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RESEARCH ARTICLE

Partial Hopf actions on generalized matrix algebras

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Abstract

Let $\Bbbk$ be a field, H a Hopf algebra over $\Bbbk$, and $R = ({}_iM_j)_{1\leq i,j\leq n}$ a generalized matrix algebra. In this work, we establish necessary and sufficient conditions for H to act partially on R. To achieve this, we introduce the concept of an opposite covariant pair and demonstrate that it satisfies a universal property. In the special case where $H = \Bbbk G$ is the group algebra of a group G, we recover the conditions given in [7] for the existence of a unital partial action of G on R.

1. Introduction

Generalized matrix algebras were first introduced in the literature by Brown in [9], where the author explored matrix rings associated with orthogonal groups. These matrix rings form a large class of rings that arise in various areas of algebra and module theory and have been extensively studied. Broadly speaking, a generalized matrix algebra is an algebra formed as a direct sum of $\Bbbk$-vector spaces,

$$R = \bigoplus_{i,j=1}^{n} {}_iM_j,$$

where the diagonal components $R_i = {}_iM_i$ are $\Bbbk$-algebras, and the off-diagonal components ${}_iM_j$ are (R_i, R_j)-bimodules. The algebra structure of R emulates the usual structure of a matrix algebra by employing a series of bimodule morphisms to define the product between entries residing solely in the bimodules.

Notably, the class of 2×2 generalized matrix algebras has been shown to correspond one-to-one with Morita contexts [15]. The study of the so-called *Morita ring* has gained significant attention in recent years, as it encapsulates key algebraic properties of the Morita context itself [13, 14, 16]. Moreover, Morita rings were used to define the concept of Morita equivalence of partial actions initially for groups and later for Hopf algebras [1, 5]. Another context in which generalized matrix algebras arise is in the study of idempotents. Specifically, any algebra A equipped with a complete set of orthogonal idempotents $\{e_i\}_{i=1}^n$ can be decomposed using Peirce decomposition. This decomposition can be organized into an $n \times n$ generalized matrix algebra A^π, which is isomorphic to A. For further details, see, for instance, [6].

Partial actions of Hopf algebras were introduced in [10], motivated by the Galois theory of partial group actions studied in [12]. Since then, numerous works on partial actions of Hopf algebras have been developed, exploring several problems that were initially addressed from a global perspective through a partial framework. For a detailed account of this topic, the interested reader is referred to [8].

The main objective of this work is to establish a connection between two significant topics: partial actions of Hopf algebras and generalized matrix algebras. Specifically, we aim to determine the

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necessary and sufficient conditions for a Hopf algebra to act partially on a generalized matrix algebra. The analogous problem was considered in [7] for the case of partial group actions.

This paper is structured as follows. In Section 2, we review the fundamental concepts of generalized matrix algebras, partial actions, partial representations of Hopf algebras, and partial smash products. We also introduce the notion of an opposite covariant pair and demonstrate that it satisfies a universal property. In Section 3, we prove Theorem 3.3, which is the main result of this work. In this theorem, we provide necessary and sufficient conditions for the existence of a partial action $\triangleright$ of a Hopf algebra H on a generalized matrix algebra such that each ${}_iM_j$ is invariant under the action $\triangleright$. Conditions for the existence of a partial action of a group G on R were previously discussed in [7]. The relationship between these results and those of Theorem 3.4 is analyzed in Section 4, focusing on the specific case where the Hopf algebra is the group algebra $\Bbbk G$, where $\Bbbk$ is a field. Finally, in Section 5, we discuss Morita-equivalent partial actions of a Hopf algebra. Additionally, we address an inconsistency in Proposition 73 of [5].

Conventions

Throughout this work, $\Bbbk$ is a field, by an algebra we mean an associative unital $\Bbbk$-algebra, and vector spaces and tensor product are over $\Bbbk$. Also, all modules will be unital modules. Given an algebra A, the opposite algebra A^{op} is the vector space A with the opposite multiplication, that is $a \cdot_{\mathrm{op}} b = ba$, for all $a, b \in A$. Given a Hopf algebra H and $h \in H$, we use the simplified Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ to denote the comultiplication of h. We will assume that every Hopf algebra H has a bijective antipode. For a coalgebra C, the coopposite coalgebra C^{cop} is the vector space C with comultiplication $\Delta_{\mathrm{cop}}(c) = c_{(2)} \otimes c_{(1)}$, where $\Delta(c) = c_{(1)} \otimes c_{(2)}$. In order to avoid confusion, we will denote $\Delta_{\mathrm{cop}}(c) = c_{[1]} \otimes c_{[2]}$, for all $c \in C^{\mathrm{cop}}$. Finally, the set $\{1, \ldots, n\}$ will be denoted by $\mathbb{I}_n$.

2. Preliminaries

In this section, we review definitions and results that will be used in the rest of the work. We introduce the concept of opposite covariant pair and prove that it satisfies a certain universal property.

2.1. Generalized matrix algebras

We begin with the following definition, which will allow us to introduce one of the key concepts of this work, namely, the notion of a generalized matrix ring.

Definition 2.1. *Let n be a positive integer. A generalized matrix datum of order n over $\Bbbk$ is a triple*

$$\mathcal{R} = \left(\{{}_iM_j\}_{i,j\in\mathbb{I}_n}, \{\theta_{ijk}\}_{i,j,k\in\mathbb{I}_n}, \{\eta_i\}_{i\in\mathbb{I}_n}\right),$$

such that, for every $i, j, k \in \mathbb{I}_n$,

(GMD1) ${}_iM_j$ *is a $\Bbbk$-vector space,*
(GMD2) $\theta_{ijk} : {}_iM_j \otimes {}_jM_k \to {}_iM_k$ *and* $\eta_i : \Bbbk \to {}_iM_i$ *are $\Bbbk$-linear maps,*
(GMD3) θ_{ijk} *and* η_i *satisfy the following compatibility identities*

$$\theta_{ikl} \circ \left(\theta_{ijk} \otimes \mathrm{id}_{{}_kM_l}\right) = \theta_{ijl} \circ \left(\mathrm{id}_{{}_iM_j} \otimes \theta_{jkl}\right),$$

$$\theta_{iij} \circ (\eta_i \otimes \mathrm{id}_{{}_iM_j}) = l_{{}_iM_j}, \qquad \theta_{ijj} \circ (\mathrm{id}_{{}_iM_j} \otimes \eta_j) = r_{{}_iM_j},$$

where $l_{{}_iM_j}$ (resp. $r_{{}_iM_j}$) is the natural $\Bbbk$-linear isomorphism from $\Bbbk \otimes {}_iM_j$ (resp. ${}_iM_j \otimes \Bbbk$) to ${}_iM_j$.

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The identities of (GMD3) can be interpreted by requiring that the following diagrams be commutative:

$$\begin{array}{ccc} {}_iM_j \otimes {}_jM_k \otimes {}_kM_l & \xrightarrow{\theta_{ijk}\otimes \mathrm{id}_{{}_kM_l}} & {}_iM_k \otimes {}_kM_l \\ \downarrow{\scriptstyle \mathrm{id}_{{}_iM_j}\otimes\theta_{jkl}} & & \downarrow{\scriptstyle \theta_{ikl}} \\ {}_iM_j \otimes {}_jM_l & \xrightarrow[\theta_{ijl}]{} & {}_iM_l \end{array} \tag{2.2}$$

and

$$\begin{array}{ccc} {}_iM_j \otimes \Bbbk \simeq \Bbbk \otimes {}_iM_j & \xrightarrow{\eta_i\otimes \mathrm{id}_{{}_iM_j}} & {}_iM_i \otimes {}_iM_j \\ \downarrow{\scriptstyle \mathrm{id}_{{}_iM_j}\otimes\eta_j} & \searrow^{\cong} & \downarrow{\scriptstyle \theta_{iij}} \\ {}_iM_j \otimes {}_jM_j & \xrightarrow[\theta_{ijj}]{} & {}_iM_j \end{array} \tag{2.3}$$

Proposition 2.4. *Let R be a generalized matrix datum of order n over $\Bbbk$. Then, for every $i, j, k \in I_n$, we have that:*

(i) *the vector spaces ${}_iM_i$ are unital $\Bbbk$-algebras, henceforth denoted as R_i,*
(ii) *the vector spaces ${}_iM_j$ are unital (R_i, R_j)-bimodules,*
(iii) *the linear maps θ_{ijk} are morphisms of (R_i, R_k)-bimodules and are balanced over R_j, that is, θ_{ijk} determines a well-defined mapping from ${}_iM_j \otimes_{R_j} {}_jM_k$ to ${}_iM_k$.*

Proof. For (i), notice that $R_i := {}_iM_i$ is an algebra with multiplication and unity given, respectively, by θ_{iii} and η_i. Also, ${}_iM_j$ has structure of left (resp. right) R_i-module (resp. R_j-module) given by θ_{iij} (resp. θ_{ijj}). Substituting j by i, k by j and l by j in (2.2), we obtain that ${}_iM_j$ is a (R_i, R_j)-bimodule. From (2.3) follows that ${}_iM_j$ is unital and (ii) is proved. The statement (iii) follows from (2.2) setting $k = j$ and $l = k$. □

From the previous proof, we have that for all $i, j \in I_n$,

- $\theta_{iii}: {}_iM_i \otimes_{R_i} {}_iM_i \to {}_iM_i$ is simply the multiplication in R_i,
- $\theta_{iij}: {}_iM_i \otimes_{R_i} {}_iM_j \to {}_iM_j$ is the left R_i-module structure on ${}_iM_j$,
- $\theta_{ijj}: {}_iM_j \otimes_{R_j} {}_jM_j \to {}_iM_j$ is the right R_j-module structure on ${}_iM_j$.

Given a generalized matrix datum, one can associate to the vector space

$$R = \bigoplus_{i,j=1}^{n} {}_iM_j$$

an algebra structure as follows. For all $i, j, k, l \in \mathbb{I}_n$, define the linear map $\mu_{ij,kl} : {}_iM_j \otimes {}_kM_l \to {}_iM_l$ by

$$\mu_{ij,kl} = \begin{cases} \theta_{ijl}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

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Hence, we have linear maps $\iota_{il} \circ \mu_{ij,kl} : {}_iM_j \otimes {}_kM_l \to R$, where $\iota_{il} : {}_iM_l \to R$ is the inclusion map. Using the universal property of the direct sum

$$R \otimes R \cong \bigoplus_{i,j,k,l=1}^{n} {}_iM_j \otimes {}_kM_l,$$

we obtain a linear map $\mu : R \otimes R \to R$. We also define the linear map $\eta : \Bbbk \to R$ by

$$\eta = \sum_{i=1}^{n} \iota_{ii} \circ \eta_i.$$

Proposition 2.5. *Given a generalized matrix datum*

$$\mathcal{R} = \left(\{{}_iM_j\}_{i,j\in\mathbb{I}_n}, \{\theta_{ijk}\}_{i,j,k\in\mathbb{I}_n}, \{\eta_i\}_{i\in\mathbb{I}_n} \right),$$

the triple (R, μ, η) *defines a unital algebra over the field* $\Bbbk$.

Proof. One can write a generalized matrix algebra of order n over $\Bbbk$ as a square array

$$R = \begin{pmatrix} R_1 & {}_1M_2 & \dots & {}_1M_n \\ {}_2M_1 & R_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & {}_{(n-1)}M_n \\ {}_nM_1 & \dots & {}_nM_{(n-1)} & R_n \end{pmatrix}. \tag{2.6}$$

A typical element of R can be written as a matrix $x = (x_{ij})$. Considering the vector space structure on the direct sum R one can see that the addition is done componentwise and the multiplication μ on R is the row-column matrix multiplication,

$$\mu(\,(x_{ij}) \otimes (y_{ij})\,) = (x_{ij})\,(y_{ij}) = \left(\sum_{k=1}^{n} x_{ik}y_{kj} \right) = \left(\sum_{k=1}^{n} \theta_{ikj}(x_{ik} \otimes y_{kj}) \right).$$

The associativity relations of the maps θ_{ijk} given in (GMD3) imply that the matrix multiplication is associative, and the linear map η is indeed a unit map. Explicitly,

$$\eta(\lambda) = \lambda \begin{pmatrix} 1_1 & 0 & \cdots & 0 \\ 0 & 1_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_n \end{pmatrix}, \quad \lambda \in \Bbbk,$$

in which 1_i denotes the unit of the corresponding algebra R_i. □

Definition 2.7. *The algebra constructed in Proposition 2.5 will be called the generalized matrix algebra defined by the generalized matrix datum* $\mathcal{R}$.

