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Reversible topological Markov shifts

JUNGSEOB LEE†, KEYWON K. PARK† and SUJIN SHIN‡

† *Department of Mathematics, Ajou University, Suwon 443-749, Korea*
(e-mail: {jslee, kkpark}@ajou.ac.kr)
‡ *Department of Mathematics, KAIST, Daejeon 305-701, Korea*
(e-mail: sjs@math.kaist.ac.kr)

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Abstract. We generalize the notion of reversible systems in symbolic dynamics, and investigate their properties. It is shown that a reversible topological Markov shift can be represented by a pair of matrices of special types. This enables us to classify the invariant measures of the reversible systems. Necessary and/or sufficient conditions for the existence of a reversal of finite order are established in terms of the adjacency matrices. We also prove that a topological Markov shift with a reversal of order two admits reversals of all orders.

1. *Introduction*

The study of a reversible flow or diffeomorphism was motivated by the classical Hamiltonian systems, such as the Stömer and the restricted three-body problems. The notion of reversibility has been extended and the qualitative behavior of these systems has been studied (see [**1, 3, 14, 15**]). One of the main conditions for the reversibility of a diffeomorphism T is the existence of an *involution* or a *flip* ϕ satisfying $T \circ \phi = \phi \circ T^{-1}$ and $\phi^2 = \mathrm{id}$. It is known that such a map is not unique. In fact, it is easy to see that there are infinitely many involutions, if there is one.

If ϕ is an involution, then the system with $\{T, \phi\}$ can be regarded as a dihedral group action. Reversible systems also naturally arise in the study of $\mathbb{Z}$-foliations. Let us consider a space X with a foliation each of whose equivalence classes is isometric to $\mathbb{Z}$. It is not hard to show that either:

(i) each equivalence class is an orbit of a $\mathbb{Z}$-action, that is, the foliation comes from a $\mathbb{Z}$-action; or
(ii) there exists a covering space $(\widehat{X}, T)$ that is a reversible system so that the foliation is isometric to $\widehat{X}/\sim$, where $x \sim \phi(x)$.

Furthermore, the isometry between two foliations is equivalent to the conjugacy between the dihedral group actions on the covering spaces [**10**].

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The notion of reversibility can be generalized by removing the condition $\phi^2 = \mathrm{id}$. We call the map satisfying the relation $T \circ \phi = \phi \circ T^{-1}$ a reversal, and are particularly interested in reversals of finite orders. In many systems the dynamical properties of T impose some constraints on a reversal ϕ. We state a few results in this direction. In measurable dynamics, ϕ^2 has to have a multiplicity function that only takes even values on the orthogonal complement of the subspace of functions invariant under ϕ^2 (see [**4**]). Moreover, if an ergodic transformation T has a simple spectrum or has finite rank, then any reversal ϕ has to satisfy $\phi^2 = \mathrm{id}$ (see [**5**]). Suppose that T_1 and T_2 are Bernoulli of the same entropy. It is well known that the isomorphism between T_1 and T_2 forces the isomorphism between two larger groups generated by $\{T_1, \phi_1\}$ and $\{T_2, \phi_2\}$ where ϕ_1 and ϕ_2 are, respectively, reversals for T_1 and T_2 of the same finite order [**13**]. However, this property does not hold for K-automorphisms. That is, there is a K-automorphism T such that $\{T, \phi_1\}$ and $\{T, \phi_2\}$ are not isomorphic where ϕ_1 and ϕ_2 are reversals of the same finite order [**11**]. Hence, they generate non-isometric foliations, unlike the Bernoulli case. In symbolic dynamics, it is easy to construct such examples using simple automorphisms. In fact, there are infinitely many flips on a full shift that are not conjugate to each other [**7**].

The main purpose of this paper is to study properties of a topological Markov shift together with a reversal. In §2, we give some preliminary definitions and prove the representation theorem, which states that any topological Markov shift with a reversal of finite order can be represented by a pair of matrices with certain properties. This fundamental theorem enables one to prove various results. In §3, we use the theorem to identify all reversal-invariant Markov measures, and then classify the invariant measures of a reversal system.

It is known that not every topological Markov shift is reversible. The first example of an irreversible Markov shift is due to Köllmer [**12**]. In §4, necessary and/or sufficient conditions for topological Markov shifts to admit reversals are sought in terms of their adjacency matrices. The orders of reversals are considered in §5. One of our main results is that if a system admits a reversal of order n, then it admits reversals of all orders that are multiples of n. The proof is by an argument using the marker. Finally, we present an example of a system admitting a reversal of order four, but no flips.

2. *Preliminaries and the representation theorem*

Definition 2.1. Let (X, T) be a topological dynamical system, where X is a topological space and $T : X \to X$ a homeomorphism. A homeomorphism $\phi : X \to X$ is called a *reversal* for (X, T) if

$$T \circ \phi = \phi \circ T^{-1}.$$

The triplet (X, T, ϕ) is called a *reversal system*. If $\phi^n = \mathrm{id}$ for $n \geq 1$ and $\phi^k \neq \mathrm{id}$ for $k = 1, \ldots, n-1$, then ϕ is said to be a *reversal of order n*. In particular, if ϕ is of order two, then it is called a *flip map* or simply a *flip*.

Let (X, T, ϕ) be a reversal system. By the definition, ϕ is a topological conjugacy between (X, T) and (X, T^{-1}). Also, $T \circ \phi^2 = \phi^2 \circ T$. Hence, ϕ^2 is an automorphism of (X, T). If n is odd, then

$$T \circ \phi^n = \phi^{n-1} \circ T \circ \phi = \phi^{n-1} \circ \phi \circ T^{-1} = \phi^n \circ T^{-1},$$

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so that ϕ^n is a reversal for (X, T). It follows that if ϕ is of order n, then n must be even, unless $T^2 = \mathrm{id}$. If $\psi : (X, T) \to (X, T)$ is an automorphism, then $\psi \circ \phi$ and $\phi \circ \psi$ are both reversals for (X, T).

We say that a reversal system (X, T, ϕ) is *conjugate* to another system (Y, S, ψ) if there is a homeomorphism $\theta : X \to Y$ such that

$$\theta \circ T = S \circ \theta \quad \text{and} \quad \theta \circ \phi = \psi \circ \theta.$$

In this case, θ is called a *conjugacy* from (X, T, ϕ) to (Y, S, ψ).

