COMPOSITIO MATHEMATICA

Iwasawa–Dieudonné theory of function fields

Bryden Cais

Compositio Math. **162** (2026), 1956–1992.

doi: 10.1017/S0010437X26103455

FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865



https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Compositio Math. 162 (2026) 1956–1992
doi:10.1017/S0010437X26103455

Iwasawa–Dieudonné theory of function fields

Bryden Cais

Abstract

Let k be a perfect field of characteristic p and Γ an infinite, first countable pro-p group. We study the behavior of the p-primary part of the 'motivic class group', i.e. the full p-divisible group of the Jacobian, in any Γ-tower of function fields over k that is unramified outside a finite (possibly empty) set of places Σ, and totally ramified at every place of Σ. When $\Sigma = \emptyset$ and Γ is a torsion-free p-adic Lie group, we obtain asymptotic formulae which show that the p-torsion class group schemes grow in a remarkably regular manner. In the ramified setting $\Sigma \neq \emptyset$, we obtain a similar asymptotic formula for the p-torsion in 'physical class groups', i.e. the k-rational points of the Jacobian, which generalizes the work of Mazur and Wiles, who studied the case $\Gamma = \mathbf{Z}_p$.

1. Introduction

Let k be a perfect field of characteristic $p > 0$ and let K be an algebraic function field in one variable over k. Let L/K be a Galois extension, unramified outside a finite (possibly empty) set Σ of places[1] of K, with $\Gamma := \mathrm{Aut}(L/K)$ an infinite pro-p group. We assume that k is algebraically closed in L, that every place in Σ totally ramifies in L, and that Γ admits a countable basis $\Gamma = \Gamma_0 \supsetneq \Gamma_1 \supsetneq \cdots$ of the identity consisting of open normal subgroups.[2] Let X_n be the unique smooth, proper, and geometrically connected curve over k whose function field is L^{Γ_n} (the fixed field of Γ_n), and for $n \geqslant m$ let $\pi_{n,m} : X_n \to X_m$ be the corresponding branched Galois covering of curves. In this way, we obtain a Γ-*tower* of curves and our aim is to understand the growth of the p-primary part of the *motivic class group* as we move up this tower.

More precisely, let $J_n := \mathrm{Pic}^0_{X_n/k}$ be the Jacobian of X_n over k, and for $n \geqslant m$ let $\pi^*_{n,m} : J_m \to J_n$ be the map on Jacobians induced from $\pi_{m,n}$ by Picard functoriality (i.e. pullback of line bundles). The main object of study in this paper is the inductive system of p-divisible groups

$$\mathscr{G}_n := J_n[p^\infty] \tag{1.1}$$

with transition maps $\pi^*_{n,m} : \mathscr{G}_m \to \mathscr{G}_n$ for every $n \geqslant m$. We view each $\mathscr{G}_n$ as the p-primary part of the *motivic class group* J_n of X_n (as opposed to the usual 'physical' class group $\mathrm{Cl}_{X_n} := J_n(k)$

Received 16 January 2023, accepted in final form 2 July 2026, published online 2 October 2026.
2020 Mathematics Subject Classification 11R23, 14L05, 11R58 (primary), 14L15, 14H30 (secondary).
Keywords: Iwasawa theory; p-divisible groups; function fields; class groups; group schemes.

© The Author(s), 2026. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original article is properly cited. *Compositio Mathematica* is © Foundation Compositio Mathematica.

[1] We require that these places are trivial on k, so that they correspond to closed points of the smooth projective algebraic curve over k associated to K. This is automatic if k is finite.

[2] This condition is needed in order to prove that certain derived limits vanish, and is automatic whenever k itself is countable; see Remarks 3.14 and 3.1.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

of k-rational points on J_n), and in the spirit of Iwasawa theory, our aim is to understand the growth, broadly construed, of $\mathscr{G}_n$ as $n \to \infty$.

To do this, we will linearize the inherently geometric objects $\mathscr{G}_n$ by passing to their (contravariant) Dieudonné modules or, equivalently, by [MM74], we will study the first crystalline cohomology groups of the curves X_n; by Dieudonné theory, this passage incurs no loss of information, as any p-divisible group may be functorially recovered from its Dieudonné module. Since k is perfect, for each n there is a functorial decomposition of p-divisible groups

$$\mathscr{G}_n = \mathscr{G}_n^{\text{ét}} \times_k \mathscr{G}_n^{\text{m}} \times_k \mathscr{G}_n^{\text{ll}}$$

into étale (reduced with local dual), multiplicative (local with reduced dual), and local–local components. For $\star \in \{\text{ét}, \text{m}, \text{ll}\}$, we introduce the *Iwasawa–Dieudonné module*

$$\mathbf{D}^\star := \varprojlim_n \mathbf{D}(\mathscr{G}_n^\star), \tag{1.2}$$

with the projective limit taken via the functorially induced transition maps. This is naturally a (left) topological module over the completed group algebra

$$\Lambda := \varprojlim_n \Lambda_n \quad \text{for } \Lambda_n := W[\Gamma/\Gamma_n]$$

with $W = W(k)$ the ring of Witt vectors of k. The Frobenius automorphism σ of W induces an automorphism of Λ that acts trivially on Γ, which we again denote by σ. Each $\mathbf{D}^\star$ comes equipped with continuous, additive maps $F, V : \mathbf{D}^\star \to \mathbf{D}^\star$ (Frobenius and Verschiebung) that are semilinear over σ and σ^{-1}, respectively, and satisfy $FV = VF = p$. In this paper, we will analyze the Λ-module structure of these Iwasawa–Dieudonné modules, proving in all but one case that they are finitely generated and very nearly free modules satisfying *control* in the sense that each finite level $\mathbf{D}(\mathscr{G}_n^\star)$ may be recovered from the corresponding Iwasawa–Dieudonné module.

Our work generalizes that of Mazur and Wiles [MW83] and Crew [Cre84], who in effect[3] studied $\mathbf{D}^{\text{ét}}$ in the case of ramified (i.e. $\Sigma \neq \emptyset$) $\Gamma = \mathbf{Z}_p$-towers and used their work to prove that, when k is finite, the $\mathbf{Z}_p[\![\Gamma]\!]$-module $\varprojlim_n \operatorname{Hom}_{\mathbf{Z}_p}(\operatorname{Cl}_{X_n}[p^\infty], \mathbf{Q}_p/\mathbf{Z}_p)$ of Pontryagin duals of p-primary components of 'physical' class groups is finitely generated and torsion; Iwasawa's celebrated formula $|\operatorname{Cl}_{X_n}[p^\infty]| = p^{\mu p^n + \lambda n + \nu}$ (for $n \gg 0$) follows. For *arbitrary* Γ, we obtain analogous structure and control theorems for $\mathbf{D}^{\text{ét}}$ and $\mathbf{D}^{\text{m}}$ in the ramified case, and for all three of $\mathbf{D}^{\text{ét}}, \mathbf{D}^{\text{m}}$, and $\mathbf{D}^{\text{ll}}$ in the case $\Sigma = \emptyset$ of everywhere unramified Γ-towers. In the particular case of the Igusa tower (wherein $\Gamma = \mathbf{Z}_p$), the Λ-modules $\mathbf{D}^{\text{ét}}$ and $\mathbf{D}^{\text{m}}$ were first studied in [Cai18b], and play a central role in the author's work [Cai18a, Cai18b] on geometric Hida theory.

In a different, but intimately connected, direction, Wan has initiated the study of zeta functions in Γ-towers of curves over finite fields. In [DWX16] and [RWX+18], he and his collaborators prove that for certain kinds of ramified $\Gamma = \mathbf{Z}_p^d$-towers with $X_0 = \mathbf{P}^1$ and $\Sigma = \{\infty\}$, the Newton slopes of the zeta functions of the curves X_n behave in a remarkably regular manner as $n \to \infty$ (see also [Xia18] and [Ren19]). These slopes are also the slopes of the F-isocrystal $\mathbf{D}(\mathscr{G}_n)[1/p]$, so by Dieudonné theory and the Dieudonné–Manin classification of isocrystals, they determine $\mathscr{G}_n$ up to isogeny. In this way, Wan's program can be interpreted as providing a beautiful analogue of classical Iwasawa theory for the *isogeny types* of the p-divisible groups $\mathscr{G}_n$ in these $\Gamma = \mathbf{Z}_p^d$-towers. However, *isogeny type* loses touch with torsion phenomena, which the Iwasawa–Dieudonné modules do capture. In addition to allowing arbitrary Γ as in the introduction, the results of this paper also apply in the case of arbitrary perfect k, whereas any study of zeta functions in Γ-towers of

[3]Neither Mazur and Wiles nor Crew explicitly work with Dieudonné modules, instead phrasing their work in terms of contravariant Tate modules, which capture only the étale parts of the p-divisible groups.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

curves would seem to be limited to *finite* base fields.[4] In particular, we are able to analyze *étale* Γ-towers (see Remark 1.2) and obtain rather complete results in this case. Using these results, when Γ is a torsion-free p-adic Lie group, we obtain an asymptotic formula (Corollary 1.4) which shows that the local–local p-torsion *class group schemes* $\mathscr{G}_n^{\mathrm{ll}}[p]$ grow in a remarkably regular manner, and provides a fundamentally new kind of Iwasawa-theoretic result for unramified p-adic Lie extensions. We also give a new proof of Shafarevich's theorem [Sha47, Teopema 2] (cf. [Cre84, Theorem 1.9]) on the structure of the maximal pro-p quotient of the étale fundamental group of a smooth and proper curve over an algebraically closed field of characteristic p.

