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On characterizing quantum stochastic evolutions

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1. *Introduction*

It has recently been shown [**7**, **8**, **4**] that Markov dilations of quantum dynamical semigroups may be constructed by solving corresponding quantum stochastic differential equations. These equations may be interpreted as describing the evolution of a quantum system with a singular coupling to a Boson reservoir, moreover, when solutions are combined with the free evolution of the reservoir, a reversible evolution results which may be interpreted as that of the system plus reservoir. It is the purpose of the present paper to characterize the above stochastic evolutions essentially by a condition of covariance with respect to the free evolution.

The plan of the paper is as follows:

After collecting various properties of Fock space, which also serves to fix our notation, in Section 3 we show that solutions of quantum stochastic differential equations (s.d.e.'s) provide stochastic evolutions in our sense. In Section 4 we establish that, conversely, all such evolutions arise as solutions of quantum s.d.e.'s.

The following notational conventions are used.

Complex inner products are linear in the second argument. $B(h)$ denotes the algebra of bounded linear operators on a Hilbert space h and for $A \subset B(h)$, A' denotes the commutant of A.

2. *Preliminaries*

For a given complex Hilbert space h, let $\Gamma(h)$ denote the Boson (symmetric) Fock space

$$C \oplus h \oplus (h \otimes_s h) \oplus \ldots,$$

where $\otimes_s$ is the symmetrized (Hilbert space) tensor product, and for each $v \in h$ let $\psi(v)$ denote the coherent (exponential) vector

$$(1, v, (2!)^{-\frac{1}{2}} v \otimes v, (3!)^{-\frac{1}{2}} v \otimes v \otimes v, \ldots).$$

The family $\{\psi(v): v \in h\}$ is total and linearly independent in $\Gamma(h)$ and therefore operators may be (densely) defined on $\Gamma(h)$ by giving their action on each $\psi(v)$. Thus, when C is a contraction on h and u an element of h, the second quantized contraction $\Gamma(C)$ and the Fock–Weyl operator $W(u)$ are given by

$$\Gamma(C)\psi(v) = \psi(Cv),$$

$$W(u)\psi(v) = \exp\{-\tfrac{1}{2}\|u\|^2 - \langle u, v\rangle\}\psi(v+u) \quad (v \in h).$$

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We shall write $\Gamma, \Gamma_+, \Gamma_-, \Gamma_{s,t}$ for $\Gamma(h)$ when $h = L^2(I)$ and $I = \mathbb{R}, \mathbb{R}^+, \mathbb{R}^-, [s,t]$ respectively, and identify $\Gamma_+ \otimes \Gamma_-$ with Γ. The isomorphism is given by first linear, then continuous extension of

$$\psi(v_+) \otimes \psi(v_-) \leftrightarrow \psi(v_+, v_-). \tag{2·1}$$

For $t \geqslant s \geqslant 0$ we denote by $\mathscr{M}_{s,t}$ the von Neumann subalgebra of $\mathscr{M} := B(\Gamma_+)$ given by

$$\{W(v) : \operatorname{supp} v \subset [s,t]\}''.$$

Under a similar identification to (2·1), this is simply

$$I_{\Gamma_{0,s}} \otimes B(\Gamma_{s,t}) \otimes I_{\Gamma_{t,\infty}}.$$

Fixing an 'initial' (complex Hilbert) space h_0, the space of the quantum system, the family $\{\mathscr{N}_{s,t} := \mathscr{M}_0 \otimes \mathscr{M}_{s,t} \mid t \geqslant s \geqslant 0\}$ forms a (double) filtration of the von Neumann algebra $\mathscr{N} := \mathscr{M}_0 \otimes \mathscr{M}$, where $\mathscr{M}_0 := B(h_0)$, in the sense that if $u \geqslant t > s \geqslant v \geqslant 0$, then $\mathscr{N}_{s,t} \subset \mathscr{N}_{v,u}$. Conditional expectations onto each of these subalgebras exist under which states of the form $\rho \otimes \omega_{\psi(0)}$ are invariant for any state ρ on $\mathscr{M}_0$. These are characterized by

$$\mathbb{E}_{s,t}[T \otimes W(f)] = \langle \psi(0), W(f\chi_{[s,t]^c})\psi(0)\rangle\, T \otimes W(f\chi_{[s,t]}),$$

where $[s,t]^c = \mathbb{R}^+ \backslash [s,t]$, and satisfy projectivity:

$$E_{s_1,t_1} \circ \mathbb{E}_{s_2,t_2} = \mathbb{E}_{s_1,t_1} \quad \text{when} \quad [s_1,t_1) \subset [s_2,t_2).$$

We write $\mathbb{E}_s$ for $\mathbb{E}_{0,s}$.