Let $\theta : (X, T) \to (Y, S)$ be a conjugacy between two topological dynamical systems and ϕ a reversal on (X, T). Defining $\psi = \theta \circ \phi \circ \theta^{-1}$, we obtain a reversal system (Y, S, ψ) that is conjugate to (X, T, ϕ). It is easy to see that $(X, T^m, T^n\phi)$ is a reversal system for any $m, n \in \mathbb{Z}$. Also, T is a conjugacy from $(X, T^m, T^n\phi)$ to $(X, T^m, T^{n+2}\phi)$ and ϕ is a conjugacy from $(X, T^m, T^n\phi)$ to $(X, T^{-m}, T^{-n}\phi)$.

Throughout the work, let $\mathcal{A}$ denote an alphabet, that is, a finite set of symbols, and let it be equipped with the discrete topology. For $x \in \mathcal{A}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, the ith coordinate of x is denoted by x_i. For $i, j \in \mathbb{Z}$ with $i < j$, the block $x_i x_{i+1} \cdots x_j$ is denoted $x_{[i,j]}$. For $x \in \mathcal{A}^{\mathbb{Z}}$, we define σx and ρx in $\mathcal{A}^{\mathbb{Z}}$ by

$$(\sigma x)_i = x_{i+1} \quad \text{and} \quad (\rho x)_i = x_{-i} \quad \text{for } i \in \mathbb{Z}.$$

Then σ and ρ are homeomorphisms on $\mathcal{A}^{\mathbb{Z}}$, and satisfy that

$$\sigma \circ \rho = \rho \circ \sigma^{-1} \quad \text{and} \quad \rho^2 = \mathrm{id}.$$

Hence, ρ is a flip for the full $\mathcal{A}$-shift $(\mathcal{A}^{\mathbb{Z}}, \sigma)$. The map σ is called the *shift map*, and ρ the *mirror map*.

Let A be an $\mathcal{A} \times \mathcal{A}$ binary matrix. The topological Markov shift X_A is the closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ defined by

$$\mathsf{X}_A = \{x = (x_i) \in \mathcal{A}^{\mathbb{Z}} : A_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\}.$$

The restriction of the shift on X_A is denoted by σ_A. The mirror map restricted on X_A, denoted by ρ_A, maps X_A onto $\mathsf{X}_{A^{\mathrm{T}}}$. If $A = A^{\mathrm{T}}$, then X_A is ρ_A-invariant, and hence ρ_A is a flip for X_A.

Suppose now that P is an $\mathcal{A} \times \mathcal{A}$ binary matrix such that

$$AP = PA^{\mathrm{T}} \quad \text{and} \quad P^n = I \tag{2.1}$$

for some $n \geq 1$. We construct a reversal $\varphi_{A,P}$ for X_A with $(\varphi_{A,P})^n = \mathrm{id}$. As $P^n = I$, it is a permutation matrix, so that $P^{-1} = P^{n-1} = P^{\mathrm{T}}$. For each $a \in \mathcal{A}$, let $\tau(a) \in \mathcal{A}$ be defined by

$$P_{a\tau(a)} = 1. \tag{2.2}$$

Then, $\tau : \mathcal{A} \to \mathcal{A}$ is a bijection and $\tau^n = \mathrm{id}$. One can easily see that

$$A_{ab} = A_{\tau(b)\tau(a)} \quad \text{for } a, b \in \mathcal{A}. \tag{2.3}$$

For $x \in \mathsf{X}_A$ define $\varphi_{A,P}x \in \mathsf{X}_A$ by

$$(\varphi_{A,P}x)_i = \tau(x_{-i}) \quad \text{for } i \in \mathbb{Z}.$$

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It follows from (2.3) that $\varphi_{A,P}$ is a reversal for X_A. Also $(\varphi_{A,P})^n = \mathrm{id}$, as $\tau^n = \mathrm{id}$. When we refer to $(\mathsf{X}_A, \sigma_A, \varphi_{A,P})$ as a reversal system, we always assume that A and P are binary matrices satisfying (2.1).

The following theorem asserts that every topological Markov shift with a reversal of finite order can be represented in this way. We will adopt and extend the proof of the case $n = 2$, which is given in [7].

THEOREM 2.1. (Representation theorem) *Let (X, σ) be a topological Markov shift with a reversal ϕ of order n. Then there are binary matrices A and P satisfying (2.1) such that (X, σ, ϕ) is conjugate to $(\mathsf{X}_A, \sigma_A, \varphi_{A,P})$.*

Proof. We construct an alphabet $\mathcal{A}$ and two $\mathcal{A} \times \mathcal{A}$ binary matrices A and P desired in the theorem. As ϕ is continuous, there is $k \in \mathbb{N}$ such that

$$(\phi^j x)_0 = (\phi^j y)_0 \quad j = 0, 1, \ldots, n-1 \tag{2.4}$$

for any $x, y \in X$ with $x_{[-k,k]} = y_{[-k,k]}$. For $x \in X$ let $\widetilde{x}$ be the mirror image of x, that is,

$$\widetilde{x} = \cdots x_2 x_1 . x_0 x_{-1} x_{-2} \cdots .$$

Given $x \in X$, we define $[x]$ an ordered n-tuple of $(2k+1)$-blocks by

$$[x] = (x_{[-k,k]}; (\widetilde{\phi x})_{[-k,k]}; \cdots ; (\phi^{n-2}x)_{[-k,k]}; (\widetilde{\phi^{n-1}x})_{[-k,k]}).$$

Note that if $x, y \in X$ and $x_{[-nk,nk]} = y_{[-nk,nk]}$, then $[x] = [y]$.

Now we define the alphabet $\mathcal{A} = \{[x] : x \in X\}$. It is clear that $\mathcal{A}$ is a finite set. For $\mathsf{a} = [x] \in \mathcal{A}$, define

$$\omega(\mathsf{a}) = ((\phi x)_{[-k,k]}; (\widetilde{\phi^2 x})_{[-k,k]}; \cdots ; (\phi^{n-1}x)_{[-k,k]}; \widetilde{x}_{[-k,k]}).$$

Then $\omega(\mathsf{a}) \in \mathcal{A}$, as $\omega([x]) = [\phi x]$ for any $x \in X$. Note that ω^n is the identity on $\mathcal{A}$.