In what follows, a generalized matrix algebra R as defined above will be denoted by $R = ({}_iM_j)_{i,j\in\mathbb{I}_n}$. Also, a diagonal matrix in R will be denoted by $r = \operatorname{diag}(r_{11}, \dots, r_{nn})$, that is, $r = (r_{ij})$ is the matrix such that $r_{ij} = 0$ if $i \neq j$.

Remark 2.8. *Let* $R = ({}_iM_j)_{i,j\in\mathbb{I}_n}$ *be a generalized matrix algebra and* $k, l \in \mathbb{I}_n$. *Notice that* R *is a left* R_k*-module with the following structure: for all* $r_k \in R_k$ *and* $(m_{ij}) \in R$,

$$r_k \cdot (m_{ij}) = (\tilde{m}_{ij}), \qquad \text{where } \tilde{m}_{ij} = \begin{cases} 0, & \text{if } i \neq k, \\ r_k \cdot m_{ij}, & \text{if } i = k. \end{cases}$$

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Similarly, R is a right R_l-module via: for all $r_l \in R_l$ and $(m_{ij}) \in R$,

$$(m_{ij}) \cdot r_l = (\tilde{m}_{ij}), \quad \text{where } \tilde{m}_{ij} = \begin{cases} 0, & \text{if } j \neq l, \\ m_{ij} \cdot r_l, & \text{if } j = l. \end{cases}$$

Example 2.9. *Let $\mathcal{C}$ be a $\Bbbk$-linear category with finitely many objects, that is, the morphism sets of $\mathcal{C}$ are $\Bbbk$-vector spaces and the composition $\mathrm{Hom}_{\mathcal{C}}(Y, X) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is $\Bbbk$-bilinear, for all objects X, Y, Z of $\mathcal{C}$. For each pair of objects $X_i, X_j \in \mathcal{C}^{(0)}$, denote ${}_iM_j := \mathrm{Hom}_{\mathcal{C}}(X_j, X_i)$. The total space*

$$A(\mathcal{C}) = \bigoplus_{i,j} {}_iM_j$$

is a generalized matrix algebra with the multiplication given by the ordinary composition of morphisms in the category $\mathcal{C}$.

Example 2.10. *Let (A, B, M, N, μ, ν) be a Morita context between the algebras A and B. Then*

$$R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

is a 2-by-2 generalized matrix algebra which is called a Morita ring. Conversely, each 2-by-2 generalized matrix algebra determines a Morita context [13].

Example 2.11. *Let A be an algebra and $e \in A$ be an idempotent element. One can construct the following generalized matrix algebra:*

$$R = \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}.$$

2.2. Partial actions and partial representations of Hopf algebras

Left partial actions of bialgebras on algebras were introduced in [10]. Below we give it with an additional symmetric condition as it appears in [4].

Definition 2.12. *A symmetric left partial action of a Hopf algebra H over a $\Bbbk$-algebra A is a linear map*

$$\begin{aligned} \cdot : H \otimes A &\to A \\ h \otimes a &\mapsto h \cdot a \end{aligned}$$

satisfying, for all $h, k \in H$ and $a, b \in A$,

(LPA1) $1_H \cdot a = a$,
(LPA2) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,
(LPA3) $h \cdot (k \cdot a) = (h_{(1)} \cdot 1_A)(h_{(2)}k \cdot a) = (h_{(1)}k \cdot a)(h_{(2)} \cdot 1_A)$.

It follows from (LPA2) and (LPA3) that for all $h, k \in H$ and $a, b \in A$,

$$\big(h_{(1)} \cdot (k \cdot a)\big)\,(h_{(2)} \cdot b) = (h_{(1)}k \cdot a)(h_{(2)} \cdot 1_A)(h_{(3)} \cdot b) = (h_{(1)}k \cdot a)(h_{(2)} \cdot b), \tag{2.13}$$

$$(h_{(1)} \cdot a)\,\big(h_{(2)} \cdot (k \cdot b)\big) = (h_{(1)} \cdot a)(h_{(2)} \cdot 1_A)\,\big(h_{(3)}k \cdot b\big) = (h_{(1)} \cdot a)(h_{(2)}k \cdot b). \tag{2.14}$$

Similarly, one can give the notion of a right partial action of a Hopf algebra as follows.

Definition 2.15. *A symmetric right partial action of a Hopf algebra H over a $\Bbbk$-algebra A is a linear map*

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$$\begin{aligned} \cdot : A \otimes H &\to A \\ a \otimes h &\mapsto a \cdot h \end{aligned}$$

satisfying, for all $h, k \in H$ and $a, b \in A$,

(RPA1) $a \cdot 1_H = a$,
(RPA2) $(ab) \cdot h = (a \cdot h_{(1)})(b \cdot h_{(2)})$,
(RPA3) $(a \cdot k) \cdot h = (a \cdot kh_{(1)})(1_A \cdot h_{(2)}) = (1_A \cdot h_{(1)})(a \cdot kh_{(2)})$.

Remark 2.16. *From now on in this work by left (right) partial action, we mean a symmetric left (right) partial action. Also, we say that an algebra A is a left (right) partial H-module algebra if it is endowed with a left (right) partial action.*

Example 2.17. *Given a unital partial action $\alpha = \left(A_g, \alpha_g\right)_{g \in G}$ of a group G on a unital algebra A (see Definition 1.1 in [11]) in which each ideal $A_g \trianglelefteq A$ is generated by a central idempotent element $1_g \in A$, one can define a left partial action of the group algebra $\Bbbk G$ on A by*

$$g \cdot a = \alpha_g(a 1_{g^{-1}}), \quad a \in A, \quad g \in G.$$

Conversely, let $H = \Bbbk G$ and $\cdot : H \otimes A \to A$ a left partial action of H on A. For each $g \in G$, consider $1_g := g \cdot 1_A$. Notice that 1_g is a central idempotent of A. Indeed, $1_g 1_g = (g \cdot 1_A)(g \cdot 1_A) = g \cdot (1_A 1_A) = g \cdot 1_A = 1_g$. Also, for all $a \in A$,

$$\begin{aligned} 1_g a &= (g \cdot 1_A)(1_{\Bbbk G} \cdot a) = (g \cdot 1_A)(gg^{-1} \cdot a) \\ &= g \cdot (g^{-1} \cdot a) = (gg^{-1} \cdot a)(g \cdot 1_A) \qquad \text{(by (LPA3))} \\ &= a 1_g. \end{aligned}$$

Then $\alpha = \left(D_g, \alpha_g\right)_{g \in G}$ is a unital partial action of G on A, where $D_g = A1_g$, and $\alpha_g(a) = g \cdot a$, $a \in D_{g^{-1}}$.

Example 2.18. *Let $\cdot : H \otimes A \to A$ be a left partial action of a Hopf algebra H over an algebra A. Consider the linear map $\triangleleft : A \otimes H^{\mathrm{op}} \to A$ defined by*

$$a \triangleleft h = h \cdot a, \quad a \in A, \ h \in H^{\mathrm{op}}. \tag{2.19}$$

It is straightforward to verify that $\triangleleft$ is a right partial action of H^{op} over A.

Remark 2.20. *According to [2], if A and B are partial left H-module algebras, one can define a morphism of partial H-module algebras as an algebra morphism $f : A \to B$ such that $f(h \cdot a) = h \cdot f(a)$, for all $a \in A$ and $h \in H$. The category of all symmetric partial left H-module algebras and the morphisms of partial H-module algebras between them is denoted as LParActH. Analogously, one defines the category RParActH of symmetric right partial H-module algebras. By the previous example, the categories LParActH and RParActH^{op} are isomorphic.*

We proceed with the next.

Definition 2.21. *Let H be a Hopf algebra and B be a unital algebra. A partial representation of H in B is a linear map $\pi : H \to B$ such that*

(PR1) $\pi(1_H) = 1_B$,
(PR2) $\pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \pi(hk_{(1)})\pi(S(k_{(2)}))$,
(PR3) $\pi(h_{(1)})\pi(S(h_{(2)}))\pi(k) = \pi(h_{(1)})\pi(S(k_{(2)})k)$,

for any $h, k \in H$.

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Remark 2.22. *Let H be a Hopf algebra, B be a unital algebra, and $\pi : H \to B$ be a partial representation. It follows by Lemma 2.11 of [3] that the following assertions also hold: for all $h, k \in H$,*

(PR4) $\pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \pi(hS(k_{(1)}))\pi(k_{(2)})$,
(PR5) $\pi(S(h_{(1)}))\pi(h_{(2)})\pi(k) = \pi(S(h_{(1)}))\pi(h_2k)$.

Example 2.23. *Let H be a Hopf algebra, B be a unital algebra and $\pi : H \to B$ be a partial representation. Consider the linear map $\overline{\pi} : H^{\text{opcop}} \to B^{\text{op}}$ given by*

$$\overline{\pi}(h) = \pi(h), \quad h \in H^{\text{opcop}}.$$

We claim that $\overline{\pi}$ is a partial representation. Notice that (PR1) is clear. To show (PR2), consider $h, k \in H^{\text{opcop}}$. Then

$$\begin{aligned}\overline{\pi}(h) \cdot_{\text{op}} \overline{\pi}(k_{[1]}) \cdot_{\text{op}} \overline{\pi}(S(k_{[2]})) &= \pi(S(k_{(1)}))\pi(k_{(2)})\pi(h)\\ &\overset{(PR5)}{=} \pi(S(k_{(1)}))\pi(k_{(2)}h)\\ &= \overline{\pi}(h \cdot_{\text{op}} k_{[1]}) \cdot_{\text{op}} \overline{\pi}(S(k_{[2]})).\end{aligned}$$

Similarly, using that π satisfies (PR4), we obtain that $\overline{\pi}$ satisfies (PR3).

Example 2.24. *Let H be a Hopf algebra and $\cdot : H \otimes A \to A$ be a left partial action of H on an algebra A. By Example 3.5 in [2], the linear map*

$$\pi : H \to \text{End}(A), \quad \pi(h)(a) = h \cdot a, \quad h \in H, \ a \in A, \tag{2.25}$$

is a partial representation of H in End(A). *It follows from the previous example that the linear map $\overline{\pi} : H^{\text{opcop}} \to \text{End}(A)^{\text{op}}$ given by*

$$\overline{\pi}(h) = \pi(h), \quad h \in H^{\text{opcop}},$$

is a partial representation.

2.3. The partial smash product

Let $\cdot : H \otimes A \to A$ be a partial action of a Hopf algebra H over an algebra A. On the vector space $A \otimes H$, one can define the following associative product

$$(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b) \otimes (h_{(2)}k), \quad a, b \in A, \ h, k \in H.$$

Following [10], the *(left) partial smash product* is the vector subspace

$$\underline{A\#H} := (A \otimes H)(1_A \otimes 1_H)$$

of $A \otimes H$ which is generated by elements of the form

$$a\#h = a(h_{(1)} \cdot 1_A) \otimes h_{(2)}, \quad a \in A, \ h \in H.$$

Notice that, by construction, $\underline{A\#H}$ is a unital algebra and $1_{\underline{A\#H}} = 1_A\#1_H$.

We need the following notion to recall the universal property of the partial smash product proved in [2].

Definition 2.26. *Let A and B be algebras, H be a Hopf algebra and $\cdot : H \otimes A \to A$ be a partial action of H on A. A left covariant pair associated to these data is a pair of maps (ψ, π), where $\psi : A \to B$ is an algebra morphism and $\pi : H \to B$ is a partial representation, that satisfies, for all $h \in H$ and $a \in A$,*

(CP1) $\psi(h \cdot a) = \pi(h_{(1)})\psi(a)\pi(S(h_{(2)}))$,
(CP2) $\psi(a)\pi(S(h_{(1)}))\pi(h_{(2)}) = \pi(S(h_{(1)}))\pi(h_{(2)})\psi(a)$.

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Consider the following linear maps

$$\phi_0 : A \to \underline{A\#H}, \quad a \mapsto a\#1_H, \qquad\qquad \pi_0 : H \to \underline{A\#H}, \quad h \mapsto 1_A\#h.$$

It was proved in [2] that ϕ_0 is an algebra monomorphism and π_0 is a partial representation. The next result is Theorem 3.9 of [2].