3. *Stochastic evolutions*

Let S denote the right shift on $L^2(\mathbb{R}^+)$, so that for $t \geqslant 0$,

$$(S_t f)(x) = \begin{cases} f(x-t) & x \geqslant t \\ 0 & \text{otherwise.} \end{cases}$$

Definition (cf. [5]). A *covariant adapted unitary evolution on* $h_0 \otimes \Gamma_+$ is a family $\{V_{s,t} : t \geqslant s \geqslant 0\}$ of unitary operators satisfying $\forall t \geqslant s \geqslant r \geqslant 0$,

$$\left.\begin{array}{ll} \text{(i)} & V_{r,s} V_{s,t} = V_{r,t}, \\ \text{(ii)} & V_{s,t} \in \mathscr{N}_{s,t}, \\ \text{(iii)} & \Gamma(S_s)^* V_{s,t} \Gamma(S_s) = V_{0,t-s}, \\ \text{(iv)} & P : t \to \mathbb{E}_s[V_{s,t}] \quad \text{is norm continuous.} \end{array}\right\} \tag{3·1}$$

(iv) is more restrictive than one would like – see the conclusion.

The three basic martingales of the quantum stochastic calculus of [7] are defined by

$$\mathscr{X}^1(t)\, x \otimes \psi(f) = \frac{d}{d\lambda}\bigg|_{\lambda=0} x \otimes \psi(f + \lambda\chi_{[0,t]}),$$

$$\mathscr{X}^2(t)\, x \otimes \psi(f) = \frac{d}{d\lambda}\bigg|_{\lambda=0} x \otimes \psi(\exp\{\lambda\chi_{[0,t]}\} f),$$

$$\mathscr{X}^3(t)\, x \otimes \psi(f) = \int_0^t f(s)\, ds\, x \otimes \psi(f) \quad (x \in h_0, f \in L^2(\mathbb{R}^+)),$$

for $t \geqslant 0$, and are called the creation, preservation (or gauge) and annihilation processes respectively. Itô type stochastic integrals with respect to these processes

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are constructed and a class of stochastic differential equations studied in [7]. The differential notation

$$dX = J_i\, d\mathscr{X}^i \quad \text{(Einstein convention)}$$

will be used as shorthand for

$$X(t)-X(s) = \int_s^t J_i(r)\, d\mathscr{X}^i(r) \quad (s<t),$$

when each J_i is an adapted locally square-integrable operator-valued process (in the sense of [7]). In general the index will run from 1 to 4 with $\mathscr{X}^4(t) := t$. Matrix elements for these integrals are given by ordinary Lebesgue integrals ([7], theorem 4·2): for $u, v \in h_0, f, g \in L^2(\mathbb{R}^+)$,

$$\left\langle u\otimes\psi(f), \int_s^t J_i(r)\, d\mathscr{X}^i(r)\, v\otimes\psi(g)\right\rangle = \int_s^t \langle u\otimes\psi(f), k^i(r)\, J_i(r)\, v\otimes\psi(g)\rangle\, dr, \quad (3\cdot2)$$

where $k^1 = \bar{f}, k^2 = \bar{f}g, k^3 = g$ and $k^4 = 1$.

We give a proof of the following result for the convenience of the reader.

Proposition 3·1 [8]. *If U satisfies the quantum stochastic differential equation*

$$dX = XL_i\, d\mathscr{X}^i; \quad X(0) = I, \quad (3\cdot3)$$

where L_i $(i = 1, \ldots, 4)$ is constant and belongs to $\mathscr{N}_0$, and

$$L_1 = L, \quad L_2 = W-I, \quad L_3 = -L^*W, \quad L_4 = iK-\tfrac{1}{2}L^*L, \quad (3\cdot4)$$

where $K^* = K$ and $W^* = W^{-1}$, then $\{V_{s,t} := U(s)^*\, U(t) : t \geqslant s \geqslant 0\}$ is a covariant adapted unitary evolution.