Given $\mathsf{a} = [x] \in \mathcal{A}$, we define ordered n-tuples of $2k$-blocks $l(\mathsf{a})$ and $r(\mathsf{a})$ as follows:

$$l(\mathsf{a}) = (x_{[-k,k-1]}; (\widetilde{\phi x})_{[-k,k-1]}; \cdots ; (\phi^{n-2}x)_{[-k,k-1]}; (\widetilde{\phi^{n-1}x})_{[-k,k-1]}),$$
$$r(\mathsf{a}) = (x_{[-k+1,k]}; (\widetilde{\phi x})_{[-k+1,k]}; \cdots ; (\phi^{n-2}x)_{[-k+1,k]}; (\widetilde{\phi^{n-1}x})_{[-k+1,k]}).$$

Define the $\mathcal{A} \times \mathcal{A}$ binary matrix A by

$$A_{\mathsf{ab}} = \delta_{r(\mathsf{a})l(\mathsf{b})} \quad \text{for } \mathsf{a}, \mathsf{b} \in \mathcal{A},$$

and the $\mathcal{A} \times \mathcal{A}$ permutation matrix P by

$$P_{\mathsf{ab}} = \delta_{\omega(\mathsf{a})\mathsf{b}} \quad \text{for } \mathsf{a}, \mathsf{b} \in \mathcal{A},$$

where δ denotes the Kronecker delta. Then it is straightforward to check that A and P satisfy (2.1).

Finally, we show that (X, σ, ϕ) is conjugate to $(\mathsf{X}_A, \sigma_A, \varphi_{A,P})$. Define $\theta : X \to \mathsf{X}_A$ by

$$(\theta x)_i = [\sigma^i x] \quad \text{for } x \in X \text{ and } i \in \mathbb{Z}.$$

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Then, θ is a sliding block code with memory and anticipation nk. Clearly, θ is injective and a direct calculation shows that $\theta \circ \phi = \varphi_{A,P} \circ \theta$. It remains only to show that θ is surjective. For $\mathsf{a} = [x] \in \mathcal{A}$, let

$$c(\mathsf{a}) = x_{[-k,k]} \quad \text{and} \quad s(\mathsf{a}) = (x_0; (\widetilde{\phi x})_0; \cdots; (\phi^{n-2}x)_0; (\widetilde{\phi^{n-1}x})_0).$$

Observe that (2.4) implies that

$$s(\mathsf{a}) = s(\mathsf{b}) \quad \text{for any } \mathsf{a}, \mathsf{b} \in \mathcal{A} \text{ with } c(\mathsf{a}) = c(\mathsf{b}). \tag{2.5}$$

Let $y = \cdots \mathsf{a}_{-2}\mathsf{a}_{-1}.\mathsf{a}_0\mathsf{a}_1\mathsf{a}_2 \cdots$ be any point in X_A. Then there is a point $x \in X$ such that for each $i \in \mathbb{Z}$,

$$x_{[-k+i,k+i]} = c(\mathsf{a}_i)$$

as X is a topological Markov shift. We claim that $\theta x = y$. To see this, write

$$\theta x = \cdots \mathsf{b}_{-2}\mathsf{b}_{-1}.\mathsf{b}_0\mathsf{b}_1\mathsf{b}_2 \cdots .$$

By the definition of θ, we obtain that $x_{[-k+i,k+i]} = c(\mathsf{b}_i)$ for each $i \in \mathbb{Z}$. Hence, $s(\mathsf{a}_i) = s(\mathsf{b}_i)$ for all $i \in \mathbb{Z}$ by (2.5). Thus, $y = \theta x$. □

Putting $n = 2$ in the theorem, we obtain the following.

COROLLARY 2.2. *Let (X, σ) be a topological Markov shift with a flip ϕ. Then there are binary matrices A and P satisfying $AP = PA^{\mathrm{T}}$ and $P^2 = I$ such that (X, σ, ϕ) is conjugate to $(\mathsf{X}_A, \sigma_A, \varphi_{A,P})$.*

As one can see in the following example, reversals on topological Markov shifts are not necessarily of finite order.

Example 2.1. Let $\mathcal{A} = \{0, 1, 2, 3\}$ and X be the full $\mathcal{A}$-shift. Define a block map Γ from the set of 2-blocks in X to $\mathcal{A}$ by

$$\Gamma : 00, 01, 20, 21 \mapsto 0 \quad 02, 03, 22, 23 \mapsto 1$$
$$10, 11, 30, 31 \mapsto 2 \quad 12, 13, 32, 33 \mapsto 3$$

and let $\gamma : X \to X$ be the sliding block code induced by Γ. Then, we can see that $\gamma^2 = \sigma$ and, hence, γ is an automorphism. Put $\phi = \rho \circ \gamma$. As ρ is a reversal for X, so is ϕ. Fix $n \geq 1$ and consider the $(n+1)$-periodic point

$$x = (10^n)^\infty = \cdots 10^n.10^n10^n1 \cdots \in X.$$

Then, we see that $\phi(x) = (20^n)^\infty$ and $\phi^2(x) = \sigma^{-1}(x)$. Thus,

$$\phi^{2n}(x) = \sigma^{-n}(x) \neq x.$$

Therefore, $\phi^{2n} \neq \mathrm{id}$ for any $n \geq 1$.

If A is a symmetric matrix, we can take $P = I$ in (2.1) and then see that $\varphi_{A,I}$ is the mirror map ρ_A. The following example provides a negative answer to R. Burton's question: if A and B are symmetric matrices, then does the conjugacy between (X_A, σ_A) and (X_B, σ_B) force the conjugacy between $(\mathsf{X}_A, \sigma_A, \rho_A)$ and $(\mathsf{X}_B, \sigma_B, \rho_B)$? When the adjacency matrix is primitive, the problem is still open.

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Example 2.2. Consider two symmetric matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is not hard to see that A is strong shift equivalent to B. However, $(\mathsf{X}_A, \sigma_A, \rho_A)$ and $(\mathsf{X}_B, \sigma_B, \rho_B)$ are not conjugate. In fact, every orbit of length 4 in X_B is fixed by the mirror map ρ_B, whereas some of those in X_A are not by ρ_A.

