$$= \frac{p_k}{3(p_k - 1)}$$

$$< 1$$

where we have used the formula for the sum to infinity of a convergent geometric series. This provides a contradiction, and our assumption that s is rational is therefore false, thereby completing the proof.

We make the final point here that the above argument does not actually require the prime number theorem, which is really rather a deep result. The number of terms in each section of the tail-end is equal to the prime gap $p_{k+j+1} - p_{k+j}$, and this can be bounded by a Chebyshev-type elementary estimate such as $3p_{i+1} < 4p_i$ for all sufficiently large i [2].

Acknowledgement

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MARTIN GRIFFITHS
Dept. of Mathematical Sciences, University of Essex, Colchester CO4 3SQ
DES MacHALE
School of Mathematical Sciences, University College Cork, Ireland

99.05 An inequality for Si(x)

The 'sine integral' is

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t} dt.$$

The aim of this note is to establish the pair of inequalities

$$\tan^{-1} x < \mathrm{Si}\,(x) < \pi - \tan^{-1} x \qquad \text{for } x > 0. \tag{1}$$

These bounds imply the well-known integral

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

Our proof assumes this result; of course, it would be more satisfying to find a proof that does not rely on it.

An unusual feature of the bounds is that the upper and lower bounds are monotone functions (whose graphs are symmetrical about the line $y = \pi/2$), whereas Si (x) is an oscillatory function: see Figure 1.

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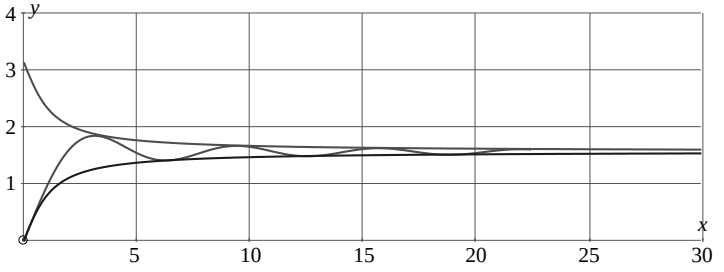


FIGURE 1: The graph of $\mathrm{Si}\,(x)$ sandwiched between $y = \tan^{-1} x$ (below) and $\pi - \tan^{-1} x$ (above).

Weaker bounds for $\mathrm{Si}\,(x)$ can be found in the literature. For example, Burckel in [1, p. 123] estimates a contour integral to establish bounds equivalent to

$$\frac{\pi}{2}\left(1 - \frac{1}{x}\right) < \mathrm{Si}\,(x) < \frac{\pi}{2}\left(1 + \frac{1}{x}\right).$$

This note had an unusual genesis. James McKee is a sixth form student of Nick Lord's. He conjectured that $\tan^{-1} x < \mathrm{Si}\,(x)$ by comparing Maclaurin expansions of each side (which works for $0 < x \leqslant 1$), then graphing both sides for larger and larger values of x. He then experimented with graphs for the upper bound, initially arriving at $\mathrm{Si}\,(x) < \sec^{-1}(-x)$. In discussion with Nick Lord, the upper bound was refined:

$$\sec^{-1}(-x) = \cos^{-1}\left(-\frac{1}{x}\right) = \frac{\pi}{2} + \sin^{-1}\frac{1}{x} > \frac{\pi}{2} + \tan^{-1}\frac{1}{x} = \pi - \tan^{-1} x.$$

At this point, highly intrigued but lacking a definitive proof, Nick Lord contacted Graham Jameson, who produced the elegant proof which follows.

Write

$$I\,(x) = \int_x^\infty \frac{\sin t}{t}\,dt,$$

$$J\,(x) = \int_x^\infty \frac{1}{t^2 + 1}\,dt.$$

Given the integrals

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{1}{t^2 + 1}\,dt = \frac{\pi}{2},$$

we have $I\,(x) = \frac{\pi}{2} - \mathrm{Si}\,(x)$ and $J\,(x) = \frac{\pi}{2} - \tan^{-1} x$, so (1) is equivalent to

$$-J\,(x) < I\,(x) < J\,(x). \tag{2}$$

We shall prove (2). We note in passing that the notation $\mathrm{si}\,(x)$ is sometimes used for $-I\,(x)$, and that $J\,(x)$ also equates to $\tan^{-1}\frac{1}{x}$.

