\geq\kappa$.*

Then $\mathsf{unbounded}^+(J^{\mathrm{bd}}[\kappa],\theta)$ holds.

[1] $J^{\mathrm{bd}}[\kappa]$ stands for the ideal of bounded subsets of κ; NS_κ stands for the ideal of nonstationary subsets of κ. Undefined terminology can be found in Section 2.

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This takes care of the remaining cases left open by our previous works. Moving on to deal with indecomposable ideals, our next result improves a theorem of Eisworth [Eis12, §3], by itself an improvement of classical theorems of Chang [Cha67, §1], Kunen and Prikry [KP71], and, of course, Ulam's original [Ula30].

Theorem C. *Suppose λ is an infinite cardinal, and J is a uniform ideal over λ^+.*
If J is $\mathrm{cf}(\lambda)$-indecomposable and $\sup\{\theta\in\mathrm{Reg}(\lambda^+)\mid J \text{ is } \theta\text{-indecomposable}\}=\lambda$, then $\mathsf{unbounded}^{\square}(J,\lambda^+)$ holds.

Compared to the classical proof approach used by Ketonen [Ket73], Prikry [Pri73], Kanamori [Kan76], as well as in the more recent work of Shelah [She94b, §3] and its extensions by Eisworth [Eis10, Eis12] that involve the notion of a *least function* modulo an ideal, the proofs here use the two-dimensional *last* function isolated by Zhang and Rinot in [RZ21, Definition 2.10] in the context of walks on ordinals.

As further evidence of the utility of our method, we have the following theorem. Its first clause improves a corollary arising from [LHR23, Theorem 4.13(2) and Lemma 3.38(3)]. Its second clause improves a theorem of Prikry and Silver [Pri73, Theorem 4] and a theorem of Eisworth [Eis10, Theorem 2]. Its third clause improves a milestone result of Hajnal [Haj69] which was the first to address limit cardinals.

Theorem D. *Suppose $\mu<\kappa$ is a pair of infinite regular cardinals, and J is a μ-complete uniform ideal over κ. Consider $T:=\{\tau<\kappa\mid J \text{ is } \mathrm{cf}(\tau)\text{-indecomposable}\}$.*

- *if $\square(\kappa,<\mu)$ holds and $T\neq\emptyset$, then $\mathsf{unbounded}^{+}(J,\kappa)$ holds;*
- *if there is a family of less than μ many stationary subsets of T that do not reflect simultaneously, then $\mathsf{unbounded}^{\square}(J,\kappa)$ holds;*
- *if there is a family of less than μ many stationary subsets of T that do not reflect simultaneously at regulars, and $\sup\{\theta\in\mathrm{Reg}(\kappa)\mid J \text{ is } \theta\text{-indecomposable}\}=\kappa$, then $\mathsf{unbounded}^{\square}(J,\kappa)$ holds.*

We now present an application to Ramsey theory of the higher infinite. In his *Cardinal Arithmetic* book, Shelah proved [She94b, Conclusion 4.8(2)] that for a weakly inaccessible κ and a regular $\chi<\kappa$, if $E^{\kappa}_{\geq\chi}$ admits a stationary set that does not reflect at regulars, then the strong anti-Ramsey principle $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ holds for every $\theta<\kappa$. Whether this may be improved to get $\theta=\kappa$ remained open, yet it was clear that an affirmative answer would follow from a particular extension of Hajnal's 1969 theorem weaker than the third clause of Theorem D.

Theorem E. *Suppose that κ is weakly inaccessible, $\chi\in\mathrm{Reg}(\kappa)$, and $E^{\kappa}_{\geq\chi}$ admits a stationary subset that does not reflect at regulars. Then $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$ holds.*

This is the final cut. A combination of [IR24, Theorem A] and [LHR21, Theorem 3.3] yields that Theorem A cannot be improved to get $\theta=\kappa$. By [IR23, Theorem 9.10], the number of colours in Theorem B cannot be more than θ. By [BM86, Theorem 1] and [KT64, Theorem 4.16], neither the first nor the second requirement of Theorem C can be dropped. By [LHRZ25, Theorem C], neither the requirement that J be μ-complete nor the focus on stationary subsets of T in Theorem D can be dropped. By [LHR21, Corollary 6.6], Theorem E cannot be improved to get $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi^+)$.

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1.1. Organization of this paper

In Section 2, we collect various notation that we will use throughout the paper. We also define here the Sierpiński-type colouring principles which are the subject of this paper and prove some basic results about them.

In Section 3, we prove Theorem A. Our focus here is the effect of the failure of tree properties on the colouring principle $\mathsf{onto}(\dots)$.

In Section 4, we prove Theorem B. Our focus here is to obtain the maximum possible number of colours for the principle $\mathsf{unbounded}^{+}(\dots)$.

In Section 5, we prove some results about the new principle $\mathsf{unbounded}^{\square}(\dots)$ which we introduce in this paper. This is where our focus shifts to ideals characterised by their indecomposability spectrum.

In Section 6, we push further, aiming for the maximum number of colours for ideals characterised by their indecomposability spectrum. Theorems C and D are proved here.

In Section 7, we give some applications to anti-Ramsey theory. Theorem E is proved here.

2. Preliminaries

2.1. Conventions

Throughout the paper, κ stands for a regular uncountable cardinal, λ stands for an infinite cardinal, and θ,μ,ν,χ stand for arbitrary cardinals. We assume the reader is comfortable with *walks on ordinals*; our conventions in this respect may be found in [IR25, §4]. The definitions of square principles such as $\square(\kappa,<\mu)$ may be found in [BR19, Definition 1.16]. $\mathrm{Reg}(\kappa)$ denotes the collection of all infinite regular cardinals below κ, and $\mathrm{Sing}(\kappa)$ denotes the collection of all limit singular ordinals below κ. For A,B sets of ordinals, we denote $A \circledast B := \{(\alpha,\beta) \in A \times B \mid \alpha < \beta\}$ and we write $[B]^2$ for $B \circledast B$. For $\theta \neq 2$, $[B]^{\theta}$ stands for the collection of all subsets of B of size θ. Let $E^{\kappa}_{\theta} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$, and define $E^{\kappa}_{\leq\theta}$, $E^{\kappa}_{<\theta}$, $E^{\kappa}_{\geq\theta}$, $E^{\kappa}_{>\theta}$, $E^{\kappa}_{\neq\theta}$ analogously. For a set of ordinals A, we write $\mathrm{ssup}(A) := \sup\{\alpha+1 \mid \alpha \in A\}$, $\mathrm{acc}^{+}(A) := \{\alpha < \mathrm{ssup}(A) \mid \sup(A \cap \alpha) = \alpha > 0\}$, $\mathrm{acc}(A) := A \cap \mathrm{acc}^{+}(A)$, and $\mathrm{nacc}(A) := A \setminus \mathrm{acc}(A)$. For a stationary $S \subseteq \kappa$, we write $\mathrm{Tr}(S) := \{\alpha \in E^{\kappa}_{>\omega} \mid S \cap \alpha \text{ is stationary in } \alpha\}$; we say that S *reflects* (resp. *reflects at regulars*) iff $\mathrm{Tr}(S)$ is nonempty (resp. contains a regular cardinal). A family $\mathcal{S}$ of stationary subsets of κ is said to *reflect* (resp. *reflect at regulars*) iff $\bigcap_{S\in\mathcal{S}} \mathrm{Tr}(S)$ is nonempty (resp. contains a regular cardinal). For two functions f,g from ordinals to ordinals, we let

$$\Delta(f,g) := \min\{\mathrm{dom}(f),\mathrm{dom}(g),\delta \mid \delta \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \ \&\ f(\delta) \neq g(\delta)\}.$$

2.2. Ideals

For an ideal J over κ, the collection of J-positive sets is denoted by $J^{+} := \mathcal{P}(\kappa) \setminus J$. We say that J is *uniform* iff $J \supseteq J^{\mathrm{bd}}[\kappa]$. For a cardinal θ, J is *weakly θ-saturated* if there is no family of θ-many pairwise disjoint elements of J^{+}. Given $B \in J^{+}$, we write $J \restriction B$ for $J \cap \mathcal{P}(B)$. The ideal J is *ν-complete* iff it is closed under unions of length less than ν; it is *θ-indecomposable* iff for every function $f : B \to \theta$ with $B \in J^{+}$, there exists a $T \in [\theta]^{<\theta}$ such that $f^{-1}[T]$ is in J^{+}.

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- $\mathcal{J}^\kappa_\nu$ denotes the collection of all ν-complete ideals over κ extending $J^{\rm bd}[\kappa]$;
- $\mathcal{I}^\kappa_\theta$ denotes the collection of all $I \in \mathcal{J}^\kappa_\omega$ that are θ-indecomposable.

Note that $\mathcal{J}^\kappa_\theta \cap \mathcal{I}^\kappa_\theta = \mathcal{J}^\kappa_{\theta^+}$. In particular, $\mathcal{I}^\kappa_\omega = \mathcal{J}^\kappa_{\omega_1}$. Also note that if $\mathcal{I}^\kappa_\theta$ contains a proper ideal, then $\mathrm{cf}(\kappa) \neq \theta$.

Example 2.1. Given a bijection $\pi : \kappa \leftrightarrow \kappa \times \kappa$, let J be the collection of all $A \subseteq \kappa$ such that $\sup\{\eta_0 < \kappa \mid \sup\{\eta_1 < \kappa \mid \pi^{-1}(\eta_0,\eta_1) \in A\} = \kappa\} < \kappa$. Then $J \in \mathcal{J}^\kappa_\kappa$.

2.3. Sierpiński-type colouring principles

In reading the next definition, recall that a colouring $c : [\kappa]^2 \to \theta$ is *upper-regressive* if $c(\alpha,\beta) < \beta$ for all $\alpha < \beta < \kappa$.

Definition 2.2. For a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ and an ideal J over κ:

(i) $\mathsf{onto}^{++}(\mathcal{A},J,\theta)$ asserts the existence of a colouring $c : [\kappa]^2 \to \theta$ with the property that, for every $A \in \mathcal{A}$ and every sequence $\langle B_\tau \mid \tau < \theta\rangle$ of elements of J^+, there is an $\eta \in A$ such that $\{\beta \in B_\tau \setminus (\eta+1) \mid c(\eta,\beta) = \tau\} \in J^+$ for every $\tau < \theta$;

(ii) $\mathsf{onto}^{+}(\mathcal{A},J,\theta)$ asserts the existence of a colouring $c : [\kappa]^2 \to \theta$ with the property that, for all $A \in \mathcal{A}$ and $B \in J^+$, there is an $\eta \in A$ such that

$$\{\tau < \theta \mid \{\beta \in B \setminus (\eta+1) \mid c(\eta,\beta) = \tau\} \in J^+\} = \theta;$$

(iii) $\mathsf{unbounded}^{+}(\mathcal{A},J,\theta)$ asserts the existence of an upper-regressive colouring $c : [\kappa]^2 \to \theta$ with the property that, for all $A \in \mathcal{A}$ and $B \in J^+$, there is an $\eta \in A$ such that

$$\mathrm{otp}(\{\tau < \theta \mid \{\beta \in B \setminus (\eta+1) \mid c(\eta,\beta) = \tau\} \in J^+\}) = \theta;$$

(iv) $\mathsf{unbounded}^{\square}(\mathcal{A},J,\theta)$ asserts the existence of an upper-regressive colouring $c : [\kappa]^2 \to \theta$ with the property that, for all $A \in \mathcal{A}$ and $B \in J^+$, there are an $\eta \in A$ and a set $T \in [\theta]^\theta$ such that, for every pair $\sigma < \tau$ of ordinals from T, $\{\beta \in B \setminus (\eta+1) \mid \sigma \leq c(\eta,\beta) < \tau\} \in J^+$.

Convention 2.3. If we omit $\mathcal{A}$, then we mean that $\mathcal{A} := \{\kappa\}$. As for the second parameter, we may also assign a family $\mathcal{J}$ of ideals instead of a single ideal, in which case we mean that the colouring c witnesses the instance for all $J \in \mathcal{J}$ simultaneously.

It is not hard to see that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) in Definition 2.2. In [IR24, §4], one can find methods for proving nontrivial implications between instances of the above. The principles appearing in Clauses (i)–(iii) were introduced in [IR23], and their connection to Ulam matrices is discussed in Section 5 of that paper. Clause (iv) is new and is still sufficient to deduce non-weak saturation, as stated in Clause (1) of the upcoming lemma. The other clauses of the lemma will only be used in Sections 5 and 6, hence we recommend that the reader skip it on a first pass.

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Lemma 2.4. *Suppose that* $\mathsf{unbounded}^{\square}(\mathcal{A},\mathcal{I},\theta)$ *holds with* $\mathcal{A}\neq\emptyset$ *and* $\mathcal{I}\subseteq\mathcal{J}^{\kappa}_{\omega}$. *Then:*

(1) *For every* $I\in\mathcal{I}$, *every* $B\in I^+$ *may be decomposed into* θ*-many* I^+*-sets.*

(2) *If* $\mathcal{A}=\{\nu\}$ *and* $\theta=\vartheta^+$ *for a regular uncountable cardinal* $\vartheta<\nu$, *then* $\mathsf{unbounded}^{+}(\{\nu\},\mathcal{I},\vartheta)$ *holds.*

(3) *Suppose* $\theta=\omega$.
- *If* $\mathcal{A}=\{\nu\}$ *and* $\mathfrak{d}\leq\nu$, *then* $\mathsf{onto}^{+}(\{\nu\},\mathcal{I},\omega)$ *holds.*
- *If* $\mathcal{A}=J^+$ *for some countably-complete proper ideal* J *over* κ *and* $\mathfrak{d}\leq\kappa$, *then* $\mathsf{onto}^{++}(\mathcal{I},\omega)$ *holds.*

(4) *Suppose* θ *is regular uncountable, and* $\vartheta\leq\theta$.
- *If* $\mathcal{A}=\{\nu\}$ *and* $\mathcal{C}(\theta,\vartheta)\leq\nu$,[2] *then* $\mathsf{onto}^{+}(\{\nu\},\mathcal{I},\vartheta)$ *holds.*
- *If* $\mathcal{A}=J^+$ *for some* ϑ^+*-complete proper ideal* J *over* κ *and* $\mathcal{C}(\theta,\vartheta)\leq\kappa$, *then* $\mathsf{onto}^{++}(\mathcal{I},\vartheta)$ *holds.*

Proof. Fix a colouring $c:[\kappa]^2\to\theta$ witnessing $\mathsf{unbounded}^{\square}(\mathcal{A},\mathcal{I},\theta)$.

(1) Given $B\in I^+$ with $I\in\mathcal{I}$, find $\eta<\kappa$ and $T\in[\theta]^{\theta}$ such that, for every pair $\sigma<\tau$ of ordinals from T, $\{\beta\in B\setminus(\eta+1)\mid\sigma\leq c(\eta,\beta)<\tau\}\in I^+$. For each $\sigma\in T$, let

$$B_\sigma:=\{\beta\in B\setminus(\eta+1)\mid\sigma\leq c(\eta,\beta)<\min(T\setminus(\sigma+1))\}.$$

Then $\langle B_\sigma\mid\sigma\in T\rangle$ is a pairwise disjoint family of I^+-subsets of B. By replacing $B_{\min(T)}$ with $B\setminus\bigcup_{\sigma\in T\setminus\{\min(T)\}}B_\sigma$, we then get a decomposition of B into θ many I^+-sets.

(2) Suppose that $\theta=\vartheta^+$ for a regular uncountable cardinal $\vartheta<\nu$. By [She94b, Claim 2.4(b)], we may fix a ϑ-bounded C-sequence $\langle C_\alpha\mid\alpha\in E^{\theta}_{\vartheta}\rangle$[3] such that, for every club $D\subseteq\theta$, there exists $\alpha\in E^{\theta}_{\vartheta}$ such that $\sup(\mathrm{nacc}(C_\alpha)\cap D)=\alpha$. Let $\langle(\varsigma_\eta,\alpha_\eta)\mid\eta<\nu\rangle$ be a listing of the elements of $\nu\times E^{\theta}_{\vartheta}$. Define a colouring $d:[\kappa]^2\to\vartheta$ by letting for all $\eta<\beta<\kappa$:

$$d(\eta,\beta):=\begin{cases}\mathrm{otp}(C_{\alpha_\eta}\cap c(\varsigma_\eta,\beta)), & \text{if } \eta<\nu \ \&\ c(\varsigma_\eta,\beta)<\alpha_\eta;\\ 0, & \text{otherwise.}\end{cases}$$

To see that d witnesses $\mathsf{unbounded}^{+}(\{\nu\},\mathcal{I},\vartheta)$, let $I\in\mathcal{I}$ and let $B\in I^+$. Pick $\varsigma<\nu$ and $T\in[\theta]^{\theta}$ such that, for every pair $\sigma<\sigma'$ of ordinals from T, $\{\beta\in B\setminus(\varsigma+1)\mid\sigma\leq c(\varsigma,\beta)<\sigma'\}\in I^+$. Let $D:=\mathrm{acc}^{+}(T)$. Find $\eta<\nu$ such that $\varsigma_\eta=\varsigma$ and $\sup(\mathrm{nacc}(C_{\alpha_\eta})\cap D)=\alpha_\eta$. Let $X:=\{\gamma\in C_{\alpha_\eta}\mid\min(C_{\alpha_\eta}\setminus(\gamma+1))\in D\}$, so that $Z:=\{\mathrm{otp}(C_{\alpha_\eta}\cap(\gamma+1))\mid\gamma\in X\}$ is in $[\vartheta]^{\vartheta}$.