3. *Invariant measures for reversals*

Throughout the section, assume that A is an $\mathcal{A} \times \mathcal{A}$ binary irreducible matrix. Let $M(\mathsf{X}_A)$ denote the set of all shift-invariant Borel probability measures on X_A. Let ϕ be a reversal for X_A. Then ϕ naturally induces a bijection from $M(\mathsf{X}_A)$ onto $M(\mathsf{X}_A)$, denoted again by ϕ, that is, for $\mu \in M(\mathsf{X}_A)$, $\phi(\mu) \in M(\mathsf{X}_A)$ is defined by $\phi(\mu)(B) = \mu(\phi^{-1}(B))$ for any Borel set B in X_A. Note that $\rho_A \circ \phi : (\mathsf{X}_A, \sigma_A) \to (\mathsf{X}_{A^{\mathrm{T}}}, \sigma_{A^{\mathrm{T}}})$ is a conjugacy. From this, we observe that ϕ maps Markov measures to Markov measures. In this section, we identify those measures in $M(\mathsf{X}_A)$ that are invariant under ϕ.

For a positive integer n, we denote by $\mathcal{B}_n = \mathcal{B}_n(\mathsf{X}_A)$ the collection of all n-blocks in X_A, and let $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{B}_n$. Each $u \in \mathcal{B}_n$ defines the cylinder sets

$$_l[u]_m = \{x \in \mathsf{X}_A : x_{[l,m]} = u\}$$

for integers l, m with $m - l = n - 1$.

An $\mathcal{A} \times \mathcal{A}$ matrix is called *compatible with* A if it has zeros exactly where A has zero entries. Let $\mathcal{S}$ denote the collection of all non-negative real matrices S that are compatible with A, and satisfy the following properties:

$$\sum_{i \in \mathcal{A}} S_{ij} = \sum_{k \in \mathcal{A}} S_{jk} \quad \text{for each } j \in \mathcal{A} \tag{3.1}$$

and

$$\sum_{i,k \in \mathcal{A}} S_{ik} = 1.$$

There is a one-to-one correspondence between the set $\mathcal{S}$ and the set of one-step Markov measures on X_A. For details, see [**8**]. For each $S \in \mathcal{S}$, let μ_S denote the unique one-step Markov measure on X_A induced by S.

Let $u \in \mathcal{B}_n$. As $\mu_S(_l[u]_m)$ are the same for all l, m with $m - l = n - 1$, we use the simple notation $\mu_S([u])$ to represent the measure of the cylinder sets. We note that for a block $b_1 \cdots b_n \in \mathcal{B}_n$, $n \geq 2$,

$$\mu_S([b_1 \cdots b_n]) = S_{b_1 b_2} \cdot \frac{S_{b_2 b_3}}{\sum_k S_{b_2 k}} \cdots \frac{S_{b_{n-1} b_n}}{\sum_k S_{b_{n-1} k}}. \tag{3.2}$$

In particular,

$$\mu_S([ij]) = S_{ij} \quad \text{for } ij \in \mathcal{B}_2. \tag{3.3}$$

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For the rest of this section we assume that P is an $\mathcal{A} \times \mathcal{A}$ permutation matrix satisfying $AP = PA^{\mathrm{T}}$, and $\phi = \varphi_{A,P}$ is the reversal as in §2. Let $\gamma : \mathcal{A} \to \mathcal{A}$ be the bijection such that $P_{\gamma(a)a} = 1$ for each $a \in \mathcal{A}$. We remark that γ is the inverse of the bijection τ defined in (2.2). For $u = b_1 b_2 \cdots b_n \in \mathcal{A}^n$, let $\tilde{u} = b_n \cdots b_2 b_1$, and $\gamma(u) = \gamma(b_1)\gamma(b_2)\cdots\gamma(b_n)$. Note that $u \in \mathcal{B}_n$ implies $\gamma(\tilde{u}) \in \mathcal{B}_n$, and that for each l,

$$\phi^{-1}({}_l[u]_{l+n-1}) = {}_{-l-n+1}[\gamma(\tilde{u})]_{-l}.$$

Consequently, for $\mu \in M(\mathsf{X}_A)$, $\phi(\mu) = \mu$ if and only if

$$\mu([u]) = \mu([\gamma(\tilde{u})]) \quad \text{for all } u \in \mathcal{B}. \tag{3.4}$$

PROPOSITION 3.1. *Let μ_S be a one-step Markov measure on X_A induced by $S \in \mathcal{S}$. Then the following are equivalent:*

(i) $\phi(\mu_S) = \mu_S$;

(ii) $\mu_S([ij]) = \mu_S([\gamma(j)\gamma(i)])$ *for all* $ij \in \mathcal{B}_2$;

(iii) $SP = PS^{\mathrm{T}}$.

Proof. Clearly (i) implies (ii) by (3.4). Assume that (ii) holds. Then, by (3.3), we see that

$$S_{ij} = \mu_S([ij]) = \mu_S([\gamma(j)\gamma(i)]) = S_{\gamma(j)\gamma(i)}$$

for all $i, j \in \mathcal{A}$. This is equivalent to $SP = PS^{\mathrm{T}}$. Now, suppose that $SP = PS^{\mathrm{T}}$. Then, $S_{ij} = S_{\gamma(j)\gamma(i)}$ for all $i, j \in \mathcal{A}$. From this and (3.2) it follows that for any block $u = b_1 \cdots b_n \in \mathcal{B}_n$,

$$\mu_S([u]) = S_{\gamma(b_2)\gamma(b_1)} \cdot \frac{S_{\gamma(b_3)\gamma(b_2)}}{\sum_k S_{\gamma(k)\gamma(b_2)}} \cdots \frac{S_{\gamma(b_n)\gamma(b_{n-1})}}{\sum_k S_{\gamma(k)\gamma(b_{n-1})}}.$$

We reverse the order of the product, and use the relation (3.1). Then, the right-hand side of the above equation is equal to

$$S_{\gamma(b_n)\gamma(b_{n-1})} \cdot \frac{S_{\gamma(b_{n-1})\gamma(b_{n-2})}}{\sum_k S_{\gamma(b_{n-1})k}} \cdots \frac{S_{\gamma(b_2)\gamma(b_1)}}{\sum_k S_{\gamma(b_2)k}} = \mu_S([\gamma(b_n)\cdots\gamma(b_1)]) = \mu_S([\gamma(\tilde{u})]).$$

Thus, $\mu_S([u]) = \mu_S([\gamma(\tilde{u})])$, which implies (i). □

Fix $n \geq 1$. Let $A^{[n]}$ denote the $\mathcal{B}_n \times \mathcal{B}_n$ binary matrix that represents the nth higher block shift of X_A. Define a $\mathcal{B}_n \times \mathcal{B}_n$ permutation matrix $P^{[n]}$ by

$$(P^{[n]})_{uv} = \delta_{u\gamma(\tilde{v})} \quad \text{for } u, v \in \mathcal{B}_n.$$

Then, $A^{[n]}P^{[n]} = P^{[n]}(A^{[n]})^{\mathrm{T}}$.

