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Characterization for the queueing system $M/G/\infty$

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Consider a queueing system $M/G/s$ with the arrival intensity λ, the service time distribution function $B(t)$ $(B(0) < 1)$ having a finite mean and the waiting room size $N \leqslant \infty$. If $s < \infty$ and $N = \infty$, we shall also assume that its relative traffic intensity is less than 1. Since the arrival process of this system is Poisson, it is immediate that in this case the distribution of the number of arrivals during an interval is infinitely divisible.

For the queueing systems $M/M/s$ with infinite waiting space and $M/G/\infty$ which are special cases of the present system, the steady state distributions of the number of departures during an interval are known to be Poisson and hence are infinitely divisible (cf. Burke(1) and Mirasol(4)). It is also clear that for the systems $M/M/1$ with infinite waiting space and $M/G/\infty$ the steady state joint distributions of the number of departures during an interval and the queue length at its termination are infinitely divisible. Recently Shanbhag and Tambouratzis(5) have established that in the system $M/G/s$ with $N = 0$ if the losses are included in the departures, then the steady state distribution of the number of departures during an interval is Poisson and hence infinitely divisible. A similar result is also valid for the $M/M/s$ system with $N > 0$.

In his recent paper, Daley(2) has shown that for a stationary $M/M/1$ queue with infinite waiting space the joint distribution of the number of arrivals and departures during an interval is not infinitely divisible. However, according to Milne's findings (see (2)) the distribution is infinitely divisible for $M/G/\infty$. In the present paper, we establish that the number of arrivals and departures during an interval for the above $M/G/s$ system have an infinitely divisible steady state distribution iff the system is $M/G/\infty$.

Let $A(t, t+T)$ and $D(t, t+T)$ denote the number of arrivals and departures during $(t, t+T]$ for our system and

$$P_{ij}(T) = \lim_{t\to\infty} P\{A(t, t+T) = i,\ D(t, t+T) = j\} \quad (i, j = 0, 1, \ldots).$$

In view of Milne's result it is sufficient if we show that for an $M/G/s$ system with $s < \infty$ and waiting room of size $N \leqslant \infty$, $\{P_{ij}(T)\}$ assuming it exists does not denote an infinitely divisible distribution. This is done in what follows.

Assume $\{p_{ij}(T)\}$ to be infinitely divisible distribution. Then

$$\sum_{j=0}^{\infty} P_{0j}(T)\, z^j = P_{0.}(T)\, e^{-\eta+\eta P(z)} \quad (|z| \leqslant 1), \tag{1}$$

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where $$P_{0.}(T) = \sum_{j=0}^{\infty} P_{0j}(T),$$

$$P(z) = \sum_{1}^{\infty} P_j z^j,$$

with $\{P_j\}$ $(j = 1, 2, \ldots)$ as a probability distribution and η is some positive constant (see Feller (3), p. 290). If $s, N < \infty$, then irrespective of whether we include losses in the departures or not, $P_{0j}(T) = 0, j = N+s+1, N+s+2, \ldots$. If this is so (1) is not valid and hence in this case we cannot have $\{P_{ij}(T)\}$ to be infinitely divisible distribution. Consider now the case $s < \infty$, $N = \infty$. It can be easily seen that

$$C_s(T)\, e^{-\lambda T}\frac{(\lambda T)^i}{i!} \leqslant P_{i0}(T) \leqslant e^{-\lambda T}\frac{(\lambda T)^i}{i!} \quad (i = 0, 1, \ldots), \tag{2}$$

where $C_s(T)$ is equal to $P_{00}(T)\,(B^*(T))^s/e^{-\lambda T}$ with $B^*(T)$ as the probability that a service time is $\geqslant T$. Using the result in (3), we can at once see that

$$\begin{aligned} C_s(T)\, e^{-\lambda T+\lambda T z} &\leqslant \sum_{i=0}^{\infty} P_{i0}(T)\, z^i = P_{.0}(T)\, e^{-\eta^*+\eta^* P^*(z)} \\ &\leqslant e^{-\lambda T+\lambda T z} \quad (0 \leqslant z < \infty), \end{aligned} \tag{3}$$

where $$P_{.0}(T) = \sum_{i=0}^{\infty} P_{i0}(T),$$

and $P^*(z)$ and η^* are as defined in the previous case. Dividing (3) by $\exp\{\lambda T z\}$ and taking the limit as $z \to \infty$, we observe that $P^*(z) = z$ and if $B^*(T) > 0$ then $\eta^* = \lambda T$. In the case of $B^*(T) > 0$, we have therefore

$$\sum_{i=0}^{\infty} P_{i0}(T)\, z^i = P_{.0}(T)\, e^{-\lambda T+\lambda T z} \quad (0 \leqslant z < \infty),$$

which gives $$P_{00}(T) = e^{-\lambda T} P_{.0}(T). \tag{4}$$

If $B^*(T)$ equals 1, then $P_{0j}(T) = 0$, $j = s+1, s+2, \ldots$ and hence (1) is not valid. Similarly, if $B(t)$ is degenerate (1) is not valid. Further if $0 < B^*(T) < 1$, then

$$\frac{P_{00}(T)}{\exp(-\lambda T)} > P_{.0}(T), \tag{5}$$

which contradicts (4). This is so because the left-hand side of the above inequality denotes the limiting probability that during an interval of length T none of the customers present in the queue at the beginning of the interval departs and the right-hand side denotes the probability of the intersection of the above event with the event that none of the incoming customers during the interval departs before the end of the interval. For the case $B^*(T) = 0$ with $B(t)$ non-degenerate, we have

$$\frac{P_{10}(T)}{P_{00}(T)} = \eta^* = \lambda \int_0^T (1 - B(y))\, dy.$$

This gives us $$P\{D(0, T) = 0 | Q(0) = 0\} = \exp\left\{-\lambda \int_0^T B(y)\, dy\right\}$$

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where $Q(t)$ denotes the queue length at t. Since this last equation is valid for $s = \infty$, it is immediate that it cannot be valid for $s < \infty$. Hence $\{P_{ij}(T)\}$ can not be infinitely divisible distribution for $s < \infty$, $N = \infty$.

Considering $B(y) > 0$ for some $y < T$ and removing the assumption that the service time mean is finite, and the relative traffic intensity is less than 1 for $s < \infty$, $N = \infty$, it can be seen that $P\{A(0,T) = i, D(0,T) = j | Q(0) = 0\}$ $(i,j = 0, 1, \ldots)$ for our system denotes an infinitely divisible distribution iff the system is $M/G/\infty$. We leave this as an exercise to the reader.

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