When $\Sigma \neq \emptyset$, the very nature of wild ramification forces the local–local p-divisible groups $\mathscr{G}_n^{\mathrm{ll}}$ to grow rapidly (see [GK88]), and using work of Wintenberger [Win79], we prove that when Γ is a p-adic Lie group, the local–local Iwasawa–Dieudonné module $\mathbf{D}^{\mathrm{ll}}$ is *not* finitely generated as a Λ-module. In fact, the situation is even worse: as F and V are topologically nilpotent on $\mathbf{D}^{\mathrm{ll}}$, it is naturally a module over the 'Iwasawa–Dieudonné' algebra $\Lambda[\![F, V]\!]$ of formal power series in commuting variables F, V satisfying $FV = p$, $F\lambda = \sigma(\lambda)F$, and $V\lambda = \sigma^{-1}\lambda V$, for $\lambda \in \Lambda$; using [BC20], we prove that $\mathbf{D}^{\mathrm{ll}}$ is not finitely generated as a module over *this* ring either! Nevertheless, in [BC23], Booher and the author investigate the structure of the F-torsion group schemes $\mathscr{G}_n^{\mathrm{ll}}[F]$ in certain $\Gamma = \mathbf{Z}_p$-towers with $X_0 = \mathbf{P}^1$ and $\Sigma = \{\infty\}$. Based on extensive computational evidence, and in perfect analogy with the asymptotic formula of Corollary 1.4 in the unramified case, we formulate precise conjectures (e.g. [BC23, Conjecture 1.2]) which posit an Iwasawa-style formula for the isomorphism type of the F-torsion group schemes $\mathscr{G}_n^{\mathrm{ll}}[F]$ in such towers. In [BCK+24], Booher, Kramer-Miller, Upton, and the author use ideas from Dwork theory to establish the main conjectures of [BC23] in many situations. These results appear to lie outside the scope of classical Iwasawa-theoretic methods (i.e. structure theory of Λ-modules), as the relevant Λ-modules are not finitely generated.

Using our structure and control theorems for $\mathbf{D}^{\text{ét}}$, we prove that when k is finite and Γ is a d-dimensional, torsion-free p-adic Lie group equipped with its lower central p-series filtration $\{\Gamma_n\}$, the p-torsion of 'physical' class groups in any ramified Γ-tower of curves $\{X_n\}$ satisfies an Iwasawa-style asymptotic formula (for the precise statement, see Corollary 1.5). When $\Gamma = \mathbf{Z}_p^d$, the analogue of our formula for Γ-extensions of number fields has been proved by Monsky [Mon83b], using the results [CM81]. The proofs of both Monsky's formula and ours ultimately rest on knowing that the relevant 'p-torsion' Iwasawa module is finitely generated, and are similar in spirit. In the abelian case $\Gamma = \mathbf{Z}_p^d$, it is possible to obtain an exact asymptotic formula, as Monsky does, by appealing to the theory of Hilbert–Kunz multiplicity initiated in [Mon83a]. For general Γ, we obtain only asymptotic upper and lower bounds, of the same order of growth but with possibly different constants, by appealing to [EP20, Proposition 2.18], which relies on Venjakob's influential work [Ven02] on the structure of Iwasawa algebras of p-adic Lie groups.

In the number-field setting, and when $\Gamma = \mathbf{Z}_p^d$, Cuoco [Cuo80] (for $d = 2$), Cuoco and Monsky [CM81] and later Monsky [Mon86a, Mon86b, Mon87, Mon89] (for general d) have established increasingly more refined asymptotic formulae for the order of the full p-primary subgroup of the class group. As noted by Li and Zhao [LZ97], their methods, which again ultimately rest upon the structure theory of Iwasawa modules and commutative algebra, carry over *mutatis mutandis* to the function-field case. Using an entirely different approach via L-functions, Wan has recently established a beautiful *exact* formula [Wan19, Theorem 1.3] for the order of the full p-primary subgroup of Cl_{X_n} in any $\Gamma = \mathbf{Z}_p^d$ tower of function fields over a finite field; in the

[4] Indeed, when k is not finite, it is not clear how to define the Hasse–Weil zeta function of a curve over k, as neither the usual Dirichlet series encoding point counts over finite extensions of k nor the cohomological definition via ℓ-adic étale or crystalline cohomology make sense in this context.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

number-field setting, the analogous exact formula has been conjectured by Greenberg (stated in [CM81, p. 236]), and remains open. Wan also conjectures [Wan19, Conjecture 5.1(2)] an analogous exact formula for the order of the p-primary subgroup of the class group in *arbitrary* (i.e. not necessarily abelian) Γ-extensions of a function field over a finite field, and our Corollary 1.5 provides new evidence for Wan's conjecture.

In order to state our main results precisely, we must first fix some notation. When $\Sigma \neq \emptyset$, we set $S := \Sigma$, and when $\Sigma = \emptyset$, we choose a closed point x_0 of X_0 and put $S := \{x_0\}$. For each n, we write $S_n := (\pi_{n,0}^{-1} S)_{\mathrm{red}}$ for the reduced scheme underlying the (scheme-theoretic) fiber of $\pi_{n,0} : X_n \to X_0$, so $S_0 = S$. We view S_n as a *modulus* on X_n, and we analyze the modules (1.2) by first studying their analogues for the 'motivic' *ray class groups* of conductor S_n. We form the generalized Jacobian J_{n,S_n} of X_n with modulus S_n which, as S_n is étale, is an extension of the usual Jacobian J_n by a torus. The inductive system $\mathscr{G}_{n,S_n} := \{J_{n,S_n}[p^n]\}_{n \geqslant 1}$ of p-power torsion group schemes is then a p-divisible group, and for each n we have two extensions of p-divisible groups

$$0 \longrightarrow \mathscr{T}_n \longrightarrow \mathscr{G}_{n,S_n} \longrightarrow \mathscr{G}_n \longrightarrow 0, \tag{1.3a}$$

$$0 \longrightarrow \mathscr{G}_n \longrightarrow \mathscr{G}_{n,S_n}^{\vee} \longrightarrow \mathscr{Q}_n \longrightarrow 0, \tag{1.3b}$$

in which $\mathscr{T}_n$ (respectively, $\mathscr{Q}_n$) is a group of multiplicative (respectively, étale) type, i.e. isomorphic to a finite number of copies of μ_{p^∞} (respectively, $\mathbf{Q}_p/\mathbf{Z}_p$) over $\overline{k}$. These extensions are exchanged by duality, using the canonical autoduality of $\mathscr{G}_n$ coming from the principal polarization on each Jacobian. We then form the Iwasawa–Dieudonné modules

$$\mathbf{D}_S^\star := \varprojlim_{n,\pi^*} \mathbf{D}(\mathscr{G}_{n,S_n}^\star) \text{ for } \star \in \{\mathrm{m}, \mathrm{ll}\} \quad \text{and} \quad \mathbf{D}_S^{\text{ét}} := \varprojlim_{n,\pi_*^\vee} \mathbf{D}((\mathscr{G}_{n,S_n}^\vee)^{\text{ét}}), \tag{1.4}$$

using Picard functoriality for the first, and Albanese functoriality and duality for the second. In this way, $\mathbf{D}_S^\star$ is naturally a (left) topological Λ-module with continuous, semilinear actions of F and V. As ramification in any p-group cover of curves in characteristic p is necessarily *wild*, the ramified case $\Sigma \neq \emptyset$ and unramified case $\Sigma = \emptyset$ are *radically* different, and in some ways our results in the unramified case are more satisfying, so we describe them first.