First, we give a simple estimation of $J\,(x)$. Since

$$\frac{1}{t^2 + 1} = \frac{1}{t^2} - \frac{1}{t^2\left(t^2 + 1\right)},$$

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we have

$$J(x) = \frac{1}{x} - \int_x^\infty \frac{1}{t^2(t^2+1)}dt > \frac{1}{x} - \int_x^\infty \frac{1}{t^4}\,dt = \frac{1}{x} - \frac{1}{3x^3}. \qquad (3)$$

Also, $J(x) < \frac{1}{x}$, so (2), once proved, will also imply the inequality $|I(x)| < \frac{1}{x}$.

We now work on the estimation of $I(x)$. It adds to clarity to describe it more generally, for an integral of the form

$$I_f(x) = \int_x^\infty f(t)\sin t\,dt.$$

We assume that $f(t)$ is non-negative and that $f(t)$, together with all its derivatives, is decreasing in magnitude and tends to 0 as $t \to \infty$. This is satisfied by $f(t) = 1/t^p$ for any $p > 0$. It implies that the odd-numbered derivatives are negative and the even-numbered ones positive (so in fact these derivatives satisfy the same hypotheses as $f(t)$ itself).

Integrating by parts, we have

$$I_f(x) = \left[-f(t)\cos t\right]_x^\infty + r_1(x) = f(x)\cos x + r_1(x),$$

where

$$r_1(x) = \int_x^\infty f'(t)\cos t\,dt.$$

We can estimate $r_1(x)$ as follows: since $f'(t) \leqslant 0$, we have $|f'(t)\cos t| \leqslant -f'(t)$, hence

$$|r_1(x)| \leqslant \int_x^\infty \left(-f'(t)\right)dt = f(x).$$

This is essentially the proof that the integral $I_f(x)$ converges in the first place. It also gives the estimation

$$f(x)(\cos x - 1) \leqslant I_f(x) \leqslant f(x)(\cos x + 1), \qquad (4)$$

so in particular $|I_f(x)| \leqslant 2f(x)$. However, this is well short of the inequality we have set out to prove. To get a better approximation, we repeat the integration by parts, obtaining

$$\begin{aligned} I_f(x) &= f(x)\cos x + \left[f'(t)\sin t\right]_x^\infty - I_{f''}(x) \\ &= f(x)\cos x - f'(x)\sin x - I_{f''}(x). \end{aligned} \qquad (5)$$

Now applying this to $f''(t)$ and substituting, we have:

$$\begin{aligned} I_f(x) &= f(x)\cos x - f'(x)\sin x - f''(x)\cos x + f^{(3)}(x)\sin x + I_{f^{(4)}}(x) \\ &= \left[f(x) - f''(x)\right]\cos x - \left[f'(x) - f^{(3)}(x)\right]\sin x + I_{f^{(4)}}(x). \end{aligned} \qquad (6)$$

Furthermore, as in the estimate of $r_1(x)$, we have $|I_{f''}(x)| \leqslant -f'(x)$ and $|I_{f^{(4)}}(x)| \leqslant -f^{(3)}(x)$.

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Of course, the process can be continued: successive derivatives of $f(x)$ appear in the expressions multiplying $\cos x$ and $\sin x$. However, this does not simply deliver ever-closer approximations, because for a fixed x, the derivatives $f^{(n)}(x)$ ultimately grow large in magnitude. For our purposes, (6) is exactly what we want.

We mention the fact (although we will not actually use it) that these results simplify pleasantly for $x = n\pi$, where n is an integer. For example, for even n, (4) shows that $I_f(n\pi) \geqslant 0$. Applying this to $I_{f''}$, we see from (5) that $I_f(n\pi) \leqslant f(n\pi)$. In both cases, the opposite holds for odd n. So we can state (for all n) that $0 \leqslant (-1)^n I_f(n\pi) \leqslant f(n\pi)$.