Claim 2.4.1. Let $\tau\in Z$. Then $B^{\eta,\tau}:=\{\beta\in B\setminus(\eta+1)\mid d(\eta,\beta)=\tau\}$ is in I^+.

Proof. Let $\gamma\in X$ be such that $\mathrm{otp}(C_{\alpha_\eta}\cap(\gamma+1))=\tau$ and let $\delta:=\min(C_{\alpha_\eta}\setminus(\gamma+1))$. As $\delta\in\mathrm{acc}^{+}(T)$, we may pick $\sigma<\sigma'$ from T such that $\gamma<\sigma<\sigma'<\delta$, so that the following set is in I^+:

[2] See Definition 5.4 below. For instance, $\mathcal{C}(\theta,n)=\theta$ for every $n<\omega$, and $\clubsuit(\vartheta^+)$ implies $\mathcal{C}(\vartheta^+,\vartheta)=\vartheta^+$.

[3] Recall that this means that for every $\alpha\in E^{\theta}_{\vartheta}$, C_α is a closed unbounded subset of α of ordertype ϑ.

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$$B' := \{\beta \in B \setminus (\eta+1) \mid \sigma \leq c(\varsigma,\beta) < \sigma'\}.$$

Now, for every $\beta \in B'$,

$$d(\eta,\beta) = \mathrm{otp}(C_{\alpha_\eta} \cap c(\varsigma_\eta,\beta)) = \mathrm{otp}(C_{\alpha_\eta} \cap (\gamma+1)) = \tau,$$

as sought. □

This completes the proof.

(3) By the definition of $\mathfrak{d}$, we may fix a family $\mathcal{F} \subseteq {}^{\omega}\omega$ of size $\mathfrak{d}$ such that for every $g : \omega \to \omega$ there exists an $f \in \mathcal{F}$ such that $g(m) \leq f(m)$ for every $m < \omega$. Let μ be the least cardinal such that $\mathfrak{d} \leq \mu$ and $\mathcal{A} \subseteq \mathcal{P}(\mu)$, and suppose μ is no more than κ. Let $\langle(\varsigma_\eta, f_\eta) \mid \eta < \mu\rangle$ be a listing of the elements of $\mu \times \mathcal{F}$. For every function $f : \omega \to \omega$, let $f^* : \omega \to \omega$ denote some strictly increasing function satisfying $f^*(n+1) > f(f^*(n)+1)$ for every $n < \omega$. Finally, fix a colouring $d : [\kappa]^2 \to \omega$ such that for all $\eta < \beta < \kappa$ with $\eta < \mu$ and $\varsigma_\eta < \beta$,

$$d(\eta,\beta) = \min\{n < \omega \mid c(\varsigma_\eta,\beta) < f^*_\eta(n+1)\}.$$

We claim that if $\mathcal{A} = \{\mu\}$, then d witnesses $\mathsf{onto}^+(\{\mu\},\mathcal{I},\omega)$, and if $\mathcal{A} = J^+$ for some countably-complete proper ideal J over κ, then d witnesses $\mathsf{onto}^{++}(\mathcal{I},\omega)$. We shall focus on verifying the latter, moreover showing that given a sequence $\langle(I_n, B_n) \mid n < \omega\rangle$ such that $I_n \in \mathcal{I}$ and $B_n \in I_n^+$ for every $n < \omega$, there exists an $\eta < \kappa$ such that $\{\beta \in B_n \setminus (\eta+1) \mid d(\eta,\beta) = n\} \in I_n^+$ for every $n < \omega$.

For each $n < \omega$, let A_n denote the set of all $\varsigma < \kappa$ for which there is a $T \in [\omega]^\omega$ such that, for every pair $\sigma < \tau$ of integers from T, $\{\beta \in B_n \setminus (\varsigma+1) \mid \sigma \leq c(\varsigma,\beta) < \tau\} \in I_n^+$. As c witnesses $\mathsf{unbounded}^{\square}(J^+,\mathcal{I},\theta)$, A_n must be in J^*. So, as J is a countably-complete proper ideal, we may pick a $\varsigma \in \bigcap_{n<\omega} A_n$. For each $n < \omega$, fix $T_n \in [\omega]^\omega$ witnessing that $\varsigma \in A_n$, and then let $g_n : \omega \leftrightarrow T_n$ be the order-preserving bijection. Define $g : \omega \to \omega$ via $g(m) := \max\{g_n(m) \mid n \leq m\}$. Find $\eta < \mu$ such that $\varsigma_\eta = \varsigma$ and $g(m) \leq f_\eta(m)$ for every $m < \omega$.

Claim 2.4.2. Let $n < \omega$. Then $B^\eta_n := \{\beta \in B_n \setminus (\eta+1) \mid d(\eta,\beta) = n\}$ is in I_n^+.

Proof. Set $\sigma := g_n(f^*_\eta(n))$ and $\tau := g_n(f^*_\eta(n)+1)$. Clearly,

$$f^*_\eta(n) \leq g_n(f^*_\eta(n)) = \sigma < \tau \leq g(f^*_\eta(n)+1) \leq f_\eta(f^*_\eta(n)+1) < f^*_\eta(n+1),$$

and hence B^η_n covers

$$\{\beta \in B_n \setminus (\max\{\varsigma,\eta\}+1) \mid \sigma \leq c(\varsigma,\beta) < \tau\},$$

which belongs to I_n^+. □

(4) Let $\mathcal{X} \subseteq [\theta]^\vartheta$ be a family of sets of ordertype ϑ witnessing the value of $\mathcal{C}(\theta,\vartheta)$, so that for every club $D \subseteq \theta$ there is an $X \in \mathcal{X}$ such that $X \subseteq D$. Let μ be the least cardinal such that $|\mathcal{X}| \leq \mu$ and $\mathcal{A} \subseteq \mathcal{P}(\mu)$, and suppose μ is no more than κ. Let $\langle(\varsigma_\eta, X_\eta) \mid \eta < \mu\rangle$ be a listing of the elements of $\mu \times \mathcal{X}$. Define a colouring $d : [\kappa]^2 \to \vartheta$ by letting for all $\eta < \beta < \kappa$ with $\varsigma_\eta < \beta$:

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$$d(\eta,\beta) := \begin{cases} \sup(\mathrm{otp}(X_\eta \cap c(\varsigma_\eta,\beta))), & \text{if } \eta<\mu \ \&\ c(\varsigma_\eta,\beta)<\sup(X_\eta);\\ 0, & \text{otherwise.}\end{cases}$$

We claim that if $\mathcal{A}=\{\mu\}$, then d witnesses $\mathsf{onto}^+(\{\mu\},\mathcal{I},\vartheta)$, and if $\mathcal{A}=J^+$ for some ϑ^+-complete proper ideal J over κ, then d witnesses $\mathsf{onto}^{++}(\mathcal{I},\vartheta)$. We shall focus on verifying the latter, moreover showing that given a sequence $\langle (I_\tau,B_\tau) \mid \tau<\vartheta\rangle$ such that $I_\tau\in\mathcal{I}$ and $B_\tau\in I_\tau^+$ for every $\tau<\vartheta$, there exists an $\eta<\kappa$ such that $\{\beta\in B_\tau\setminus(\eta+1) \mid d(\eta,\beta)=\tau\}\in I_\tau^+$ for every $\tau<\vartheta$. As in the proof of Clause (3), fix $\varsigma<\kappa$ and a sequence $\langle T_\tau \mid \tau<\vartheta\rangle$ of elements of $[\theta]^\theta$ such that, for every $\tau<\vartheta$, for every pair $\sigma<\sigma'$ of ordinals from T_τ, $\{\beta\in B_\tau\setminus(\varsigma+1) \mid \sigma\le c(\varsigma,\beta)<\sigma'\}\in I_\tau^+$. Denote $D_\tau:=\mathrm{acc}^+(T_\tau)$. If $\vartheta<\theta$, then let $D:=\bigcap_{\tau<\vartheta}D_\tau$, and if $\vartheta=\theta$, then let $D:=\triangle_{\tau<\vartheta}D_\tau$. Find $\eta<\mu$ such that $\varsigma_\eta=\varsigma$ and $X_\eta\subseteq D$.

Claim 2.4.3. Let $\tau<\vartheta$. Then $B_\tau^\eta:=\{\beta\in B_\tau\setminus(\eta+1) \mid d(\eta,\beta)=\tau\}$ is in I_τ^+.

Proof. Let $\gamma\in X_\eta$ be such that $\mathrm{otp}(X_\eta\cap\gamma)=\tau$ and let $\delta:=\min(X_\eta\setminus(\gamma+1))$. As $\delta\in D\setminus(\tau+1)$, we infer that $\delta\in D_\tau=\mathrm{acc}^+(T_\tau)$. Thus we may pick $\sigma<\sigma'$ from T_τ such that $\gamma<\sigma<\sigma'<\delta$. Consequently, the following set is in I_τ^+:

$$B':=\{\beta\in B_\tau\setminus(\max\{\varsigma,\eta\}+1) \mid \sigma\le c(\varsigma,\beta)<\sigma'\}.$$

Now, for every $\beta\in B'$,

$$d(\eta,\beta)=\sup(\mathrm{otp}(X_\eta\cap c(\varsigma_\eta,\beta)))=\sup(\mathrm{otp}(X_\eta\cap(\gamma+1)))=\sup(\tau+1)=\tau,$$

as sought. □

This completes the proof. □

Remark 2.5. Various results from [IR23, IR24] have analogues for this new colouring principle $\mathsf{unbounded}^{\square}(\ldots)$ with similar proofs making the obvious changes. As two concrete examples, we point to [IR23, Lemma 5.12(2)] and [IR23, Lemma 6.15(2)].

Lemma 2.6. *Let* p *be one of the four colouring principles of Definition 2.2, and suppose* $\mu<\kappa$.

(1) *If* $\nu\in[\mu,\kappa)$ *is a cardinal such that* $\mathsf{p}(\mathcal{A},\mathcal{J}_\mu^\nu,\theta)$ *holds, then for every* $I\in\mathcal{J}_\mu^\kappa\setminus\mathcal{I}_\nu^\kappa$, *there is a* $B\in I^+$ *such that* $\mathsf{p}(\mathcal{A},I\restriction B,\theta)$ *holds;*

(2) *If there exists a cardinal* $\nu\in[\mu,\kappa)$ *such that* $\mathsf{p}(\mathcal{I}_\nu^\kappa,\theta)$ *and* $\mathsf{p}(\mathcal{J}_\mu^\nu,\theta)$ *both hold, then every* $I\in\mathcal{J}_\mu^\kappa$ *is not weakly* θ*-saturated.*

Proof. (1) Suppose that $\nu\in[\mu,\kappa)$ is a cardinal and $c:[\nu]^2\to\theta$ is a map witnessing that $\mathsf{p}(\mathcal{A},\mathcal{J}_\mu^\nu,\theta)$ holds. Now given $I\in\mathcal{J}_\mu^\kappa\setminus\mathcal{I}_\nu^\kappa$, fix a function $f:B\to\nu$ such that $B\in I^+$ but $f^{-1}[H]\in I$ for all $H\in[\nu]^{<\nu}$. Clearly, the f-image of any J-positive subset of B has size ν and hence $J:=\{f[X] \mid X\in I\restriction B\}$ belongs to $\mathcal{J}_\mu^\nu$. Finally, define $d:[\kappa]^2\to\theta$ via $d(\eta,\beta):=c(\eta,f(\beta))$, or $d(\eta,\beta):=0$ in case the former requirement would be ill-defined or contradict a needed instance of upper-regressiveness. Then d witnesses that $\mathsf{p}(\mathcal{A},I\restriction B,\theta)$ holds.

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(2) Suppose that $\nu \in [\mu,\kappa)$ is a cardinal such that $\mathsf{p}(\mathcal{I}^\kappa_\nu,\theta)$ and $\mathsf{p}(\mathcal{J}^\nu_\mu,\theta)$ both hold. Now, given $I \in \mathcal{J}^\kappa_\mu$, there are two options. If $I \in \mathcal{I}^\kappa_\nu$, then, by $\mathsf{p}(\mathcal{I}^\kappa_\nu,\theta)$, every I-positive set may be decomposed into θ-many I-positive sets, and we are done. Otherwise, by Clause (1), there exists some $B \in I^+$ that may be decomposed into θ-many I-positive sets. □

Corollary 2.7. *Suppose that $\theta < \kappa$ is a pair of regular uncountable cardinals and for every $\mu \in \mathrm{Reg}(\kappa) \setminus \theta$ one of the following holds:*

- *$\chi(\mu) > 1$;*[4]
- *μ is not greatly Mahlo;*
- *μ is not weakly compact in L;*
- *$\mu > \theta$ and there is a μ-Aronszajn tree.*

Then every θ-complete κ-incomplete uniform ideal over κ is not weakly θ-saturated.

Proof. Note that by [LHR21, Lemma 2.12], the second and third items imply the first item.

Let $I \in \mathcal{J}^\kappa_\theta \setminus \mathcal{J}^\kappa_\kappa$ and then let λ be the least cardinal such that I is not λ-complete. Note that λ must be a successor cardinal $\lambda = \mu^+$ for some $\mu \in [\theta,\kappa)$. So, $I \in \mathcal{J}^\kappa_\mu \setminus \mathcal{I}^\kappa_\mu$. By Lemma 2.6(1), it thus suffices to prove that $\mathsf{unbounded}^+(\mathcal{J}^\mu_\mu,\theta)$ holds.

▶ If $\chi(\mu) > 1$, then $\mathsf{unbounded}^+(\mathcal{J}^\mu_\mu,\mu)$ holds by [IR24, Theorem A], which is enough by monotonicity [IR24, Corollary 4.18(1)].

▶ If $\mu > \theta$ and there is a μ-Aronszajn tree, then by the upcoming Theorem 3.1, $\mathsf{onto}(J^{\mathrm{bd}}[\mu],\theta)$ holds, and by [IR24, Theorem 4.1(1)] this implies that $\mathsf{onto}^+(\mathcal{J}^\mu_\mu,\theta)$ holds. □

3. The tree property and the ineffable tree property

In this section we prove Theorem A, and then draw conclusions.

Theorem 3.1. *Suppose that κ is a regular uncountable cardinal admitting a κ-Aronszajn tree. For every $\theta \in \mathrm{Reg}(\kappa)$, $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\theta)$ holds.*

Proof. The case $\kappa = \theta^+$ is covered by [IR23, Corollary 7.3], so suppose that we are given a $\theta \in \mathrm{Reg}(\kappa)$ with $\theta^+ < \kappa$. By Shelah's club-guessing theorem [She94b, Claim 2.7], we may pick a sequence $\vec{C} = \langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$ such that:

- For every $\delta \in E^\kappa_\theta$, C_δ is a club in δ of order-type θ;
- For every club $C \subseteq \kappa$, there exists a $\delta \in E^\kappa_\theta$ such that $C_\delta \subseteq C$.

As there is a κ-Aronszajn tree, a standard argument (see [BR17, Appendix]) yields a κ-Aronszajn subtree of $({}^{<\kappa}2,\subseteq)$,[5] i.e., a $T \subseteq {}^{<\kappa}2$ satisfying all of the following:

[4]The cardinal characteristic $\chi(\mu)$ was defined in [LHR21, Definition 1.6] as a rough measure of how close is μ to being weakly compact. By [LHR21, Theorem 3.3], it is consistent (from large cardinals) that $\chi(\mu) = 1$ and yet there is a μ-Aronszajn tree.

[5]The shift into working with a binary tree is just for convenience.

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(i) For every $t \in T$ and $\alpha < \mathrm{dom}(t)$, $t \restriction \alpha \in T$;

(ii) For every $\alpha < \kappa$, the set $T_\alpha := \{t \in T \mid \mathrm{dom}(t) = \alpha\}$ satisfies $0 < |T_\alpha| < \kappa$;

(iii) For every $f : \kappa \to 2$, there exists an $\alpha < \kappa$ such that $f \restriction \alpha \notin T$.

For every $\gamma < \kappa$, denote $T \restriction \gamma := \bigcup_{\alpha<\gamma} T_\alpha$. Fix an injective enumeration $\langle a_\eta \mid \eta < \kappa\rangle$ of T. Finally, define $c : [\kappa]^2 \to \theta$ as follows. Given $\eta < \beta < \kappa$, if $\mathrm{dom}(a_\eta) \in E^\kappa_\theta$ and $a_\eta \not\sqsubseteq a_\beta$, then let

$$c(\eta,\beta) := \sup(\mathrm{otp}(C_{\mathrm{dom}(a_\eta)} \cap \Delta(a_\eta, a_\beta)));$$

otherwise, just let $c(\eta,\beta) := 0$.