Theorem A. *Assume that $\Sigma = \emptyset$ and let g and γ be the genus and p-rank of X_0, respectively.*

(i) *The Λ-module $\mathbf{D}_S^\star$ is free of rank γ for $\star = \text{ét}, \mathrm{m}$ and free of rank $2(g-\gamma)$ for $\star = \mathrm{ll}$.*

(ii) *The canonical projection maps yield isomorphisms of Λ_n-modules*

$$\Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^\star \simeq \mathbf{D}(\mathscr{G}_{n,S_n}^\star) \text{ for } \star \in \{\mathrm{m}, \mathrm{ll}\} \quad \text{and} \quad \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^{\text{ét}} \simeq \mathbf{D}((\mathscr{G}_{n,S_n}^\vee)^{\text{ét}})$$

for all n, compatibly with F and V.

(iii) *There are canonical, Λ-bilinear perfect pairings*

$$(\cdot,\cdot) : \mathbf{D}_S^{\mathrm{m}} \times \mathbf{D}_S^{\text{ét}} \to \Lambda \quad \text{and} \quad (\cdot,\cdot) : \mathbf{D}_S^{\mathrm{ll}} \times \mathbf{D}_S^{\mathrm{ll}} \to \Lambda$$

with respect to which F and V are adjoint, and which identify each of $\mathbf{D}_S^{\text{ét}}$ and $\mathbf{D}_S^{\mathrm{m}}$ with the Λ-dual (Definition 2.6) of the other, and $\mathbf{D}_S^{\mathrm{ll}}$ with its own Λ-dual.

From Theorem A, we deduce analogous structure and control theorems for the Iwasawa–Dieudonné modules $\mathbf{D}^\star$. Here and in what follows, we put $I_n := \ker(\Lambda \twoheadrightarrow \Lambda_n)$, and set $I := I_0$.

Theorem B. *With the notation and hypotheses of Theorem A, we have the following.*

(i) *There is a canonical isomorphism of Λ-modules with F- and V-action*

$$\mathbf{D}^{\mathrm{ll}} \simeq \mathbf{D}_S^{\mathrm{ll}}.$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

In particular, $\mathbf{D}^{\mathrm{ll}}$ *is a free, self-dual* Λ*-module of rank* $2(g-\gamma)$*. For each* n*, the projection maps yield canonical isomorphisms of* Λ_n*-modules with* F*- and* V*-action*

$$\Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\mathrm{ll}} \simeq \mathbf{D}(\mathscr{G}_n^{\mathrm{ll}}).$$

(ii) *Assume that the finite étale* k*-scheme* $S_n = \pi_{n,0}^{-1}(S)$ *splits completely over* k *for all* $n \geqslant 0$*. Then there are canonical short exact sequences of* Λ*-modules with* F*- and* V*-action*

$$0 \longrightarrow \Lambda \longrightarrow \mathbf{D}_S^{\text{ét}} \longrightarrow \mathbf{D}^{\text{ét}} \longrightarrow 0,$$
$$0 \longrightarrow \mathbf{D}^{\mathrm{m}} \longrightarrow \mathbf{D}_S^{\mathrm{m}} \longrightarrow I \longrightarrow \mathbf{D}^{\mathrm{m},1} \longrightarrow 0,$$

where $\mathbf{D}^{\mathrm{m},1} = \varprojlim_n{}^1 \mathbf{D}_n^{\mathrm{m}}$*. Here,* $F = \sigma$ *and* $V = p\sigma^{-1}$ *on* Λ *and* $F = p\sigma$ *and* $V = \sigma^{-1}$ *on* I*.*

(iii) *Under the hypothesis of (ii), for each* n *there are canonical isomorphisms of* Λ_n*-modules*

$$\Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\text{ét}} \simeq \mathbf{D}(\mathscr{G}_n^{\text{ét}}) \quad \text{and} \quad \mathrm{Tor}_1^{\Lambda}(\Lambda_n, \mathbf{D}^{\text{ét}}) \simeq W.$$

In general, the derived limit $\mathbf{D}^{\mathrm{m},1}$ appearing in the second exact sequence of Theorem B(ii) seems rather mysterious; as a result, the kernel and cokernel of the canonical map $\Lambda_n \otimes_\Lambda \mathbf{D}^{\mathrm{m}} \to \mathbf{D}_n^{\mathrm{m}}$ are likewise mysterious (for more discussion, see Remark 3.15). However, when k is algebraically closed and L/K is the maximal unramified p-extension, we prove that both $\mathbf{D}^{\mathrm{m}}$ and $\mathbf{D}^{\mathrm{m},1}$ vanish, and thereby give a new proof of Shafarevich's theorem [Sha47, Teopema 2] on the structure of the maximal pro-p quotient of the étale fundamental group of a projective curve over an algebraically closed field of characteristic $p > 0$.

Corollary 1.1. *With the notation and hypotheses of Theorem A, assume that* k *is algebraically closed and that* Γ *is the maximal pro-*p *quotient of* $\pi_1^{\text{ét}}(X_0)$*. Then* $\mathbf{D}^{\mathrm{m}} = 0 = \mathbf{D}^{\mathrm{m},1}$*, and* I *is a free* Λ*-module of rank* γ*; in particular,* Γ *is a free pro-*p *group on* γ *generators.*

Remark 1.2. The existence of an everywhere unramified Γ-extension L/K with Γ an infinite pro-p group forces k to be rather large. For example, if k is finitely generated (as a field) over $\mathbf{F}_p$ there are *no* geometric, everywhere unramified, infinite abelian extensions L/K whatsoever! See Theorem 2 (and cf. Theorem 5) of [KL81]. On the other hand, if k is algebraically closed, then by Shafarevich's theorem, if Γ is *any* pro-p group that can be (topologically) generated by at most γ generators, then there exists an everywhere unramified Γ-extension of K.

Remark 1.3. The hypothesis that S_n split completely over k in (ii)–(iii) of Theorem B is needed in order to obtain the relatively simple descriptions of $\ker(\mathbf{D}_S^{\text{ét}} \to \mathbf{D}^{\text{ét}})$ and $\mathrm{coker}(\mathbf{D}^{\mathrm{m}} \to \mathbf{D}_S^{\mathrm{m}})$ implicit in (ii); without it, these Λ-modules depend on the finer arithmetic properties of the étale schemes S_n, and seem to be considerably harder to describe. This hypothesis may at first appear somewhat restrictive, but in practice it is rather innocuous. It is certainly verified if k is algebraically closed. For general k, if the chosen point x_0 is k-rational and K admits an everywhere unramified geometric Γ-extension L/K with Γ an infinite abelian pro-p group, then there is such an extension L'/K with $L' \otimes_k \overline{k} \simeq L \otimes_k \overline{k}$ in which x_0 is split completely (see [KL81, § 0]).

Recall that the *order* $|G|$ of a finite k-group scheme G is the k-dimension of its Hopf algebra. If $\mathscr{G}$ is a p-divisible group over k, and J is any proper ideal of $\mathbf{Z}_p[\![F, V]\!]$ containing p, we define

$$\mathscr{G}[J] := \bigcap_{f \in J} \ker(f : \mathscr{G} \to \mathscr{G}) \quad \text{(scheme-theoretic intersection)}.$$

This is a finite k-subgroup scheme of $\mathscr{G}[p]$, and so, in particular, is killed by p. As a consequence of Theorem B(i), for any such J, we deduce an Iwasawa-style formula for the orders $|\mathscr{G}_n^{\mathrm{ll}}[J]|$ of J-torsion local–local group schemes in unramified p-adic Lie extensions.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

COROLLARY 1.4. *Let Γ be a p-adic Lie group of dimension d without p-torsion, and let $\{X_n\}$ be an étale Γ-tower of curves with X_n corresponding to the nth subgroup in the lower central p-series of Γ (Definition 2.2). Let J be a proper ideal of $\mathbf{Z}_p[\![F,V]\!]$ containing p, and write $\delta \in [0,d]$ for the dimension of the $\Lambda/p\Lambda$-module $\mathbf{D}^{\mathrm{ll}}/J\mathbf{D}^{\mathrm{ll}}$ in the sense of [Ven02, Definition 3.1]. Then there exist real constants $\nu \geqslant \mu \geqslant (1/\delta!)$ such that*

$$\mu p^{\delta n} + O(p^{(\delta-1)n}) \leqslant \log_p |\mathscr{G}_n^{\mathrm{ll}}[J]| \leqslant \nu p^{\delta n} + O(p^{(\delta-1)n}).$$

If $\Gamma = \mathbf{Z}_p^d$ is abelian, we may, moreover, take $\nu = \mu$.

