to

$$\sum_{k=1}^{N} k^2 = \frac{(N+1)(2N+1)N}{6}. \tag{11}$$

The proof presented in our Note is an alternative to that found, for example, in [1] and it uses the properties of the multiplication table.

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100.03 Evaluating the probability integral by annular partition

The evaluation of the probability integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{1}$$

can be achieved in numerous ways including the methods based on double integration and solids of revolution (for example, see [1, 2, 3, 4]). A well-known approach of double integral in numerous textbooks is to compute

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \pi$$

using Fubini's theorem and the change of variable theorem to transform the double integral from rectangular coordinates to polar coordinates. In this note we use double integration with annular partition to evaluate the probability integral.

Consider the double integral of $z = f(x, y) = e^{-(x^2+y^2)}$ over three different regions respectively. Region 1 is the disk, denoted by D_a, with radius a and centre at the origin. Region 2 is the square, denoted by S_a, with $-a \leqslant x \leqslant a$ and $-a \leqslant y \leqslant a$. Region 3 is the disk, denoted by $D_{\sqrt{2}a}$, with radius $\sqrt{2}a$ and centre at the origin. The construction of these three regions is offered in [2]. Since $f(x, y)$ is positive and the square S_a lies between the disk D_a and the disk $D_{\sqrt{2}a}$, we obtain

$$V_1 = \iint_{D_a} f(x,y)dA < V_2 = \iint_{S_a} f(x,y)dA < V_3 = \iint_{D_{\sqrt{2}a}} f(x,y)dA. \tag{2}$$

Each double integral in (2) represents the volume, V_i, of the solid over its respective region i between the surfaces $z = 0$ and $z = f(x, y)$.

In particular, V_2 has two properties. On the one hand, Fubini's theorem gives

$$V_2 = \iint_{S_a} f(x, y)\, dA = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)} dx\, dy = \left[\int_{-a}^{a} e^{-x^2} dx\right]^2.$$

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On the other hand, the double integral V_2 over the square S_a is the limit of double Riemann sums using rectangular partition. This suggests that we can partition the disk D_a using non-overlapping annuli or rings due to the nature of its shape. For example, a small element of ring is given by the region

$$R_i = \left\{(x, y) \in \mathbb{R}^2 \middle| r_{i-1} \leqslant \sqrt{x^2 + y^2} \leqslant r_i,\ r_i = \frac{ia}{n}\right\},$$

where $1 \leqslant i \leqslant n$ and $r_0 = 0$.

For a point (x, y) in R_i, its height is denoted by

$$h = f(x, y) = e^{-(x^2+y^2)} = e^{-r^2}.$$

It follows that r can be regarded as a function of height h and expressed as

$$(r(h))^2 = -\ln h. \tag{3}$$

Furthermore, the area of the region R_i is equal to $\pi\left(r_i^2 - r_{i-1}^2\right)$, which simplifies to

$$2\pi r_i^* \Delta r_i, \text{ where } r_i^* = \left(\frac{r_i + r_{i-1}}{2}\right) \text{ and } \Delta r_i = \left(r_i - r_{i-1}\right).$$

Let the height of a sample point (x_i, y_i) selected from the ring R_i be ξ_i. For simplicity, we choose (x_i, y_i) to satisfy

$$\sqrt{x_i^2 + y_i^2} = r_i^*.$$

It follows that $\xi_i = e^{-r_i^{*2}}$. Hence, the Riemann-Stieltjes sum for approximating the volume V_1 is

$$\sum_{i=1}^{n} \xi_i \left(2\pi r_i^* \Delta r_i\right).$$

The limit of Riemann-Stieltjes sum is obtained from evaluating the Riemann-Stieltjes integral

$$\int_1^{e^{-a^2}} h \cdot 2\pi r(h)\, dr(h) = \pi \int_1^{e^{-a^2}} (-1)\, dh = \pi\left(1 - e^{-a^2}\right).$$

The first equality is obtained from (3). The integration limits $h = 1$ and $h = e^{-a^2}$ correspond to $r = 0$ and $r = a$.

The annular partition method suggests that the double integral $V_1 = \int\int_{D_a} f(x, y)\, dA$ is directly reduced to a simple single-variable integral without using coordinate transformation. Similarly, the evaluation for the double integral V_3 gives $\pi\left(1 - e^{-2a^2}\right)$. Hence, (2) leads to

$$\pi\left(1 - e^{-a^2}\right) < \left[\int_{-a}^{a} e^{-x^2} dx\right]^2 < \pi\left(1 - e^{-2a^2}\right). \tag{4}$$

As radius a tends to infinity, applying the squeeze (or sandwich) theorem to the inequality (4) yields the probability integral (1).

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100.04 Converses of a 'curious' theorem of Cesàro on conditionally convergent series

In [1], Bromwich describes a theorem of Cesàro as 'A curious theorem, to some extent a kind of converse of Art. 19' which is Leibniz's alternating series test (§23, p. 61). Hardy (see [2] and also [3], where [2] is reproduced with notes) describes the result as 'an exceedingly curious and interesting one'. Rephrasing it in modern terms, we have:

Cesàro's Theorem: If $\sum a_r$ is conditionally convergent, $(|a_r|)$ is a decreasing sequence, and p_n denotes the number of positive terms, q_n the number of negative terms among the first n terms of the series, and $\lim p_n/q_n$ exists, then the limit is unity. (Proofs may be found in [1 , 2, or 3].)

The purpose of this note is to obtain some other converses of Cesàro's theorem. We give three such converses, the first an elementary result, the second due to Rademacher [4], and the third involving a condition, not only sufficient but also necessary for the convergence of the series. Finally, we give a numerical example to illustrate these results.

We consider here three converses of Cesàro's theorem. The first is a simple and easily proved result, generalising Leibniz's alternating series test.

Theorem 1: If the sequence $(p_n - q_n)$ is bounded, i.e. $|p_n - q_n| \leqslant K$ for all n (K a positive constant), and $(|a_n|)$ decreases to zero, then $\sum a_r$ converges.

Proof: Since $\sum_{r=1}^{n} \operatorname{sgn} a_r = p_n - q_n$ is bounded (where $\operatorname{sgn} a_r$ is +1 or −1 according as the sign of a_r is + or −) and $(|a_n|)$ decreases to zero, then $\sum \operatorname{sgn} a_r (|a_r|) \equiv \sum a_r$ converges by Dirichlet's theorem (see [1, p. 59] or [5, p. 315]).

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