To see that c witnesses $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\theta)$, let an arbitrary $B \in [\kappa]^\kappa$ be given. Denote

$$T^{\rightsquigarrow B} := \{t \in T \mid |\{\beta \in B \mid t \sqsubseteq a_\beta\}| = \kappa\}.$$

Claim 3.1.1. $T^{\rightsquigarrow B}$ maintains properties (i)–(iii), and there exists a club $C \subseteq \kappa$ such that for every $\gamma \in C$ and every $t \in (T^{\rightsquigarrow B}) \restriction \gamma$, there are two incompatible extensions of t in $(T^{\rightsquigarrow B})_\gamma$.

Proof. It is clear that Clauses (i) and (iii) hold for $T^{\rightsquigarrow B}$. To see that Clause (ii) holds, it suffices to verify that $(T^{\rightsquigarrow B})_\alpha$ is nonempty for every $\alpha < \kappa$. As $\emptyset$ is clearly in $T^{\rightsquigarrow B}$, we opt to showing that $T^{\rightsquigarrow B}$ is in fact *normal*, i.e., for every $t \in T^{\rightsquigarrow B}$ and every $\alpha < \kappa$ above $\mathrm{dom}(t)$, there exist an $s \in (T^{\rightsquigarrow B})_\alpha$ with $t \subseteq s$. Thus, given t and α as above, since $|T \restriction \alpha| < \kappa = |\{\beta \in B \mid t \sqsubseteq a_\beta\}|$, the set $B' := \{\beta \in B \mid t \sqsubseteq a_\beta \ \&\ a_\beta \notin T \restriction \alpha\}$ has size κ. As $|T_\alpha| < \kappa = |B'|$, the pigeonhole principle implies the existence of an $s \in T_\alpha$ for which $|\{\beta \in B' \mid a_\beta \restriction \alpha = s\}| = \kappa$. Clearly, $t \subseteq s \in (T^{\rightsquigarrow B})_\alpha$.

Combining normality together with Clause (iii), we infer that every node of $T^{\rightsquigarrow B}$ admits two incompatible extensions in $T^{\rightsquigarrow B}$. Using Clause (ii), we then infer that there exists a club $C \subseteq \kappa$ such that for all $\gamma \in C$ and $t \in (T^{\rightsquigarrow B}) \restriction \gamma$, there are two incompatible extensions s_0, s_1 of t in $(T^{\rightsquigarrow B}) \restriction \gamma$. By normality, for each $i < 2$, we may find a $t_i \in (T^{\rightsquigarrow B})_\gamma$ extending s_i. This shows that C is as sought. □

Let C be as given by the claim. Recalling the choice of $\vec{C}$, we now fix a $\delta \in E^\kappa_\theta$ such that $C_\delta \subseteq C$. Pick $t \in (T^{\rightsquigarrow B})_\delta$. For every $\gamma \in C_\delta$, as $\gamma^+ := \min(C_\delta \setminus (\gamma+1))$ belongs to C, we may pick a node $t_\gamma \in (T^{\rightsquigarrow B})_{\gamma^+}$ such that $t_\gamma \restriction (\gamma+1) = t \restriction (\gamma+1)$ and $t_\gamma \neq t \restriction \gamma^+$, so that $\gamma < \Delta(t, t_\gamma) < \gamma^+$. Pick $\eta < \kappa$ such that $t = a_\eta$.

For each $\gamma \in C_\delta$, as $t_\gamma \in T^{\rightsquigarrow B}$, we may pick some $\beta_\gamma \in B$ above η with $t_\gamma \sqsubseteq a_{\beta_\gamma}$. Then $\Delta(a_\eta, a_{\beta_\gamma}) = \Delta(t, t_\gamma)$ and so recalling that $\gamma < \Delta(t, t_\gamma) < \min(C_\delta \setminus (\gamma+1))$, we infer that

$$c(\eta,\beta) = \sup(\mathrm{otp}(C_{\mathrm{dom}(a_\eta)} \cap \Delta(a_\eta, a_{\beta_\gamma}))) = \sup(\mathrm{otp}(C_\delta \cap (\gamma+1))) = \mathrm{otp}(C_\delta \cap \gamma).$$

Therefore, $c[\{\eta\} \circledast B] = \{\mathrm{otp}(C_\delta \cap \gamma) \mid \gamma \in C_\delta\} = \theta$. □

Remark 3.2. The preceding is optimal and cannot be improved to get $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\kappa)$. Indeed, by [IR24, Corollary 3.18], $\mathsf{unbounded}(J^{\mathrm{bd}}[\kappa],\kappa)$ holds iff κ admits a nontrivial C-sequence, whereas, by [LHR21, Theorem 3.3], it is consistent (from large cardinals) that all C-sequences over κ are trivial, and yet, there is a κ-Souslin tree.

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Theorem 3.3. *Suppose that κ is a regular uncountable cardinal and the ineffable tree property fails. Then for every $\theta \in \mathrm{Reg}(\kappa)$, $\mathsf{onto}(\mathrm{NS}_\kappa,\theta)$ holds.*

Proof. As in the beginning of the proof of Theorem 3.1, we may assume that we are given a $\theta \in \mathrm{Reg}(\kappa)$ with $\theta^+ < \kappa$, and pick a θ-bounded club-guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$. As the ineffable tree property fails, we may fix a subset $T \subseteq {}^{<\kappa}2$ satisfying all of the following:

(i) For every $t \in T$ and $\alpha < \mathrm{dom}(t)$, $t \restriction \alpha \in T$;
(ii) For every $\alpha < \kappa$, the set $T_\alpha := \{t \in T \mid \mathrm{dom}(t) = \alpha\}$ satisfies $0 < |T_\alpha| < \kappa$;
(iii) There exists a sequence $\langle b_\beta \mid \beta < \kappa \rangle \in \prod_{\beta<\kappa} T_\beta$ such that for every stationary $S \subseteq \kappa$, there are $(\alpha,\beta) \in [S]^2$ with $b_\alpha \not\subseteq b_\beta$.

For every $\gamma < \kappa$, denote $T \restriction \gamma := \bigcup_{\alpha<\gamma} T_\alpha$. Fix an injective enumeration $\langle a_\eta \mid \eta < \kappa \rangle$ of T. Define $c : [\kappa]^2 \to \theta$ as follows. Given $\eta < \beta < \kappa$, if $\mathrm{dom}(a_\eta) \in E^\kappa_\theta$ and $a_\eta \not\subseteq b_\beta$, then let

$$c(\eta,\beta) := \sup(\mathrm{otp}(C_{\mathrm{dom}(a_\eta)} \cap \Delta(a_\eta,b_\beta)));$$

otherwise, just let $c(\eta,\beta) := 0$.

To see that c witnesses $\mathsf{onto}(\mathrm{NS}_\kappa,\theta)$ holds, let a stationary $B \subseteq \kappa$ be given. Denote

$$T^{\rightsquigarrow B} := \{t \in T \mid \{\beta \in B \mid t \sqsubseteq b_\beta\} \text{ is stationary}\}.$$

Claim 3.3.1. $T^{\rightsquigarrow B}$ maintains properties (i) and (ii), and there exists a club $C \subseteq \kappa$ such that for every $\gamma \in C$ and every $t \in (T^{\rightsquigarrow B}) \restriction \gamma$, there are two incompatible extensions of t in $(T^{\rightsquigarrow B})_\gamma$.

Proof. It is clear that Clause (i) holds for $T^{\rightsquigarrow B}$. To see that Clause (ii) holds, it suffices to show that $T^{\rightsquigarrow B}$ is normal, and this follows in a proof similar to that of Claim 3.1.1, this time using the fact that NS_κ is a κ-complete ideal.

Next, as seen in the proof of Claim 3.1.1, in order to find a club C as in the statement of the claim, it suffices to prove that every node $t \in T^{\rightsquigarrow B}$ admits two incompatible extensions in $T^{\rightsquigarrow B}$. Towards a contradiction, suppose that this is not the case, as witnessed by a node t. This means that $Z := \{z \in T^{\rightsquigarrow B} \mid z \subseteq t \text{ or } t \subseteq z\}$ is a chain. Since $T^{\rightsquigarrow B}$ is normal, it follows that for every $\beta < \kappa$, we may let z_β denote the unique element of $Z \cap {}^\beta 2$. Recalling Clause (iii), we infer that $\{\beta \in B \mid b_\beta = z_\beta\}$ is nonstationary. As $t \in T^{\rightsquigarrow B}$, altogether,

$$\{\beta \in B \mid t \sqsubseteq b_\beta \ \&\ b_\beta \neq z_\beta\} = \{\beta \in B \mid \mathrm{dom}(t) \leq \Delta(b_\beta,z_\beta) < \beta\}$$

is stationary. By Fodor's lemma, pick $\epsilon \geq \mathrm{dom}(t)$ such that $\{\beta \in B \mid \mathrm{dom}(t) \leq \Delta(b_\beta,z_\beta) = \epsilon\}$ is stationary. As $|T_{\epsilon+1}| < \kappa$, we may pick $z \in T_{\epsilon+1}$ such that

$$B' := \{\beta \in B \mid \mathrm{dom}(t) \leq \Delta(b_\beta,z_\beta) = \epsilon < \beta \ \&\ b_\beta \restriction (\epsilon+1) = z\}$$

is stationary. Then B' witnesses that $z \in T^{\rightsquigarrow B}$. Furthermore, picking a $\beta \in B'$, we see that $\mathrm{dom}(t) \leq \Delta(z,z_\beta) < \epsilon+1 = \mathrm{dom}(z)$, which means that $t \subseteq z$ and $z \neq z_{\epsilon+1}$, contradicting the fact that Z is a chain. □

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Let C be as given by the claim. Fix a $\delta \in E^\kappa_\theta$ such that $C_\delta \subseteq C$. Pick $t \in (T^{\rightsquigarrow B})_\delta$. For every $\gamma \in C_\delta$, as $\gamma^+ := \min(C_\delta \setminus (\gamma+1))$ belongs to C, we may pick a node $t_\gamma \in (T^{\rightsquigarrow B})_{\gamma^+}$ such that $\gamma < \Delta(t,t_\gamma) < \gamma^+$. Pick $\eta < \kappa$ such that $t = a_\eta$.

For each $\gamma \in C_\delta$, as $t_\gamma \in T^{\rightsquigarrow B}$, we may find some $\beta_\gamma \in B$ above η with $t_\gamma \sqsubseteq b_{\beta_\gamma}$. Then $\Delta(a_\eta,b_{\beta_\gamma}) = \Delta(t,t_\gamma)$ and hence

$$c(\eta,\beta) = \sup(\mathrm{otp}(C_{\mathrm{dom}(a_\eta)} \cap \Delta(a_\eta,b_{\beta_\gamma}))) = \sup(\mathrm{otp}(C_\delta \cap (\gamma+1))) = \mathrm{otp}(C_\delta \cap \gamma).$$

Therefore, $c[\{\eta\} \circledast B] = \theta$. □

Theorem B of [IR24] deals with the instance $\mathsf{onto}^{++}(\mathcal{J}^\kappa_\kappa,\theta)$. Unfortunately, its statement omits a regularity requirement for θ, where the bug is in the proof of [IR24, Corollary 5.5] that incorrectly infers that at successors of singulars, the existence of a stationary set nonreflecting at regulars yields a stationary set that does not reflect at all. The correct statement reads as follows.

Fact 3.4 (IR24, Theorem B). Suppose that κ is a regular uncountable cardinal admitting a stationary set that does not reflect at regulars. Then, for every $\theta \in \mathrm{Reg}(\kappa)$, $\mathsf{onto}^{++}(\mathcal{J}^\kappa_\kappa,\theta)$ holds.

The next corollary provides a characterization when $\mathsf{onto}^{++}(\mathcal{J}^\kappa_\kappa,\theta)$ holds for κ strongly inaccessible. This cannot be extended to κ weakly inaccessible, since [IR23, Theorem 9.10] shows that $\mathsf{unbounded}(J^{\mathrm{bd}}[\kappa],\aleph_1)$ may fail at κ a weakly inaccessible.

Corollary 3.5. *Suppose that κ is a strongly inaccessible cardinal.*
Then the following are equivalent:

(1) *κ is not weakly compact;*
(2) *$\mathsf{onto}^{++}(\mathcal{J}^\kappa_\kappa,\theta)$ holds for every $\theta < \kappa$;*
(3) *$\mathsf{onto}(J^{\mathrm{bd}}[\kappa],3)$ holds.*

Proof. (1) $\Longrightarrow$ (2): By possibly enlarging θ, we may assume it is an infinite regular cardinal. By Theorem 3.1 together with [IR24, Theorem 4.1(1)], $\mathsf{onto}^{+}(\mathcal{J}^\kappa_\kappa,\theta^+)$ holds. Then, by [IR24, Corollary 4.18(4)], $\mathsf{onto}^{++}(\mathcal{J}^\kappa_\kappa,\theta)$ holds.
(2) $\Longrightarrow$ (3): This is trivial.
(3) $\Longrightarrow$ (1): By [IR23, Corollary 10.8]. □

Corollary 3.6. *Suppose that κ is a strongly inaccessible cardinal.*
Then the following are equivalent:

(1) *κ is not ineffable;*
(2) *$\mathsf{onto}^{++}(\mathrm{NS}_\kappa,\theta)$ holds for every $\theta < \kappa$;*
(3) *$\mathsf{onto}(\mathrm{NS}_\kappa,2)$ holds.*

Proof. (1) $\Longrightarrow$ (2): By possibly enlarging θ, we may assume it is an infinite regular cardinal. By Theorem 3.3 together with [IR23, Proposition 2.26(4)], $\mathsf{onto}^{+}(\mathrm{NS}_\kappa,\theta^+)$ holds. Then, by [IR24, Corollary 4.18(4)], $\mathsf{onto}^{++}(\mathrm{NS}_\kappa,\theta)$ holds.

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(2) $\implies$ (3): This is trivial.
(3) $\implies$ (1): By [IR23, Lemma 11.1]. □

4. Getting the maximal number of colours

In this section we prove Theorem B. In reading the statement of the next lemma, recall that $\log_\lambda(\kappa)$ stands for the least cardinal $\theta \leq \kappa$ such that $\lambda^\theta \geq \kappa$.

Lemma 4.1. *Suppose that κ is a regular uncountable that is not strongly inaccessible. If $\log_2(\kappa)$ is a singular cardinal, then there exists a singular cardinal ν with $\log_2(\kappa) \leq \nu < \kappa$ such that $\lambda^{\mathrm{cf}(\nu)} < \kappa \leq \nu^{\mathrm{cf}(\nu)}$ for every $\lambda < \nu$.*

Proof. Suppose that $\theta := \log_2(\kappa)$ is a singular cardinal.

Claim 4.1.1. There are cardinals μ,ν such that $\mu < \theta \leq \nu < \kappa$ and $\nu^\mu \geq \kappa$.

Proof. Let χ be the least cardinal such that $\theta^\chi \geq \kappa$. If $\chi < \theta$, then the pair $(\nu,\mu) := (\theta,\chi)$ is as sought. Otherwise, it is the case that $\chi = \theta$. Let $\langle \vartheta_j \mid j < \mathrm{cf}(\theta)\rangle$ be some strictly increasing sequence of infinite cardinals, converging to θ. As κ is regular and as each ϑ_j is smaller than χ, it follows that $\nu := \sup_{j<\mathrm{cf}(\theta)} \theta^{\vartheta_j}$ is some cardinal lying in $[\theta,\kappa)$. Finally, $\mu := \mathrm{cf}(\theta)$ is smaller than θ and it is easy to see that $\nu^\mu \geq \theta^\theta \geq \kappa$. □

Fix the least cardinal μ for which there exists a cardinal $\nu \in (\mu,\kappa)$ with $\nu^\mu \geq \kappa$. By the preceding claim, μ exists and is smaller than θ. Fix the least $\nu \in (\mu,\kappa)$ such that $\nu^\mu \geq \kappa$. In particular, $\nu^\nu \geq \kappa$, so that $\nu \geq \theta$.

Claim 4.1.2. ν is a singular cardinal of cofinality μ.

Proof. Suppose not.

▶ If $\nu = \chi^+$ for some cardinal χ, then Hausdorff's formula implies that $\kappa \leq \nu^\mu = \max\{\nu,\chi^\mu\}$, so since $\nu < \kappa$, we get that $\chi^\mu \geq \kappa$. However, $\mu < \theta \leq \nu = \chi^+$ and θ is a limit cardinal, so $\mu < \chi$, and then χ contradicts the minimality of ν.

▶ If ν is a limit cardinal and $\mathrm{cf}(\nu) > \mu$, then $\nu^\mu = \sup_{\mu<\chi<\nu} \chi^\mu$. However, by the minimality of ν, $\chi^\mu < \kappa$ whenever $\mu < \chi < \nu$, and then the regularity of κ implies that $\sup_{\mu<\chi<\nu} \chi^\mu$ is smaller than κ. This is a contradiction.