Let $\mathcal{S}_n$ be the collection of all non-negative real matrices S that are compatible with $A^{[n]}$ and satisfy

$$\sum_{u \in \mathcal{B}_n} S_{uv} = \sum_{w \in \mathcal{B}_n} S_{vw} \quad \text{for all } v \in \mathcal{B}_n \quad \text{and} \quad \sum_{u,w \in \mathcal{B}_n} S_{uw} = 1.$$

There is a one-to-one correspondence between the set $\mathcal{S}_n$ and the set of n-step Markov measures on X_A, since each one-step Markov measure on $\mathsf{X}_{A^{[n]}}$ uniquely determines an n-step Markov measure on X_A. For each $S \in \mathcal{S}_n$, let μ_S denote the unique n-step Markov measure on X_A induced by S. Imitating the proof of Proposition 3.1, we obtain the following.

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PROPOSITION 3.2. *Let μ_S be an n-step Markov measure on X_A induced by $S \in \mathcal{S}_n$. Then the following are equivalent:*

(i) $\phi(\mu_S) = \mu_S$;

(ii) $\mu_S([u]) = \mu_S([\gamma(\tilde{u})])$ *for all* $u \in \mathcal{B}_{n+1}$;

(iii) $SP^{[n]} = P^{[n]}S^{\mathrm{T}}$.

Let $\mathcal{I}_n$ denote the set of n-step Markov measures on X_A that are ϕ-invariant, and set

$$\mathcal{I} = \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{n \geq k} \mathcal{I}_n} \right).$$

It is obvious that each measure in $\mathcal{I}$ is ϕ-invariant.

We prove that the converse is also true. Suppose that $\mu \in M(\mathsf{X}_A)$, $\phi(\mu) = \mu$ and that μ is fully supported. Fix $n \geq 1$. Consider the n-step Markovization μ_n of μ, that is, the (unique) n-step Markov measure on X_A such that $\mu_n(B) = \mu(B)$ for all $B \in \mathcal{B}_{n+1}$. From Proposition 3.2 it follows that $\phi(\mu_n) = \mu_n$. Thus, $\mu_n \in \mathcal{I}_n$ for all $n \geq 1$. As μ is the weak* limit point of $\{\mu_n\}$, we have $\mu \in \mathcal{I}$. Now, we consider the general case where $\mu \in M(\mathsf{X}_A)$ and $\phi(\mu) = \mu$. Then we can see that for each n, there exists an n-step Markov measure $\mu_n \in \mathcal{I}_n$ such that $|\mu_n(B) - \mu(B)| < 1/N^{2n}$ for all $B \in \mathcal{B}_{n+1}$ where N is the number of 1-blocks. It can be shown that μ is the weak* limit point of $\{\mu_n\}$. Thus $\mu \in \mathcal{I}$. This completes the proof of the following.

THEOREM 3.3. *The set $\mathcal{I}$ exactly consists of ϕ-invariant measures in $M(\mathsf{X}_A)$.*

Example 3.1. Let $\mathcal{A} = \{1, 2\}$ and consider the $\mathcal{A} \times \mathcal{A}$ matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It is not difficult to check that every two-step Markov measure on X_A is ρ_A-invariant. We construct a three-step Markov measure that is not ρ_A-invariant. Define a matrix $S = (S_{uv}) \in \mathcal{S}_3$ as follows. If u and v do not overlap progressively, we set $S_{uv} = 0$, and if they do, we set

$$S_{uv} = \begin{cases} \dfrac{1}{24} & \text{if either } u = 121,\ v = 211 \text{ or } u = 221,\ v = 212, \\ \dfrac{1}{12} & \text{if either } u = 121,\ v = 212 \text{ or } u = 221,\ v = 211, \\ \dfrac{1}{16} & \text{otherwise.} \end{cases}$$

As

$$\mu_S([1121]) = \frac{1}{16} \neq \frac{1}{24} = \mu_S([1211]),$$

we conclude that $\rho_A(\mu_S) \neq \mu_S$.

4. *Existence of a reversal*

Let A be a non-negative integer square matrix. Throughout this section let X_A denote the edge shift defined by the graph whose adjacency matrix is A, and be accompanied by the shift map, denoted σ_A, to represent a topological dynamical system.

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Definition 4.1. Let $\mathcal{M} = \{1, \ldots, m\}$ and let P be an $\mathcal{M} \times \mathcal{M}$ permutation matrix. An $\mathcal{M} \times \mathcal{M}$ non-negative integer matrix A is said to be *P-symmetric* if $A^{\mathrm{T}} = PA = AP$.

Let $\tau : \mathcal{M} \to \mathcal{M}$ denote the bijection such that $P_{i\tau(i)} = 1$ for each $i \in \mathcal{M}$. Then A is P-symmetric if and only if $A_{ij} = A_{\tau(j)i}$ for all $i, j \in \mathcal{M}$. We remark that a matrix is symmetric if and only if it is I-symmetric.

THEOREM 4.1. *Let $A = RS$ where R and S^{T} are P-symmetric for some permutation matrix P with $P^n = I$. Then there is a reversal ϕ for X_A with $\phi^{2n} = \mathrm{id}$.*

Proof. We suppose that P has size $m \geq 1$ and let $\mathcal{M} = \{1, \ldots, m\}$. Define $\tau : \mathcal{M} \to \mathcal{M}$ to be the bijection such that $P_{i\tau(i)} = 1$ for each $i \in \mathcal{M}$. Then $\tau^n = \mathrm{id}$. As R and S^{T} are P-symmetric, we have, for each $i, j \in \mathcal{M}$,

$$R_{ij} = R_{\tau(j)i} = R_{\tau(i)\tau(j)} = R_{\tau^2(j)\tau(i)} = \cdots = R_{\tau^n(j)\tau^{n-1}(i)} \tag{4.1}$$

and

$$S_{ij} = S_{j\tau(i)} = S_{\tau(i)\tau(j)} = S_{\tau(j)\tau^2(i)} = \cdots = S_{\tau^{n-1}(j)\tau^n(i)}. \tag{4.2}$$