Theorem 2.27. *Let A and B be unital algebras and $\cdot : H \otimes A \to A$ be a left partial action of the Hopf algebra H on A. Suppose that (ψ, π) is a covariant pair associated to these data. Then the map*

$$\Phi : \underline{A\#H} \to B, \quad a\#h \to \psi(a)\pi(h), \quad a \in A, \quad h \in H,$$

is the unique algebra morphism such that $\psi = \Phi \circ \phi_0$ and $\pi = \Phi \circ \pi_0$.

Consider now a right partial action $\cdot : A \otimes H \to A$ of a Hopf algebra H on an algebra A. As in the left side case, one can define the right partial smash product. Precisely, one endows the vector space $H \otimes A$ with an associative product

$$(k \otimes b)(h \otimes a) = kh_{(1)} \otimes (b \cdot h_{(2)})a, \quad a, b \in A, \ \ h, k \in H.$$

Definition 2.28. *Let $\cdot : A \otimes H \to A$ be a right partial action of a Hopf algebra H on an algebra A. The (right) partial smash product is the vector subspace*

$$\underline{H\#A} := (1_H \otimes 1_A)(H \otimes A)$$

of $H \otimes A$ which is generated by typical elements of the form

$$h\#a = h_{(1)} \otimes (1_A \cdot h_{(2)})a, \quad a \in A, \ h \in H. \tag{2.29}$$

The right partial smash product $\underline{H\#A}$ is an associative unital algebra and its unity element is $1_{\underline{H\#A}} = 1_H\#1_A$.

Example 2.30. *Suppose that $\alpha = \left(A_g, \alpha_g\right)_{g\in G}$ is a unital partial action of a group G on an algebra A. As we saw in Example 2.17, we have a left partial action of $\Bbbk G$ on A given by $g \cdot a = \alpha_g(a1_{g^{-1}})$, for all $a \in A$ and $g \in G$. We recall from [11] that the partial crossed product by α is the associative algebra $A \rtimes_\alpha G$ whose elements are finite formal sums in the form $\sum_{g\in G} a_g\delta_g$, where $a_g \in A_g$, and the multiplication is given by*

$$a_g\delta_g \cdot a_h\delta_h = a_g\alpha_g(a_h1_{g^{-1}})\delta_{gh}, \quad g, h \in G, \ \ a_g \in A_g, \ \ a_h \in A_h.$$

Using (2.29) we get that the map

$$\eta : \underline{A\#\Bbbk G} \to A \rtimes_\alpha G, \ \ a\#g \mapsto a1_g\delta_g, \quad g \in G, \ \ a \in A, \tag{2.31}$$

is an algebra isomorphism with inverse $b\delta_g \mapsto b\#g$, $b \in A_g$, $g \in G$. Moreover, it follows by Example 2.18, that the partial action α of G on A induces a right partial action of $(\Bbbk G)^{\rm op} = (\Bbbk G)^{\rm opcop}$ on $A^{\rm op}$ via $a \cdot g = \alpha_g(a1_{g^{-1}})$, for all $a \in A$ and $g \in G$. Also, by (2.29) it is not difficult to verify that

$$\lambda : \underline{(\Bbbk G)^{\rm op}\#A} \to (A^{\rm op} \ltimes_\alpha G)^{\rm op}, \quad g\#a \mapsto a1_g\delta_g, \quad g \in G, \ \ a \in A, \tag{2.32}$$

is an algebra isomorphism.

Definition 2.33. *Let A and B be algebras, H be a Hopf algebra and $\cdot : A \otimes H^{\rm op} \to A$ be a right partial action of $H^{\rm op}$ on A. A opposite covariant pair associated to these data is a pair of maps (φ, γ), where $\varphi : A \to B$ is an algebra morphism and $\gamma : H^{\rm opcop} \to B$ is a partial representation, that satisfies, for all $h \in H^{\rm opcop}$ and $a \in A$,*

(OCP1) $\varphi(a \cdot h) = \gamma(S^{-1}(h_{[2]}))\varphi(a)\gamma(h_{[1]}))$,
(OCP2) $\varphi(a)\gamma(h_{[2]})\gamma(S^{-1}(h_{[1]})) = \gamma(h_{[2]})\gamma(S^{-1}(h_{[1]}))\varphi(a)$.

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Lemma 2.34. *Let H be a Hopf algebra and A be a right partial H^{op}-module. Then the following assertions hold:*

(i) *the linear map $\varphi_0 : A \to \underline{H^{\mathrm{op}}\#A}$, $a \mapsto 1_H\#a$, is an algebra monomorphism,*
(ii) *the linear map $\gamma_0 : H^{\mathrm{opcop}} \to \underline{H^{\mathrm{op}}\#A}$, $h \mapsto h\#1_A$, is a partial representation,*
(iii) *(φ_0, γ_0) is an opposite covariant pair.*

Proof. It is straightforward to verify that (i) is true. To prove (ii), we observe that $\gamma_0(1_H) = 1_H\#1_A = 1_{\underline{H^{\mathrm{op}}\#A}}$. Also, given $h, k \in H^{\mathrm{opcop}}$, we have that

$$\begin{aligned}
\gamma_0(h)\gamma_0(k_{[1]})\gamma_0(S(k_{[2]})) &= (h\#1_A)(k_{(2)}\#1_A)(S(k_{(1)})\#1_A)\\
&= (k_{(2)}h\#1_A \cdot k_{(3)})(S(k_{(1)})\#1_A)\\
&= S(k_{(2)})k_{(3)}h\#(1_A \cdot k_{(4)}) \cdot S(k_{(1)})\\
&= h\#(1_A \cdot k_{(2)}) \cdot S(k_{(1)})\\
&= h\#(1_A \cdot S(k_{(2)})k_{(3)})(1_A \cdot S(k_{(1)})) \qquad \text{(by (RPA3))}\\
&= h\#1_A \cdot S(k).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma_0(h \cdot_{op} k_{[1]})\gamma_0(S(k_{[2]})) &= (k_{(2)}h\#1_A)(S(k_{(1)})\#1_A)\\
&= S(k_{(2)})k_{(3)}h\#1_A \cdot S(k_{(1)})\\
&= h\#1_A \cdot S(k),
\end{aligned}$$

and consequently (PR2) holds. Finally, notice that

$$\begin{aligned}
\gamma_0(h_{[1]})\gamma_0(S(h_{[2]}))\gamma_0(k) &= (h_{(2)}\#1_A)(S(h_{(1)})\#1_A)(k\#1_A)\\
&= (S(h_{(2)})h_{(3)}\#1_A \cdot S(h_{(2)}))(k\#1_A)\\
&= (1_H\#1_A \cdot S(h))(k\#1_A)\\
&= k_{(1)}\#(1_A \cdot S(h)) \cdot k_{(2)},
\end{aligned}$$

and

$$\begin{aligned}
\gamma_0(h_{[1]})\gamma_0(S(h_{[2]}) \cdot_{op} k) &= (h_{(2)}\#1_A)(kS(h_{(1)})\#1_A)\\
&= k_{(1)}S(h_{(2)})h_{(3)}\#1_A \cdot k_{(2)}S(h_{(1)})\\
&= k_{(1)}\#1_A \cdot k_{(2)}S(h)\\
&= k_{(1)}\#1_A \cdot (S(h) \cdot_{\mathrm{op}} k_{(2)})\\
&= k_{(1)}\#(1_A \cdot k_{(2)})(1_A \cdot (S(h) \cdot_{\mathrm{op}} k_{(3)}) \qquad \text{(by (2.29))}\\
&= k_{(1)}\#(1_A \cdot S(h)) \cdot k_{(2)}, \qquad \text{(by (RPA3))}
\end{aligned}$$

which implies that (PR3) holds, and hence (ii) is proved. To prove (iii) we note that

$$\begin{aligned}
\gamma_0(S^{-1}(h_{[2]}))\varphi_0(a)\gamma_0(h_{[1]}) &= (S^{-1}(h_{(1)})\#1_A)(1_H\#a)(h_{(2)}\#1_A)\\
&= (S^{-1}(h_{(1)})\#a)(h_{(2)}\#1_A)\\
&= h_{(2)}S^{-1}(h_{(1)})\#a \cdot h_{(3)}\\
&= 1_H\#a \cdot h\\
&= \varphi_0(a \cdot h).
\end{aligned}$$

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Also,

$$\begin{aligned}\varphi_0(a)\gamma_0(h_{[2]})\gamma_0(S^{-1}(h_{[1]})) &= (1_H\#a)(h_{(1)}\#1_A)(S^{-1}(h_{(2)})\#1_A)\\ &= (h_{(1)}\#a\cdot h_{(2)})(S^{-1}(h_{(3)})\#1_A)\\ &= S^{-1}(h_{(4)})h_{(1)}\#(a\cdot h_{(2)})\cdot S^{-1}(h_{(3)})\\ &\overset{(\mathrm{RPA3})}{=} S^{-1}(h_{(5)})h_{(1)}\#(1_A\cdot S^{-1}(h_{(4)}))(a\cdot(h_{(2)}\cdot_{\mathrm{op}} S^{-1}(h_{(3)})))\\ &= S^{-1}(h_{(5)})h_{(1)}\#(1_A\cdot S^{-1}(h_{(4)}))(a\cdot S^{-1}(h_{(3)})h_{(2)})\\ &= S^{-1}(h_{(3)})h_{(1)}\#(1_A\cdot S^{-1}(h_{(2)}))a,\end{aligned}$$

and

$$\begin{aligned}\gamma_0(h_{[2]})\gamma_0(S^{-1}(h_{[1]}))\varphi_0(a) &= (h_{(1)}\#1_A)(S^{-1}(h_{(2)})\#1_A)(1_H\#a)\\ &= (h_{(1)}\#1_A)(S^{-1}(h_{(2)})\#a)\\ &= S^{-1}(h_{(3)})h_{(1)}\#(1_A\cdot S^{-1}(h_{(2)}))a,\end{aligned}$$

then $\gamma_0(h_{[2]})\gamma_0(S^{-1}(h_{[1]}))\varphi_0(a)=\varphi_0(a)\gamma_0(h_{[2]})\gamma_0(S^{-1}(h_{[1]}))$, which finishes the proof. □

We now establish a universal property for opposite covariant pairs, analogous to the version for covariant pairs presented in Theorem 2.25.

Theorem 2.35. *Let A and B be unital algebras and $\cdot : A\otimes H^{\mathrm{op}}\to A$ be a right partial action of the Hopf algebra H on A. Suppose that (φ,γ) is an opposite covariant pair associated with these data. Then there exists a unique algebra morphism $\Gamma : \underline{H^{\mathrm{op}}\#A}\to B$ such that $\varphi=\Gamma\circ\varphi_0$ and $\gamma=\Gamma\circ\gamma_0$.*

Proof. Consider the linear map

$$\Gamma : \underline{H^{\mathrm{op}}\#A}\to B,\qquad \Gamma(h\#a)=\gamma(h)\varphi(a),\quad h\in H,\ a\in A. \tag{2.36}$$

It is straightforward to see that $\varphi=\Gamma\circ\varphi_0$ and $\gamma=\Gamma\circ\gamma_0$. Given $k,h\in H$ and $a,b\in A$, we have

$$\begin{aligned}\Gamma((k\#b)(h\#a)) &= \Gamma(k\cdot_{\mathrm{op}} h_{(1)}\#(b\cdot h_{(2)})a)\\ &= \gamma(k\cdot_{\mathrm{op}} h_{(1)})\varphi((b\cdot h_{(2)})a)\\ &= \gamma(k\cdot_{\mathrm{op}} h_{[2]})\varphi(b\cdot h_{[1]})\varphi(a)\\ &= \gamma(k\cdot_{\mathrm{op}} h_{[3]})\gamma(S^{-1}(h_{[2]}))\varphi(b)\gamma(h_{[1]})\varphi(a) && \text{(by (OCP1))}\\ &= \gamma(k)\gamma(h_{[3]})\gamma(S^{-1}(h_{[2]}))\varphi(b)\gamma(h_{[1]})\varphi(a) && \text{(by (PR4))}\\ &= \gamma(k)\varphi(b)\gamma(h_{[3]})\gamma(S^{-1}(h_{[2]}))\gamma(h_{[1]})\varphi(a) && \text{(by (OCP2))}\\ &= \gamma(k)\varphi(b)\gamma(h_{[3]})\gamma(S^{-1}(h_{[2]}))\gamma(S(S^{-1}(h_{[1]})))\varphi(a)\\ &= \gamma(k)\varphi(b)\gamma(h_{[3]})\gamma(S^{-1}(h)_{[1]})\gamma(S(S^{-1}(h)_{[2]}))\varphi(a)\\ &= \gamma(k)\varphi(b)\gamma(h_{[3]}\cdot_{\mathrm{op}} S^{-1}(h_{[2]}))\gamma(h_{[1]})\varphi(a) && \text{(by (PR2))}\\ &= \gamma(k)\varphi(b)\varphi(a)\gamma(h)\varphi(a)\\ &= \Gamma(k\#b)\circ\Gamma(h\#a).\end{aligned}$$

Since $\Gamma(1_H\#1_A)=1_B$ it follows that Γ is a morphism of algebras. The proof of the uniqueness of Γ is similar to the proof given in [2, Theorem 3.9]. □

3. The main result

Throughout this section, H denotes a Hopf algebra with inversible antipode S and $R=({}_iM_j)_{i,j\in\mathbb{I}_n}$ denotes a generalized matrix algebra such that ${}_iM_i=R_i$ is a unital algebra. The identity element of R_i is denoted

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by 1_i. This section is dedicated to prove the main result of this work, which provides necessary and sufficient conditions for the existence of a partial action of H on R.