▶ If ν is a limit cardinal and $\mathrm{cf}(\nu) < \mu$, then let $\langle \chi_j \mid j < \mathrm{cf}(\nu)\rangle$ be a strictly increasing sequence of cardinals greater than μ that converges to ν. By the minimality of ν, $(\chi_j)^\mu < \kappa$ for every $j < \mathrm{cf}(\nu)$. So the regularity of κ implies that $\chi := \sup_{j<\mathrm{cf}(\nu)}(\chi_j)^\mu$ is smaller than κ. However, $\chi^{\mathrm{cf}(\nu)} \geq \nu^\mu \geq \kappa$. So the fact that $\mathrm{cf}(\nu) < \mu$ contradicts the minimality of μ. □

By the minimality of ν, $\lambda^\mu < \kappa$ for every $\lambda < \nu$. □

Breaking our convention for a moment, in the next lemma κ is not assumed to be regular. In reading it for the first time, one may want to keep in mind the relaxed case of $\mu = \mathrm{cf}(\nu) < \nu < \varkappa = \mathrm{cf}(\kappa) = \kappa$ that arises by the preceding lemma.

Lemma 4.2. *Suppose that $\mu < \nu < \varkappa \leq \kappa$ are cardinals such that $\lambda^\mu < \varkappa \leq \kappa \leq \nu^\mu$ for every $\lambda < \nu$. Then:*

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(1) *For every* $\theta \in \mathrm{Reg}(\nu)$, $\mathsf{unbounded}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta)$ *holds;*

(2) *For every* $\theta \in \mathrm{Reg}(\nu)$ *such that* $\theta^{+} < \nu$, $\mathsf{onto}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta)$ *holds;*

(3) *If* $\mathrm{cf}(\nu) < \nu < \mathrm{cf}(\varkappa)$ *and* $\mu^{\mathrm{cf}(\nu)} < \varkappa$, *then* $\mathsf{unbounded}(\{\nu'\},[\kappa]^{<\varkappa},\nu)$ *holds for* $\nu' := \max\{\nu,\mu^{\mathrm{cf}(\nu)}\}$;

(4) *If* $\mu^{\mathrm{cf}(\nu)} \leq \nu < \mathrm{cf}(\kappa)$ *and* $2^{\nu} = \kappa$, *then* $\mathsf{onto}^{+}([\kappa]^{<\kappa},\nu)$ *holds.*

Proof. Let $\langle g_\beta \mid \beta < \kappa\rangle$ be an injective enumeration of a family of functions from μ to ν. We commence with a trivial observation.

Claim 4.2.1. Let $B \in [\kappa]^{\varkappa}$ and $\theta < \nu$. Then there exists an $i < \mu$ such that $|\{g_\beta(i) \mid \beta \in B\}| > \theta$.

Proof. Suppose not, and let B be a counterexample, so that $X := \{g_\beta(i) \mid i < \mu, \beta \in B\}$ has size no more than $\lambda := \max\{\theta,\mu\}$ which is smaller than ν. As B is a subset of ${}^{\mu}X$ of size $\varkappa$, it follows that $\lambda^{\mu} \geq \varkappa$, contradicting the hypothesis. □

We now turn to prove each of the clauses.

(1) Let $\theta \in \mathrm{Reg}(\nu)$. Let $\langle C_\delta \mid \delta \in E^{\nu}_{\theta}\rangle$ be some θ-bounded C-sequence. Fix a bijection $\pi : \nu \leftrightarrow \mu \times \nu$. Finally pick $c : [\kappa]^2 \to \theta$ such that for every $\eta < \nu < \beta < \kappa$, letting $(i,\delta) := \pi(\eta)$,

$$c(\eta,\beta) := \begin{cases} \mathrm{otp}(C_\delta \cap g_\beta(i)), & \text{if } \delta \in E^{\nu}_{\theta} \text{ and } g_\beta(i) < \delta;\\ 0, & \text{otherwise.}\end{cases}$$

As $\theta \leq \nu < \mathrm{cf}(\varkappa)$, the proof of [IR24, Proposition 2.9(1)] shows that any witness to $\mathsf{unbounded}(\{\nu\},[\kappa]^{<\varkappa},\theta)$ moreover witnesses $\mathsf{unbounded}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta)$. Thus, we verify that c witnesses $\mathsf{unbounded}(\{\nu\},[\kappa]^{<\varkappa},\theta)$. To this end, let $B \in [\kappa]^{\varkappa}$ and we shall show that there exists an $\eta < \nu$ such that $c[\{\eta\} \circledast B]$ has size θ.

Without loss of generality, $\min(B) \geq \nu$. Using Claim 4.2.1, fix an $i < \mu$ such that $X := \{g_\beta(i) \mid \beta \in B\}$ has size $> \theta$. Then fix the least $\delta < \nu$ such that $\mathrm{otp}(X \cap \delta) = \theta$. Consequently, $\delta \in E^{\nu}_{\theta}$, $\sup(X \cap \delta) = \delta$, and $Y := \{\mathrm{otp}(C_\delta \cap \xi) \mid \xi \in X \cap \delta\}$ is a subset of θ of size θ. Pick $\eta < \nu$ such that $\pi(\eta) = (i,\delta)$. Then $c[\{\eta\} \circledast B]$ covers Y, as sought.

(2) Suppose $\theta \in \mathrm{Reg}(\nu)$ is such that $\theta^{+} < \nu$. By Clause (1), $\mathsf{unbounded}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta^{+})$ holds. As θ is regular, by [IR24, Corollary 4.12], $\mathsf{projection}(\theta^{+},\theta^{+},\theta,1)$ holds. Then, by [IR24, Lemma 4.17(2)], $\mathsf{onto}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta)$ holds, as well.

(3) Denote $\theta := \mathrm{cf}(\nu)$. Suppose that $\theta < \nu < \mathrm{cf}(\varkappa)$ and that $\nu' := \max\{\nu,\mu^{\theta}\}$ is smaller than $\varkappa$. Let $\langle \nu_j \mid j < \theta\rangle$ be a strictly increasing sequence of cardinals, converging to ν, and fix a bijection $\pi : \nu' \leftrightarrow \nu \times {}^{\theta}\mu$. Use Clause (2) to fix a colouring $c : [\kappa]^2 \to \theta$ witnessing $\mathsf{onto}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta)$. Define $d : [\kappa]^2 \to \theta$ as follows. Given $\eta < \nu' < \beta < \kappa$, let $(\bar\eta, f) := \pi(\eta)$, and then let $d(\eta,\beta) := g_\beta(f(c(\bar\eta,\beta)))$.

To see that d witnesses $\mathsf{unbounded}(\{\nu'\},[\kappa]^{<\varkappa},\nu)$, let $B \in [\kappa]^{\varkappa}$, and we shall find an $\eta < \nu'$ such that $c[\{\eta\} \circledast B]$ has size ν.

Without loss of generality, $\min(B) \geq \nu'$. As c witnesses $\mathsf{onto}^{+}(\{\nu\},[\kappa]^{<\varkappa},\theta)$, we may pick some $\bar\eta < \nu$ such that for every $j < \theta$, $B_j := \{\beta \in B \mid c(\bar\eta,\beta) = j\}$ is in $([\kappa]^{<\varkappa})^{+}$. By Claim 4.2.1, for each $j < \theta$, there exists some $i_j < \mu$ such that $X_j := \{g_\beta(i_j) \mid \beta \in B_j\}$ has

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size $>\nu_j$. Define $f:\theta\to\mu$ via $f(j):=i_j$. Pick an $\eta<\nu'$ such that $\pi(\eta)=(\bar{\eta},f)$. Then, for every $j<\theta$,

$$d[\{\eta\}\circledast B_j]=\{g_\beta(f(c(\bar{\eta},\beta)))\mid\beta\in B_j\}=\{g_\beta(f(j))\mid\beta\in B_j\}=X^j$$

is a subset of ν of size $>\nu_j$. As $\sup_{j<\theta}\nu_j=\nu$, $d[\{\eta\}\circledast B]$ is a subset of ν of size ν, as sought.

(4) Suppose that $\mu^{\mathrm{cf}(\nu)}\leq\nu<\mathrm{cf}(\kappa)$. It follows that $\mathrm{cf}(\nu)<\nu<\mathrm{cf}(\kappa)$ and $\mu^{\mathrm{cf}(\nu)}\leq\nu<\kappa$, so that $\mathsf{unbounded}(\{\nu\},[\kappa]^{<\kappa},\nu)$ holds by Clause (3). As $\nu<\mathrm{cf}(\kappa)$, it is the case that $[\kappa]^{<\kappa}\in\mathcal{J}^\kappa_{\nu^+}$, so, by [IR24, Proposition 2.9(1)], moreover $\mathsf{unbounded}^+(\{\nu\},[\kappa]^{<\kappa},\nu)$ holds. In particular, we may fix a colouring $c:[\kappa]^2\to\nu$ witnessing $\mathsf{unbounded}^+([\kappa]^{<\kappa},\nu)$. Suppose now that $2^\nu=\kappa$, and fix a bijection $\pi:\kappa\leftrightarrow\kappa\times{}^\nu\nu$. Define $d:[\kappa]^2\to\nu$ as follows. Given $\eta<\beta<\kappa$, let $(\bar{\eta},f):=\pi(\eta)$, and then let $d(\eta,\beta):=f(c(\bar{\eta},\beta))$. To see that d witnesses $\mathsf{onto}^+([\kappa]^{<\kappa},\nu)$, let $B\in[\kappa]^\kappa$ be given. By the choice of c, we may fix $\bar{\eta}<\kappa$ and $X\in[\nu]^\nu$ such that for every $\xi\in X$, the set $B_\xi:=\{\beta\in B\setminus(\bar{\eta}+1)\mid c(\bar{\eta},\beta)=\xi\}$ is in $[\kappa]^\kappa$. Fix $f\in{}^\nu\nu$ such that $f[X]=\nu$. Find $\eta<\kappa$ such that $\pi(\eta)=(\bar{\eta},f)$. Then, given $\tau<\nu$, pick $\xi\in X$ such that $f(\xi)=\tau$, and note that $\{\beta\in B\setminus(\eta+1)\mid d(\eta,\beta)=\tau\}$ covers $B_\xi\setminus(\eta+1)$ and hence it is in $[\kappa]^\kappa$. □

Corollary 4.3. *Suppose that κ is not strongly inaccessible.*
Then $\mathsf{unbounded}^+(\mathcal{J}^\kappa_\kappa,\theta)$ holds for $\theta:=\log_2(\kappa)$.

Proof. By [IR24, Theorem 4.1(2)], it suffices to prove that $\mathsf{unbounded}^+(J^{\mathrm{bd}}[\kappa],\theta)$ holds. Meanwhile, by [LHR18, Lemma 2.7], $\mathrm{U}(\kappa,\kappa,\theta,2)$ holds. In particular, $\mathrm{U}(\kappa,2,\mathrm{cf}(\theta),2)$ holds. Then by [IR24, Lemma 5.12(1)], $\mathsf{unbounded}^+(J^{\mathrm{bd}}[\kappa],\mathrm{cf}(\theta))$ holds. So, if θ is regular, then we are done.

Hereafter, suppose that θ is singular. By Lemma 4.1, we may fix a singular cardinal ν with $\theta\leq\nu<\kappa$ such that $\lambda^{\mathrm{cf}(\nu)}<\kappa\leq\nu^{\mathrm{cf}(\nu)}$ for every $\lambda<\nu$.

▶ If $\theta<\nu$, then appealing to Lemma 4.2(2) with $\mu:=\mathrm{cf}(\nu)$, $\varkappa:=\kappa$ and the successor of θ, we get that $\mathsf{onto}^+(\{\nu\},[\kappa]^{<\kappa},\theta^+)$ holds. In particular, $\mathsf{onto}^+(J^{\mathrm{bd}}[\kappa],\theta)$ holds.

▶ If $\theta=\nu$, then appealing to Lemma 4.2(3) with $\mu:=\mathrm{cf}(\nu)$ and $\varkappa:=\kappa$, we get that $\mathsf{unbounded}(\{\nu'\},[\kappa]^{<\kappa},\nu)$ holds for some cardinal $\nu'<\kappa$. In particular, by [IR24, Proposition 2.9(1)], $\mathsf{unbounded}^+(J^{\mathrm{bd}}[\kappa],\theta)$ holds. □

Remark 4.4. It remains open whether $\mathsf{unbounded}^+$ may be improved to onto in the preceding. This is connected to the problem of whether $\mathsf{onto}(J^{\mathrm{bd}}[\theta^+],\theta)$ must hold for every singular cardinal θ.

5. Small number of colours, but for indecomposable ideals

In this short section, we provide two sufficient conditions for $\mathsf{unbounded}^\square(\mathcal{I}^\kappa_\theta,\theta)$ to hold. Our first result improves [LHR23, Lemma 3.38(3)], and has to do with subadditive colourings. Recall that a colouring $c:[\kappa]^2\to\theta$ is *subadditive of the second kind* iff $c(\alpha,\beta)\leq\max\{c(\alpha,\gamma),c(\beta,\gamma)\}$ for all $\alpha<\beta<\gamma<\kappa$.

Definition 5.1 (special case of [LHR18, Definition 1.2]). $\mathrm{U}(\kappa,2,\theta,2)$ asserts the existence of a colouring $c:[\kappa]^2\to\theta$ such that, for every $A\in[\kappa]^\kappa$, $\sup(c``[A])=\theta$.

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By [LHR23, Theorem 4.13], for every pair $\theta < \kappa$ of infinite regular cardinals, a weak form of $\square(\kappa)$ is sufficient to get a fully subadditive witness to $\mathrm{U}(\kappa,2,\theta,2)$.

Proposition 5.2. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$, and $c : [\kappa]^2 \to \theta$ is a witness for $\mathrm{U}(\kappa,2,\theta,2)$ which is subadditive of the second kind.*

Then c witnesses $\mathsf{unbounded}^{\square}([\kappa]^\kappa,\mathcal{I}^\kappa_\theta,\theta)$.

Proof. Let $A \in [\kappa]^\kappa$, $I \in \mathcal{I}^\kappa_\theta$ and $B \in I^+$ be given. For all $\eta < \kappa$ and $\tau < \theta$, denote $B^{\eta,\geq\tau} := \{\beta \in B \mid c(\eta,\beta) \geq \tau\}$.

Claim 5.2.1. There exists an $\eta \in A$ such that $B^{\eta,\geq\tau} \in I^+$ for all $\tau < \theta$.

Proof. Otherwise, we may find a function $g : A \to \theta$ such that $B^{\eta,\geq g(\eta)} \in I$ for every $\eta \in A$. Fix $\tau < \theta$ for which the following set is cofinal in κ:

$$A' := \{\eta \in A \mid g(\eta) = \tau\}.$$

As c witnesses $\mathrm{U}(\kappa,2,\theta,2)$, we may pick $(\eta_0,\eta_1) \in [A']^2$ such that $c(\eta_0,\eta_1) > \tau$. As $B \in I^+$ and $B^{\eta_0,\geq g(\eta_0)} \cup B^{\eta_1,\geq g(\eta_1)} \in I$, we may pick a $\beta \in B \setminus B^{\eta_0,\geq g(\eta_0)} \cup B^{\eta_1,\geq g(\eta_1)}$ above η_1. As $\eta_0 < \eta_1 < \beta$ and c is subadditive of the second kind,

$$\tau < c(\eta_0,\eta_1) \leq \max\{c(\eta_0,\beta),c(\eta_1,\beta)\} < \max\{g(\eta_0),g(\eta_1)\} = \tau.$$

This is a contradiction. □

Let η be given by the preceding claim, and then define a function $f : \kappa \to \theta$ via $f(\beta) := c(\eta,\beta)$. By the choice of η, for every $\tau < \theta$, $f \restriction B^{\eta,\geq\tau}$ is a function from an I^+-set to $\theta \setminus \tau$, so by the θ-indecomposability of I, there exists some $h(\tau) \in (\tau,\theta)$ such that the following set is in I^+:

$$B_\tau := \{\beta \in B^{\eta,\geq\tau} \mid f(\beta) < h(\tau)\}.$$

Fix a sparse enough $T \in [\theta]^\theta$ such that, for all $\tau \in T$ and $\sigma < \tau$, $h(\sigma) < \tau$. Then η and T are as sought. □

The next corollary is a variant of Theorem 3.1.

Corollary 5.3. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$, and there exists a tree of height κ with a θ-ascent path but no branch of size κ. Then $\mathsf{unbounded}^{\square}([\kappa]^\kappa,\mathcal{I}^\kappa_\theta,\theta)$ holds.*

Proof. By Proposition 5.2 and [LHR23, Lemma 3.7]. □

Definition 5.4. $\mathcal{C}(\theta,\vartheta)$ denotes the least cardinality of a family of sets $\mathcal{X} \subseteq [\theta]^\vartheta$ such that for every club subset $D \subseteq \theta$, there is an $X \in \mathcal{X}$ such that $X \subseteq D$.