When $J=(p)$, $J=(F)$, or $J=(V)$, it is straightforward to deduce an *exact* formula for $|\mathscr{G}_n^{\mathrm{ll}}[J]|$ using the Riemann–Hurwitz and Deuring–Shafarevich formulae (see Remark 3.16). For other choices of J, the group schemes $\mathscr{G}_n^{\mathrm{ll}}[J]$ are much more mysterious; e.g., the integer $a_n := \log_p |\mathscr{G}_n^{\mathrm{ll}}[(F,V)]|$ is called the *a-number* of X_n, and is a subtle numerical invariant that has been studied extensively in many different contexts [WK86, Vol88, KW88, Re01, EP07, Elk11, FGM+13, DF14, MS18, Fre18, BC20]. Examples show that there can be no exact analogue of the Riemann–Hurwitz or Deuring–Shafarevich formulae for the growth of a-numbers in (branched) Galois coverings of curves [BC20, Example 7.2], so the existence of an asymptotic formula for the a_n is surprising. In this way, Corollary 1.4 provides a fundamentally new kind of Iwasawa-theoretic result for p-adic Lie extensions of function fields.

Let us now turn to the ramified case $\Sigma \neq \emptyset$. Here, our hypothesis that every point of $\Sigma = S_0$ totally ramifies in X_n means that $\pi_{n,0}$ induces an isomorphism of finite étale k-schemes $S_n \simeq S_0$ for all n, and we will henceforth make this identification, writing S for this common k-scheme. These identifications then induce isomorphisms $\mathscr{T}_n \simeq \mathscr{T}_0$ and $\mathscr{Q}_n \simeq \mathscr{Q}_0$ for all n, and we will similarly write $\mathscr{T}_S$ and $\mathscr{Q}_S$ for these common p-divisible groups. We will likewise simply write $\mathscr{G}_{n,S}$ in place of $\mathscr{G}_{n,S_n}$.

THEOREM C. *Assume that $\Sigma \neq \emptyset$ and let $d := \gamma + \deg S - 1$, where γ is the p-rank of X_0.*

(i) *Both of $\mathbf{D}_S^{\mathrm{m}}$ and $\mathbf{D}_S^{\text{ét}}$ are free Λ-modules of rank d.*

(ii) *The canonical projection maps yield isomorphisms of Λ_n-modules*

$$\Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^{\mathrm{m}} \simeq \mathbf{D}(\mathscr{G}_{n,S}^{\mathrm{m}}) \quad \text{and} \quad \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^{\text{ét}} \simeq \mathbf{D}((\mathscr{G}_{n,S}^{\vee})^{\text{ét}})$$

for all n, compatibly with F and V.

(iii) *There is a canonical, Λ-bilinear perfect pairing*

$$(\cdot,\cdot) : \mathbf{D}_S^{\mathrm{m}} \times \mathbf{D}_S^{\text{ét}} \to \Lambda$$

with respect to which F and V are adjoint, and which identifies each of $\mathbf{D}_S^{\text{ét}}$ and $\mathbf{D}_S^{\mathrm{m}}$ with the Λ-dual of the other.

Analogously to Theorem B, we have the following theorem.

THEOREM D. *With the notation and assumptions of Theorem C, we have the following.*

(i) *There are a canonical isomorphism of Λ-modules*

$$\mathbf{D}^{\text{ét}} \simeq \mathbf{D}_S^{\text{ét}}$$

and a canonical short exact sequence of Λ-modules

$$0 \longrightarrow \mathbf{D}^{\mathrm{m}} \longrightarrow \mathbf{D}_S^{\mathrm{m}} \longrightarrow \mathbf{D}(\mathscr{T}_S) \longrightarrow 0$$

that are compatible with F and V. In particular, $\mathbf{D}^{\text{ét}}$ is free of rank d over Λ.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

(ii) *For each n, there are canonical short exact sequences of Λ_n-modules with F- and V-action*

$$0 \longrightarrow \mathbf{D}(\mathscr{Q}_S) \longrightarrow \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\text{ét}} \longrightarrow \mathbf{D}(\mathscr{G}_n^{\text{ét}}) \longrightarrow 0,$$

$$0 \longrightarrow \frac{I_n}{I \cdot I_n} \underset{W}{\otimes} \mathbf{D}(\mathscr{T}_S) \longrightarrow \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\text{m}} \longrightarrow \mathbf{D}(\mathscr{G}_n^{\text{m}}) \longrightarrow 0.$$

(iii) *There are canonical isomorphisms of Λ-modules with F- and V-action*

$$\mathbf{D}(\mathscr{Q}_S) \simeq \operatorname{coker}\Big(W \xrightarrow{\Delta} \bigoplus_{s\in S} W(k(s))\Big) \quad \text{and} \quad \mathbf{D}(\mathscr{T}_S) \simeq \ker\Big(\bigoplus_{s\in S} W(k(s)) \xrightarrow{\sum \operatorname{tr}_{W(k(s))/W}} W\Big),$$

with $W(k(s))$ viewed as a Λ-module via the augmentation map $\Lambda \twoheadrightarrow W \hookrightarrow W(k(s))$. In the first (respectively, second) identification, $F = \sigma$ (respectively, $F = p\sigma$) on $W(k(s))$, and $V = p\sigma^{-1}$ (respectively, $V = \sigma^{-1}$).

In the spirit of Iwasawa theory, and in analogy with Theorem B, one might expect (hope?) that $\mathbf{D}^{\text{ll}}$ is finitely generated over Λ in the ramified setting as well. Unfortunately, this turns out *not* to be the case. In fact, the influence of wild ramification is so severe that the situation is even worse: Noting that F and V are topologically nilpotent on $\mathbf{D}^{\text{ll}}$, we may consider the Λ-module $\mathbf{D}^{\text{ll}}$ as a left module over the larger 'Iwasawa–Dieudonné' ring $\Lambda[\![F, V]\!]$, and in rather general situations it is not finitely generated over *this* ring either.

THEOREM E. *If Γ is a p-adic Lie group and $\Sigma \neq \emptyset$, the $\Lambda[\![F, V]\!]$-module $\mathbf{D}^{\text{ll}}$ is not finitely generated.*

To relate our work to the growth of p-primary components of 'physical' class groups in a given Γ-tower $\{X_n\}$, we write $(\cdot)^* := \operatorname{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ for the Pontryagin dual, and form the $\mathbf{Z}_p[\![\Gamma]\!]$-module

$$M := \varprojlim_n \operatorname{Cl}_{X_n}[p^\infty]^* = \varprojlim_n \mathscr{G}_n^{\text{ét}}(k)^* = \varprojlim_n \mathscr{G}_n(k)^*,$$

with transition maps the Pontryagin duals of the pullback maps π^* on p-divisible groups. We write $M_W := \Lambda \otimes_{\mathbf{Z}_p[\![\Gamma]\!]} M = W \otimes_{\mathbf{Z}_p} M$ for the (left) Λ-module obtained by extension of scalars.

THEOREM F. *Assume that $|k| = p^r$ and set $\wp := 1 - F^r$. Then there is a canonical short exact sequence of Λ-modules*

$$0 \longrightarrow \mathbf{D}^{\text{ét}} \xrightarrow{\wp} \mathbf{D}^{\text{ét}} \longrightarrow M_W \longrightarrow 0.$$

For each n, there are canonical short exact sequences of Λ_n-modules

$$0 \longrightarrow W \underset{\mathbf{Z}_p}{\otimes} \mathscr{Q}_S(k)^* \longrightarrow \Lambda_n \underset{\Lambda}{\otimes} M_W \longrightarrow W \underset{\mathbf{Z}_p}{\otimes} \mathscr{G}_n(k)^* \longrightarrow 0.$$

COROLLARY 1.5. *Let Γ be a torsion-free p-adic Lie group of dimension d and let $\{X_n\}$ be a ramified (i.e. $\Sigma \neq \emptyset$) Γ-tower of curves over a finite field k, with X_n corresponding to the nth subgroup in the lower central p-series of Γ. Let $\delta \in [0, d]$ be the dimension of the $\mathbf{F}_p[\![\Gamma]\!]$-module M/pM. Then there exist real constants $\nu \geqslant \mu \geqslant 1/\delta!$ with*

$$\mu p^{\delta n} + O(p^{(\delta-1)n}) \leqslant \log_p |\operatorname{Cl}_{X_n}[p]| \leqslant \nu p^{\delta n} + O(p^{(\delta-1)n}).$$

If $\Gamma = \mathbf{Z}_p^d$ is abelian, we may, moreover, take $\nu = \mu$.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

2. Preliminaries

2.1 Iwasawa algebras and modules

Here, we record some basic facts and notation for completed group rings and modules over them, using [NSW08] and [Ven02] as our guides. Throughout, we fix an infinite pro-p group Γ equipped with a countable basis $\Gamma = \Gamma_0 \supsetneq \Gamma_1 \supsetneq \cdots$ of the identity consisting of open normal subgroups. As in § 1, we write $W := W(k)$ for the Witt ring of k.