The reader might like to work out the corresponding expressions for $\int_x^\infty f(t)\cos t\,dt$.

Now let us return to the case we want, given by $f(t) = 1/t$. Statement (6) becomes

$$I(x) = \left(\frac{1}{x} - \frac{2}{x^3}\right)\cos x + \left(\frac{1}{x^2} - \frac{6}{x^4}\right)\sin x + r_4(x), \tag{7}$$

where

$$|r_4(x)| \leqslant \frac{6}{x^4}.$$

Write this as $F(x) + r_4(x)$. Clearly, $F(x)$ is a highly accurate approximation to $I(x)$ for large x. We now estimate its absolute value. By the Cauchy-Schwarz inequality, $|a\cos x + b\sin x| \leqslant (a^2 + b^2)^{1/2}$. Assume that $x \geqslant 2$ and write $y = 1/x$, so that $0 < y \leqslant \frac{1}{2}$. Then

$$\begin{aligned} F(x)^2 &\leqslant \left(y - 2y^3\right)^2 + \left(y^2 - 6y^4\right)^2 \\ &= y^2 - 3y^4 - 8y^6 + 36y^8 \\ &= \left(y - \tfrac{3}{2}y^3\right)^2 - \tfrac{41}{4}y^6 + 36y^8 \\ &< \left(y - \tfrac{3}{2}y^3\right)^2. \end{aligned}$$

So $|F(x)| < y - \frac{3}{2}y^3$, and hence

$$|I(x)| < \frac{1}{x} - \frac{3}{2x^3} + \frac{6}{x^4}. \tag{8}$$

With (3), this gives

$$J(x) - |I(x)| > \frac{7}{6x^3} - \frac{6}{x^4} = \frac{1}{6x^4}(7x - 36),$$

which is positive for $x \geqslant \frac{36}{7}$. This proves (2) for such x.

It remains to show that (2) also holds for $x \leqslant \frac{36}{7}$. (In fact, we consider $x \leqslant 2\pi$.) Unfortunately, with all the details included, this requires nearly as much work as the main case! However, readers have the option of leaving out some of the details, as indicated. Write

$$f(t) = \frac{\sin t}{t}, \qquad g(t) = \frac{1}{t^2 + 1}, \qquad h(t) = \frac{t}{t^2 + 1}.$$

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We need two Lemmas comparing $f(t)$ with $g(t)$ and $-g(t)$. These Lemmas are illustrated in Figures 2 and 3, and readers happy with the graphical evidence may prefer to skip the formal proofs and go straight to the proofs of $I(x) < J(x)$ and $I(x) + J(x) > 0$.

Lemma 1: There is a point z_0 in $(0, \pi)$ such that $f(t) > g(t)$ on $(0, z_0)$ and $f(t) < g(t)$ on $(z_0, 2\pi)$.

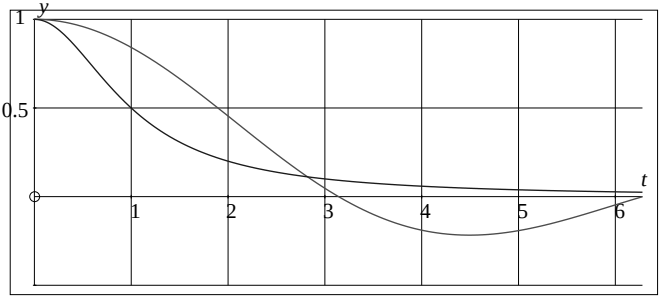


FIGURE 2: The oscillating graph of $f(t) = \dfrac{\sin t}{t}$ and the graph of $g(t) = \dfrac{1}{t^2 + 1}$, crossing at $t = z_0 \approx 2.8213$.