Recall that for $f,g \in {}^\theta\theta$, one writes $f \leq^{\mathrm{bd}} g$ iff $\{\eta < \theta \mid f(\eta) > g(\eta)\} \in J^{\mathrm{bd}}[\theta]$, and for $\tau < \theta$, one writes $f \leq^\tau g$ iff $f(\eta) \leq g(\eta)$ for every $\eta \in [\tau,\theta)$.

- $\mathfrak{b}_\theta$ stands for the least size of an unbounded family in $({}^\theta\theta,\leq^{\mathrm{bd}})$;
- $\mathfrak{d}_\theta$ stands for the least size of a cofinal family in $({}^\theta\theta,\leq^{\mathrm{bd}})$.

The next lemma must be well-known, but we could not find a reference for it.

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Lemma 5.5. *Suppose that θ is a regular uncountable cardinal. Then $\mathcal{C}(\theta,\theta)=\mathfrak{d}_\theta$.*

Proof. Denote $\nu:=\mathcal{C}(\theta,\theta)$.

▶ To see that $\nu\leq\mathfrak{d}_\theta$, fix a sequence $\langle f_\beta\mid\beta<\mathfrak{d}_\theta\rangle$ of functions from θ to θ that is cofinal in $({}^\theta\theta,\leq^{\mathrm{bd}})$. For all $\tau<\theta$ and $\beta<\mathfrak{d}_\theta$, let $C_{\tau,\beta}:=\{\gamma\in\mathrm{acc}(\theta\setminus\tau)\mid f_\beta[\gamma]\subseteq\gamma\}$. Now, given any club $D\subseteq\theta$, let $\pi:\theta\to D$ denote the inverse collapsing map. Pick $\tau<\theta$ and $\beta<\mathfrak{d}_\theta$ such that $\pi\leq^\tau f_\beta$. Towards a contradiction, suppose that $C_{\tau+1,\beta}\nsubseteq D$ and fix $\gamma\in C_{\tau+1,\beta}\setminus D$. In particular, $\bar\gamma:=\sup(D\cap\gamma)$ and τ are both smaller than γ. Put $\eta:=\max\{\bar\gamma,\tau\}+1$ so that $\bar\gamma<\eta<\gamma$. Then $\bar\gamma<\eta\leq\pi(\eta)\leq f_\beta(\eta)<\gamma$. Altogether, $\pi(\eta)$ is an element of D lying in the interval $(\bar\gamma,\gamma)$. This is a contradiction.

▶ To see that $\mathfrak{d}_\theta\leq\nu$, fix a sequence $\langle X_\beta\mid\beta<\nu\rangle$ of cofinal subsets of θ, such that, for every club D in θ, for some $\beta<\nu$, $X_\beta\subseteq D$. For each $\beta<\nu$, define $f_\beta:\theta\to X_\beta$ via $f_\beta(\eta):=\min(X_\beta\setminus(\eta+1))$. Now, given any function $f:\theta\to\theta$, consider the club $D:=\{\delta<\theta\mid f[\delta]\subseteq\delta\}$. Fix $\beta<\nu$ such that $X_\beta\subseteq D$, and we shall show that $f\leq^0 f_\beta$. Indeed, for every $\eta<\theta$, $\delta:=f_\beta(\eta)$ is an element of D above η and hence $f(\eta)<\delta=f_\beta(\eta)$. □

Proposition 5.6. *Suppose that θ is an infinite regular cardinal, and $\mathfrak{b}_\theta=\mathfrak{d}_\theta=\kappa$. Then $\mathsf{unbounded}^{\square}([\theta]^\theta,\mathcal{I}^\kappa_\theta,\theta)$ and $\mathsf{onto}^+(\mathcal{I}^\kappa_\theta,\theta)$ both hold.*

Proof. By Clauses (3) and (4) of Lemma 2.4 together with Lemma 5.5, in our case, $\mathsf{unbounded}^{\square}(\mathcal{I}^\kappa_\theta,\theta)$ implies $\mathsf{onto}^+(\mathcal{I}^\kappa_\theta,\theta)$. Thus, we may focus on proving that $\mathsf{unbounded}^{\square}([\theta]^\theta,\mathcal{I}^\kappa_\theta,\theta)$ holds.

As $\mathfrak{b}_\theta=\mathfrak{d}_\theta=\kappa$, we may fix a sequence $\vec g=\langle g_\beta\mid\beta<\kappa\rangle$ that is increasing and cofinal in $({}^\theta\theta,\leq^{\mathrm{bd}})$. Define a colouring $c:[\kappa]^2\to\theta$ by letting for all $\eta<\theta\leq\beta<\kappa$:

$$c(\eta,\beta):=g_\beta(\eta).$$

To see that c witnesses $\mathsf{unbounded}^{\square}([\theta]^\theta,\mathcal{I}^\kappa_\theta,\theta)$, let $A\in[\theta]^\theta$, $I\in\mathcal{I}^\kappa_\theta$ and $B\in I^+$ be given. For all $\eta,\tau<\theta$, denote $B^{\eta,\geq\tau}:=\{\beta\in B\mid g_\beta(\eta)\geq\tau\}$.

Claim 5.6.1. There exists an $\eta\in A$ such that $B^{\eta,\geq\tau}\in I^+$ for all $\tau<\theta$.

Proof. Otherwise, we may define a strictly increasing function $g:\theta\to\theta$ such that $B^{\eta,\geq g(\eta)}\in I$ for every $\eta\in A$. Pick $\alpha<\kappa$ such that $g\leq^{\mathrm{bd}}g_\alpha$. Since $\vec g$ is increasing, it follows that we may define a function $f:B\setminus\alpha\to\theta$ via

$$f(\beta):=\min\{\eta<\theta\mid g\leq^\eta g_\beta\}.$$

As I is θ-indecomposable, we may now fix a large enough $\eta\in A$ such that the following set is in I^+:

$$B':=\{\beta\in B\setminus\alpha\mid f(\beta)\leq\eta\}.$$

For every $\beta\in B'$, since $f(\beta)\leq\eta$, it is the case that $g\leq^\eta g_\beta$, in particular, $g_\beta(\eta)\geq g(\eta)$, and hence $B^{\eta,\geq g(\eta)}$ contains the positive set B'. This is a contradiction. □

Let η be given by the preceding claim, and then define a function $f:\kappa\to\theta$ via $f(\beta):=g_\beta(\eta)$. By the choice of η, for every $\tau<\theta$, $f\restriction B^{\eta,\geq\tau}$ is a function from an I^+-set to $\theta\setminus\tau$,

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so by the θ-indecomposability of I, there exists some $h(\tau) \in (\tau,\theta)$ such that the following set is in I^+:

$$B_\tau := \{\beta \in B^{\eta,\geq\tau} \mid f(\beta) < h(\tau)\}.$$

Fix a sparse enough $T \in [\theta]^\theta$ such that, for all $\tau \in T$ and $\sigma < \tau$, $h(\sigma) < \tau$. Then η and T are as sought. □

Remark 5.7. A tweak of the proceeding proof establishes that for every infinite regular cardinal θ with $\mathfrak{b}_\theta = \mathfrak{d}_\theta = \kappa$, $\mathsf{onto}^{++}(\mathcal{I}^\kappa_\theta, \vartheta)$ holds for every cardinal $\vartheta < \theta$.

6. Weak saturation of indecomposable ideals

The study of indecomposable ideals on a cardinal κ started in the 1960s and 1970s, focusing on ultrafilters, that is, weakly 2-saturated ideals. The next batch of results from the 1990's to the 2010's expanded to weak θ-saturation for θ typically smaller than κ. In this section, we focus on weak κ-saturation. In particular, we shall prove here Theorems C and D.

For the purpose of this section, we introduce a variant of $\mathsf{unbounded}^+(\mathcal{A},J,\theta)$ that arises naturally in our context. Note that whenever $\mathcal{T}$ is an ideal over θ extending $[\theta]^{<\theta}$, the following principle implies the former.

Definition 6.1. $\mathsf{unbounded}^+(\mathcal{A},J,\mathcal{T})$ asserts the existence of an upper-regressive colouring $c : [\kappa]^2 \to \theta$ with the property that, for all $A \in \mathcal{A}$ and $B \in J^+$, there is an $\eta \in A$ such that

$$\{\tau < \theta \mid \{\beta \in B \setminus (\eta+1) \mid c(\eta,\beta) = \tau\} \in J^+\} \notin \mathcal{T}.$$

Convention 2.3 applies to this principle as well. The utility of the new principle is that, in the presence of an instance of onto over θ, it may enable us to pump up $\mathsf{unbounded}^+$ to onto^+ over κ. The proof of the following is similar to that of [IR23, Lemma 5.12] and [IR24, Lemma 4.14(2)] and is left to the interested reader.

Proposition 6.2. *Suppose that* $\mathsf{unbounded}^+(\mathcal{A},\mathcal{J},I)$ *holds with* $\mathcal{J} \subseteq \mathcal{J}^\kappa_\omega$ *and* I *an ideal over* θ*. Let* ϑ *be another cardinal.*

(1) *If there are cardinals* $\mu \leq \nu$ *such that* $\mathcal{A} = \{\nu\}$ *and* $\mathsf{onto}(\{\mu\},I,\vartheta)$ *holds, then* $\mathsf{onto}^+(\{\nu\},\mathcal{J},\vartheta)$ *holds;*

(2) *If* $\mathcal{A} = J^+$ *for some* $J \in \mathcal{J}^\kappa_{\vartheta^+}$*, and* $\mathsf{onto}^{++}(I,\vartheta)$ *holds, then so does* $\mathsf{onto}^{++}(\mathcal{J},\vartheta)$. □

Note however that in the case that θ is a successor cardinal the new principle hardly conveys any new information. This is due to the next proposition whose proof is similar to that of [IR23, Lemma 6.15] and is left to the reader.

Proposition 6.3. *For* $\nu \leq \lambda$ *and a collection* $\mathcal{J} \subseteq \mathcal{J}^\kappa_\omega$*, the following are equivalent:*

(1) $\mathsf{unbounded}^+(\{\nu\},\mathcal{J},\lambda^+)$ *holds;*

(2) *there is a colouring simultaneously witnessing* $\mathsf{unbounded}^+(\{\nu\},\mathcal{J},I)$ *for all* $I \in \mathcal{J}^{\lambda^+}_{\lambda^+}$. □

In stating our results, we make use of the following piece of notation.

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Definition 6.4. For $I \in \mathcal{J}^\kappa_\omega$, let $T^*(I) := \{\tau < \kappa \mid I \text{ is } \mathrm{cf}(\tau)\text{-indecomposable}\}$.

We commence with a warm-up. We remind the reader that our conventions concerning walks on ordinals [Tod87, Tod07] can be found in [IR25, §4]. In particular, familiarity with the functions $\mathrm{tr}(\cdot,\cdot)$, $\mathrm{Tr}(\cdot,\cdot)$, $\lambda(\cdot,\cdot)$, $\lambda_2(\cdot,\cdot)$, $\rho_2(\cdot,\cdot)$ and $\eth_{\cdot,\cdot}$ will be assumed hereafter.

Lemma 6.5. *Suppose that S is a nonreflecting stationary subset of κ. Consider the collection $\mathcal{I} := \{I \in \mathcal{J}^\kappa_\omega \mid S \cap T^*(I) \text{ is stationary}\}$. Then:*

(1) *There is a proper ideal $J \in \mathcal{J}^\kappa_\kappa$ and a colouring $c : [\kappa]^2 \to \kappa$ witnessing $\mathsf{unbounded}^+(J^+, I, \mathrm{NS}_\kappa \restriction (S \cap T^*(I)))$ for every $I \in \mathcal{I}$;*

(2) *$\mathsf{onto}^{++}(\mathcal{I}, \vartheta)$ holds for every cardinal $\vartheta < \kappa$;*

(3) *If $\diamondsuit(S)$ holds, then so does $\mathsf{onto}^+(\mathcal{I}, \kappa)$.*

Proof. (1) Fix a bijection $\pi : \kappa \leftrightarrow \kappa \times \kappa$ and let J be the corresponding ideal from Example 2.1. Fix a C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa \rangle$ that avoids S, i.e., $\mathrm{acc}(C_\delta) \cap S = \emptyset$ for every $\delta < \kappa$. We shall walk along $\vec{C}$, and let tr, λ, etc. be the induced characteristic maps. Define a colouring $c : [\kappa]^2 \to \kappa$ as follows. Given $\eta < \beta < \kappa$, consider $(\eta_0, \eta_1) := \pi(\eta)$, and then set $c(\eta, \beta) := \min(\mathrm{Im}(\mathrm{tr}(\eta_0, \beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1, \beta)))$, provided that the latter is well-defined. Otherwise, just let $c(\eta, \beta) := 0$.

Let $I \in \mathcal{I}$, and we shall show that c witnesses $\mathsf{unbounded}^+(J^+, I, \mathrm{NS}_\kappa \restriction (S \cap T^*(I)))$. Let $A \in J^+$ and $B \in I^+$ be given. Consider $A_0 := \{\eta_0 < \kappa \mid \sup\{\eta_1 < \kappa \mid \pi^{-1}(\eta_0, \eta_1) \in A\} = \kappa\}$, and note that, since $A \in J^+$, the following set is a club in κ:

$$E := \{\delta \in \mathrm{acc}^+(A_0) \mid \forall \eta_0 \in A_0 \cap \delta \sup\{\eta_1 < \delta \mid \pi^{-1}(\eta_0, \eta_1) \in A \cap \delta\} = \delta\}.$$

Claim 6.5.1. The following holds:

$$S \cap T^*(I) \cap E \subseteq \{\delta < \kappa \mid \exists \eta \in A \cap \delta \{\beta \in B \mid c(\eta, \beta) = \delta\} \in I^+\}.$$

Proof. Fix $\delta \in S \cap T^*(I) \cap E$. As I is $\mathrm{cf}(\delta)$-indecomposable, we may fix some $\epsilon < \delta$ for which the following set is in I^+:

$$B' := \{\beta \in B \setminus (\delta + 1) \mid \lambda(\delta, \beta) \leq \epsilon\}.$$

To justify, note that as $\vec{C}$ avoids S and as $\delta \in S$, for every $\beta \in B \setminus (\delta + 1)$, $\lambda(\delta, \beta) < \delta$; and secondly, as $B \setminus (\delta + 1)$ is in I^+ and I is $\mathrm{cf}(\delta)$-indecomposable, such an $\epsilon < \delta$ can indeed be found.

Pick $\eta_0 \in A_0 \cap \delta$ above ϵ, and then pick an η_1 with $\min(C_\delta \setminus \eta_0) < \eta_1 < \delta$ such that $\eta := \pi^{-1}(\eta_0, \eta_1)$ is in $A \cap \delta$. For every $\beta \in B'$,

$$\lambda(\delta, \beta) \leq \epsilon < \eta_0 < \eta_1 < \delta < \beta$$

and hence $\delta \in \mathrm{Im}(\mathrm{tr}(\eta_0, \beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1, \beta))$. In addition,

$$\min(C_\delta \setminus \eta_0) < \eta_1 \leq \min(C_\delta \setminus \eta_1),$$

and hence $c(\eta, \beta) = \delta$. □

The conclusion now follows from Fodor's lemma.

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(2) By Clause (1) and Proposition 6.2(2), it suffices to prove that $\mathsf{onto}^{++}(J^{\mathrm{bd}}[\kappa],\vartheta)$ holds for every cardinal $\vartheta<\kappa$. Now, since κ admits a nonreflecting stationary set, the main result of [Rin14] implies that $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\omega)$ holds (see Definition 7.3). In particular, $\kappa\nrightarrow[\kappa;\kappa]^2_\vartheta$ holds for every $\vartheta\leq\kappa$. By [IR24, Theorem 5.4(2)], then, $\mathsf{onto}^{++}(J^{\mathrm{bd}}[\kappa],\vartheta)$ holds for every $\vartheta<\kappa$, as sought.

(3) By Clause (1), and Proposition 6.2(1), recalling [IR23, Corollary 8.6]. □

Our next result establishes the first clause of Theorem D.

Lemma 6.6. *Suppose that $\square(\kappa,<\mu)$ holds with $\mu<\kappa$. Then there exists a proper ideal $J\in\mathcal{J}^\kappa_\kappa$ such that, for every $\Theta\in[\mathrm{Reg}(\kappa)]^{<\kappa}$, $\mathsf{unbounded}^+(J^+,\bigcup_{\theta\in\Theta}\mathcal{I}^\kappa_\theta\cap\mathcal{J}^\kappa_\mu,\mathrm{NS}_\kappa)$ holds.*

Proof. Fix a bijection $\pi:\kappa\leftrightarrow\kappa\times\kappa$ and let J be the corresponding ideal from Example 2.1. Let $\Theta\in[\mathrm{Reg}(\kappa)]^{<\kappa}$. By [BR19, Lemma 2.5], we may fix a witness $\vec{\mathcal{C}}=\langle\mathcal{C}_\delta\mid\delta<\kappa\rangle$ for $\square(\kappa,<\mu)$ such that for every club $D\subseteq\kappa$, for every $\theta\in\Theta$, there exists a $\delta\in E^\kappa_\theta$ such that $\sup(\mathrm{nacc}(C)\cap D)=\delta$ for all $C\in\mathcal{C}_\delta$. Let $\vec{C}=\langle C_\delta\mid\delta<\kappa\rangle$ be a transversal for $\vec{\mathcal{C}}$. We shall walk along $\vec{C}$, and let tr, λ_2, etc. be the induced characteristic maps. Define a colouring $c:[\kappa]^2\to\kappa$ as follows. Given $\eta<\beta<\kappa$, consider $(\eta_0,\eta_1):=\pi(\eta)$. Let $\gamma:=\min(\mathrm{Im}(\mathrm{tr}(\eta_0,\beta))\cap\mathrm{Im}(\mathrm{tr}(\eta_1,\beta)))$, and then set $c(\eta,\beta):=\min(C_\gamma\setminus\eta_1)$, provided that the latter is well-defined. Otherwise, just let $c(\eta,\beta):=0$.