DEFINITION 2.1. For each n, we give $\Lambda_n := W[\Gamma/\Gamma_n]$ (respectively, $\Omega_n := k[\Gamma/\Gamma_n]$) the p-adic (respectively, discrete) topology, and equip

$$\Lambda := \varprojlim_n W[\Gamma/\Gamma_n] \quad \text{and} \quad \Omega := \varprojlim_n \Omega_n$$

with the inverse limit topology. We set $I_n := \ker(\Lambda \twoheadrightarrow \Lambda_n)$, and simply write $I := I_0$ for the augmentation ideal of Λ.

The I_n form a decreasing chain of two-sided, closed ideals of Λ, and each I_n is generated (as either a left or right ideal) by all expressions of the form $\gamma - 1$ with $\gamma \in \Gamma_n$. Furthermore, the collection of ideals $\{p^m\Lambda + I_n\}_{n,m\geqslant 0}$ forms a fundamental system of neighborhoods of $0 \in \Lambda$.

DEFINITION 2.2. Assume that Γ is topologically finitely generated. The *lower central p-series of* Γ is defined recursively by

$$P_0 := \Gamma \quad \text{and} \quad P_{n+1} := P_n^p[P_n, \Gamma],$$

where $P_n^p := \langle g^p \ : \ g \in P_n\rangle$ is the subgroup of P_n generated by all pth powers, and $[P_n, \Gamma]$ is the commutator subgroup of Γ (see [DdSM+99, Definition 1.15 and Corollary 1.20]).

We note that the P_n form a basis of the identity in Γ consisting of open normal subgroups [DdSM+99, Proposition 1.16].

LEMMA 2.3. *Assume that Γ is topologically finitely generated and let $\Gamma_n := P_n$ for all n. Then $pI_n \subseteq I_nI + I_{n+1}$.*

Proof. As a Λ-module, I_n is generated by elements of the form $u := x - 1$ with $x \in \Gamma_n$, so it suffices to prove that pu lies in $I_nI + I_{n+1}$. We have

$$x^p - 1 = (1 + u)^p - 1 = pu + u^2w$$

for some $w \in \Lambda$. Now, $x^p \in \Gamma_{n+1}$, so $x^p - 1 \in I_{n+1}$ and clearly $u^2w \in I_n^2 \subseteq I_nI$; the result follows. □

LEMMA 2.4. *The ring Λ is local, with unique maximal ideal $p\Lambda + I$. In particular, for every n, the quotient Λ_n is local with unique maximal ideal $p\Lambda_n + I_{\Gamma/\Gamma_n}$.*

Proof. This follows immediately from [NSW08, Proposition 5.2.16(iii)]. □

PROPOSITION 2.5. *If Γ is p-adic analytic, then Λ is both left and right noetherian. If, moreover, Γ has no elements of finite order, then Λ has no zero divisors.*

Proof. The first assertion is [Laz65, V, 2.2.4], while the second is [Neu88, Theorem 1]. □

By a left (respectively, right) Λ-*module*, we will always mean a separated, topological left (respectively, right) Λ-module. If M is any left Λ-module, we define M^{op} to be the right Λ-module with the same underlying abelian group as M and right Γ-action given by $m \cdot \gamma := \gamma^{-1}m$ for

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

$m \in M$. This enables us to define a duality functor from the category of left Λ-modules to itself.

Definition 2.6. If M is any left Λ-module, We define the *dual* of M to be the left Λ-module

$$M^\vee := \mathrm{Hom}_\Lambda(M^{\mathrm{op}}, \Lambda).$$

This is the abelian group of all continuous homomorphisms of right Λ-modules $\varphi : M^{\mathrm{op}} \to \Lambda$ with the left Λ-module structure induced by the canonical Λ-Λ-bimodule structure on Λ: $(\lambda \cdot \varphi)(m) := \lambda\varphi(m)$.

Remark 2.7. Since Ω admits a homomorphism to the field k, both Ω and Λ have *invariant basis number* [Lam99, Definition 1.3, Remark 1.5], and the *rank* of a free module is well-defined.

2.2 Towers of modules

In this section, we generalize the commutative algebra formalism developed in [Cai18a, § 3.1] for certain projective systems of Λ-modules when $\Gamma = \mathbf{Z}_p$ to the case of arbitrary pro-p groups Γ. This machinery is at the heart of our proofs of Theorems A and C.

Definition 2.8. A Γ-*tower of* W-*modules* consists of the following data:

(i) for each nonnegative integer n, a left Λ_n-module M_n; and
(ii) for each pair of integers $n \geqslant m \geqslant 0$, a Γ-equivariant map of W-modules $\rho_{n,m} : M_n \to M_m$.

Given a Γ-tower of W-modules $\{M_n, \rho_{n,m}\}$, we simply write $M := \varprojlim_n M_n$ for the projective limit taken with respect to the maps $\rho_{n,m}$; it is naturally a left Λ-module.

Proposition 2.9. *Let $\{M_n, \rho_{n,m}\}$ be a Γ-tower of W-modules with M_n a finite and free W-module for all n. Assume that the following conditions hold for all $n \geqslant 0$:*

(i) *$\overline{M}_n := k \otimes_W M_n$ is a free Ω_n-module of rank d that is independent of n; and*
(ii) *for all $m \leqslant n$, the map*

$$\overline{\rho}_{n,m} : \overline{M}_n \longrightarrow \overline{M}_m$$

induced from $\rho_{n,m}$ by reduction modulo p is surjective.

Then, for all $n \geqslant 0$:

(i) *M_n is a free Λ_n-module of rank d; and*
(ii) *the induced maps of left Λ_m-modules,*

$$\Lambda_m \underset{\Lambda_n}{\otimes} M_n \longrightarrow M_m, \tag{2.1}$$

are isomorphisms for all $m \leqslant n$.

Moreover, M is a finite and free $\Lambda_W(\Gamma)$-module of rank d, and for each n the canonical map $\Lambda_n \otimes_\Lambda M \to M_n$ is an isomorphism of Λ_n-modules.

Proof. Fix $n \geqslant 0$ and choose $m_1, \ldots, m_d \in M_n$ whose images in $\overline{M}_n$ freely generate $\overline{M}_n$ as a left Ω_n-module. The images in $\overline{M}_n$ of the $[\Gamma : \Gamma_n]d$ elements gm_i for $g \in \Gamma/\Gamma_n$ and $1 \leqslant i \leqslant d$ are then a basis for $\overline{M}_n$ as a k-vector space, and the free W-module M_n must have rank $[\Gamma : \Gamma_n]d$. By Nakayama's lemma (in the usual commutative case), the set $\{gm_i : g \in \Gamma/\Gamma_n,\ 1 \leqslant i \leqslant d\}$ is then a minimal set of generators of the free W-module M_n whose cardinality is equal to the W-rank of

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

M_n. It follows that this set freely generates M_n as a W-module, and hence that $\{m_i : 1 \leqslant i \leqslant d\}$ freely generates M_n as a left Λ_n-module.

Reducing the map (2.1) modulo the ideal (p) yields a map of left Ω_m-modules

$$\Omega_m \underset{\Omega_n}{\otimes} \overline{M}_n \longrightarrow \overline{M}_m, \tag{2.2}$$

through which the surjective map $\overline{\rho}_{n,m}$ factors, whence (2.2) is surjective. As (p) is contained in the radical of Λ_m, it follows from Nakayama's lemma [Lam01, (4.22) (3)] that (2.1) is surjective as well. By our assumption (i), the map (2.1) is then a surjective map of free Λ_m-modules of the same rank, whence it must be an isomorphism.[5]

For any $m \leqslant n$, the kernel of the canonical surjection $\Lambda_n \twoheadrightarrow \Lambda_m$ is contained in the radical of Λ_n. It follows from (i)–(ii) and Nakayama's lemma that any lift to M_n of a Λ_m-basis of M_m is a Λ_n-basis of M_n. We may therefore choose, for each $n \geqslant 0$, a basis $e_{n,1}, \ldots, e_{n,d}$ of the free Λ_n-module M_n with the property that $\rho_{n,m} : M_n \to M_m$ carries $e_{n,i}$ to $e_{m,i}$ for all $m \leqslant n$ and $1 \leqslant i \leqslant d$. These choices yield isomorphisms of left Λ_n-modules $\Lambda_n^{\oplus d} \simeq M_n$ for all n that are compatible with change in n via the maps $\rho_{n,m}$. Passing to projective limits, we conclude that $M = \varprojlim_n M_n$ is free of rank d over $\Lambda_W(\Gamma)$. The canonical map $\Lambda_n \otimes_\Lambda M \to M_n$ is then a surjective map of free Λ_n-modules of the same rank, whence it is an isomorphism. □

We next wish to investigate duality for towers of Γ-modules.