Let G and G^{T} denote the graphs associated to A and A^{T} respectively. There is a natural bijection between states of G and states of G^{T}. For each state i in G, i^* denotes the corresponding state in G^{T}. Also, for each edge a in G from a state i to k, we denote by a^* the unique reversed edge in G^{T} from k^* to i^*. To construct a graph $G_{R,S}$, start with the disjoint union of G and G^{T}. For each state i in G and each state j^* in G^{T}, add R_{ij} edges from i to j^* (called R-edges), and S_{ji} edges from j^* to i (called S-edges), completing the graph. An (R, S)-path is defined as an R-edge followed by an S-edge forming a path in $G_{R,S}$. Let i and k be states in G. As $A = RS$, there is a bijection from the set of A-edges from i to k and the set of (R, S)-paths from i to k. Denote a fixed choice of bijection by $a \mapsto \mathbf{r}(a)\mathbf{s}(a)$ and its inverse by $rs \mapsto \mathbf{a}(rs)$.

We now define a bijection ζ from the set of all R-edges onto itself as follows. Let e be an R-edge from a state i in G to a state j^* in G^{T} (so $R_{ij} > 0$). Then $\zeta(e)$ is defined to be an R-edge from the state $\tau(j)$ in G to the state i^* in G^{T}. By (4.1), the map ζ can be chosen to be a bijection with $\zeta^{2n} = \mathrm{id}$. Using (4.2), one can similarly define a bijection η from the set of all S-edges onto itself such that $\eta^{2n} = \mathrm{id}$: if e is an S-edge from a state i^* in G^{T} to a state j in G, then $\eta(e)$ is an S-edge from j^* to $\tau(i)$.

Define a map $\phi : \mathsf{X}_A \to \mathsf{X}_A$ by

$$\phi(x)_i = \mathbf{a}(\zeta(\mathbf{r}(x_{-i+1}))\eta(\mathbf{s}(x_{-i})))$$

for $x = (x_i) \in \mathsf{X}_A$. As the terminal vertex of $\zeta(\mathbf{r}(x_{-i+1}))$ coincides with the initial vertex of $\eta(\mathbf{s}(x_{-i}))$, the map ϕ is well defined. It is trivial to check that $\phi \circ \sigma_A^{-1} = \sigma_A \circ \phi$. From the definitions of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{a}$, it follows that for each $i \in \mathbb{Z}$, $\mathbf{r}(\phi(x)_i) = \zeta(\mathbf{r}(x_{-i+1}))$ and $\mathbf{s}(\phi(x)_i) = \eta(\mathbf{s}(x_{-i}))$. Hence

$$\phi^2(x)_i = \mathbf{a}(\zeta^2(\mathbf{r}(x_i))\eta^2(\mathbf{s}(x_i))).$$

Inductively, we have

$$\phi^{2n}(x)_i = \mathbf{a}(\zeta^{2n}(\mathbf{r}(x_i))\eta^{2n}(\mathbf{s}(x_i))) = \mathbf{a}(\mathbf{r}(x_i)\mathbf{s}(x_i)) = x_i$$

so that $\phi^{2n} = \mathrm{id}$. This completes the proof. □

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COROLLARY 4.2. *Let A be a non-negative integer matrix. Then there is a reversal ϕ for X_A with $\phi^{2n} = \mathrm{id}$, $n \geq 1$, if and only if A is strong shift equivalent to a matrix of the form RS where R and S^{T} are P-symmetric for some permutation matrix P with $P^n = I$.*

Proof. First, assume that there is a reversal ϕ for X_A with $\phi^{2n} = \mathrm{id}$. By Theorem 2.1, we may assume that there is a permutation matrix Q such that $AQ = QA^{\mathrm{T}}$ and $Q^{2n} = I$. Let $P = Q^2$. Then P is a permutation matrix with $P^n = I$. We take $R = Q^{\mathrm{T}}$ and $S = QA$. It is trivial to see that $A = RS$ and that R, S^{T} are P-symmetric. The converse is an immediate consequence of Theorem 4.1. □

COROLLARY 4.3. *For a non-negative integer matrix A, there is a flip for X_A if and only if A is strong shift equivalent to a product of two symmetric matrices.*

The following example shows that the existence of a flip for X_A does not necessarily imply A being strong shift equivalent to a symmetric matrix.

Example 4.1. Let A be a product of two symmetric matrices:

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

As the matrix A has non-real eigenvalues, it is not strong shift equivalent to any symmetric matrix.

Example 4.2. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is not hard to see that A is strong shift equivalent to B. As B is symmetric, X_B admits a flip and so does X_A. One can show, however, that no pair of symmetric matrices R, S satisfies $A = RS$.

The following theorem asserts that if a topological Markov shift X_A admits a flip, some power of the matrix A must be a product of two symmetric matrices.

THEOREM 4.4. *Let A be a non-negative integer matrix such that X_A admits a flip map. Then there exist symmetric matrices K, L and an integer l satisfying the relations*

$$\begin{aligned} AK &= KA^{\mathrm{T}}, \quad LA = A^{\mathrm{T}}L, \\ A^l &= KL, \quad LK = (A^{\mathrm{T}})^l. \end{aligned} \tag{4.3}$$

Proof. In view of Corollary 2.2, we may assume that A is elementary equivalent to a matrix B such that

$$\begin{aligned} BM &= MB^{\mathrm{T}}, \quad NB = B^{\mathrm{T}}N, \\ B^n &= MN, \quad NM = (B^{\mathrm{T}})^n \end{aligned}$$

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for some symmetric non-negative integer matrices M, N and an integer n. As A and B are elementary equivalent, we have R and S for which $A = RS$ and $B = SR$. We set $K = RMR^{\mathrm{T}}$ and $L = S^{\mathrm{T}}NS$. Then it is straightforward to check that K and L satisfy the required properties in (4.3) with $l = n + 2$. □

5. *Orders of reversals*

Suppose that a topological Markov shift admits a reversal of finite order. Then does it admit reversals of other orders? The main result of this section (Theorem 5.3) indicates that reversals of higher orders are more likely to exist than those of lower orders. To prove the theorem, we introduce some technical lemmas and use the method of 'markers'.

Let τ be a permutation on an alphabet $\mathcal{A}$. For $\omega = b_1 \cdots b_m \in \mathcal{A}^m$, $m \geq 1$, we put $\tau(\omega) = \tau(b_1) \cdots \tau(b_m)$ and $\widetilde{\omega} = b_m \cdots b_2 b_1$.