To see that c witnesses $\mathsf{unbounded}^+(J^+,\bigcup_{\theta\in\Theta}\mathcal{I}^\kappa_\theta\cap\mathcal{J}^\kappa_\mu,\mathrm{NS}_\kappa)$, let $A\in J^+$, $\theta\in\Theta$, $I\in\mathcal{I}^\kappa_\theta\cap\mathcal{I}^\kappa_\mu$ and $B\in I^+$ be given. Derive from A the sets A_0 and E as in the proof of Lemma 6.5.

Claim 6.6.1. The following set is stationary:

$$T:=\{\tau<\kappa\mid\exists\eta\in A\cap\tau\,\{\beta\in B\mid c(\eta,\beta)=\tau\}\in I^+\}.$$

Proof. Let D be an arbitrary club in κ, and we shall find a τ in $D\cap T$. Fix $\delta\in E^\kappa_\theta$ such that $\sup(\mathrm{nacc}(C)\cap D\cap E)=\delta$ for all $C\in\mathcal{C}_\delta$. As I is μ-complete, we may fix a $C\in\mathcal{C}_\delta$ for which the following set is in I^+:

$$B':=\{\beta\in B\setminus(\delta+1)\mid C_{\mathfrak{d}_{\delta,\beta}}\cap\delta=C\}.$$

As $\mathrm{cf}(\delta)=\theta$ and as I is θ-indecomposable, we may fix some $\epsilon<\delta$ for which the following set is in I^+:

$$B'':=\{\beta\in B'\mid\lambda_2(\delta,\beta)\leq\epsilon\}.$$

Pick $\eta_0\in A_0\cap\delta$ above ϵ. Fix $\tau\in\mathrm{nacc}(C)\cap D\cap E$ above $\min(C\setminus\eta_0)$. As $\tau\in E$, we may then fix an η_1 with $\sup(C\cap\tau)<\eta_1<\tau$ such that $\eta:=\pi^{-1}(\eta_0,\eta_1)$ is in $A\cap\tau$. For every $\beta\in B''$,

$$\lambda(\mathfrak{d}_{\delta,\beta},\beta)\leq\epsilon<\eta_0<\eta_1<\tau<\delta<\beta,$$

and hence $\mathfrak{d}_{\delta,\beta}\in\mathrm{Im}(\mathrm{tr}(\eta_0,\beta))\cap\mathrm{Im}(\mathrm{tr}(\eta_1,\beta))$. In addition,

$$\min(C_{\mathfrak{d}_{\delta,\beta}}\setminus\eta_0)\leq\sup(C\cap\tau)<\eta_1\leq\min(C_{\mathfrak{d}_{\delta,\beta}}\setminus\eta_1),$$

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and hence $\min(\mathrm{Im}(\mathrm{tr}(\eta_0,\beta))\cap\mathrm{Im}(\mathrm{tr}(\eta_1,\beta)))=\eth_{\delta,\beta}$ from which it follows that

$$c(\eta,\beta)=\min(C_{\eth_{\delta,\beta}}\setminus\eta_1)=\min(C\setminus\eta_1)=\tau,$$

as sought. □

The conclusion now follows from Fodor's lemma. □

We now derive two corollaries. The first is the following result whose original proof went through generic elementary embeddings.

Corollary 6.7 (RZ23, Lemma 4.6). *Suppose that λ is a regular uncountable cardinal and $\square(\lambda^+,<\lambda)$ holds. Then every λ-complete uniform filter over λ^+ is not weakly λ-saturated.* □

Corollary 6.8. *Suppose that $\square(\kappa,<\omega)$ holds.*

(1) *For all $\theta,\vartheta\in\mathrm{Reg}(\kappa)$, $\mathsf{onto}^{++}(\mathcal{I}^\kappa_\theta,\vartheta)$ holds;*

(2) *If $\kappa=\vartheta^+$, then $\mathsf{onto}^{++}(\mathcal{I},\vartheta)$ holds for $\mathcal{I}:=\{I\in\mathcal{J}^\kappa_\omega\mid T^*(I)\neq\emptyset\}$;*

(3) *If $\kappa=\vartheta^+$ and $\pitchfork(\kappa)$ holds, then $\mathsf{onto}^{+}(\mathcal{I},\kappa)$ holds for $\mathcal{I}$ of Clause (2);*

(4) *If $\diamondsuit(S)$ holds for some stationary $S\subseteq\kappa$ that does not reflect at regulars, then $\mathsf{onto}^{+}(\mathcal{I}^\kappa_\theta,\kappa)$ holds for all $\theta\in\mathrm{Reg}(\kappa)$.*

Proof. (1) Let $\theta,\vartheta\in\mathrm{Reg}(\kappa)$. By the upcoming Clause (2), we may assume that $\kappa>\vartheta^+$. So, by Shelah's club-guessing theorem [She94b, Claim 2.7] $\mathcal{C}(\kappa,\vartheta)=\kappa$. Now, the conclusion follows from Lemma 6.6 and Lemma 2.4(4).

(2) By Lemma 6.6, in particular, $\mathsf{unbounded}^{+}(J^{+},\mathcal{I},J^{\mathrm{bd}}[\kappa])$ holds for some ideal $J\in\mathcal{J}^\kappa_\kappa$. So, by Proposition 6.2(2), it suffices to prove that $\mathsf{onto}^{++}(J^{\mathrm{bd}}[\kappa],\vartheta)$ holds. The latter follows from [IR24, Theorem 5.3(2)] and Lemma 7.2 below.

(3) By Proposition 6.2(1), it suffices to prove that $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\kappa)$ holds. By [IR23, Lemma 8.3], for every successor cardinal κ, $\pitchfork(\kappa)$ implies $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\kappa)$.

(4) Suppose that $\diamondsuit(S)$ holds for some stationary $S\subseteq\kappa$ that does not reflect at regulars. By [IR23, Corollary 8.6], then, $\mathsf{onto}(J^{\mathrm{bd}}[\kappa],\kappa)$ holds. Thus, letting $\theta\in\mathrm{Reg}(\kappa)$, we get from Lemma 6.6 and Proposition 6.2(1) that $\mathsf{onto}^{+}(\mathcal{I}^\kappa_\theta,\kappa)$ holds. □

Remark 6.9. An extra hypothesis in Clause (3) is necessary, as $\square(\omega_1,<\omega)$ trivially holds, whereas, by [Lar07, Corollary 6.8], $\mathsf{onto}(\mathrm{NS}_{\omega_1},\omega_1)$ may consistently fail.

We next deal with the failure of simultaneous reflection, getting non-weak saturation via $\mathsf{unbounded}^{\square}$ instead of $\mathsf{unbounded}^{+}$. By a theorem of Prikry and Silver [Pri73, Theorem 4], whenever I is the dual ideal of a normal ultrafilter over κ, every finite family of stationary subset of $T^*(I)$ reflects simultaneously. We now generalize it, along the way, establishing the second clause of Theorem D.

Lemma 6.10. *Suppose $I\in\mathcal{J}^\kappa_\mu$, and there is a family of less than μ many stationary subsets of $T^*(I)$ that do not reflect simultaneously.*

Then $\mathsf{unbounded}^{\square}(J^{+},I,\kappa)$ holds for some proper ideal $J\in\mathcal{J}^\kappa_\kappa$.

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Proof. Fix a bijection $\pi : \kappa \leftrightarrow \kappa \times \kappa$ and let J be the corresponding ideal from Example 2.1. Fix a C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa \rangle$, a cardinal $\nu < \mu$, and a sequence $\langle S_i \mid i < \nu \rangle$ of stationary subsets of $T^*(I)$ such that for every $\delta \in \mathrm{acc}(\kappa)$, for some $i < \nu$, $\mathrm{acc}(C_\delta) \cap S_i = \emptyset$. We shall walk along $\vec{C}$, and let tr, λ_2, etc. be the induced characteristic maps. Define $c : [\kappa]^2 \to \kappa$ as follows. Given $\eta < \beta < \kappa$, consider $(\eta_0, \eta_1) := \pi(\eta)$. Let $\gamma := \min(\mathrm{Im}(\mathrm{tr}(\eta_0, \beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1, \beta)))$, and then set $c(\eta, \beta) := \min(C_\gamma \setminus \eta_1)$, provided that the latter is well-defined. Otherwise, just let $c(\eta, \beta) := 0$.

To see that c witnesses $\mathsf{unbounded}^{\square}(J^+, I, \kappa)$, let $A \in J^+$ and $B \in I^+$ be given. Derive from A the sets A_0 and E as in the proof of Lemma 6.5.

Claim 6.10.1. The following set is stationary:

$$T := \{\tau < \kappa \mid \exists \eta \in A \cap \tau \, \exists \delta < \kappa \, \{\beta \in B \mid \tau \leq c(\eta, \beta) < \delta\} \in I^+\}.$$

Proof. Let D be an arbitrary club in κ, and we shall find a τ in $D \cap T$. By possibly shrinking, we may assume that $D \subseteq E$. Pick $\delta \in S_0 \cap \bigcap_{i<\nu} \mathrm{acc}^+(D \cap S_i)$. For every $\beta \in B$ above δ, there is an $i < \nu$ such that $\mathrm{acc}(C_{\mathfrak{d}_{\delta,\beta}}) \cap S_i = \emptyset$. As $\nu < \mu$ and I is μ-complete, we may fix $i < \nu$ and $B' \subseteq B \setminus (\delta + 1)$ in I^+ such that, for every $\beta \in B'$, $\mathrm{acc}(C_{\mathfrak{d}_{\delta,\beta}}) \cap S_i = \emptyset$. As $\delta \in S_0$, I is $\mathrm{cf}(\delta)$-indecomposable, so we may find a large enough $\eta_0 \in A_0 \cap \delta$ such that the following set is in I^+:

$$B'' := \{\beta \in B' \mid \lambda_2(\delta, \beta) < \eta_0\}.$$

As I is $\mathrm{cf}(\delta)$-indecomposable, we may next find some $\varepsilon < \delta$ such that the following set is in I^+:

$$B''' := \{\beta \in B'' \mid \min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_0) \leq \varepsilon\}.$$

Pick $\tau \in S_i \cap D \cap \delta$ above ε. For every $\beta \in B'''$, as $\mathrm{acc}(C_{\mathfrak{d}_{\delta,\beta}}) \cap S_i = \emptyset$, it is the case that $\sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \tau$. As $\tau \in S_i \cap E$, I is $\mathrm{cf}(\tau)$-indecomposable and we may find a large enough $\eta_1 < \tau$ with $\pi^{-1}(\eta_0, \eta_1) \in A \cap \tau$ such that the following set is in I^+:

$$B'''' := \{\beta \in B''' \mid \sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \eta_1\}.$$

Let $\eta < \tau$ be such that $\pi(\eta) = (\eta_0, \eta_1)$. For every $\beta \in B''''$,

$$\lambda(\mathfrak{d}_{\delta,\beta}, \beta) < \eta_0 < \eta_1 < \tau < \delta \leq \mathfrak{d}_{\delta,\beta} \leq \beta$$

and hence $\mathfrak{d}_{\delta,\beta} \in \mathrm{Im}(\mathrm{tr}(\eta_0, \beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1, \beta))$. In addition,

$$\min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_0) \leq \varepsilon < \tau \leq \min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_1) < \delta,$$

and hence $c(\eta, \beta) \in [\tau, \delta)$, as sought. □

Let T be given by the claim. Pick two functions $f : T \to A$ and $g : T \to \kappa$ such that for every $\tau \in T$, $f(\tau) < \tau < g(\tau)$ and

$$\{\beta \in B \mid \tau \leq c(f(\tau), \beta) < g(\tau)\} \in I^+.$$

Find a stationary $T' \subseteq \{\tau \in T \mid g[\tau] \subseteq \tau\}$ on which f is constant with value, say, η. Then for every pair $\sigma < \tau$ of elements of T', the set $\{\beta \in B \mid \sigma \leq c(\eta, \beta) < \tau\}$ covers $\{\beta \in B \mid \sigma \leq c(f(\sigma), \beta) < g(\sigma)\}$, and hence it is in I^+. □

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Corollary 6.11 (Eisworth, [Eis10, Theorem 2]). *Suppose $I \in \mathcal{J}^\kappa_\mu$ and let $S^*(I) := \{\tau \in T^*(I) \cap \mathrm{Sing}(\kappa) \mid I$ is weakly $\mathrm{cf}(\tau)$-saturated$\}$. If there is a $\theta \in \mathrm{Reg}(\kappa)$ such that I is weakly θ-saturated and θ-indecomposable then every family of less than μ many stationary subsets of $S^*(I)$ reflects simultaneously.*

Proof. Suppose there is a $\theta \in \mathrm{Reg}(\kappa)$ such that I is weakly θ-saturated and θ-indecomposable. In particular, I is weakly κ-saturated. In particular, by Lemma 2.4(1), $\mathsf{unbounded}^{\square}(I,\kappa)$ fails. Then, by Lemma 6.10, every family of less than μ many stationary subsets of $T^*(I)$ reflects simultaneously. In particular, every family of less than μ many stationary subsets of $S^*(I)$ reflects simultaneously. □

The proof of Lemma 6.10 can be tweaked to weaken the hypothesis '$I \in \mathcal{J}^\kappa_\mu$' to '$I \in \mathcal{I}^\kappa_\nu$' provided that the form of non-simultaneous-reflection of the sequence $\langle S_i \mid i < \nu\rangle$ of stationary subsets of $T^*(I)$ is more tamed. To keep this paper accessible, we avoid formulating this fact in full generality, and settle for demonstrating this variation through the upcoming proof of Theorem C.

Lemma 6.12. *Suppose that $\kappa = \lambda^+$ for an infinite cardinal λ. Let*

$$\mathcal{I} := \{I \in \mathcal{I}^\kappa_{\mathrm{cf}(\lambda)} \mid \sup\{\theta \in \mathrm{Reg}(\kappa) \mid I \text{ is } \theta\text{-indecomposable}\} = \lambda\}.$$

Then $\mathsf{unbounded}^{\square}(J^+,\mathcal{I},\kappa)$ holds for some proper ideal $J \in \mathcal{J}^\kappa_\kappa$.

Proof. By Lemma 6.5, we may assume that λ is a singular cardinal. Fix a bijection $\pi : \kappa \leftrightarrow \kappa \times \kappa$ and let J be corresponding ideal from Example 2.1. Fix a C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa\rangle$ such that $\mathrm{otp}(C_\delta) < \lambda$ for every $\delta < \kappa$. We shall walk along $\vec{C}$, and let tr, λ_2, etc. be the induced characteristic maps. Define $c : [\kappa]^2 \to \kappa$ as follows. Given $\eta < \beta < \kappa$, consider $(\eta_0,\eta_1) := \pi(\eta)$. Let $\gamma := \min(\mathrm{Im}(\mathrm{tr}(\eta_0,\beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1,\beta)))$, and then set $c(\eta,\beta) := \min(C_\gamma \setminus \eta_1)$, provided that the latter is well-defined. Otherwise, just let $c(\eta,\beta) := 0$.

To see that c witnesses $\mathsf{unbounded}^{\square}(J^+,\mathcal{I},\kappa)$, let $A \in J^+$, $I \in \mathcal{I}$ and $B \in I^+$ be given. Derive from A the sets A_0 and E as in the proof of Lemma 6.5.