PROPOSITION 2.10. *Let $\{M_n, \rho_{n,m}\}$ and $\{M'_n, \rho'_{n,m}\}$ be two Γ-towers of W-modules satisfying the hypotheses of Proposition 2.9. Suppose that for each n, there are W-bilinear perfect pairings*

$$\langle \cdot, \cdot \rangle_n : M_n \times M'_n \longrightarrow W, \tag{2.3}$$

with respect to which γ and γ^{-1} are adjoint for all $\gamma \in \Gamma$, and which satisfy

$$\langle \rho_{n,m} x, \rho'_{n,m} y \rangle_m = \sum_{\gamma \in \Gamma_m / \Gamma_n} \langle x, \gamma^{-1} y \rangle_n$$

for all $m \leqslant n$. Then for each n, the pairings $(\cdot, \cdot)_n : M_n \times M'_n \longrightarrow \Lambda_n$ defined by

$$(x, y)_n := \sum_{\gamma \in \Gamma / \Gamma_n} \langle x, \gamma^{-1} y \rangle_n \cdot \gamma \tag{2.4}$$

compile to induce isomorphisms of left Λ-modules

$$M \xrightarrow{\simeq} (M')^\vee \quad \text{and} \quad M' \xrightarrow{\simeq} M^\vee. \tag{2.5}$$

Proof. The proof of [Cai18a, Lemma 3.4] carries over *mutatis mutandis* to the present setting. □

We end this section with a variant of the Mittag–Leffler criterion [Sta22, Tag0598], which will enable us to show that certain (derived) inverse limits of Γ-towers vanish.

LEMMA 2.11. *Let R be a commutative noetherian ring, complete with respect to an ideal J that is contained in the radical of R, and let $\{M_n\}_n$ be an inverse system of finitely generated R-modules. If every transition map $\rho : M_{n+1} \to M_n$ has image contained in JM_n, then $\varprojlim_n M_n = \varprojlim^1_n M_n = 0$.*

[5] Indeed, let R be any local ring with residue field k and let $\rho : M \to M'$ be a surjective map of finite and free left R-modules of the same rank. As M' is free, the surjection ρ splits, giving an isomorphism of left R-modules $M \simeq M' \oplus \ker(\rho)$. Applying $k \otimes_R (\cdot)$ to this isomorphism and using the fact that M and M' are free of the same finite rank forces, $k \otimes_R \ker(\rho) = 0$. The splitting of ρ yields a surjection $M \twoheadrightarrow \ker(\rho)$, whence $\ker(\rho)$ is finitely generated as M is, and Nakayama's lemma [Lam01, (4.22) (2)] then gives $\ker(\rho) = 0$, as desired.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Proof. As R is noetherian and J-adically complete and each M_n is finitely generated, M_n is also J-adically complete [Mat89, Theorem 8.7]. Since $J \subseteq \operatorname{Rad} R$, we likewise know that M_n is J-adically separated by the Krull intersection theorem [Mat89, Theorem 8.10(i)]. By hypothesis, for all $r \geqslant 0$, we have that the image of $\rho^r : M_{m+r} \to M_m$ is contained in $J^r M_m$, so if $\{x_n\}_n \in \varprojlim_n M_n$, we have $x_n \in \bigcap_{r \geqslant 1} J^r M_n = 0$. On the other hand, by [Sta22, Tag 091D], the derived limit $\varprojlim_n^1 M_n$ is isomorphic to the cokernel of the map

$$\tau : \prod_n M_n \longrightarrow \prod_n M_n \quad \text{given by} \quad \tau(\alpha_n) := (\alpha_n - \rho(\alpha_{n+1})). \tag{2.6}$$

We claim that τ is surjective: given any $(\beta_n)_n \in \prod_n M_n$, for each fixed n, we set $\alpha_n := \sum_{\ell \geqslant n} \rho^{\ell - n}(\beta_\ell)$. By hypothesis, $\rho^{\ell-n}(\beta_\ell) \in J^{\ell-n} M_n$, so as M_n is J-adically complete, this sum converges for each n, and by construction we have $\tau((\alpha_n)_n) = (\beta_n)_n$. Thus, $\varprojlim_n^1 M_n = 0$. □

2.3 Generalized Jacobians

In this section, we fix a smooth, proper, and geometrically connected curve X over k.

Definition 2.12. A *modulus* $\mathfrak{m}$ on X is an effective Cartier divisor. If $\deg_k(\mathfrak{m}) > 1$, we denote by $X_\mathfrak{m}$ the projective and geometrically integral curve associated to X and the modulus $\mathfrak{m}$ as in [Ser88, IV § 1 No. 4]. If $\deg_k(\mathfrak{m}) \leqslant 1$, we set $X_\mathfrak{m} := X$. In all cases, we write $\nu : X \to X_\mathfrak{m}$ for the canonical map, via which X is the normalization of $X_\mathfrak{m}$.

When $\deg_k(\mathfrak{m}) > 0$, we note that $X_\mathfrak{m}$ has a distinguished k-rational point $x : \operatorname{Spec} k \to X_\mathfrak{m}$ with the property that $X_\mathfrak{m}$ is k-smooth outside of x, the divisor $\mathfrak{m}$ on X is the pullback of x along ν, and $\nu : X \to X_\mathfrak{m}$ is initial among all k-maps $X \to Y$ to k-schemes Y such that $\mathfrak{m}$ scheme-theoretically factors through $Y(k)$. It follows at once that for any $\mathfrak{m}' \geqslant \mathfrak{m}$, the normalization map $\nu' : X \to X_{\mathfrak{m}'}$ factors uniquely through $\nu : X \to X_\mathfrak{m}$.

Lemma 2.13. *There is a functorial identification $\omega_{X_\mathfrak{m}/k} \simeq \nu_* \Omega^1_X(\mathfrak{m})$ of the relative dualizing sheaf of $X_\mathfrak{m}$ with the pushforward of the sheaf of relative differential forms on X with poles no worse than $\mathfrak{m}$. In particular, $X_\mathfrak{m}$ has (arithmetic) genus $g_{X_\mathfrak{m}} := \dim_k H^0(X, \Omega^1_{X/k}(\mathfrak{m}))$.*

Proof. This follows by Galois descent from Rosenlicht's explicit description [Con00, Theorem 5.2.3] of the relative dualizing sheaf of a proper reduced curve over an algebraically closed field. □

Definition 2.14. For $n \in \mathbf{Z}$, we write $J^n_{X,\mathfrak{m}} := \operatorname{Pic}^n_{X_\mathfrak{m}/k}$ for the connected component of the relative Picard scheme $\operatorname{Pic}_{X_\mathfrak{m}/k}$ classifying degree-n line bundles on $X_\mathfrak{m}$, and simply write $J_{X,\mathfrak{m}} := \operatorname{Pic}^0_{X_\mathfrak{m}/k}$ for the *generalized Jacobian* of X relative to $\mathfrak{m}$. We will drop $\mathfrak{m}$ from the notation when $\mathfrak{m} = 0$.

The k-group structure on $\operatorname{Pic}_{X_\mathfrak{m}/k}$ (functorially induced by tensor product of line bundles) makes $J_{X,\mathfrak{m}}$ into a commutative k-group scheme and $J^n_{X,\mathfrak{m}}$ into a principal homogeneous space for $J_{X,\mathfrak{m}}$ for all n. Moreover, for any $\mathfrak{m}' \geqslant \mathfrak{m}$, pullback of line bundles along $X_\mathfrak{m} \to X_{\mathfrak{m}'}$ yields a canonical surjection of k-groups $J_{X,\mathfrak{m}'} \to J_{X,\mathfrak{m}}$ with affine kernel. The structure of $J_{X,\mathfrak{m}}$ can be made quite explicit [BLR90, § 9.2].