LEMMA 5.1. *Let $u = i_0 i_1 \cdots i_k \in \mathcal{A}^{k+1}$ and $j_1 j_2 \cdots j_l \in \mathcal{A}^l$ for $k, l \geq 1$, with all distinct i_r and j_q. Put:*

(i) $v = i_0$;

(ii) $v = i_0 j_1 \cdots j_l i_0 i_1 \cdots i_k$ *where* $l > k$; *or*

(iii) $v = i_0 j_1 \cdots j_l i_s i_{s+1} \cdots i_k$ *where* $1 \leq s \leq l$.

Define

$$w_1 = u^2 v^2 u v^3, \quad w_2 = u^2 v^3 u v^2, \quad w_3 = u^2 v^4 u v. \tag{5.1}$$

Put $\theta = w_3 w_{s_1} \cdots w_{s_m} w_3$ for $s_1, \ldots, s_m \in \{1, 2\}$. Let τ be a permutation on $\mathcal{A}$. If $\tau(w_3)$ occurs in θ, then $\tau(w_3) = w_3$. Furthermore, the block $\tau(\widetilde{w_3})$ cannot occur in θ.

Proof. We will only prove the last assertion of the lemma as the first can be proved similarly. Let $\pi = i_1 \cdots i_k$ and put $i = \tau(i_0)$.

Case (i): $v = i_0$. Note that $w_3 = i_0 \pi i_0 \pi i_0^5 \pi i_0$ and that the block i^5 cannot occur in $w_{s_1} \cdots w_{s_m}$. Hence, if

$$\tau(\widetilde{w_3}) = i\,\tau(\widetilde{\pi}) i^5 \tau(\widetilde{\pi}) i\,\tau(\widetilde{\pi}) i$$

occurs in θ, then $i = i_0$ and $\tau(\widetilde{\pi}) = \pi$. So, $\tau(\widetilde{w_3}) = i_0 \pi i_0^5 \pi i_0 \pi i_0$. It follows that w_{s_1} starts with π. That is absurd, as $i_0 \neq i_1$.

Case (ii): $v = i_0 j_1 \cdots j_l i_0 i_1 \cdots i_k$ *where* $l > k$. Let $\eta = j_1 \cdots j_l$. We can write $u = i_0 \pi$ and $v = i_0 \eta i_0 \pi$. Consider the subblock $\omega = i(\tau(\widetilde{\pi}) i \tau(\widetilde{\eta}) i)^4$ of $\tau(\widetilde{w_3})$. The block ω cannot occur in $w_{s_1} \cdots w_{s_m}$, as i_r and j_q are all distinct and the length of $\tau(\eta)$ is greater than that of $\tau(\pi)$. If $\tau(\widetilde{w_3})$ occurs in θ, then ω occurs in w_3, so $i = i_0$, $\tau(\widetilde{\pi}) = \pi$ and $\tau(\widetilde{\eta}) = \eta$. Comparing w_3 with

$$\tau(\widetilde{w_3}) = \pi i_0 \eta i_0 \pi i_0 (\pi i_0 \eta i_0)^4 (\pi i_0)^2,$$

we find that w_{s_m} must end with $\pi i_0 \eta$. This is a contradiction, as $i_k \neq j_l$.

Case (iii): $v = i_0 j_1 \cdots j_l i_s i_{s+1} \cdots i_k$ *where* $1 \leq s \leq l$. The proof is similar to that for *Case (ii)*. □

LEMMA 5.2. *Let X be a mixing topological Markov shift that has no fixed points. Then there are $i_0 i_1 \cdots i_k i_0 \in \mathcal{B}_{k+2}(X)$, $k \geq 1$, and $\omega = i_0 j_1 j_2 \cdots j_l \in \mathcal{B}_{l+1}(X)$, $l \geq 1$, with all distinct i_r, j_q such that either $\omega i_0 \in \mathcal{B}_{l+2}(X)$ with $l > k$, or $\omega i_s \in \mathcal{B}_{l+2}(X)$ with $1 \leq s \leq l$.*

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As the above lemma is straightforward, we omit its proof.

THEOREM 5.3. *Let X be a mixing topological Markov shift that admits a reversal of order $n \geq 1$. Then, for each $l \geq 1$, there is a reversal for X of order nl.*

Proof. Let ϕ be a reversal of order n. Due to Theorem 2.1 we may assume that $X \subset \mathcal{A}^{\mathbb{Z}}$ and there exists a bijection $\tau : \mathcal{A} \to \mathcal{A}$ of order n such that for all $x \in X$,

$$\phi(x)_i = \tau(x_{-i}) \quad \text{for } i \in \mathbb{Z}.$$

To construct a 'marker isomorphism', we define $u, v \in \mathcal{B}(X)$ as follows. First, suppose that X has a fixed point $(i_0)^\infty$. We can take a simple cycle $i_0 i_1 \cdots i_k i_0 \in \mathcal{B}_{k+2}(X)$ for some $k \geq 1$, as X is mixing. Put $v = i_0$ and $u = i_0 i_1 \cdots i_k$. Now, suppose that X has no fixed points. Let $i_0 i_1 \cdots i_k i_0$ and $\omega = i_0 j_1 \cdots j_l$ be as in Lemma 5.2. If $\omega i_0 \in \mathcal{B}_{l+2}(X)$, we put

$$v = i_0 j_1 \cdots j_l i_0 i_1 \cdots i_k,$$

and if $\omega i_s \in \mathcal{B}_{l+2}(X)$, we put

$$v = i_0 j_1 \cdots j_l i_s i_{s+1} \cdots i_k.$$

In either case, let $u = i_0 i_1 \cdots i_k$. We now define w_1, w_2 and w_3 as in (5.1).