Given $i, j \in \mathbb{I}_n$, we denote by $\iota_{ij}: {}_iM_j \to R$ the natural inclusion of ${}_iM_j$ in R, that is, $\iota_{ij}(m)$ is the matrix whose entries are all zero except the (i,j)-entry which is equal to m, for all $m \in {}_iM_j$. By Remark 2.8, R is an (R_i, R_j)-bimodule, and it is easy to see that ι_{ij} (resp. ι_{ii}) is a monomorphism of (R_i, R_j)-bimodules (resp. of algebras). From now on, we will denote by ${}_i\tilde{M}_j = \iota_{ij}({}_iM_j)$ the (R_i, R_j)-subbimodule of R which is isomorphic to ${}_iM_j$.

Definition 3.1. *Let $\triangleright : H \otimes R \to R$ be a left partial action of H on R and $i, j \in \mathbb{I}_n$. The subbimodule ${}_i\tilde{M}_j$ of R is said to be H-invariant if $h \triangleright m \in {}_i\tilde{M}_j$, for all $h \in H$ and $m \in {}_i\tilde{M}_j$.*

Lemma 3.2. *Let $\triangleright : H \otimes R \to R$ be a left partial action of H on R such that ${}_i\tilde{M}_j$ is invariant, for all $i, j \in \mathbb{I}_n$. Then:*

(i) *the linear map $\rightharpoonup_i : H \otimes R_i \to R_i$ given by*

$$h \rightharpoonup_i r = \iota_{ii}^{-1}\big(h \triangleright \iota_{ii}(r)\big), \quad h \in H, \ r \in R_i,$$

is a partial action,

(ii) *the linear map $\pi_{ij} : H \to \mathrm{End}\,({}_iM_j)$ given by*

$$\pi_{ij}(h)(m) = \iota_{ij}^{-1}\big(h \triangleright \iota_{ij}(m)\big), \quad h \in H, \ m \in {}_iM_j,$$

is a partial representation,

(iii) *the linear map $\gamma_{ij} : H^{\mathrm{opcop}} \to \mathrm{End}\,({}_iM_j)^{\mathrm{op}}$ given by*

$$\gamma_{ij}(h)(m) = \iota_{ij}^{-1}\big(h \triangleright \iota_{ij}(m)\big), \quad h \in H, \ m \in {}_iM_j,$$

is a partial representation.

Proof. (i) Let $r \in R_i$. Then

$$1_H \rightharpoonup_i r = \iota_{ii}^{-1}\big(1_H \triangleright \iota_{ii}(r)\big) = \iota_{ii}^{-1}\big(\iota_{ii}(r)\big) = r,$$

which implies (LPA1). For (LPA2), consider $r, s \in R_i$ and $h \in H$. Thus,

$$\begin{aligned} h \rightharpoonup_i (rs) &= \iota_{ii}^{-1}\big(h \triangleright \iota_{ii}(rs)\big) \\ &= \iota_{ii}^{-1}\big(h \triangleright (\iota_{ii}(r)\iota_{ii}(s))\big) \\ &= \iota_{ii}^{-1}\big(\big(h_{(1)} \triangleright \iota_{ii}(r)\big)\big(h_{(2)} \triangleright \iota_{ii}(s)\big)\big) \\ &= \iota_{ii}^{-1}\big(h_{(1)} \triangleright \iota_{ii}(r)\big)\,\iota_{ii}^{-1}\big(h_{(2)} \triangleright \iota_{ii}(s)\big) \\ &= \big(h_{(1)} \rightharpoonup_i r\big)\big(h_{(2)} \rightharpoonup_i r\big). \end{aligned}$$

Finally, in order to verify (LPA3), consider $i \in \mathbb{I}_n$, $h \in H$ and $r \in R_i$. Then,

$$(h \triangleright 1_R)\iota_{ii}(r) = (h \triangleright \iota_{ii}(1_i))\iota_{ii}(r), \qquad \iota_{ii}(r)(h \triangleright 1_R) = \iota_{ii}(r)(h \triangleright \iota_{ii}(1_i)).$$

Then, for every $h, k \in H$ and $r \in R_i$

$$\begin{aligned} h \rightharpoonup_i (k \rightharpoonup_i r) &= \iota_{ii}^{-1}\big(h \triangleright \big(k \triangleright \iota_{ii}(r)\big)\big) \\ &= \iota_{ii}^{-1}\big(\big(h_{(1)} \triangleright 1_R\big)\big(h_{(2)}k \triangleright \iota_{ii}(r)\big)\big) \\ &\overset{(*)}{=} \iota_{ii}^{-1}\big(\big(h_{(1)} \triangleright \iota_{ii}(1_i)\big)\big(h_{(2)}k \triangleright \iota_{ii}(r)\big)\big) \\ &= \iota_{ii}^{-1}\big(h_{(1)} \triangleright \iota_{ii}(1_i)\big)\,\iota_{ii}^{-1}\big(h_{(2)}k \triangleright \iota_{ii}(r)\big) \\ &= \big(h_{(1)} \rightharpoonup_i 1_i\big)\big(h_{(2)}k \rightharpoonup_i r\big), \end{aligned}$$

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where $(*)$ follows from the H-invariance of $\tilde{R}_i$. Similarly, one verifies that

$$h \rightharpoonup_i (k \rightharpoonup_i r) = \big(h_{(1)}k \rightharpoonup_i r\big)\ \big(h_{(2)} \rightharpoonup_i 1_i\big)$$

(ii) For any $m \in {}_iM_j$, we have that

$$\pi_{ij}(1_H)(m) = \iota_{ij}^{-1}\big(1_H \rhd \iota_{ij}(m)\big) = \iota_{ij}^{-1}\big(\iota_{ij}(m)\big) = m,$$

and hence (PR1) holds. Also, given $h, k \in H$, we have that

$$\begin{aligned}\pi_{ij}(h)\pi_{ij}(k_{(1)})\pi_{ij}(S(k_{(2)}))(m) &= \iota_{ij}^{-1}\big(h \rhd \big(k_{(1)} \rhd (S(k_{(2)}) \rhd \iota_{ij}(m))\big)\big)\\ &= \iota_{ij}^{-1}\big(h \rhd \big((k_{(1)} \rhd 1_R)(k_{(2)}S(k_{(3)}) \rhd \iota_{ij}(m))\big)\big) \text{ (by (LPA3))}\\ &= \iota_{ij}^{-1}\big(h \rhd \big((k \rhd 1_R)\iota_{ij}(m)\big)\big),\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\pi_{ij}(hk_{(1)})\pi_{ij}(S(k_{(2)}))(m) &= \iota_{ij}^{-1}\big(hk_{(1)} \rhd (S(k_{(2)}) \rhd \iota_{ij}(m))\big)\\ &= \iota_{ij}^{-1}\big((h_{(1)}k_{(1)} \rhd 1_R)(h_{(2)}k_{(2)}S(k_{(3)}) \rhd \iota_{ij}(m))\big) \quad \text{(by (LPA3))}\\ &= \iota_{ij}^{-1}\big((h_{(1)}k \rhd 1_R)(h_{(2)} \rhd \iota_{ij}(m))\big)\\ &= \iota_{ij}^{-1}\big((h_{(1)}k \rhd 1_R)(h_{(2)} \rhd (1_R \cdot \iota_{ij}(m)))\big)\\ &= \iota_{ij}^{-1}\big((h_{(1)}k \rhd 1_R)(h_{(2)} \rhd 1_R)(h_{(3)} \rhd \iota_{ij}(m))\big) \quad \text{(by (LPA2))}\\ &= \iota_{ij}^{-1}\big((h_{(1)} \rhd (k \rhd 1_R))(h_{(2)} \rhd \iota_{ij}(m))\big) \quad \text{(by (LPA3))}\\ &= \iota_{ij}^{-1}\big(h \rhd \big((k \rhd 1_R)\iota_{ij}(m)\big)\big) \quad \text{(by (LPA2))}.\end{aligned}$$

Thus, (PR2) is true. Similarly, we can verify that (PR3) also holds, and consequently (ii) is proved. Item (iii) follows from (ii) and Example 2.23. □

Theorem 3.3. *The following assertions are equivalent:*

(i) there exists a left partial action $\rhd : H \otimes R \to R$ of H on R such that ${}_i\tilde{M}_j$ is invariant, for all $i, j \in \mathbb{I}_n$,

(ii) the following conditions hold:

(a) for each $i \in \mathbb{I}_n$, there is a left partial action $\rightharpoonup_i : H \otimes R_i \to R_i$ of H on R_i,

(b) ${}_iM_j$ is a left $\underline{R_i\#H}$-module and a right $\underline{H^{\mathrm{op}}\#R_j}$-module such that

$$\begin{aligned}(1_i\#h)\cdot m &= m\cdot(h\#1_j), & (1_i\#h)\cdot r &= h \rightharpoonup_i r,\\ (r_i\#1_H)\cdot m &= r_i\cdot m, & m\cdot(1_H\#r_j) &= m\cdot r_j,\end{aligned}$$

for all $h \in H$, $m \in {}_iM_j$, $r, r_i \in R_i$, $r_j \in R_j$ and $i, j \in \mathbb{I}_n$,

(c) $(1_i\#h)\cdot mn = \big((1_i\#h_{(1)})\cdot m\big)\big((1_j\#h_{(2)})\cdot n\big)$, for all $h \in H$, $m \in {}_iM_j$, $n \in {}_jM_k$.