Claim 6.12.1. The following set is stationary:

$$T := \{\tau < \kappa \mid \exists \eta \in A \cap \tau\, \exists \delta < \kappa\, \{\beta \in B \mid \tau \leq c(\eta,\beta) < \delta\} \in I^+\}.$$

Proof. Let D be an arbitrary subclub of E, and we shall find a τ in $D \cap T$. Fix a strictly increasing sequence $\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ of cardinals from $\{\theta \in \mathrm{Reg}(\lambda) \mid I \text{ is } \theta\text{-indecomposable}\}$ such that $\sup_{i<\mathrm{cf}(\lambda)} \lambda_i = \lambda$. Pick

$$\delta \in E^\kappa_{\mathrm{cf}(\lambda)} \cap \bigcap_{i<\mathrm{cf}(\lambda)} \mathrm{acc}^+(D \cap E^\kappa_{\lambda_i}).$$

As I is $\mathrm{cf}(\lambda)$-indecomposable, we may find an $i < \mathrm{cf}(\lambda)$ such that the following set is in I^+:

$$B' := \{\beta \in B \setminus (\delta+1) \mid \mathrm{otp}(C_{\mathfrak{d}_{\delta,\beta}}) < \lambda_i\}.$$

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As I is $\mathrm{cf}(\delta)$-indecomposable, we may find a large enough $\eta_0 \in A_0 \cap \delta$ such that the following set is in I^+:

$$B'' := \{\beta \in B' \mid \lambda_2(\delta,\beta) < \eta_0\}.$$

As I is $\mathrm{cf}(\delta)$-indecomposable, we may next find some $\varepsilon < \delta$ such that the following set is in I^+:

$$B''' := \{\beta \in B'' \mid \min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_0) \leq \varepsilon\}.$$

Pick $\tau \in D \cap E^\delta_{\lambda_i}$ above ε. For every $\beta \in B'''$, as $\mathrm{otp}(C_{\mathfrak{d}_{\delta,\beta}}) < \lambda_i = \mathrm{cf}(\tau)$, it is the case that $\sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \tau$. As I is $\mathrm{cf}(\tau)$-indecomposable, we may find a large enough $\eta_1 < \tau$ with $\pi^{-1}(\eta_0,\eta_1) \in A \cap \tau$ such that the following set is in I^+:

$$B'''' := \{\beta \in B''' \mid \sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \eta_1\}.$$

Let $\eta < \tau$ be such that $\pi(\eta) = (\eta_0,\eta_1)$. For every $\beta \in B''''$,

$$\lambda(\mathfrak{d}_{\delta,\beta},\beta) < \eta_0 < \eta_1 < \tau < \delta \leq \mathfrak{d}_{\delta,\beta} \leq \beta$$

and hence $\mathfrak{d}_{\delta,\beta} \in \mathrm{Im}(\mathrm{tr}(\eta_0,\beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1,\beta))$. In addition,

$$\min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_0) \leq \varepsilon < \tau \leq \min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_1) < \delta,$$

and hence $c(\eta,\beta) \in [\tau,\delta)$, as sought. □

The conclusion now follows. □

Remark 6.13. It is also true that for every $\chi < \lambda$, $\mathsf{unbounded}^+(\mathcal{I}^{\lambda^+}_{\mathrm{cf}(\lambda)} \cap \mathcal{I}^{\lambda^+}_{\geq\chi},\lambda^+)$ holds, where $\mathcal{I}^{\lambda^+}_{\geq\chi} := \bigcap_{\theta\in\mathrm{Reg}(\lambda^+)\setminus\chi}\mathcal{I}^{\lambda^+}_\theta$. While $\mathsf{unbounded}^+(\ldots)$ is stronger than $\mathsf{unbounded}^{\square}(\ldots)$, the class $\mathcal{I}$ addressed by Lemma 6.12 is considerably richer than $\mathcal{I}^{\lambda^+}_{\mathrm{cf}(\lambda)} \cap \mathcal{I}^{\lambda^+}_{\geq\chi}$, so we omit the (rather long) proof for the smaller class.

Corollary 6.14 (Kunen–Prikry, [KP71, Theorem 0.2]). *Suppose that I is a $\mathrm{cf}(\lambda)$-indecomposable ideal over λ^+ whose dual is an ultrafilter. Then λ is singular and $\sup\{\theta \in \mathrm{Reg}(\lambda) \mid I \text{ is } \theta\text{-indecomposable}\} < \lambda$.*

Proof. λ must be singular, since otherwise $E^{\lambda^+}_\lambda$ is a non-reflecting stationary set, contradicting Lemma 6.5. Next, appeal to Lemma 6.12, bearing in mind Lemma 2.4(1). □

Remark 6.15. This result of Kunen and Prikry was first proved by Chang [Cha67, §1] under the additional hypothesis that $2^\lambda = \lambda^+$.

Corollary 6.16 (Eisworth, [Eis12, Lemmas 3.2 and 3.3]). *Suppose that I is a $\mathrm{cf}(\lambda)$-indecomposable weakly λ-saturated ideal over the successor of a singular cardinal λ. Then $\sup\{\theta \in \mathrm{Reg}(\lambda) \mid I \text{ is } \theta\text{-indecomposable}\} < \lambda$.*

Proof. By Lemma 6.12, bearing in mind Lemma 2.4(1). □

We now arrive at the third clause of Theorem D. Its proof simply combines the ideas of Lemmas 6.10 and 6.12.

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Theorem 6.17. *Suppose:*

(1) $I \in \mathcal{J}^\kappa_\mu$;

(2) *there is a family of less than* μ *many stationary subsets of* $T^*(I)$ *that do not reflect simultaneously at regulars;*

(3) $\sup\{\theta \in \mathrm{Reg}(\kappa) \mid I \text{ is } \theta\text{-indecomposable}\} = \kappa$.

Then $\mathsf{unbounded}^{\square}(J^+,I,\kappa)$ *holds for some proper ideal* $J \in \mathcal{J}^\kappa_\kappa$.

Proof. Fix a bijection $\pi : \kappa \leftrightarrow \kappa \times \kappa$ and let J be corresponding ideal from Example 2.1. Using Clause (2), we fix a C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa\rangle$ with $\nu < \mu$, and a sequence $\langle S_i \mid i < \nu\rangle$ of stationary subsets of $T^*(I)$ such that:

- for every $\delta \in \mathrm{Sing}(\kappa)$, $\min(C_\delta) = \mathrm{otp}(C_\delta)$, and
- for every $\delta \in \mathrm{Reg}(\kappa)$, $\mathrm{acc}(C_\delta) \cap S_i = \emptyset$ for some $i < \nu$.

We shall walk along $\vec{C}$, and let tr, λ_2, etc. be the induced characteristic maps. Define $c : [\kappa]^2 \to \kappa$ as follows. Given $\eta < \beta < \kappa$, consider $(\eta_0,\eta_1) := \pi(\eta)$. Let $\gamma := \min(\mathrm{Im}(\mathrm{tr}(\eta_0,\beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1,\beta)))$, and then set $c(\eta,\beta) := \min(C_\gamma \setminus \eta_1)$, provided that the latter is well-defined. Otherwise, just let $c(\eta,\beta) := 0$.

To see that c witnesses $\mathsf{unbounded}^{\square}(J^+,I,\kappa)$, let $A \in J^+$ and $B \in I^+$ be given. Derive from A the sets A_0 and E as in the proof of Lemma 6.5.

Claim 6.17.1. Let $B \in I^+$. Then the following set is stationary:

$$T := \{\tau < \kappa \mid \exists \eta \in A \cap \tau\, \exists \delta < \kappa\, \{\beta \in B \mid \tau \leq c(\eta,\beta) < \delta\} \in I^+\}.$$

Proof. Let D be an arbitrary subclub of E, and we shall find a τ in $D \cap T$. Denote $\Theta := \{\theta \in \mathrm{Reg}(\kappa) \mid I \text{ is } \theta\text{-indecomposable}\}$. Fix an elementary submodel $M \prec H_{\kappa^+}$ with $\{D,\Theta,\langle S_i \mid i < \nu\rangle\} \in M$ such that $\delta := M \cap \kappa$ is in $T^*(I) \setminus \nu$.

As I is $\mathrm{cf}(\delta)$-indecomposable, we may fix a large enough $\eta_0 \in A \cap \delta$ such that the following set is in I^+:

$$B_0 := \{\beta \in B \setminus (\delta+1) \mid \lambda_2(\delta,\beta) < \eta_0\}.$$

As I is $\mathrm{cf}(\delta)$-indecomposable, we may then fix a large enough $\theta \in \Theta \cap \delta$ such that the following set is in I^+:

$$B_1 := \{\beta \in B_0 \mid \min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_0) < \theta\}.$$

The definition of $B_2 \subseteq B_1$ now splits into two cases:

- If $\{\beta \in B_1 \mid \mathfrak{d}_{\delta,\beta} \in \mathrm{Sing}(\kappa)\}$ is in I^+, then denote that set by B_2. Recalling that $\delta = M \cap \kappa$, pick a $\tau \in D \cap E^\kappa_\theta$ below δ. For every $\beta \in B_2$, $\mathrm{otp}(C_{\mathfrak{d}_{\delta,\beta}}) = \min(C_{\mathfrak{d}_{\delta,\beta}}) < \theta = \mathrm{cf}(\tau)$, and hence it is the case that $\sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \tau$.
- Otherwise, for almost all $\beta \in B_1$, there is an $i < \nu$ such that $\mathrm{acc}(C_{\mathfrak{d}_{\delta,\beta}}) \cap S_i = \emptyset$. As $\nu < \mu$ and I is μ-complete, we may fix $i < \nu$ and $B_2 \subseteq B_1$ in I^+ such that, for every $\beta \in B_2$, $\mathrm{acc}(C_{\mathfrak{d}_{\delta,\beta}}) \cap S_i = \emptyset$. Recalling that $\delta = M \cap \kappa$, pick a $\tau \in D \cap S_i \cap [\theta,\delta)$. For every $\beta \in B_2$, $\tau \notin C_{\mathfrak{d}_{\delta,\beta}}$ and hence it is the case that $\sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \tau$.

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As I is $\mathrm{cf}(\tau)$-indecomposable, we may now find a large enough $\eta_1 < \tau$ such that $\pi^{-1}(\eta_0,\eta_1) \in A \cap \tau$ and the following set is in I^+:

$$B_3 := \{\beta \in B_2 \mid \sup(C_{\mathfrak{d}_{\delta,\beta}} \cap \tau) < \eta_1\}.$$

Let $\eta < \tau$ be such that $\pi(\eta) = (\eta_0,\eta_1)$. For every $\beta \in B_3$,

$$\lambda(\mathfrak{d}_{\delta,\beta},\beta) < \eta_0 < \eta_1 < \tau < \delta \leq \mathfrak{d}_{\delta,\beta} \leq \beta$$

and hence $\mathfrak{d}_{\delta,\beta} \in \mathrm{Im}(\mathrm{tr}(\eta_0,\beta)) \cap \mathrm{Im}(\mathrm{tr}(\eta_1,\beta))$. In addition,

$$\min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_0) < \theta \leq \tau \leq \min(C_{\mathfrak{d}_{\delta,\beta}} \setminus \eta_1) < \delta,$$

and hence $c(\eta,\beta) \in [\tau,\delta)$, as sought. □

The conclusion now follows. □

Remark 6.18. Similarly to Remark 6.13, for κ weakly inaccessible, and a stationary $S \subseteq E^\kappa_{\geq\chi}$ that does not reflect at regulars, $\mathsf{unbounded}^+(\mathcal{I}^\kappa_{\geq\chi},\mathrm{NS}_\kappa \restriction S)$ holds; and if furthermore $\diamondsuit(S)$ holds, then so does $\mathsf{onto}^+(\mathcal{I}^\kappa_{\geq\chi},\kappa)$. As the class of ideals addressed by the preceding theorem is considerably richer, we omit the proof for the smaller class.

Remark 6.19. Theorem 6.17 is optimal as You and Yuan recently announced that the first ω-Mahlo cardinal κ may consistently carry a uniform ultrafilter that is θ-indecomposable for every uncountable cardinal $\theta < \kappa$.

Theorem 6.17 improves results of Chudnovsky, Prikry and Silver (see Theorems 3, 3.1, 6 and 7 of [Pri73]). It also improves Clauses (α) and (β) of Shelah's [She94b, Claim 3.3]. We close this section by revisiting Shelah's proof of Clause (γ), verifying that it in fact gives an instance of one of our colouring principles. It has to do with a certain system of almost disjoint sets, as follows.

Definition 6.20 (Shelah, [She82, p. 440]). ADS_λ asserts the existence of a sequence $\langle A_\tau \mid \tau < \lambda^+\rangle$ of cofinal subsets of λ of order-type $\mathrm{cf}(\lambda)$, with the property that, for every $\beta < \lambda^+$, there is an $f_\beta : \beta \to \mathrm{cf}(\lambda)$ such that $\langle A_\tau \setminus A_\tau(f_\beta(\tau)) \mid \tau < \beta\rangle$ consists of pairwise disjoint sets.

It is easy to see that ADS_λ holds whenever λ is regular. To connect the upcoming result with the results of Section 3, we mention that ADS_λ follows from the existence of a special λ^+-Aronszajn tree [CFM01, Theorem 4.1].

Proposition 6.21. *If* ADS_λ *holds then so does* $\mathsf{unbounded}^+(\{\lambda\},\mathcal{I}^{\lambda^+}_{\mathrm{cf}(\lambda)},\lambda^+)$.

Proof. Suppose $\langle A_\tau \mid \tau < \lambda^+\rangle$ and $\langle f_\beta \mid \beta < \lambda^+\rangle$ witness together that ADS_λ holds. Define an upper-regressive map $c : \lambda \times \lambda^+ \to \lambda^+$, as follows. Given $\eta < \lambda \leq \beta < \lambda^+$, if there exists a $\tau < \beta$ such that $\eta \in A_\tau \setminus A_\tau(f_\beta(\tau))$, then let $c(\eta,\beta)$ be equal to this unique τ. Otherwise, set $c(\eta,\beta) := 0$.

To show that c witnesses $\mathsf{unbounded}^+(\{\lambda\},\mathcal{I}^{\lambda^+}_{\mathrm{cf}(\lambda)},\lambda^+)$, let $I \in \mathcal{I}^{\lambda^+}_{\mathrm{cf}(\lambda)}$ and $B \in I^+$ be given. For each $\tau < \lambda^+$, we may find $i_\tau < \mathrm{cf}(\lambda)$ such that $B^{i_\tau}_\tau := \{\beta \in B \setminus (\tau+1) \mid f_\beta(\tau) \leq i_\tau\}$

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is in I^+. Pick $i<\mathrm{cf}(\lambda)$ for which $T:=\{\tau<\lambda^+\mid i_\tau=i\}$ has size λ^+. Then find $\eta<\lambda$ for which $T':=\{\tau\in T\mid A_\tau(i)=\eta\}$ has size λ^+.

Now, let $\tau\in T'$. For every $\beta\in B^i_\tau$, $f_\beta(\tau)\leq i$ and $A_\tau(i)=\eta$, and hence $c(\eta,\beta)=\tau$. So $B_\tau:=\{\beta\in B\mid c(\eta,\beta)=\tau\}$ covers $B^{i_\tau}_\tau$ and hence it is in I^+. □

In summary, in the context of successors of regulars, we have identified three ZFC generalizations of Ulam's theorem, each having its own merit.

Corollary 6.22. *For every infinite regular cardinal λ, all of the following hold:*

(1) $\mathsf{unbounded}^{\square}([\lambda^+]^{\lambda^+},\mathcal{I}^{\lambda^+}_\lambda,\lambda)$;

(2) $\mathsf{unbounded}^{+}(\{\lambda\},\mathcal{I}^{\lambda^+}_\lambda,\lambda^+)$;

(3) $\mathsf{onto}^{++}(\mathcal{I}^{\lambda^+}_\lambda,\lambda)$.

Proof. (1) By Proposition 5.2 and [LHR23, Lemma 3.14(1)].

(2) By Proposition 6.21, since λ is regular.

(3) By Lemma 6.5(2), since $E^{\lambda^+}_\lambda$ is a nonreflecting stationary set. □

7. Contributions to the theory of strong colourings

Definition 7.1 (EHR65, §18). For a pair $\theta\leq\kappa$ of cardinals:

- $\kappa\nrightarrow[\kappa]^2_\theta$ asserts the existence of a colouring $c:[\kappa]^2\to\theta$ such that, for every $B\in[\kappa]^\kappa$, $c``[B]^2=\theta$.
- $\kappa\nrightarrow[\kappa;\kappa]^2_\theta$ asserts the existence of a colouring $c:[\kappa]^2\to\theta$ such that, for all A, $B\in[\kappa]^\kappa$, $c``A\circledast B=\theta$.

The following lemma was promised in the proof of Corollary 6.8 above.

Lemma 7.2. $\square(\kappa,<\omega)$ *implies* $\kappa\nrightarrow[\kappa;\kappa]^2_\kappa$.

Proof. First, note that by [RT13], we may assume that κ is not the successor of a regular cardinal. In particular, $\kappa\geq\aleph_3$. Now, appeal to [Rin14, Theorem 2.6] with $\mu:=\aleph_0$ in the notation there to get a colouring $d:{}^{<\omega}\omega_1\to\omega$ such that for every sequence $\langle(\eta_i,\sigma_i)\mid i\in I\rangle$ with

(i) I is an uncountable subset of ω_1;

(ii) η_i and σ_i are elements of ${}^{<\omega}\omega_1$;

(iii) $i\in\mathrm{Im}(\eta_i)\cap\mathrm{Im}(\sigma_i)$ for every $i\in I$,

there is a pair $i<j$ of ordinals in I such that $d(\eta_i{}^\frown\sigma_j)=\ell(\eta_i)$, the length of η_i.