Proposition 2.15. *The commutative k-group scheme $J_{X,\mathfrak{m}}$ is smooth over k of dimension $g_{X_\mathfrak{m}}$. Pullback of line bundles along $\nu : X \to X_\mathfrak{m}$ induces a surjective map of commutative k-group schemes $J_{X,\mathfrak{m}} \twoheadrightarrow J_X$ with kernel that is canonically a product $U_\mathfrak{m} \times T_\mathfrak{m}$ of a unipotent group $U_\mathfrak{m}$ and a torus $T_\mathfrak{m}$. Furthermore, there is a canonical identification $T_\mathfrak{m} \simeq \operatorname{Res}_{\mathfrak{m}_{\mathrm{red}}/k} \mathbf{G}_m / \mathbf{G}_m$ of $T_\mathfrak{m}$ with the quotient of the Weil restriction $\operatorname{Res}_{\mathfrak{m}_{\mathrm{red}}/k} \mathbf{G}_m$ by the diagonally embedded copy of $\mathbf{G}_m$,*

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

and $U_{\mathfrak{m}} = 0$ if and only if $\mathfrak{m} = \mathfrak{m}_{\mathrm{red}}$ is reduced; in particular, $J_{X,\mathfrak{m}_{\mathrm{red}}}$ is the maximal semiabelian quotient of $J_{X,\mathfrak{m}}$.

We now turn to the functoriality of $J_{X,\mathfrak{m}}$ in X.

PROPOSITION 2.16. *Let $\pi : Y \to X$ be a finite and generically étale map of proper, smooth, and geometrically connected curves over k. For any modulus $\mathfrak{m}$ on X with support a finite set S of closed points in X, there exists a unique maximal (respectively, minimal) modulus $\mathfrak{n}$ (respectively, $\mathfrak{n}'$) on Y that is supported on $\pi^{-1}S$, and canonical homomorphisms of k-group schemes*

$$\pi^* : J_{X,\mathfrak{m}} \longrightarrow J_{Y,\mathfrak{n}} \quad \text{and} \quad \pi_* : J_{Y,\mathfrak{n}'} \longrightarrow J_{X,\mathfrak{m}}$$

making the diagrams

$$\begin{array}{ccc} J_{X,\mathfrak{m}} & \xrightarrow{\pi^*} & J_{Y,\mathfrak{n}} \\ \downarrow & & \downarrow \\ J_X & \xrightarrow[\pi^*]{} & J_Y \end{array} \qquad \begin{array}{ccc} J_{X,\mathfrak{m}} & \xleftarrow{\pi_*} & J_{Y,\mathfrak{n}'} \\ \downarrow & & \downarrow \\ J_X & \xleftarrow[\pi_*]{} & J_Y \end{array} \tag{2.7}$$

commute, where $\pi^* = \mathrm{Pic}^0(\pi) : J_X \to J_Y$ and $\pi_* = \mathrm{Alb}(\pi) : J_Y \to J_X$ are the usual maps on Jacobians associated to the finite (flat!) map $\pi : Y \to X$ by Picard (pullback of line bundles) and Albanese (norm of line bundles) functoriality. Considering S and $\pi^{-1}S$ as reduced Cartier divisors on X and Y, these maps induce morphisms $\pi^* : J_{X,S} \to J_{Y,\pi^{-1}S}$ and $\pi_* : J_{Y,\pi^{-1}S} \to J_{X,S}$ of semiabelian varieties whose composition $\pi_*\pi^* = \deg(\pi)$ is multiplication by $\deg(\pi)$. If, moreover, π is generically Galois with group G, then $\pi^*\pi_* = \sum_{g \in G} g^*$, where g^* is the automorphism of $J_{Y,\pi^{-1}S}$ induced by applying the preceding construction to $g : Y \to Y$ and the modulus $\pi^{-1}S = g^{-1}(\pi^{-1}S)$.

The respective restrictions of π^* and π_* to the tori $T_{\mathfrak{m}}$ and $T_{\mathfrak{n}'}$ are induced by the unique descents to k of the $\overline{k}$ maps

$$\mathrm{Res}_{S/k}\, \mathbf{G}_m \times_k \overline{k} = \prod_{s \in S_{\overline{k}}} \mathbf{G}_m \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \prod_{s \in S_{\overline{k}}} \prod_{t \in \pi_{\overline{k}}^{-1}(s)} \mathbf{G}_m = \mathrm{Res}_{\pi^{-1}S/k}\, \mathbf{G}_m \times_k \overline{k}$$

given on the factor with index $s \in S_{\overline{k}}$ by the diagonal $\pi^* = \Delta : \mathbf{G}_m \to \prod_{t \in \pi_{\overline{k}}^{-1}(s)} \mathbf{G}_m$ and the norm map $\pi_* = \mathrm{Nm} : \prod_{t \in \pi_{\overline{k}}^{-1}(s)} \mathbf{G}_m \to \mathbf{G}_m$ functorially determined on R-valued points for any $\overline{k}$-algebra R by $\mathrm{Nm}((a_t)_t) := \prod_t a_t^{e_t}$, where e_t is the ramification degree of $\mathcal{O}_{X,s} \to \mathcal{O}_{Y,t}$.

Proof. We may assume that $\deg_k \mathfrak{m} > 1$. As $\pi : Y \to X$ certainly carries $\pi^{-1}S$ into S, it follows from the universal property of $\nu : Y \to Y_{\mathfrak{n}}$ that in order for the composite $\nu \circ \pi : Y \to X \to X_{\mathfrak{m}}$ to factor through $Y_{\mathfrak{n}}$, it is necessary and sufficient that the closed subscheme $\mathfrak{n}$ of Y scheme-theoretically factor through the distinguished point $x \in X(k)$. Writing $\mathscr{I} \subseteq \mathcal{O}_X$ for the ideal sheaf of $\mathfrak{m}$ and $\mathscr{J} \subseteq \mathcal{O}_Y$ for the ideal sheaf of $\mathfrak{n}$, such factorization holds if and only if $\mathscr{J} \subseteq \pi^*\mathscr{I}$ inside $\mathcal{O}_Y$, and $\mathscr{J} = \pi^*\mathscr{I}$ is patently maximal with respect to this condition. Assuming that $\mathfrak{n} \leqslant \pi^*\mathfrak{m}$, we thus obtain a canonical map $\pi : Y_{\mathfrak{n}} \to X_{\mathfrak{m}}$ whose strict transform under the normalization map $X \to X_{\mathfrak{m}}$ *is* $\pi : Y \to X$, and we denote by $\pi^* : J_{X,\mathfrak{m}} \to J_{Y,\mathfrak{n}}$ the map induced by pullback of line bundles along $\pi : Y_{\mathfrak{n}} \to X_{\mathfrak{m}}$. It is clear from this construction that the first diagram in (2.7) commutes.

Denote by $\theta_{X,\mathfrak{m}} : X - S \to J^1_{X,\mathfrak{m}}$ the canonical map of k-schemes functorially determined by sending a Z-valued point P of $X_Z - S_Z$ to the inverse ideal sheaf $\mathcal{O}(P)$ associated to the closed subscheme P of $(X_{\mathfrak{m}})_Z$. Consider the map of k-schemes

$$\rho : Y - \pi^{-1}S \xrightarrow{\pi} X - S \xrightarrow{\theta_{X,\mathfrak{m}}} J^1_{X,\mathfrak{m}}. \tag{2.8}$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

By the universal mapping property of generalized Jacobians [Ser88, V § 4 No. 22, Proposition 13], there is a modulus $\mathfrak{n}'$ on Y with support $\pi^{-1}S$, a map of k-schemes $\rho^1_{\mathfrak{n}'} : J^1_{Y,\mathfrak{n}'} \to J^1_{X,\mathfrak{m}}$, and a map of k-group schemes $\rho_{\mathfrak{n}'} : J_{Y,\mathfrak{n}'} \to J_{X,\mathfrak{m}}$ such that $\rho^1_{\mathfrak{n}'}$ is equivariant with respect to $\rho_{\mathfrak{n}'}$ and $\rho = \rho^1_{\mathfrak{n}'} \circ \theta_{Y,\mathfrak{n}'}$. Each such modulus uniquely determines $\rho_{\mathfrak{n}'}$ and $\rho^1_{\mathfrak{n}'}$, and there is a unique minimal modulus verifying these conditions. We simply write $\pi_* : J_{Y,\mathfrak{n}'} \to J_{X,\mathfrak{m}}$ for the map $\rho_{\mathfrak{n}'}$ associated to the minimal modulus $\mathfrak{n}'$. As pullback of line bundles along ν preserves degree (see, e.g., [BLR90, § 9.1]), for any $n \in \mathbf{Z}$, we have a canonical map of k-schemes $J^n_{Y,\mathfrak{n}'} \to J^n_Y$ whose fibers are affine; as J^n_X is proper, it follows that the composite

$$J^n_{Y,\mathfrak{n}'} \xrightarrow{\pi_*} J^n_{X,\mathfrak{m}} \xrightarrow{\nu^*} J^n_X \tag{2.9}$$