Proof: Clearly, $f(\pi) = 0$, while $g(\pi) > 0$. Also, for $\pi \leqslant t \leqslant 2\pi$, we have $f(t) \leqslant 0 < g(t)$. By the sine series, $f(t) > 1 - \frac{1}{6}t^2$ for $0 < t < \sqrt{5}$, while direct multiplication shows that $g(t) \leqslant 1 - \frac{1}{6}t^2$ for $0 \leqslant t \leqslant \sqrt{5}$. So there is a point z_0 between $\sqrt{5}$ and π with $f(z_0) = g(z_0)$. To show the uniqueness of z_0, we compare the gradients of $\sin t$ and $h(t)$. Now $\cos\sqrt{5} < \cos\frac{2\pi}{3} = -\frac{1}{2}$, so for $\sqrt{5} \leqslant t \leqslant \pi$, the magnitude of the gradient of $\sin t$ is at least $\frac{1}{2}$. Meanwhile, $h'(t) = \dfrac{1 - t^2}{(1 + t^2)^2}$, so $|h'(t)| < \frac{1}{t^2} < \frac{1}{5}$. Hence z_0 is unique.

Proof of $I(x) < J(x)$ *for* $0 \leqslant x \leqslant 2\pi$: We have shown that $I(2\pi) < J(2\pi)$, and of course $I(0) = J(0) = \frac{1}{2}\pi$. By Lemma 1, $J(x) - I(x)$ is increasing on $[0, z_0]$ and decreasing on $[z_0, 2\pi]$, since its derivative is $f(x) - g(x)$. Hence $J(x) - I(x) \geqslant 0$ on $[0, 2\pi]$.

The proof that $I(x) + J(x) > 0$ is along similar lines, but it requires slightly more work.

Lemma 2: There are points x_1, z_1 in $(\pi, 2\pi)$ at which $f(t) + g(t) = 0$, with $f(t) + g(t) > 0$ on $[0, x_1]$ and $[z_1, 2\pi]$, while $f(t) + g(t) < 0$ on $[x_1, z_1]$.

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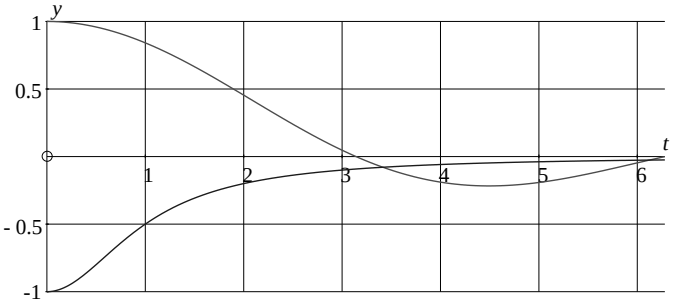


FIGURE 3: The oscillating graph of $f(t) = \dfrac{\sin t}{t}$ and the graph of $-g(t) = -\dfrac{1}{t^2 + 1}$, crossing at $t = x_1 \approx 3.4147$ and $t = z_1 \approx 6.1234$.

Proof: Write $H(t) = t\big[f(t) + g(t)\big] = \sin t + h(t)$. Clearly, $H(\pi) = h(\pi) > 0$, $H(2\pi) > 0$ and $H\left(\frac{3}{2}\pi\right) = h\left(\frac{3}{2}\pi\right) - 1 < 0$. So there exist $x_1 < \frac{3}{2}\pi$ and $z_1 > \frac{3}{2}\pi$ with $H(x_1) = H(z_1) = 0$. Also, $H(t)$ is strictly decreasing on $\left[\pi, \frac{3}{2}\pi\right]$, so x_1 is unique. For $\frac{3}{2}\pi \leqslant t \leqslant \frac{7}{4}\pi$, $\sin t \leqslant -\frac{1}{\sqrt{2}}$ while $h(t) < \frac{1}{\pi}$, so $H(t) < 0$. Between $\frac{7}{4}\pi$ and 2π, the gradient of $\sin t$ is at least $\frac{1}{\sqrt{2}}$, while $\left|h'(t)\right| < \frac{1}{t^2} < \frac{1}{\pi^2}$, so $H'(t) > 0$. Hence z_1 is unique.