Next, assuming $\square(\kappa,<\omega)$, by [LHR21, Theorem 5.27], we may fix a $\square(\kappa,<\omega)$-sequence $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ such that, for every $\tau<\kappa$, the following set is stationary:

$$H_\tau:=\{\alpha\in E^\kappa_{\omega_2}\mid\forall C\in\mathcal{C}_\alpha\,[\min(C)=\tau]\}.$$

By systematically replacing various ordinals by their successors, we may also assume that $C\cap\mathrm{acc}(\kappa)\subseteq\mathrm{acc}(C)\cup\{\min(C)\}$ for every $C\in\bigcup_{\alpha<\kappa}\mathcal{C}_\alpha$.

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Let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ be a transversal of $\vec{\mathcal{C}}$, and let tr, ρ_2, etc. be the induced characteristic maps of walking along $\vec{C}$. By [LHR21, Lemma 4.12], $\chi(\vec{C}) \geq \sup(\mathrm{Reg}(\kappa)) \geq \aleph_2$. By [LHR21, Definition 4.2], this means that for every $\Delta \in [\kappa]^\kappa$, there exists some $\alpha < \kappa$ such that $\Delta \cap \alpha \not\subseteq \bigcup_{\beta\in b} C_\beta$ for every $b \in [\kappa]^{\aleph_1}$.

Define a function $h : \kappa \to \omega_1$ via:

$$h(\alpha) := \begin{cases} \min(C_\alpha), & \text{if } 0 < \alpha \ \&\ \min(C_\alpha) < \omega_1;\\ 0, & \text{otherwise.}\end{cases}$$

Define another function $o : [\kappa]^2 \to \omega$ via

$$o(\alpha,\beta) := \begin{cases} d(h\circ \mathrm{tr}(\alpha,\beta)) - 1, & \text{if } d(h\circ\mathrm{tr}(\alpha,\beta)) > 0;\\ 0, & \text{otherwise,}\end{cases}$$

Finally, our colouring $c : [\kappa]^2 \to \kappa$ is defined via

$$c(\alpha,\beta) := \min(C_{\mathrm{Tr}(\alpha,\beta)(o(\alpha,\beta))}).$$

To see that c witnesses $\kappa \nrightarrow [\kappa;\kappa]^2_\kappa$, let sets $A,B \in [\kappa]^\kappa$ and a prescribed colour $\tau < \kappa$ be given. For every $\gamma < \kappa$, let $\beta_\gamma := \min(B \setminus (\gamma+1))$. For each $i < \omega_1$, find some $\varepsilon_i < \kappa$ for which the following set is stationary

$$G_i := \{\gamma \in H_i \mid \lambda_2(\gamma,\beta_\gamma) = \varepsilon_i\}.$$

Recalling that $\chi(\vec{C}) \geq \aleph_2$, and as the following set is in $[\kappa]^\kappa$:

$$\Delta := (H_\tau \cap \bigcap\nolimits_{i<\omega_1} \mathrm{acc}^+(\mathrm{acc}^+(A)\cap H_i)) \setminus ((\sup\nolimits_{i<\omega_1} \varepsilon_i)+1),$$

we may fix a large enough $\alpha^* < \kappa$ such that $\Delta \cap \alpha^* \not\subseteq \bigcup_{\beta\in b} C_\beta$ for every $b \in [\kappa]^{\aleph_1}$.

For each $i < \omega_1$, let $\gamma_i := \min(G_i \setminus (\alpha^*+1))$. Letting $b := \{\eth_{\gamma_i,\beta_{\gamma_i}} \mid i < \omega_1\}$, we then pick $\xi \in \Delta \cap \alpha^* \setminus \bigcup_{\beta\in b} C_\beta$. As $\mathrm{cf}(\xi) = \omega_2$, $\epsilon := \sup_{i<\omega_1} \lambda_2(\xi,\beta_{\gamma_i})$ is smaller than ξ. Consequently, for each $i < \omega_1$, we may pick a $\delta_i \in \mathrm{acc}^+(A) \cap H_i \cap \xi$ above $\max\{\epsilon,\tau\}$.

Next, as $\mathcal{C}_\xi$ is finite, we may pick some $C \in \mathcal{C}_\xi$ for which the following set is uncountable:

$$I := \{i \in \omega_1 \setminus \{\tau\} \mid C_{\eth_{\xi,\beta_{\gamma_i}}} \cap \xi = C\}.$$

Claim 7.2.1. $\{\delta_i \mid i \in I\} \cap C = \emptyset$.

Proof. Suppose not, and pick $i \in I$ with $\delta_i \in C$. So $\delta_i \in C \cap E^\kappa_{\omega_2} \subseteq C \cap \mathrm{acc}(\kappa) \subseteq \mathrm{acc}(C) \cup \{\min(C)\}$.

▶ If $\delta_i \in \mathrm{acc}(C)$, then $C \cap \delta_i \in \mathcal{C}_{\delta_i}$, and since $\delta_i \in H_i$, this would mean that $\min(C) = i$, contradicting the fact that $\min(C) = \tau$.

▶ Otherwise, $\delta_i = \min(C)$, contradicting the fact that $\delta_i > \tau$. □

It follows that for each $i \in I$, $\zeta_i := \min(C \setminus \delta_i)$ is an element of $\mathrm{nacc}(C)$ bigger than δ_i. For each $i \in I$, pick $\alpha_i \in A \cap \delta_i$ above $\max\{\epsilon, \sup(C\cap\delta_i), \lambda_2(\delta_i,\zeta_i)\}$, and finally set:

- $\eta_i := h \circ (\mathrm{tr}(\eth_{\xi,\beta_{\gamma_i}},\beta_{\gamma_i})^\frown \langle \eth_{\xi,\beta_{\gamma_i}}\rangle)$;
- $\sigma_i := h \circ \mathrm{tr}(\alpha_i,\zeta_i)$.

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Claim 7.2.2. Let $i \in I$. Then:

(i) $i \in \mathrm{Im}(\eta_i)$;
(ii) $i \in \mathrm{Im}(\sigma_i)$.

Proof. (i) As $\lambda_2(\gamma_i,\beta_{\gamma_i}) = \varepsilon_i < \xi < \alpha^* < \gamma_i < \beta_{\gamma_i}$, it is the case that

$$\mathrm{tr}(\eth_{\xi,\beta_{\gamma_i}},\beta_{\gamma_i}) = \mathrm{tr}(\eth_{\gamma_i,\beta_{\gamma_i}},\beta_{\gamma_i})^\frown \mathrm{tr}(\eth_{\xi,\beta_{\gamma_i}},\eth_{\gamma_i,\beta_{\gamma_i}}).$$

By the choice of ξ, we have $\xi \notin C_{\eth_{\gamma_i,\beta_{\gamma_i}}}$, so that the sequence $\mathrm{tr}(\eth_{\xi,\beta_{\gamma_i}},\eth_{\gamma_i,\beta_{\gamma_i}})$ is nonempty. Altogether, $h(\eth_{\gamma_i,\beta_{\gamma_i}}) \in \mathrm{Im}(\eta_i)$. As $\sup(C_{\eth_{\gamma_i,\beta_{\gamma_i}}} \cap \gamma_i) = \gamma_i \in H_i$, it is the case that $h(\eth_{\gamma_i,\beta_{\gamma_i}}) = i$.

(ii) As $\lambda_2(\delta_i,\zeta_i) < \alpha_i < \delta_i < \zeta_i$, it is the case that $h(\eth_{\delta_i,\zeta_i}) \in \mathrm{Im}(\sigma_i)$. As $\sup(C_{\eth_{\delta_i,\zeta_i}} \cap \delta_i) = \delta_i \in H_i$, it is the case that $h(\eth_{\delta_i,\zeta_i}) = i$. □

Using the choice of d, pick a pair $i < j$ of ordinals in I such that $d(\eta_i{}^\frown \sigma_j) = \ell(\eta_i)$.

Claim 7.2.3. $\mathrm{tr}(\alpha_j,\beta_{\gamma_i}) = \mathrm{tr}(\eth_{\xi,\beta_{\gamma_i}},\beta_{\gamma_i})^\frown \langle \eth_{\xi,\beta_{\gamma_i}} \rangle^\frown \mathrm{tr}(\alpha_j,\zeta_j)$.

Proof. As $C_{\eth_{\xi,\beta_i}} \cap \xi = C$ with $\min(C \setminus \delta_i) = \zeta_j$ and as

$$\lambda_2(\xi,\beta_{\gamma_i}) \leq \max\{\epsilon, \sup(C \cap \delta_j), \lambda_2(\delta_j,\zeta_j)\} < \alpha_j < \delta_j < \zeta_j < \xi < \gamma_i < \beta_{\gamma_i},$$

it is the case that

$$\mathrm{tr}(\alpha_j,\beta_{\gamma_i}) = \mathrm{tr}(\eth_{\xi,\beta_{\gamma_i}},\beta_{\gamma_i})^\frown \mathrm{tr}(\alpha_j,\eth_{\xi,\beta_{\gamma_i}}),$$

and

$$\mathrm{tr}(\alpha_j,\eth_{\xi,\beta_{\gamma_i}}) = \langle \eth_{\xi,\beta_{\gamma_i}} \rangle^\frown \mathrm{tr}(\alpha_j,\zeta_j),$$

as needed to show. □

Consequently, $d(h \circ \mathrm{tr}(\alpha_j,\beta_{\gamma_i})) = \rho_2(\eth_{\xi,\beta_{\gamma_i}},\beta_{\gamma_i}) + 1$, so that $\mathrm{Tr}(\alpha_j,\beta_i)(o(\alpha_j,\beta_i)) = \eth_{\xi,\beta_{\gamma_i}}$ and $c(\alpha_j,\beta_i) = \min(C_{\eth_{\xi,\beta_{\gamma_i}}})$. Finally, as $\sup(C_{\eth_{\xi,\beta_{\gamma_i}}} \cap \xi) = \xi \in H_\tau$, we infer that $c(\alpha_j,\beta_i) = \tau$, as sought. □

Our next result has to do with the following generalization of Definition 7.1.

Definition 7.3 (Shelah, [She88]). $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ asserts the existence of a colouring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{B} \subseteq [\kappa]^\sigma$ of size κ, and every $\tau < \theta$, there are $a,b \in \mathcal{A}$ with $\sup(a) < \min(b)$ such that $c[a \times b] = \{\tau\}$.

By [She94b, Conclusion 4.8(2)], for every weakly inaccessible κ, if $E^\kappa_{\geq\chi}$ admits a stationary set that does not reflect at regulars, then for every $\theta < \kappa$, $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ holds. We now improve it to get $\theta = \kappa$. The following is Theorem E.

Theorem 7.4. *Suppose that κ is weakly inaccessible, $\chi \in \mathrm{Reg}(\kappa)$, and $E^\kappa_{\geq\chi}$ admits a stationary set that does not reflect at regulars. Then $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$ holds.*

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Proof. By the main result of [Rin14], we may assume that every stationary subset of $E^\kappa_{\geq\chi}$ reflects. In particular, by [RZ21, Lemma 6.1], we may fix regular uncountable cardinals $\sigma^1<\sigma^0$ smaller than κ together with stationary sets $S^1\subseteq E^\kappa_{\sigma^1}\cap\mathrm{Sing}(\kappa)$ and $S^0\subseteq E^\kappa_{\sigma^0}\cap\mathrm{Sing}(\kappa)$ such that neither of them reflect at inaccessibles. Following [RZ21, §6], we fix a σ^1-bounded C-sequence $\vec{e}=\langle e_\delta\mid\delta\in S^1\rangle$ and another C-sequence $\vec{C}=\langle C_\alpha\mid\alpha<\kappa\rangle$ such that all of the following hold:

(i) for every $\delta\in S^1$, $\langle\mathrm{cf}(\gamma)\mid\gamma\in\mathrm{nacc}(e_\delta)\rangle$ is strictly increasing, converging to δ;

(ii) for every club $D\subseteq\kappa$, there exists a $\delta\in S^1$ with $e_\delta\subseteq D$;

(iii) for every $\alpha<\kappa$, $|C_\alpha|=\mathrm{cf}(\alpha)$;

(iv) for every $\alpha<\kappa$, if $\mathrm{acc}(C_\alpha)\cap S^1\neq\emptyset$, then $\min(C_\alpha)\geq\mathrm{cf}(\alpha)>\sigma^1$;

(v) for every $\alpha<\kappa$, for every $\delta\in(\mathrm{acc}(C_\alpha)\cup\{\alpha\})\cap S^1$, $\sup(e_\delta\setminus C_\alpha)<\delta$.

By [RZ21, Lemma 6.4], for every $(\delta,\beta)\in S^1\circledast\kappa$, there exists an ordinal $\Lambda(\delta,\beta)<\delta$ such that the following two hold:

(a) $\mathrm{nacc}(e_\delta)\setminus\Lambda(\delta,\beta)\subseteq\mathrm{nacc}(C_{\mathfrak{d}_{\delta,\beta}})$;

(b) for every $\gamma\in\mathrm{nacc}(C_{\mathfrak{d}_{\delta,\beta}})\cap[\Lambda(\delta,\beta),\delta)$, $\sup(e_\delta\cap\gamma)\leq\lambda(\gamma,\beta)<\gamma$.

Let I be the collection of all $X\subseteq\kappa$ such that, for some club $D\subseteq\kappa$, for every $\delta\in S^1$, $\sup(\mathrm{nacc}(e_\delta)\cap D\cap X)<\delta$. It is not hard to see that I is an ideal. In fact, I is a countably-complete ideal, since $S^1\subseteq E^\kappa_{>\omega}$. It is clear that $J^{\mathrm{bd}}[\kappa]\subseteq\mathrm{NS}_\kappa\subseteq I$, and by the choice of $\vec{e}$, I is proper.

Claim 7.4.1. I is not weakly κ-saturated.

Proof. As $S^1\subseteq E^\kappa_{<\sigma^0}$, I is θ-indecomposable for every $\theta\in\mathrm{Reg}(\kappa)\setminus\sigma^0$; indeed, it is easy to see that if $\langle X_i\mid i<\theta\rangle$ is a $\subseteq$-increasing sequence of sets in I whose membership in I is witnessed, respectively, by the sequence $\langle D_i\mid i<\theta\rangle$ of clubs in κ, then $D:=\bigcap_{i<\theta}D_i$ witnesses that $X:=\bigcup_{i<\theta}X_i$ is in I. It follows that S^0 is a stationary subset of $T^*(I)$ that does not reflect at regulars. So, by appealing to Theorem 6.17, we infer that I is not weakly κ-saturated. □

Next, as $E^\kappa_{\geq\chi}$ admits a stationary set that does not reflect at regulars, [RZ21, Theorem C(3)] implies in particular that $\mathrm{P}\ell_1(\kappa,1,\chi)$ holds. This has two implications relevant to us:

- By [RZ21, Lemma 2.18(2)], to prove that $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$ holds, it suffices to prove that $\kappa\nrightarrow[\kappa]^2_\kappa$ holds;
- By [RZ21, Corollary 2.22], we may fix a colouring $o:[\kappa]^2\to\omega$ witnessing $\kappa\nrightarrow[\kappa]^2_\omega$.

Fix a map $h:\kappa\to\kappa$ witnessing that I is not weakly κ-saturated. We shall walk along $\vec{C}$, and let $\mathrm{Tr},\mathrm{tr},\lambda$ and ρ_2 be the induced characteristic maps. Define $c:[\kappa]^2\to\kappa$ via

$$c(\alpha,\beta):=h(\mathrm{Tr}(\alpha,\beta)(o(\alpha,\beta))).$$

To see that c witnesses $\kappa\nrightarrow[\kappa]^2_\kappa$, let $B\in[\kappa]^\kappa$ be given along with a prescribed colour $\tau<\kappa$. By [IR24, Claim 3.15.1], we may fix a club $D\subseteq\kappa$ such that, for every $\gamma\in D$, for every $\beta\in B\setminus\gamma$, for every $n<\omega$, $\sup\{\alpha\in B\cap\gamma\mid o(\alpha,\beta)=n\}=\gamma$.

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Now, by the choice of h, let us pick a $\delta \in S^1$ such that $\sup\{\gamma \in \mathrm{nacc}(e_\delta) \cap D \mid h(\gamma) = \tau\} = \delta$. Pick $\beta \in B$ above δ and then pick $\gamma \in \mathrm{nacc}(e_\delta) \cap D$ above $\Lambda(\delta,\beta)$ such that $h(\gamma) = \tau$. Denote $n := \rho_2(\gamma,\beta)$. By Clauses (a) and (b), $\lambda(\gamma,\beta) < \gamma$, so since $\gamma \in D$, we may find some $\alpha \in B$ with $\lambda(\gamma,\beta) < \alpha < \gamma$ such that $o(\alpha,\beta) = n$. As $\lambda(\gamma,\beta) < \alpha < \gamma < \delta < \beta$, it follows that $\mathrm{tr}(\alpha,\beta) = \mathrm{tr}(\gamma,\beta)^\frown \mathrm{tr}(\alpha,\gamma)$ and hence

$$c(\alpha,\beta) = h(\mathrm{Tr}(\alpha,\beta)(o(\alpha,\beta))) = h(\mathrm{Tr}(\alpha,\beta)(\rho_2(\gamma,\beta))) = h(\gamma) = \tau,$$

as sought. □

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