Proof. The differential equation (3·3) has a unique (adapted) solution which moreover is unitary-valued under the conditions (3·4) by [7], theorem 7·1. This implies (3·1 (i)). Now fix $s \geqslant 0$ and let $U_s(x) = U(s)^*\, U(s+x)$ $(x \geqslant 0)$, so that

$$\begin{aligned} U_s(x) &= I + \int_s^{s+x} U_s(u-s)\, L_i\, d\mathscr{X}^i(u) \\ &= I + \int_0^x U_s(y)\, L_i\, d\mathscr{X}^i_s(y), \end{aligned}$$

where $\mathscr{X}^i_s(\,.\,) := \mathscr{X}^i(s+\,.\,) - \mathscr{X}^i(s)$ $(i = 1, 2, 3)$ are new creation, preservation and annihilation processes. But the unique solution of this integral equation may be obtained by Picard iteration [7], from which it is clear that $U_s(x) \in \mathscr{N}_{s,\, s+x}$ (since each of the iterates belong to this algebra), giving (3·1 (ii)). Since $\Gamma(S_s)^*\, \mathscr{N}_{s,\,t}\, \Gamma(S_s) = \mathscr{N}_{0,\,t-s}$,

$$W : x \to \Gamma(S_s)^*\, U(s)^*\, U(s+x)\, \Gamma(S_s)$$

is an adapted process, and by (3·2) for $f, g \in L^2(\mathbb{R}^+), u, v \in h_0$,

$$\begin{aligned} &\left\langle u\otimes\psi(f), \Gamma(S_s)^* \int_s^{s+t} U(s)^*\, U(x)\, L_i\, d\mathscr{X}^i(x)\, \Gamma(S_s)\, v\otimes\psi(g)\right\rangle \\ &\qquad = \int_s^{s+t} \langle u\otimes\psi(S_s f), U(s)^*\, U(x)\, L_i\, k^i(x-s)\, v\otimes\psi(S_s g)\rangle\, dx, \end{aligned}$$

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where $k^1 = \bar{f}$, $k^2 = \bar{f}g$, $k^3 = g$ and $k^4 = 1$,

$$= \int_0^t \langle u \otimes \psi(S_s f), U(s)^* U(y+s) L_i k^i(y) v \otimes \psi(S_s g) \rangle \, dy$$

$$= \left\langle u \otimes \psi(f), \int_0^t \Gamma(S_s)^* U(s)^* U(s+y) \Gamma(S_s) L_i \, d\mathscr{X}^i(y) v \otimes \psi(g) \right\rangle,$$

so that W also satisfies the differential equation

$$dX = XL_i \, d\mathscr{X}^i; \quad X(0) = I$$

and hence by uniqueness ([7], theorem 7·1) $W = U$, which gives (3·1 (iii)). The proof is completed by the observation

$$\|P(t) - P(s)\| \leqslant \|L_4\|(t-s). \quad \blacksquare$$

4. *From covariant adapted unitary evolutions to stochastic evolutions*

We now turn to the converse problem of characterizing every covariant adapted unitary evolution. Of crucial importance is the representability of martingales as stochastic integrals of regular Fock martingales [11].

Definition 4·1. A bounded operator-valued process M is a *martingale* if

$$\mathbb{E}_s[M(t)] = M(s) \quad \forall s < t$$

and a *regular* martingale if there is a Borel measure μ on $\mathbb{R}^+$ for which

$$\|[M(t) - M(s)] u\|^2 + \|[M^*(t) - M^*(s)] u\|^2 \leqslant \mu([s,t)) \, \|u\|^2, \tag{4·1}$$

whenever $s < t$ and $u \in \Gamma_{0,s} \otimes \psi(0)_{\Gamma_{s,\infty}}$.

It is shown in [11] that each bounded operator-valued regular Fock martingale M is expressible as a sum of stochastic integrals of bounded operator-valued processes J_i $(i = 1, 2, 3)$ with respect to the basic (unbounded operator-valued) martingales $\mathscr{X}^i$ $(i = 1, 2, 3)$

$$M(b) - M(a) = \int_a^b J_i(s) \, d\mathscr{X}^i(s).$$

THEOREM 4·2. *If V is a covariant adapted unitary evolution, then there are $L_i \in \mathscr{N}_0$ $(i = 1, \ldots, 4)$ such that $U(\cdot) := V_0$, satisfies the quantum stochastic differential equation*