Proof of $I(x) + J(x) > 0$ *for* $0 \leqslant x \leqslant 2\pi$: By Lemma 2, $I(x) + J(x)$ is decreasing on $[0, x_1]$ and $[z_1, 2\pi]$, and increasing on $[x_1, z_1]$. Since $I(2\pi) + J(2\pi) > 0$, it is enough to show that $I(x_1) + J(x_1) > 0$. We show that $x_1 < \pi + \frac{3}{10}$. (A more accurate estimate is 3.4147, as stated above.) Write $\pi + \frac{3}{10} = u$. From the inequality $\sin t > t - \frac{1}{6}t^3$, we have $-\sin u = \sin\frac{3}{10} > 0.295$, while $h(u) < \frac{1}{u} < 0.291$. So $\sin u + h(u) < 0$, hence $f(u) + g(u) < 0$, which shows that $x_1 < u$, as stated.

Since I is increasing and J is decreasing on $[\pi, 2\pi]$, it is enough to show that $I(\pi) + J\left(\pi + \frac{3}{10}\right) > 0$. From (3), we find $J\left(\pi + \frac{3}{10}\right) > 0.28238$. To evaluate $I(\pi)$, we use the identity

$$\mathrm{Si}(x) = x - \frac{x^3}{3!\,3} + \frac{x^5}{5!\,5} - \ \dots\ ,$$

obtained by integrating the series for $\sin t / t$. This gives $\mathrm{Si}(\pi) \approx 1.85194$, hence $I(\pi) \approx -0.28114$ to five d.p., so the required inequality holds.

A stronger inequality for $x \geqslant 1$

We are indebted to the referee for the following observation: $\left|I(x)\right| < K(x)$ for $x \geqslant 1$, where $K(x) = \dfrac{1}{(x^2 + 1)^{1/2}}$. This is stronger than (2), because $K(x) < J(x)$, as one can see by writing $K(x)$ as $-\int_x^\infty K'(t)\,dt$. At the same time, it does not hold for x close to 0, since $K(0) = 1$.

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We sketch the proof. By multiplying out, one sees that $(1 + u)^{-1/2} \geqslant 1 - \frac{1}{2}u$ for $0 \leqslant u \leqslant 1$, from which we deduce

$$K(x) \geqslant \frac{1}{x} - \frac{1}{2x^3}$$

for $x \geqslant 1$. By (8), it follows that $|I(x)| < K(x)$ for $x \geqslant 6$. Calculation shows that $I(1) < K(1)$. A suitable modification of the steps above now shows that the inequality holds for $1 \leqslant x \leqslant 6$.

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GRAHAM JAMESON
Dept. of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF
e-mail: *g.jameson@lancaster.ac.uk*
NICK LORD and JAMES McKEE
Tonbridge School, Kent TN9 1JP
e-mails: *njl@tonbridge-school.org; McKeeJ@tonbridge-school.org*

99.06 A peculiar proof of an identity of Euler

Introduction

The history of complex analysis is a fascinating subject. Many mathematicians took part in the development of complex function theory. As one would expect, Euler was among them. At the age of 70, he became particularly interested in the use of complex variables in integration. According to Bottazzini and Gray [1, p. 93]

> 'In 1777 he presented a series of nine papers on this topic to the St. Petersburg Academy which, however, had little impact, if any, because they were published only posthumously between 1793 and 1805.'

These papers were devoted to the evaluation of certain definite integrals by making an appropriate change of variables. Euler believed that, in some cases at least, complex substitutions need not change the range of integration $[0, \infty)$ to a path in the complex plane. To a certain extent, he was correct. Allow us to elaborate more constructively.

Let u and z be real. Recall that the usual integration by substitution consists of computing $\int_a^b f(z)\,dz$ by setting $z = \varphi(u)$. Euler, on the other hand, rather presumptuously, used substitutions of the form $z = x + iy$, where x and y are real and $i^2 = -1$. For instance, in [2, pp. 337-345] Euler considered the integral $\int_0^\infty x^{n-1}e^{-x}dx$. In the notation due to Legendre, this is the gamma function $\Gamma(n)$, but Euler designated this integral by Δ. He made the substitution $x = ky$, with $k > 0$, to get

$$\int_0^\infty y^{n-1}e^{-ky}dy = \frac{\Gamma(n)}{k^n}, \tag{1}$$

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