Proof. (i) $\Rightarrow$ (ii) Let $i \in \mathbb{I}_n$. Since ${}_i\tilde{M}_i$ is invariant by the partial action of H on R, it follows from Lemma 3.2 (i) that the restriction $\rightharpoonup_i : H \otimes R_i \to R_i$ is a partial action of H on R_i and hence (a) is proved. To prove (b), we recall that ${}_iM_j$ an (R_i, R_j)-bimodule, for all $i, j \in \mathbb{I}_n$. In particular, ${}_iM_j$ is a left R_i-module and consequently there is an algebra morphism $\psi_{ij} : R_i \to \mathrm{End}\,({}_iM_j)$, given by

$$\psi_{ij}(r)(m) = \iota_{ij}^{-1}\big(\iota_{ii}(r)\iota_{ij}(m)\big) = \iota_{ij}^{-1}\big(\iota_{ij}(r\cdot m)\big) = r\cdot m, \quad \text{for all } r \in R_i,\ m \in {}_iM_j.$$

Moreover, by Lemma 3.2 (ii), $\pi_{ij} : H \to \mathrm{End}\,({}_iM_j)$ given by

$$\pi_{ij}(h)(m) = \iota_{ij}^{-1}\big(h \rhd \iota_{ij}(m)\big), \qquad h \in H,\ m \in {}_iM_j,$$

is a partial representation. One needs to verify that (ψ_{ij}, π_{ij}) is a covariant pair. In order to prove (CP1), take $r \in R_i$, $h \in H$ and $m \in {}_iM_j$. Then

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$$\begin{aligned}
\psi_{ij}(h \rightharpoonup_i r)(m) &= \iota_{ij}^{-1}\left(\iota_{ii}(h \rightharpoonup_i r)\iota_{ij}(m)\right)\\
&= \iota_{ij}^{-1}\left(\iota_{ii}\iota_{ii}^{-1}(h \triangleright \iota_{ii}(r))\iota_{ij}(m)\right)\\
&= \iota_{ij}^{-1}\left((h \triangleright \iota_{ii}(r))\iota_{ij}(m)\right)\\
&= \iota_{ij}^{-1}\left((h_{(1)} \triangleright \iota_{ii}(r))(h_{(2)}S(h_{(3)}) \triangleright \iota_{ij}(m))\right)\\
&= \iota_{ij}^{-1}\left((h_{(1)} \triangleright \iota_{ii}(r))(h_{(2)} \triangleright (S(h_{(3)}) \triangleright \iota_{ij}(m)))\right)\\
&= \iota_{ij}^{-1}\left(h_{(1)} \triangleright \left(\iota_{ii}(r)(S(h_{(2)}) \triangleright \iota_{ij}(m))\right)\right)\\
&= \iota_{ij}^{-1}\left(h_{(1)} \triangleright \left(\iota_{ii}(r)\iota_{ij}\iota_{ij}^{-1}(S(h_{(2)}) \triangleright \iota_{ij}(m))\right)\right)\\
&= \iota_{ij}^{-1}\left(h_{(1)} \triangleright \left(\iota_{ii}(r)\iota_{ij}(\pi_{ij}(S(h_{(2)}))(m))\right)\right)\\
&= \iota_{ij}^{-1}\left(h_{(1)} \triangleright \iota_{ij}\iota_{ij}^{-1}\left(\iota_{ii}(r)\iota_{ij}(\pi_{ij}(S(h_{(2)}))(m))\right)\right)\\
&= \iota_{ij}^{-1}\left(h_{(1)} \triangleright \iota_{ij}\left(\psi_{ij}(r)(\pi_{ij}(S(h_{(2)}))(m))\right)\right)\\
&= \pi_{ij}(h_{(1)})\left(\psi_{ij}(r)(\pi_{ij}(S(h_{(2)}))(m))\right)\\
&= \pi_{ij}(h_{(1)}) \circ \psi_{ij}(r) \circ \pi_{ij}(S(h_{(2)}))(m).
\end{aligned}$$

For (CP2), on the one hand we have

$$\begin{aligned}
\psi_{ij}(r) \circ \pi_{ij}(S(h_{(1)})) \circ \pi_{ij}(h_{(2)})(m) &= \psi_{ij}(r) \circ \pi_{ij}(S(h_{(1)}))\left(\iota_{ij}^{-1}(h_{(2)} \triangleright \iota_{ij}(m))\right)\\
&= \psi_{ij}(r)\left(\iota_{ij}^{-1}\left(S(h_{(1)}) \triangleright (h_{(2)} \triangleright \iota_{ij}(m))\right)\right)\\
&= \psi_{ij}(r)\left(\iota_{ij}^{-1}\left(\left(S(h_{(2)})h_{(3)} \triangleright \iota_{ij}(m)\right)\left(S(h_{(1)}) \triangleright 1_R\right)\right)\right)\\
&= \psi_{ij}(r)\left(\iota_{ij}^{-1}\left(\iota_{ij}(m)\,(S(h) \triangleright 1_R)\right)\right)\\
&= \iota_{ij}^{-1}\left(\iota_{ii}(r)\iota_{ij}(m)\,(S(h) \triangleright 1_R)\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\pi_{ij}(S(h_{(1)})) \circ \pi_{ij}(h_{(2)}) \circ \psi_{ij}(r)(m) &= \pi_{ij}(S(h_{(1)})) \circ \pi_{ij}(h_{(2)})\left(\iota_{ij}^{-1}(\iota_{ii}(r)\iota_{ij}(m))\right)\\
&= \pi_{ij}(S(h_{(1)}))\left(\iota_{ij}^{-1}\left(h_{(2)} \triangleright (\iota_{ii}(r)\iota_{ij}(m))\right)\right)\\
&= \iota_{ij}^{-1}\left(S(h_{(1)}) \triangleright \left(h_{(2)} \triangleright (\iota_{ii}(r)\iota_{ij}(m))\right)\right)\\
&= \iota_{ij}^{-1}\left(\left(S(h_{(1)})h_{(2)} \triangleright (\iota_{ii}(r)\iota_{ij}(m))\right)\left(S(h_{(1)}) \triangleright 1_R\right)\right)\\
&= \iota_{ij}^{-1}\left(\iota_{ii}(r)\iota_{ij}(m)\,(S(h) \triangleright 1_R)\right).
\end{aligned}$$

Therefore, (ψ_{ij}, π_{ij}) is a covariant pair. It follows from Theorem 2.27 that

$$\Phi_{ij} : R_i\#H \to \mathrm{End}\,({}_iM_j), \quad \Phi_{ij}(r\#h) = \psi_{ij}(r)\pi_{ij}(h), \;\; r \in R_i, \;\; h \in H,$$

is an algebra morphism. Therefore, ${}_iM_j$ is a left $R_i\#H$-module.

Now, we will denote by $\cdot : {}_iM_j \otimes R_j \to {}_iM_j$ the right action of R_j on ${}_iM_j$. Then, $\varphi_{ij} : R_j \to \mathrm{End}\,({}_iM_j)^{\mathrm{op}}$ given by $\varphi_{ij}(r)(m) = m \cdot r$, $r \in R_j$ and $m \in {}_iM_j$, is an algebra morphism. Also, by Lemma 3.2 (ii), the map $\gamma_{ij} : H^{\mathrm{opcop}} \to \mathrm{End}\,({}_iM_j)^{\mathrm{op}}$ given by

$$\gamma_{ij}(h)(m) = \iota_{ij}^{-1}\big(h \triangleright \iota_{ij}(m)\big), \qquad h \in H, \;\; m \in {}_iM_j,$$

is a partial representation. We claim that $(\varphi_{ij}, \gamma_{ij})$ is an opposite covariant pair. Firstly, by Example 2.18, the linear map $\leftharpoonup_j : R_j \otimes H^{\mathrm{op}} \to R_j$, given by

$$r \leftharpoonup_j h := h \rightharpoonup_j r = \iota_{jj}^{-1}(h \triangleright \iota_{jj}(r)), \;\; \text{for all } \; r \in R_j, \;\; h \in H^{\mathrm{op}},$$

defines a right partial action of H^{op} in R_j. For all $r \in R_j$ and $m \in {}_iM_j$, we will denote by $u(m, r) := \gamma_{ij}(S^{-1}(h_{[2]})) \cdot_{\mathrm{op}} \varphi_{i,j}(r) \cdot_{\mathrm{op}} \gamma_{ij}(h_{[1]})(m)$ the element of ${}_iM_j$ obtained using the opposite product of $\mathrm{End}\,({}_iM_j)$.

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Then

$$\begin{aligned}
u(m,r) &= \gamma_{ij}(h_{[1]})\varphi_{i,j}(r)\gamma_{ij}(S^{-1}(h_{[2]}))(m)\\
&= \iota_{ij}^{-1}\Big(h_{(2)} \rhd \iota_{ij}\big(\big(\iota_{ij}^{-1}\big(S^{-1}(h_{(1)}) \rhd \iota_{ij}(m)\big) \cdot r\big)\big)\Big)\\
&= \iota_{ij}^{-1}\big(h_{(2)} \rhd \big(S^{-1}(h_{(1)}) \rhd \iota_{ij}(m)\big) \cdot r\big)\\
&= \iota_{ij}^{-1}\big(h_{(2)} \rhd \big(S^{-1}(h_{(1)}) \rhd \iota_{ij}(m)\big)\iota_{jj}(r)\big)\\
&= \iota_{ij}^{-1}\Big(\big(h_{(2)} \rhd \big(S^{-1}(h_{(1)}) \rhd \iota_{ij}(m)\big)\big(h_{(3)} \rhd \iota_{jj}(r)\big)\Big)\\
&\overset{(2.13)}{=} \iota_{ij}^{-1}\big(\big(h_{(2)}S^{-1}(h_{(1)}) \rhd \iota_{ij}(m)\big)\big(h_{(3)} \rhd \iota_{jj}(r)\big)\big)\\
&= \iota_{ij}^{-1}\big(\iota_{ij}(m)\big(h \rhd \iota_{jj}(r)\big)\big)\\
&\overset{(*)}{=} m \cdot \iota_{jj}^{-1}\big(h \rhd \iota_{jj}(r)\big)\big)\\
&= \varphi_{ij}(r \leftharpoonup_j h)(m),
\end{aligned}$$

where $(*)$ holds because there exists a unique $r' \in R_j$ such that $h \rhd \iota_{jj}(r) = \iota_{jj}(r')$ and consequently

$$\iota_{ij}^{-1}\big(\iota_{ij}(m)\big(h \rhd \iota_{jj}(r)\big)\big) = \iota_{ij}^{-1}\big(\iota_{ij}(m)\iota_{jj}(r')\big) = \iota_{ij}^{-1}\big(\iota_{ij}(m \cdot r')\big) = m \cdot r'.$$

Thus, (OCP1) is true. In order to prove (OCP2) , given $r \in R_j$ and $m \in {}_iM_j$ we will denote $v(m,r) = \varphi_{ij}(r) \cdot_{\mathrm{op}} \gamma_{ij}(h_{[2]}) \cdot_{\mathrm{op}} \gamma_{ij}(S^{-1}(h_{[1]}))(m) \in {}_iM_j$. Notice that

$$\begin{aligned}
v(m,r) &= \gamma_{ij}(S^{-1}(h_{[1]}))\gamma_{ij}(h_{[2]})\varphi_{ij}(r)(m)\\
&= \iota_{ij}^{-1}\big(S^{-1}(h_{(2)}) \rhd (h_{(1)} \rhd \iota_{ij}(m \cdot r))\big)\\
&= \iota_{ij}^{-1}\big(S^{-1}(h_{(2)}) \rhd (h_{(1)} \rhd \iota_{ij}(m)\iota_{jj}(r))\big)\\
&= \iota_{ij}^{-1}\Big(S^{-1}(h_{(3)}) \rhd \big((h_{(1)} \rhd \iota_{ij}(m))(h_{(2)} \rhd \iota_{jj}(r))\big)\Big)\\
&= \iota_{ij}^{-1}\Big(\big(S^{-1}(h_{(4)}) \rhd (h_{(1)} \rhd \iota_{ij}(m))\big)\big(S^{-1}(h_{(3)}) \rhd (h_{(2)} \rhd \iota_{jj}(r))\big)\Big)\\
&\overset{(2.14)}{=} \iota_{ij}^{-1}\Big(\big(S^{-1}(h_{(4)}) \rhd (h_{(1)} \rhd \iota_{ij}(m))\big)\big(h_{(2)}S^{-1}(h_{(3)}) \rhd \iota_{jj}(r)\big)\Big)\\
&= \iota_{ij}^{-1}\Big(\big(S^{-1}(h_{(2)}) \rhd (h_{(1)} \rhd \iota_{ij}(m))\big)\iota_{jj}(r)\Big)\\
&= \iota_{ij}^{-1}\Big(\big(S^{-1}(h_{[1]}) \rhd (h_{[2]} \rhd \iota_{ij}(m))\big)\iota_{jj}(r)\Big)\\
&= \varphi_{ij}(r)\gamma_{ij}(S^{-1}(h_{[1]})\gamma_{ij}(h_{[2]})(m)\\
&= \gamma_{ij}(h_{[2]}) \cdot_{\mathrm{op}} \gamma_{ij}(S^{-1}(h_{[1]})) \cdot_{\mathrm{op}} \varphi_{ij}(r)(m).
\end{aligned}$$

Hence, by Theorem 2.35, the map $\Gamma_{ij} : \underline{H^{\mathrm{op}}\#R_j} \to \mathrm{End}\,({}_iM_j)^{\mathrm{op}}$ given by

$$\Gamma_{ij}(h\#r) = \gamma_{ij}(h) \cdot_{\mathrm{op}} \varphi_{ij}(r), \quad h \in H,\ r \in R_j,$$

is an algebra morphism. Thus, ${}_iM_j$ is a right $\underline{H^{\mathrm{op}}\#R_j}$-module. Now for $h \in H$, $m \in {}_iM_j$, and $r \in R_i$, we have

$$\begin{aligned}
(1_i\#h) \cdot m &= \Phi_{ij}(1_i\#h)(m) = (\psi_{ij}(1_i)\pi_{ij}(h))(m)\\
&= \iota_{ij}^{-1}\big(h \rhd \iota_{ij}(m)\big) = \gamma_{ij}(h)(m)\\
&= (\gamma_{ij}(h) \cdot_{\mathrm{op}} \varphi_{ij}(1_j))(m) = \Gamma_{ij}(h\#1_j)(m)\\
&= m \cdot (h\#1_j),
\end{aligned}$$