$$dU = UL_i \, d\mathscr{X}^i.$$

Proof. First observe that P is in fact a semigroup [5]: for $s, t \geqslant 0$,

$$P_{s+t} = \mathbb{E}_0[V_{0,s+t}] = \mathbb{E}_0[V_{0,s} \mathbb{E}_s[V_{s,s+t}]]$$

$$= \mathbb{E}_0[V_{0,s} \mathbb{E}_0[V_{s,s+t}]]$$

$$= \mathbb{E}_0[V_{0,s}] \mathbb{E}_0[\Gamma(S_s)^* V_{s,s+t} \Gamma(S_s)] = P_s P_t,$$

where the projectivity of conditional expectations, the covariance of the evolution and the fact that $\mathbb{E}_s$ and $\mathbb{E}_0$ agree on $\mathscr{N}_{s,\infty}$ have been used. Since this semigroup is

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norm continuous, its generator N, say, is bounded. Let X be the bounded adapted process

$$t \to V_{0,t} - \int_0^t V_{0,s} N\, ds.$$

This is a martingale since, for $s \leqslant t$,

$$\begin{aligned}
\mathbb{E}_s[X(t)-X(s)] &= V_{0,s}\,\mathbb{E}_s\left[V_{s,t} - I - \int_s^t V_{s,r} N\, dr\right] \\
&= V_{0,s}\,\mathbb{E}_0\left[\Gamma(S_s)^*\left\{V_{s,t} - I - \int_s^t V_{s,r} N\, dr\right\}\Gamma(S_s)\right] \\
&= V_{0,s}\,\mathbb{E}_0\left[V_{0,t-s} - I - \int_s^t V_{0,r-s} N\, dr\right] \\
&= V_{0,s}\left\{P_{t-s} - I - \int_0^{t-s} P_r N\, dr\right\} = 0.
\end{aligned}$$

For $a < b \leqslant T$, $u \in \Gamma_{0,a} \otimes \psi(0)_{\Gamma_{a,\infty}}$,

$$\begin{aligned}
\|[X(b)-X(a)]u\|^2 &= \left\|\left[V_{a,b} - I - \int_a^b V_{a,s} N\, ds\right]u\right\|^2 \\
&\leqslant 2\left\{\|(V_{a,b}-I)u\|^2 + \left\|\int_a^b V_{a,s} N u\, ds\right\|^2\right\}
\end{aligned}$$

and, since P is contractive,

$$\begin{aligned}
|\langle u, (V_{a,b}-I)u\rangle| &= |\langle u, \mathbb{E}_a[V_{a,b}-I]u\rangle| \\
&= |\langle u, \mathbb{E}_0[V_{a,b}-I]u\rangle| \\
&= |\langle u, \mathbb{E}_0[V_{0,b-a}-I]u\rangle| \\
&= \left|\int_0^{b-a} \langle u, P_s N u\rangle\, ds\right| \\
&\leqslant (b-a)\,\|N\|\,\|u\|^2;
\end{aligned}$$

moreover,

$$\begin{aligned}
\langle V_{a,b}u, (V_{a,b}-I)u\rangle &= \langle u(I - V_{a,b}^{-1})u\rangle \\
&= \langle (I - V_{a,b})u, u\rangle,
\end{aligned}$$

so that,

$$\begin{aligned}
\|[X(b)-X(a)]u\|^2 &\leqslant 4(b-a)\,\|N\|\,\|u\|^2 + 2(b-a)^2\,\|N\|^2\,\|u\|^2 \\
&\leqslant (k_1 + Tk_2)(b-a)\,\|u\|^2,
\end{aligned}$$

where k_1, k_2 are constants independent of a, b and T. Now martingale increments over disjoint intervals, acting on vectors from the past, are orthogonal, so if $s < t$ and $u \in \Gamma_{0,s} \otimes \psi(0_{\Gamma_{s,\infty}})$,

$$\begin{aligned}
\|[X(t)-X(s)]u\|^2 &= \|[X(t)-X(m)]u\|^2 + \|[(X(m))-X(m-1)]u\|^2 + \ldots \\
&\quad \ldots + \|[X(n+1)-X(n)]u\|^2 + \|[X(n)-X(s)]u\|^2 \\
&\leqslant \sum_i (k_1 + ik_2)\,|[s,t] \cap [i-1,i)|\,\|u\|^2 \\
&= \mu([s,t])\,\|u\|^2,
\end{aligned}$$

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where $\mu = \Sigma_{i=1}^{\infty} (k_1 + ik_2)\lambda_i$ and λ_i is Lebesgue measure restricted to $[i-1, i)$. Since a similar inequality holds for increments of the adjoint process, X is regular. By [11] we have the representation

$$U(t) = I + \int_0^t U(s) F_i(s) \, d\mathscr{X}^i(s),$$

where F_i $(i = 1, 2, 3)$ are bounded operator-valued adapted processes and F_4 is the constant process identically equal to N. The proof is complete once we have shown that each F_i is constant.