Let l be a given positive integer. Fix $m \geq 1$ with $2^m \geq nl$. Let π be any permutation on $\mathcal{S} = \{1, 2\}^m$ of order nl. Put $q = (m+2)|w_1| - 1$ where $|w_1|$ is the length of w_1. For $\mathbf{s} = (s_1, \ldots, s_m) \in \mathcal{S}$, define

$$w_{\mathbf{s}} = w_3 w_{s_1} \cdots w_{s_m} w_3 \in \mathcal{B}_{q+1}(X).$$

For $\mathbf{s} \in \mathcal{S}$, let $\bar{\pi}(w_{\mathbf{s}}) = w_{\pi(\mathbf{s})}$ and let $\bar{\pi}(\widetilde{w_{\mathbf{s}}}) = \widetilde{\bar{\pi}(w_{\mathbf{s}})} = \widetilde{w_{\pi(\mathbf{s})}}$. Define an automorphism ψ of X as follows. Let $x \in X$. Suppose that there exist $r \in \mathbb{Z}$, $p \geq 1$ and $\mathbf{s} \in \mathcal{S}$ such that $x_{[r,r+q]} = \tau^p(\eta)$ where $\eta \in \{w_{\mathbf{s}}, \widetilde{w_{\mathbf{s}}}\}$. Let

$$\psi(x)_{[r,r+q]} = \tau^p(\bar{\pi}(\eta))$$

and let ψ have no other actions. By Lemma 5.1, ψ is a well-defined automorphism on X and so $\phi \circ \psi$ is a reversal. Note that ψ and $\phi \circ \psi$ both permute the set of blocks of the form $\tau^p(\eta)$.

We will show that $(\phi \circ \psi)^{nl} = \mathrm{id}$. Let $x \in X$. Assume that $x_{[r,r+q]} = \tau^p(\eta)$. Then,

$$\phi(\psi(x))_{[-r-q,-r]} = \tau^{p+1}\widetilde{(\bar{\pi}(\eta))} = \tau^{p+1}(\widetilde{\bar{\pi}(\eta)}) = \tau^{p+1}(\bar{\pi}(\widetilde{\eta})),$$

and so

$$(\phi \circ \psi)^2(x)_{[r,r+q]} = \tau^{p+2}(\bar{\pi}^2(\eta)).$$

We inductively find that for $k \geq 1$,

$$(\phi \circ \psi)^{2k}(x)_{[r,r+q]} = \tau^{p+2k}(\bar{\pi}^{2k}(\eta)).$$

As $\tau^n = \mathrm{id}$ and $\bar{\pi}^{nl} = \mathrm{id}$, we have

$$(\phi \circ \psi)^{nl}(x)_{[r,r+q]} = \tau^p(\eta) = x_{[r,r+q]}.$$

We conclude that $(\phi \circ \psi)^{nl} = \mathrm{id}$.

Finally, it remains to show that $\phi \circ \psi$ has order nl. Let k, $1 \leq k < nl$, be a given integer. We choose $\mathbf{s} = s_1 s_2 \cdots s_m \in \mathcal{S}$ such that $\pi^k(\mathbf{s}) \neq \mathbf{s}$ and that $\pi^k(\mathbf{s}) \neq s_1' s_2' \cdots s_m'$ where $s_i' = 2$ if $s_i = 1$ and $s_i' = 1$ if $s_i = 2$. Let $x = (w_3 w_{\mathbf{s}})^\infty \in X$. Then, $(\phi \circ \psi)^k(x) \neq x$. □

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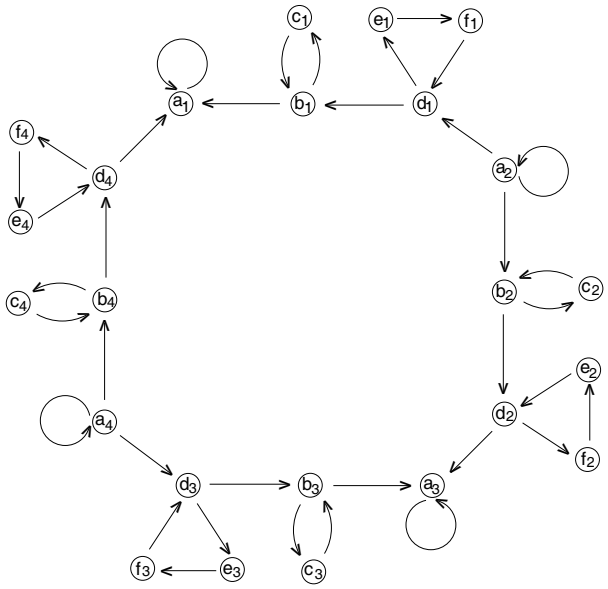


FIGURE 1. A reversal system with no flips.

COROLLARY 5.4. *If X is a mixing topological Markov shift that admits a flip, then there are reversals for X of all possible orders.*

While the existence of low-order reversals implies the existence of high-order reversals as we see in Theorem 5.3, the converse is not true in general. In the following, we provide an example of a reducible matrix A for which X_A admits a reversal of order four, but no flips. Such irreducible examples are not known yet.

Example 5.1. Let A be the adjacency matrix of the graph in Figure 1, and $\mathcal{A}$ denote the set of vertices of the graph. Define a bijection $\tau : \mathcal{A} \to \mathcal{A}$ by $\tau(w_i) = w_{i+1}$ for $w = a, b, c, d, e, f$ and $i = 0, 1, 2, 3$ with $w_0 = w_4$. Let P be the $\mathcal{A} \times \mathcal{A}$ permutation matrix induced by τ, that is, $P_{v\tau(v)} = 1$ for $v \in \mathcal{A}$. Then A and P satisfy (2.1) with $n = 4$, and so $\varphi_{A,P}$ is a reversal for X_A.

We claim that X_A admits no flips. Assume that there is a flip ϕ for X_A. For $i = 1, \ldots, 4$, let $\mathcal{P}_i$ be the set of all periodic points in X_A that are contained in $\{a_i, b_i, c_i, d_i, e_i, f_i\}^{\mathbb{Z}}$. Note that the symbols a_i, b_j, d_k occur in a point of X_A in this order if and only if either $i = j = k = 2$ or $i = j = k = 4$. As ϕ preserves the least period, it follows that a_i, b_j, d_k occur in the ϕ-image of the point

$$(e_1 f_1 d_1)^{\infty} (b_1 c_1)^{2m} b_1 a_1^{\infty}$$

in that order. Hence, either $\phi(\mathcal{P}_1) = \mathcal{P}_2$ or $\phi(\mathcal{P}_1) = \mathcal{P}_4$. If $\phi(\mathcal{P}_1) = \mathcal{P}_2$, then ϕ maps the point $a_2^{\infty}(d_1 e_1 f_1)^{\infty}$ to a point that contains the blocks $d_2 f_2 e_2$ and a_1 in this order. That is absurd. If $\phi(\mathcal{P}_1) = \mathcal{P}_4$, then a similar argument leads to a contradiction.

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