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and

$$(1_i\#h)\cdot r=\iota_{ii}^{-1}\big(h\rhd\iota_{ii}(r)\big)=h\rightharpoonup_i r.$$

The other two identities of (b) follow directly from the definitions of Φ_{ij} and Γ_{ij}. Finally,

$$\begin{aligned}(1_i\#h)\cdot mn&=\iota_{ik}^{-1}\big(h\rhd\iota_{ik}(mn)\big)\\&=\iota_{ik}^{-1}\big(h\rhd\iota_{ij}(m)\iota_{jk}(n)\big)\\&=\iota_{ik}^{-1}\big((h_{(1)}\rhd\iota_{ij}(m))(h_{(2)}\rhd\iota_{jk}(n))\big)\\&=\iota_{ij}^{-1}(h_{(1)}\rhd\iota_{ij}(m))\iota_{jk}^{-1}(h_{(2)}\rhd\iota_{jk}(n))\\&=\big((1_i\#h_{(1)})\cdot m\big)\big((1_j\#h_{(2)})\cdot n\big),\end{aligned}$$

for all $h\in H$, $m\in {}_iM_j$ and $n\in {}_jM_k$. Hence, (c) is true and the implication (i) $\Rightarrow$ (ii) is proved.
(ii) $\Rightarrow$ (i) Let $h\in H$ and $r=({}_im_j)_{i,j\in\mathbb{I}_n}\in R$. Using that ${}_iM_j$ is a left $\underline{R_i\#H}$-module, we will define the linear map $\rhd: H\otimes R\to R$ by

$$h\rhd r=\big((1_i\#h)\cdot {}_im_j\big)_{i,j\in\mathbb{I}_n}.$$

It is clear that (LPA1) is true. Also, the condition (LPA2) follows directly from (c). Now, consider $h,k\in H$ and $r=\big({}_im_j\big)_{i,j\in\mathbb{I}_n}\in {}_iM_j$. Then

$$\begin{aligned}h\rhd\big(k\rhd r\big)&=h\rhd\Big(\big((1_i\#k)\cdot {}_im_j\big)_{i,j\in\mathbb{I}_n}\Big)\\&=\Big((1_i\#h)\cdot\big((1_i\#k)\cdot {}_im_j\big)\Big)_{i,j\in\mathbb{I}_n}\\&=\Big((1_i\#h)(1_i\#k)\cdot {}_im_j\Big)_{i,j\in\mathbb{I}_n}\\&=\Big(\big(h_{(1)}\rightharpoonup_i 1_i\#h_{(2)}k\big)\cdot {}_im_j\Big)_{i,j\in\mathbb{I}_n},\end{aligned}$$

$$\begin{aligned}(h_{(1)}\rhd 1_R)\big(h_{(2)}k\rhd r\big)&=\operatorname{diag}\left((1_1\#h_{(1)})\cdot 1_1,\dots,(1_n\#h_{(1)})\cdot 1_n\right)(h_{(2)}k\rhd r)\\&\overset{(b)}{=}\operatorname{diag}\left(h_{(1)}\rightharpoonup_1 1_1,\dots,h_{(1)}\rightharpoonup_n 1_n\right)(h_{(2)}k\rhd r)\\&=\Big(\big((h_{(1)}\rightharpoonup_i 1_i)(1_i\#h_{(2)}k)\big)\cdot {}_im_j\Big)_{i,j\in\mathbb{I}_n}\\&=\Big(\big(h_{(1)}\rightharpoonup_i 1_i\#h_{(2)}k\big)\cdot {}_im_j\Big)_{i,j\in\mathbb{I}_n}.\end{aligned}$$

Denote by $D=\operatorname{diag}\left(h_{(2)}\rightharpoonup_j 1_1,\dots,h_{(2)}\rightharpoonup_j 1_n\right)\in R$. Then

$$\begin{aligned}\big(h_{(1)}k\rhd({}_im_j)\big)_{i,j\in\mathbb{I}_n}(h_{(2)}\rhd 1_R)&\overset{(b)}{=}\big((1_i\#h_{(1)}k)\cdot {}_im_j\big)_{i,j\in\mathbb{I}_n}D\\&\overset{(b)}{=}\big({}_im_j\cdot(h_{(1)}k\#1_j)\big)_{i,j\in\mathbb{I}_n}D\\&=\big({}_im_j\cdot(h_{(1)}k\#h_{(2)}\rightharpoonup_j 1_j)\big)_{i,j\in\mathbb{I}_n}\\&=\big({}_im_j\cdot(h_{[2]}k\#h_{[1]}\rightharpoonup_j 1_j)\big)_{i,j\in\mathbb{I}_n}\\&=\big({}_im_j\cdot(k\#1_j)(h\#1_j)\big)_{i,j\in\mathbb{I}_n}\\&\overset{(b)}{=}\big((1_i\#h)(1_i\#k)\cdot {}_im_j\big)_{i,j\in\mathbb{I}_n}\\&=\big((h_{(1)}\rightharpoonup_i 1_i)\#h_{(2)}k\cdot {}_im_j\big)_{i,j\in\mathbb{I}_n},\end{aligned}$$

which implies (LPA3). Thus, (ii) $\Rightarrow$ (i) is proved. □

We provide a couple of remarks related to Theorem 3.3.

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Remark 3.4. *We notice that conditions (b) and (c) of Theorem 3.3 are equivalent to (b) and (c′), where*

$$(\mathrm{c}') \qquad mn \cdot (h\#1_k) = \big(m \cdot (h_{(1)}\#1_j)\big)\big(n \cdot (h_{(2)}\#1_k)\big), \quad \text{for all } h \in H,\ m \in {}_iM_j,\ n \in {}_jM_k.$$

Indeed, assume that (b) and (c) are hold. Then

$$\begin{aligned} mn \cdot (h\#1_k) &= (1_i\#h) \cdot mn \\ &= \big((1_i\#h_{(1)}) \cdot m\big)\big((1_j\#h_{(2)}) \cdot n\big) \\ &= \big(m \cdot (h_{(1)}\#1_j)\big)\big(n \cdot (h_{(2)}\#1_k)\big). \end{aligned}$$

Analogously, it can be verified that (b) and (c′) imply (b) and (c).

Remark 3.5. *Let H be a Hopf algebra acting partially on the generalized matrix algebra $R = ({}_iM_j)_{i,j\in\mathbb{I}_n}$. By Theorem 3.3, ${}_iM_j$ is a left $\underline{R_i\#H}$-module and a right $\underline{H^{op}\#R_j}$-module. However, ${}_iM_j$ is not necessarily a $(\underline{R_i\#H}, \underline{H^{op}\#R_j})$-bimodule. Indeed, since $(1_i\#h) \cdot m = m \cdot (h\#1_j)$, for all $h \in H$ and $m \in {}_iM_j$, then we have*

$$((1_i\#h) \cdot m) \cdot (k\#1_j) = (1_i\#k) \cdot ((1_i\#h) \cdot m) = ((k_{(1)} \rightharpoonup_i 1_i)\#k_{(2)}h) \cdot m,$$

while

$$(1_i\#h) \cdot (m \cdot (k\#1_j)) = (1_i\#h) \cdot ((1_i\#k) \cdot m) = ((h_{(1)} \rightharpoonup_i 1_i)\#h_{(2)}k) \cdot m.$$

Example 3.6. *Let A be a unital algebra and suppose that $e \in A$ is an idempotent such that $A = AeA$. It is easily seen that $(A, eAe, Ae, eA, \mu, \nu)$, where $\mu : Ae \otimes_{eAe} eA \to A$ and $\nu : eA \otimes_A Ae \to eAe$ are the multiplication maps, is a strict Morita context. Thus, we have the generalized matrix algebra*

$$R = \begin{pmatrix} A & Ae \\ eA & eAe \end{pmatrix}.$$

In this case, ${}_1M_1 = R_1 = A$, ${}_1M_2 = Ae$, ${}_2M_1 = eA$, and ${}_2M_2 = R_2 = eAe$. Let H be a Hopf algebra and $\cdot : H \otimes A \to A$ a left partial action of H on A such that $h \cdot e = \varepsilon(h)e$, for all $h \in H$. Then

$$h \cdot (ea) = (h_{(1)} \cdot e)(h_{(2)} \cdot a) = \varepsilon(h_{(1)})e(h_{(2)} \cdot a) = e(h \cdot a),$$

and similarly $h \cdot (ae) = (h \cdot a)e$, for all $a \in A$ and $h \in H$. Thus, $h \cdot (eae) = e(h \cdot a)e \in eAe$, for all $a \in A$, and we have a partial action of H on eAe obtained by restriction. It is straightforward to check that A and Ae are left $\underline{A\#H}$-modules with actions given, respectively, by

$$a\#h \rightharpoonup b = a(h \cdot b), \qquad a, b \in A,\ h \in H, \tag{3.7}$$

$$a\#h \rightharpoonup be = a(h \cdot b)e, \qquad a, b \in A,\ h \in H. \tag{3.8}$$

On the other hand, A is a right $\underline{H^{\mathrm{op}}\#A}$-module and Ae is a right $\underline{H^{\mathrm{op}}\#(eAe)}$-module via the actions given, respectively, by

$$b \leftharpoonup h\#a = (h \cdot b)a, \qquad a, b \in A,\ h \in H^{op}, \tag{3.9}$$

$$be \leftharpoonup h\#eae = (h \cdot b)eae, \qquad a, b \in A,\ h \in H^{op}. \tag{3.10}$$

Observe that

$$1_A\#h \rightharpoonup be = (h \cdot b)e = be \leftharpoonup h\#e, \qquad 1_A\#h \rightharpoonup b = h \cdot b = b \leftharpoonup h\#1_A, \quad b \in A,\ h \in H.$$

Hence, the bimodules ${}_1M_1 = A$ and ${}_1M_2 = Ae$ satisfy the condition (b) of Theorem 3.3. In a similar way, eA and eAe are left $\underline{(eAe)\#H}$-modules with structures

$$eae\#h \rightharpoonup eb = eae(h \cdot b), \qquad a, b \in A,\ h \in H, \tag{3.11}$$

$$eae\#h \rightharpoonup ebe = eae(h \cdot b)e, \qquad a, b \in A,\ h \in H. \tag{3.12}$$

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while eA is a right $\underline{H^{op}\#A}$-module and eAe is a right $\underline{H^{op}\#eAe}$-module with structures

$$eb \leftharpoonup h\#a = e(h \cdot b)a, \qquad a, b \in A,\ k \in H,\ h \in H^{op}, \tag{3.13}$$

$$ebe \leftharpoonup h\#eae = e(h \cdot b)eae, \qquad a, b \in A,\ k \in H,\ h \in H^{op}. \tag{3.14}$$

The bimodules ${}_2M_1 = eA$ and ${}_2M_2 = eAe$ also satisfy the condition (b) of Theorem 3.3. By a direct computation, we can verify that the condition (c) of Theorem 3.3 is valid. Thus, by Theorem 3.3, the linear map $\triangleright : H \otimes R \to R$ given by

$$h \triangleright \begin{pmatrix} a & be \\ ec & ede \end{pmatrix} = \begin{pmatrix} h \cdot a & (h \cdot b)e \\ e(h \cdot c) & e(h \cdot d)e \end{pmatrix}$$

is a partial action of H on the Morita ring R.

4. The group case revisited

In what follows in this section, G is a group and $R = ({}_iM_j)_{i,j\in\mathbb{I}_n}$ is a generalized matrix algebra as in (2.6). It follows from [10] that having a left unital partial action of G on R is the same as having a left partial action of the Hopf algebra $\Bbbk G$ on R. In Section 4 of [7], the authors give sufficient conditions to define a unital partial action of G on R. In what follows we relate these conditions with the conditions (a), (b) and (c) in (ii) of Theorem 3.3. For the convenience of the reader, we will recall some definitions and results presented in [7].