Define new operators on $h_0 \otimes \Gamma = h_0 \otimes \Gamma_- \otimes \Gamma_+$ by

$$\tilde{V}_{s,t} = I_{\Gamma_-} \otimes V_{s,t}; \quad \tilde{F}_i(t) = I_{\Gamma_-} \otimes F_i(t); \quad \tilde{\mathscr{X}}^i(t) = I_{\Gamma_-} \otimes \mathscr{X}^i(t),$$

$\Gamma_s = I_{h_0} \otimes \Gamma(T_s)$, where T is the right shift on $L^2(\mathbb{R})$, so that

$$\tilde{U}(t) = I + \int_0^t \tilde{U}(s) \tilde{F}_i(s) \, d\tilde{\mathscr{X}}^i(s); \quad \Gamma_s^* \tilde{V}_{s,t} \Gamma_s = \tilde{V}_{0,t-s} \quad (t \geqslant s \geqslant 0)$$

which we consider as operators on $\tilde{h}_0 \otimes \Gamma_+$ where $\tilde{h}_0 = h_0 \otimes \Gamma_-$; thus

$$\begin{aligned}
\tilde{U}(t-s) &= \Gamma_s^* \left\{ I + \int_s^t \tilde{V}_{s,r} \tilde{F}_i(r) \, d\tilde{\mathscr{X}}^i(r) \right\} \Gamma_s \\
&= I + \int_s^t \tilde{U}(r-s) \, \Gamma_s^* \tilde{F}_i(r) \, \Gamma_s \, d\tilde{\mathscr{X}}^i_s(r) \quad (\tilde{\mathscr{X}}^i_s(r) = \tilde{\mathscr{X}}^i(r-s)) \\
&= I + \int_0^{t-s} \tilde{U}(r) \{\Gamma_s^* \tilde{F}_i(r+s) \, \Gamma_s\} \, d\tilde{\mathscr{X}}^i(r) \quad (t \geqslant s \geqslant 0).
\end{aligned}$$

Comparing these we obtain the relations

$$\tilde{F}_i(x) = \Gamma_s^* \tilde{F}_i(x+s) \, \Gamma_s \quad \text{for almost all } x \quad (\text{fixed } s).$$

By Fubini's theorem,

$$\Gamma_s^* \tilde{F}_i(t) \, \Gamma_s = \tilde{F}_i(t-s) \quad \text{for almost all} \quad \{(s,t) : s \leqslant t\}. \tag{4·2}$$

Now if $T > 0$ then there is $t \geqslant T$ for which (4·2) holds for almost all $s \leqslant t$, so that there is a version of $\tilde{F}_t$ for which

$$\Gamma_{t-u}^* \tilde{F}_i(t) \, \Gamma_{t-u} = \tilde{F}_i(u) \quad \forall u \leqslant t.$$

Putting $u = 0$ first and then using the fact that Γ_t is unitary and commutes with $\tilde{F}_i(0)$, we conclude that there is a version of $\tilde{F}_t$, and thus of F_i, which is constant on $[0, T]$. Since T may be taken arbitrarily large, the proof is complete. ∎

5. *Conclusion*

What if the filtration $\{\mathscr{N}_{s,t}\}$ is that of a general quantum Brownian motion [3], [1] rather than, as here, that of a unit variance Boson quantum Brownian motion? In the unit variance Fermi case ([2]), Theorem 4·2 remains true by virtue of the argument employed here together with the results of [9] – in particular the identification of Bose and Fermi martingales. In the non-unit variance cases the theorem has already been established in [6], [10].

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It would be more satisfactory if the continuity condition in the definition of a covariant adapted unitary evolution were weakened to strong continuity of $t \to V_{0,t}$. A characterization of such evolutions would then amount to a quantum stochastic Stone's theorem. We have received a preprint from J. L. Journé: 'Structure des cocycles markoviens sur l'espace de Fock' (to be published in *Probability Theory and Related Fields*), in which considerable progress on this problem is made.

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