Definition 4.1. *Let n be a positive integer, $R = ({}_iM_j)_{i,j\in\mathbb{I}_n}$ be a generalized matrix algebra, and $\mathcal{I} = \{I_j \subset R_j : I_j$ is an ideal of R_j, for all $j \in \mathbb{I}_n\}$. The (R_i, R_j)-bimodule ${}_iM_j$ will be called $\mathcal{I}$-symmetric if $I_i \cdot {}_iM_j = {}_iM_j \cdot I_j$. When ${}_iM_j$ is $\mathcal{I}$-symmetric for all $i, j \in \mathbb{I}_n$, we say that R is $\mathcal{I}$-symmetric.*

The next Lemma is Corollary 3.8 of [7].

Lemma 4.2. *Let n be a positive integer, $R = ({}_iM_j)_{i,j\in\mathbb{I}_n}$ be a generalized matrix algebra, and $\mathcal{I} = \{I_j \subset R_j : I_j$ is an ideal of R_j, for all $j \in \mathbb{I}_n\}$ be a finite family of ideals. If R is $\mathcal{I}$-symmetric, then*

$$I = \begin{pmatrix} I_{11} & I_{12} & \dots & I_{1n} \\ I_{21} & I_{22} & \dots & I_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ I_{n1} & I_{n2} & \dots & I_{nn} \end{pmatrix}, \quad \text{where } I_{jk} := I_j \cdot {}_jM_k + {}_jM_k \cdot I_k,$$

is an ideal of R.

Assume that, for each $i \in \mathbb{I}_n$, there is a partial action $\alpha^{(i)} = \left(D_g^{(i)}, \alpha_g^{(i)}\right)_{g\in G}$ of G on R_i. Consider $\mathcal{I}_g := \left\{D_g^{(i)} : i \in \mathbb{I}_n\right\}$ and suppose that R is $\mathcal{I}_g$-symmetric, for all $g \in G$. By Lemma 4.2, the subset I_g of R given by

$$I_g = \left(D_g^{(i)}\, {}_iM_j\right)_{i,j\in\mathbb{I}_n}, \tag{4.3}$$

is an ideal of R.

Now we recall the notion of datum for R introduced in Definition 4.2 of [7].

Definition 4.4. *Let $i \in \mathbb{I}_n$ and $g \in G$. Suppose that $\alpha^{(i)}$ and $\mathcal{I}_g$ are as above and that R is $\mathcal{I}_g$-symmetric. For each $g \in G$, assume that there exists a collection of linear bijections*

$$\gamma_g = \left\{\gamma_g^{(ij)} : D_{g^{-1}}^{(i)}\, {}_iM_j \to D_g^{(i)}\, {}_iM_j\right\}_{i,j\in\mathbb{I}_n}.$$

The pair $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ is called a datum for R if the following statements hold: for all $g, h \in G$, $i, j, k \in \mathbb{I}_n$,

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$$\gamma_g^{(ii)} = \alpha_g^{(i)}, \qquad \gamma_e^{(ij)} = id_{M_{ij}}, \tag{4.5}$$

$$\gamma_g^{(ik)}(u)\gamma_g^{(kj)}(v) = \gamma_g^{(ij)}(uv), \quad \textit{for all } u \in D^{(i)}_{g^{-1}\ i}M_k,\ v \in D^{(k)}_{g^{-1}\ k}M_j, \tag{4.6}$$

$$\left(\gamma_h^{(ij)}\right)^{-1}\left(D^{(i)}_{g^{-1}\ i}M_j \cap D^{(i)}_{h\ i}M_j\right) \subset D^{(i)}_{(gh)^{-1}\ i}M_j, \tag{4.7}$$

$$\gamma_g^{(ij)}\left(\gamma_h^{(ij)}(a)\right) = \gamma_{gh}^{(ij)}(a), \quad \textit{for all } a \in \left(\gamma_h^{(ij)}\right)^{-1}\left(D^{(i)}_{g^{-1}\ i}M_j \cap D^{(i)}_{h\ i}M_j\right). \tag{4.8}$$

It was proved in Theorem 4.3 of [7] that there is a partial action of G on R associated with each datum for R.

Remark 4.9. *Let $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ be a datum for R. By Remark 4.10 of [7], for all $g, h \in G$ and $i, j \in \mathbb{I}_n$, we have that*

(i) $\left(\gamma_g^{(ij)}\right)^{-1} = \gamma_{g^{-1}}^{(ij)}$;

(ii) $\left(\gamma_h^{(ij)}\right)^{-1}\left(D^{(i)}_{g^{-1}\ i}M_j \cap D^{(i)}_{h\ i}M_j\right) = D^{(i)}_{(gh)^{-1}\ i}M_j \cap D^{(i)}_{h^{-1}\ i}M_j$;

Remark 4.10. *Let $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ be a datum for R. Assume that $\alpha^{(i)}$ is unital, for all $i \in \mathbb{I}_n$, and denote by $1_g^{(i)}$ the identity element of $D_g^{(i)}$, $g \in G$. For each $g \in G$, consider the diagonal matrix*

$$1_g := \operatorname{diag}\left(1_g^{(1)}, \ldots, 1_g^{(n)}\right) = \begin{pmatrix} 1_g^{(1)} & 0 & \ldots & 0 \\ 0 & 1_g^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1_g^{(n)} \end{pmatrix} \in R. \tag{4.11}$$

It was proved in Lemma 4.5 of [7] that 1_g is a central idempotent of R if and only if

$$1_g^{(i)} m = m 1_g^{(j)}, \quad \text{for all } m \in {}_iM_j \text{ and } i, j \in \mathbb{I}_n. \tag{4.12}$$

In this case, by Lemma 4.5 of [7], we have that $I_g = R1_g$.

Definition 4.13. *A datum $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ for R is called unital, if the ideal of R given in (4.3) is unital with identity element 1_g given by (4.11), for all $g \in G$.*

In order to relate the conditions of a datum for R with the conditions given in (ii) of Theorem 3.3, we consider the following statement:

(a)′ there is a left partial action $\rightharpoonup_i : \Bbbk G \otimes R_i \to R_i$ of $\Bbbk G$ on R_i such that $g \rightharpoonup_i 1_{R_i}$ satisfies (4.12), for all $g \in G$ and $i \in \mathbb{I}_n$.

Proposition 4.14. *Let $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ be a unital datum for R. Then the Hopf algebra $H = \Bbbk G$ satisfies the conditions (a)′ and the conditions (b)–(c) of Theorem 3.3 (ii).*

Proof. Let $i \in \mathbb{I}_n$. Consider the linear map $\rightharpoonup_i : \Bbbk G \otimes R_i \to R_i$ given by

$$g \rightharpoonup_i x = \gamma_g^{(ii)}(x 1^{(i)}_{g^{-1}}), \quad g \in G,\ x \in R_i.$$

Since $\gamma_g^{(ii)} = \alpha_g^{(i)}$, it follows that $\rightharpoonup_i$ is a left partial action of $\Bbbk G$ on R_i and $g \rightharpoonup_i 1_{R_i} = \alpha_g^{(i)}(1^{(i)}_{g^{-1}}) = 1_g^{(i)}$. Hence, (a)′ is proved. To prove (b) of Theorem 3.3, we recall from (2.31) that $R_i \# \Bbbk G \simeq R_i \rtimes_{\alpha_i} G$ as algebras. We define the linear map $\cdot : (R_i \rtimes_{\alpha_i} G) \otimes {}_iM_j \to {}_iM_j$ by

$$(r\delta_g) \cdot m = r\gamma_g^{(ij)}(1^{(i)}_{g^{-1}} m) = r\gamma_g^{(ij)}(m 1^{(j)}_{g^{-1}}), \quad r \in D_g^{(i)},\ g \in G,\ m \in {}_iM_j.$$

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We claim that this linear map defines a left $R_i \rtimes_{\alpha_i} G$-module structure on ${}_iM_j$. Using that $\gamma_e^{(ij)} = \mathrm{id}_{M_{ij}}$ it is immediate that $(1_i\delta_e)\cdot m = m$, for all $m \in {}_iM_j$. Moreover, for all $g, h \in G$, $r \in D_g^{(i)}$ and $s \in D_h^{(i)}$, we have

$$\begin{aligned}
\big((r\delta_g)(s\delta_h)\big)\cdot m &= r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\delta_{gh}\cdot m\\
&= r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})\\
&= r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})1_g^{(i)}\gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})\\
&\overset{(4.12)}{=} r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})1_g^{(j)}\\
&= r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})1_g^{(j)}1_{gh}^{(j)}\\
&\overset{(4.5)}{=} r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})\gamma_{gh}^{(jj)}(1_{h^{-1}}^{(j)}1_{(gh)^{-1}}^{(j)})\\
&\overset{(4.6)}{=} r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)}1_{h^{-1}}^{(j)})\\
&\overset{(4.8)}{=} r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{g}^{(ij)}(\gamma_h^{(ij)}(m1_{(gh)^{-1}}^{(j)}1_{h^{-1}}^{(j)}))\\
&= r\alpha_g^{(i)}(s1_{g^{-1}}^{(i)})\gamma_{g}^{(ij)}(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})1_{g^{-1}}^{(j)})\\
&= r\gamma_g^{(ii)}(s1_{g^{-1}}^{(i)})\gamma_{g}^{(ij)}(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})1_{g^{-1}}^{(j)})\\
&= r\gamma_g^{(ij)}(s\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})1_{g^{-1}}^{(j)})\\
&= (r\delta_g)\cdot((s\delta_h)\cdot m).
\end{aligned}$$

To complete the proof of (b), we need to verify that ${}_iM_j$ is a right $\overline{\Bbbk G^{\mathrm{op}}\#R_j}$-module. By (2.32), $\overline{\Bbbk G^{\mathrm{op}}\#R_j} \simeq (R_j^{\mathrm{op}} \ltimes_{\alpha_j} G)^{\mathrm{op}}$ as algebras. Thus, we will prove that ${}_iM_j$ is a left $(R_j^{\mathrm{op}} \rtimes_{\alpha_j} G)$-module. For that, define the linear map $\cdot : R_j^{\mathrm{op}} \rtimes_{\alpha_j} G \otimes {}_iM_j \to {}_iM_j$ by

$$(r\delta_g)\cdot m = \gamma_g^{(ij)}(m1_{g^{-1}}^{(j)})r, \quad r \in D_g^{(j)},\ g \in G,\ m \in {}_iM_j.$$

Notice that by 4.12 this action is well defined. Also, by (4.5), we have $(1_{R_j}\delta_e)\cdot m = \gamma_e^{(ij)}(m1_e^{(j)})1_{R_j} = \gamma_e^{(ij)}(m1_{R_j})1_{R_j} = m$. Finally, observe that

$$\begin{aligned}
\big((r\delta_g)(s\delta_h)\big)\cdot m &= \alpha_g^{(j)}(s1_{g^{-1}}^{(j)})r\delta_{gh}\cdot m\\
&= \gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})\alpha_g^{(j)}(s1_{g^{-1}}^{(j)})r\\
&= \gamma_{gh}^{(ij)}(m1_{(gh)^{-1}}^{(j)})1_g^{(j)}1_{gh}^{(j)}\alpha_g^{(j)}(s1_{g^{-1}}^{(j)})r\\
&= \gamma_{g}^{(ij)}(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})1_{g^{-1}}^{(j)})\alpha_g^{(j)}(s1_{g^{-1}}^{(j)})r\\
&= \gamma_{g}^{(ij)}(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})s1_{g^{-1}}^{(j)})r,
\end{aligned}$$

and

$$\begin{aligned}
(r\delta_g)\cdot\big((s\delta_h)\cdot m\big) &= (r\delta_g)\cdot\big(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})s\big)\\
&= \gamma_g^{(ij)}\Big(\big(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})s\big)1_{g^{-1}}^{(j)}\Big)r\\
&= \gamma_g^{(ij)}\big(\gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})s1_{g^{-1}}^{(j)}\big)r.
\end{aligned}$$

Given $h \in G$ and $m \in {}_iM_j$, from (2.31) follows that

$$(1_i\#h)\cdot m = (1_h^{(i)}\delta_h)\cdot m = \gamma_h^{(ij)}(m1_{h^{-1}}^{(j)}),$$

and (2.32) implies

$$m\cdot(h\#1_j) = (1_h^{(j)}\delta_h)\cdot m = \gamma_h^{(ij)}(m1_{h^{-1}}^{(j)}).$$

Thus, (b) is proved.

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Now, we will see that (c) holds. In fact, if $n \in {}_jM_k$, then

$$\begin{aligned}(1_i\#h)\cdot mn &= \gamma_h^{(ik)}(mn1_{h^{-1}}^{(k)})\\ &\overset{(4.12)}{=} \gamma_h^{(ik)}(m1_{h^{-1}}^{(j)}n1_{h^{-1}}^{(k)})\\ &\overset{(4.6)}{=} \gamma_h^{(ij)}(m1_{h^{-1}}^{(j)})\gamma_h^{(jk)}(n1_{h^{-1}}^{(k)})\\ &= \big((1_i\#h)\cdot m\big)\big((1_j\#h)\cdot n\big),\end{aligned}$$

which finishes the proof. □

We proceed with the next.

Proposition 4.15. *Assume that $\Bbbk G$ satisfies the conditions (a)′ and the conditions (b)–(c) of Theorem 3.3 (ii). Then:*

- *given $i \in \mathbb{I}_n$, there exists a unital partial action $\alpha^{(i)} = \big(D_g^{(i)}, \alpha_g^{(i)}\big)_{g\in G}$ of G on R_i,*
- *there exists a family of $\Bbbk$-linear bijections $\gamma_g = \Big\{\gamma_g^{(ij)}{:}D_{g^{-1}}^{(i)}\,{}_iM_j \to D_g^{(i)}\,{}_iM_j\Big\}_{i,j\in\mathbb{I}_n}$, for all $g \in G$,*

such that $\mathcal{D} = \big(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\big)$ is a unital datum for R.

Proof. From Example 2.17, we obtain that $\alpha^{(i)} = \big(D_g^{(i)}, \alpha_g^{(i)}\big)_{g\in G}$ is a unital partial action of G on R_i, where

$$D_g^{(i)} = R_i 1_g^{(i)}, \qquad 1_g^{(i)} := g \rightharpoonup_i 1_{R_i}, \qquad \alpha_g^{(i)}(y) = g \rightharpoonup_i y, \quad y \in D_{g^{-1}}^{(i)}.$$

Notice that (4.12) implies that $D_g^{(i)}\,{}_iM_j = {}_iM_j\,D_g^{(j)}$, for all $i, j \in \mathbb{I}_n$ and $g \in G$. Hence, R is $\mathcal{I}_g$-symmetric, where $\mathcal{I}_g = \big\{D_g^{(i)} : i \in \mathbb{I}_n\big\}$. Also, by (b) in (ii) of Theorem 3.3, for each $(i, j) \in \mathbb{I}_n \times \mathbb{I}_n$, we have that ${}_iM_j$ is a left $R_i \rtimes_{\alpha_i} G$-module. Using this left action, we define the following $\Bbbk$-linear morphisms

$$\gamma_g^{(ij)} : D_{g^{-1}}^{(i)}\,{}_iM_j \to D_g^{(i)}\,{}_iM_j, \quad \gamma_g^{(ij)}(x) = (1_g^{(i)}\delta_g)\cdot x, \quad x \in D_{g^{-1}}^{(i)}\,{}_iM_j. \tag{4.16}$$

Notice that $\gamma_g^{(ij)}$ is well defined. In fact, it follows from (b) of Theorem 3.3 (ii) that $r_i\delta_e \cdot m = r_i \cdot m$, for all $r_i \in R_i$ and $m \in {}_iM_j$. Hence, given $x \in D_{g^{-1}}^{(i)}\,{}_iM_j$,

$$(1_g^{(i)}\delta_g)\cdot x = ((1_g^{(i)}\delta_e)\cdot(1_g^{(i)}\delta_g))\cdot x = (1_g^{(i)}\delta_e)\cdot[\big(1_g^{(i)}\delta_g\big)\cdot x] = 1_g^{(i)}\cdot\big((1_g^{(i)}\delta_g)\cdot x\big),$$

which implies that $(1_g^{(i)}\delta_g)\cdot x \in D_g^{(i)}\,{}_iM_j$. Also,

$$\gamma_{g^{-1}}^{(ij)}\gamma_g^{(ij)}(x) = 1_{g^{-1}}^{(i)}\delta_e \cdot x = 1_{g^{-1}}^{(i)}\cdot x = x.$$

Similarly, $\gamma_g^{(ij)}\gamma_{g^{-1}}^{(ij)} = \mathrm{id}_{D_g^{(i)}\,{}_iM_j}$ and consequently $\gamma_g^{(ij)}$ is a $\Bbbk$-linear isomorphism with inverse $\gamma_{g^{-1}}^{(ij)}$. Clearly, $\gamma_e^{(ij)} = \mathrm{id}_{{}_iM_j}$. Moreover, if Φ_{ii}, ψ_{ii} and π_{ii} are the maps defined in the proof of (i) ⇒ (ii) of Theorem 3.3, then

$$\gamma_g^{(ii)}(a) = (1_g\delta_g)\cdot a = \Phi_{ii}(1_g\#g)(a) = (\psi_{ii}(1_g)\circ\pi_{ii}(g))(a) = \alpha_g(a),$$

for all $a \in D_{g^{-1}}^{(i)}$. Hence, (4.5) is proved. In order to prove (4.6), consider $u \in D_{g^{-1}}^{(i)}\,{}_iM_j$ and $v \in D_{g^{-1}}^{(j)}\,{}_jM_k$. We recall that $1_i\#g = 1_i(g \rightharpoonup_i 1_i) \otimes g = 1_g^{(i)}\#g$. Then, it follows from (c) of Theorem 3.3 (ii) that

$$\begin{aligned}\gamma_g^{(ij)}(u)\gamma_g^{(jk)}(v) &= \big((1_g^{(i)}\delta_g)\cdot u\big)\big((1_g^{(j)}\delta_g)\cdot v\big) = \big((1_g^{(i)}\#g)\cdot u\big)\big((1_g^{(j)}\#g)\cdot v\big)\\ &= \big((1_i\#g)\cdot u\big)\big(1_j\#g)\cdot v\big) = (1_i\#g)\cdot(uv)\\ &= (1_g^{(i)}\#g)\cdot(uv) = (1_g^{(i)}\delta_g)\cdot(uv) = \gamma_g^{(ik)}(uv).\end{aligned}$$

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Now, we prove (4.7). Consider $x \in (\gamma_h^{(ij)})^{-1}\left(D^{(i)}_{g^{-1}}\,{}_iM_j \cap D^{(i)}_h\,{}_iM_j\right)$. Then, observe that $y = \gamma_h^{(ij)}(x) \in D^{(i)}_{g^{-1}}\,{}_iM_j \cap D^{(i)}_h\,{}_iM_j$. Since $y = 1^{(i)}_{g^{-1}}1^{(i)}_h y$, it follows that

$$x = \gamma^{(ij)}_{h^{-1}}(y) = \gamma^{(ij)}_{h^{-1}}(1^{(i)}_{g^{-1}}1^{(i)}_h y) = \alpha_{h^{-1}}(1^{(i)}_{g^{-1}}1^{(i)}_h)\gamma^{(ij)}_{h^{-1}}(y) = 1^{(i)}_{(gh)^{-1}}1^{(i)}_{h^{-1}}\gamma^{(ij)}_{h^{-1}}(y),$$

and hence $x \in D^{(i)}_{h^{-1}}\,{}_iM_j \cap D^{(i)}_{(gh)^{-1}}\,{}_iM_j \subseteq D^{(i)}_{(gh)^{-1}}\,{}_iM_j$. Finally, we prove (4.8). Let $a \in \gamma^{(ij)}_{h^{-1}}(D^{(i)}_{g^{-1}}\,{}_iM_j \cap D^{(i)}_h\,{}_iM_j)$. Then

$$\gamma^{(ij)}_g\left(\gamma^{(ij)}_h(a)\right) = (1^{(i)}_g\delta_g)\cdot\left((1^{(i)}_h\delta_h)\cdot a\right) = (1^{(i)}_g 1^{(i)}_{gh}\delta_{gh})\cdot a = 1^{(i)}_g\gamma^{(ij)}_{gh}(a).$$

However, using that $a \in D^{(i)}_{h^{-1}}\,{}_iM_j$, we obtain

$$\begin{aligned}\gamma^{(ij)}_{gh}(a) &= \gamma^{(ij)}_{gh}(1^{(i)}_{h^{-1}}a) = \gamma^{(ii)}_{gh}(1^{(i)}_{h^{-1}}1^{(i)}_{(gh)^{-1}})\gamma^{(ij)}_{gh}(a)\\ &= \alpha^{(i)}_{gh}(1^{(i)}_{h^{-1}}1^{(i)}_{(gh)^{-1}})\gamma^{(ij)}_{gh}(a) = 1^{(i)}_g\gamma^{(ij)}_{gh}(a).\end{aligned}$$

Thus, $\gamma^{(ij)}_g\left(\gamma^{(ij)}_h(a)\right) = \gamma^{(ij)}_{gh}(a)$. Hence, $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ is a unital datum for R. □

We finish this section with the following result.

Theorem 4.17. *The following assertions are equivalent:*

(i) There is a unital datum $\mathcal{D} = \left(\{\alpha^{(i)}\}_{i\in\mathbb{I}_n}, \{\gamma_g\}_{g\in G}\right)$ for R.
(ii) The Hopf algebra $\Bbbk G$ satisfies the conditions (a)′ and the conditions (b)-(c) of Theorem 3.3 (ii).

Proof. It follows directly from Propositions 4.14 and 4.15. □

5. Morita equivalent partial Hopf actions

We start by recalling from [5] the notion of Morita equivalence for partial actions of Hopf algebras.

Definition 5.1. ([5], Definition 69) *Let H be a Hopf algebra and A and B two partial H-module algebras with partial actions $\rightharpoonup_A$ and $\rightharpoonup_B$, respectively. We say that these two partial actions are Morita equivalent if*

(i) there exists a strict Morita context $(A, B, {}_AM_B, {}_BN_A, \tau, \sigma)$ between A and B,
(ii) there exists a partial action of H on the Morita ring $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ such that the restrictions to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and to $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ coincide with $\rightharpoonup_A$ and $\rightharpoonup_B$, respectively.

The next result characterizes Morita equivalent Hopf partial actions.

Proposition 5.2. *Let H be a Hopf algebra and A and B two partial H-module algebras with partial actions $\rightharpoonup_A$ and $\rightharpoonup_B$, respectively. The following statements are equivalent:*

(1) $\rightharpoonup_A$ and $\rightharpoonup_B$ are Morita equivalent,
(2) The following assertions hold:

(i) there exists a strict Morita context (A, B, M, N, μ, ν),
(ii) M is a left $\underline{A\#H}$-module and a right $\underline{H^{\mathrm{op}}\#B}$-module such that

$$(1_A\#h)\cdot m = m\cdot(h\#1_B), \qquad (1_A\#h)\cdot a = h \rightharpoonup_A a,$$
$$(a'\#1_H)\cdot m = a'm, \qquad m\cdot(1_H\#b) = mb,$$

for all $h \in H$, $m \in M$, $a, a' \in A$, $b \in B$,

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(iii) N is a left $\underline{B\#H}$-module and a right $\underline{H^{\mathrm{op}}\#A}$-module such that

$$(1_B\#h)\cdot n = n\cdot(h\#1_A), \qquad (1_B\#h)\cdot b = h\rightharpoonup_B b,$$
$$(b'\#1_H)\cdot n = b'n, \qquad n\cdot(1_H\#a) = na,$$

for all $h\in H$, $n\in N$, $b,b'\in B$, $a\in A$.

Proof. The proof follows applying Theorem 3.3 to the Morita ring $R=\begin{pmatrix} A & M \\ N & B\end{pmatrix}$. □

Now we recall the following notion.

Definition 5.3. *([5], Definition 72) Let H be a Hopf algebra and B be a right partial H-module algebra. A vector space N is a right partial (B,H)-module if N is a right B-module together with a linear map $N\otimes H\to N$, $n\otimes h\mapsto nh$ such that*

(i) $n1_H = n$,
(ii) $((nk)b)h = (n(kh_{(1)}))(b\cdot h_{(2)})$,

for all $h,k\in H$, $b\in B$ and $n\in N$.

We conclude this work by highlighting an inconsistency in the statement of Proposition 73 of [5]. In fact, the authors assume that H partially acts on B from the left. However, since N is a right (B,H)-module, H indeed acts partially on B from the right.

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