factors through $J^n_{Y,\mathfrak{n}'} \to J^n_Y$ via a unique map $\mu^n : J^n_Y \to J^n_X$ which must, moreover, be a map of k-groups when $n = 0$. Consider the resulting diagram

$$\begin{array}{ccccccccc}
 & & & & \overset{\pi}{\frown} & & & & \\
Y - \pi^{-1}S & \hookrightarrow & Y & \xrightarrow[\theta_Y]{} & J^1_Y & \xrightarrow[\mu^1]{} & J^1_X & \xleftarrow[\theta_X]{} & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \nu^*} & & \downarrow{\scriptstyle \nu^*} & & \downarrow \\
Y - S & \hookrightarrow & Y & \xrightarrow[\theta_Y]{} & J^1_Y & \xrightarrow[\mu^1]{} & J^1_X & \xleftarrow[\theta_X]{} & X \\
 & & & & \underset{\theta_{Y,\mathfrak{n}'}}{\frown} & & \underset{\rho^1_{\mathfrak{n}'}}{\frown} & & \underset{\theta_{X,\mathfrak{m}}}{\frown}
\end{array}$$

in which the middle square commutes by the very definition of μ^1 and the outer squares are readily seen to commute from the definition of $\theta_\star$ and the fact that the normalization maps are isomorphisms over the smooth locii. As the top region commutes by the very definition of $\rho^1_{\mathfrak{n}'}$, we conclude that $\mu^1 \circ \theta_Y$ and $\theta_X \circ \pi$ agree on the dense open subscheme $Y - \pi^{-1}S$ of Y; since J^1_X is separated, we deduce that they agree on Y. By the uniqueness aspect of ρ^1, it follows that $\rho^1 = \mu^1$, giving the commutativity of the second diagram in (2.7).

To prove that the restrictions of the maps that we have constructed to the tori $T_{\mathfrak{m}}$ and $T_{\mathfrak{n}'}$ have the specified descriptions, we may assume that k is algebraically closed. As two maps from a reduced k-scheme to a separated k-scheme agree if and only if they agree on k-points, it suffices to check that the proposed descriptions hold at the level of k-points, which we now do.

Let P, Q be two k-points of Y not in $\pi^{-1}S$ and let $\mathscr{L}_Y$ (respectively, $\mathscr{L}_X$) be the line bundle on $Y_{\mathfrak{n}'}$ (respectively, $X_{\mathfrak{m}}$) corresponding to the degree-zero Cartier divisor $Q - P$ (respectively, $\pi(Q) - \pi(P)$). It follows easily from the universal property of $\rho^1_{\mathfrak{n}'}$ and $\rho_{\mathfrak{n}'}$ that $\pi_*\mathscr{L}_Y = \mathscr{L}_X$, and one concludes that if $D = \sum n_Q Q$ is *any* degree-zero Cartier divisor on Y supported away from $\mathfrak{n}'$, then $\pi_*\mathscr{L}(D) = \mathscr{L}(\pi(D))$, where $\pi(D) = \sum n_Q \pi(Q)$ is the direct image of D. Thus, if $g \in k(Y)$ is any function, we conclude that

$$\pi_*(\mathscr{L}(\mathrm{div}(g))) = \mathscr{L}(\pi(\mathrm{div}(g))) = \mathscr{L}(\mathrm{div}(\mathrm{Nm}_\pi(g))),$$

where $\mathrm{Nm}_\pi : k(Y) \to k(X)$ is the norm function associated to the extension of function fields $k(Y)/k(X)$ (cf. the proof of [Ser88, III § 1 No. 2, Proposition 4]). Choose a k-valued point ξ of $T_{\mathfrak{n}'} = \prod_{s \in S} \prod_{t \in \pi^{-1}(s)} \mathbf{G}_m / \mathbf{G}_m$, and let $(a_t) \in \prod_{s \in S} \prod_{t \in \pi^{-1}(s)} k^\times$ be any representative of it. Let $g \in k(Y)$ be any function with $\mathrm{ord}_t(a_t - g) > 0$ at every point $t \in \pi^{-1}S$. The image of ξ under the inclusion $T_{\mathfrak{n}'}(k) \to J_{Y,\mathfrak{n}'}(k)$ is the line bundle $\mathscr{L}(\mathrm{div}(g))$ on $Y_{\mathfrak{n}'}$, with image under π_* the line bundle $\mathscr{L}(\mathrm{div}(\mathrm{Nm}_\pi(g)))$ on $X(\mathfrak{m}')$. For any point $s \in S$, the function g is a local unit at every $t \in \pi^{-1}(s)$ by construction, and we have

$$\mathrm{Nm}_\pi(g) = \prod_{t \in \pi^{-1}(s)} \mathrm{Nm}_{\widehat{\mathcal{O}}_{Y,t}/\widehat{\mathcal{O}}_{X,s}}(g) \equiv \prod_{t \in \pi^{-1}(s)} a_t^{e_t} \pmod{\mathfrak{p}_s},$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

with $\mathfrak{p}_s$ the maximal ideal of $\mathcal{O}_{X,s}$ and e_t the ramification index of $\mathcal{O}_{X,s} \to \mathcal{O}_{Y,t}$. The given description of π_* follows. Likewise, if $h \in k(X)$, then $\pi^* \mathscr{L}(\operatorname{div}_X(h)) = \mathscr{L}(\pi^* \operatorname{div}_X(h)) = \mathscr{L}(\operatorname{div}_Y(\pi^* h))$, so if $a_s \in k^\times$ and h satisfies $\operatorname{ord}_s(a_s - h) > 0$, then $\operatorname{ord}_t(a_s - \pi^* h) > 0$ for every $t \in \pi^{-1} s$ since the pullback map $\pi^* : k(X) \to k(Y)$ is a map of k-algebras. The given description of π^* follows.

Finally, π^* and π_* induce maps on maximal semiabelian quotients as any k-homomorphism from a unipotent group to a semiabelian variety is zero, and the fact that these induced maps satisfy $\pi_* \pi^* = \deg(\pi)$ and (when π is generically Galois with group G) $\pi^* \pi_* = \sum_{g \in G} g^*$ follows from the corresponding fact on (usual) Jacobians and the given descriptions of these maps on torii. □

Definition 2.17. For a smooth, proper, and geometrically connected curve X over k and a modulus $\mathfrak{m}$ on X, we write $\mathscr{G}_{X,\mathfrak{m}} := \{J_{X,\mathfrak{m}}[p^n]\}_{n \geqslant 0}$ for the inductive system of kernels of multiplication by p^n on $J_{X,\mathfrak{m}}$.

Corollary 2.18. *Assume that $\mathfrak{m}$ is reduced, and write $\mathscr{T}_{\mathfrak{m}} := \operatorname{Res}_{\mathfrak{m}/k}(\mu_{p^\infty})/\mu_{p^\infty}$ for the multiplicative type p-divisible group given by the quotient of the Weil restriction by its diagonally embedded copy of μ_{p^∞}. Then $\mathscr{G}_{X,\mathfrak{m}}$ is a p-divisible group, and is canonically an extension of $\mathscr{G}_X = J_X[p^\infty]$ by $\mathscr{T}_{\mathfrak{m}}$.*

Proof. As $\operatorname{Res}_{\mathfrak{m}/k}(\cdot)$ is right adjoint to base change $(\cdot) \otimes_k \mathfrak{m}$, it is left exact and, in particular, it preserves kernels. It follows that the p-divisible group of the torus $T_{\mathfrak{m}}$ is naturally identified with $\mathscr{T}_{\mathfrak{m}}$ (as, say, an *fppf*-abelian sheaf). Multiplication by p^n is *fppf* surjective on any torus, so the snake lemma shows that the canonical map $\mathscr{G}_{X,\mathfrak{m}}[p^n] \to \mathscr{G}_X[p^n]$ is surjective with kernel $\mathscr{T}_{\mathfrak{m}}[p^n]$ for all n, which completes the proof. □

2.4 Dieudonné modules and de Rham cohomology

We begin with a brief recall of Dieudonné theory for finite k-group schemes of p-power order and p-divisible groups. The standard reference for this material is [Fon77, II–III]. For a modern synopsis, we recommend [CCO14, § 1.4]. Throughout, we will simply write *finite k-group* for a finite, commutative k-group scheme of p-power order.

As k is *perfect*, any finite k-group or p-divisible group G admits a functorial decomposition

$$G = G^{\text{ét}} \times G^{\text{m}} \times G^{\text{ll}}, \tag{2.10}$$

with $G^{\text{ét}}$ étale (reduced with connected dual), G^{m} multiplicative (connected with reduced dual), and G^{ll} local–local (connected with connected dual). If G is a finite k-group, then $G = \operatorname{Spec} R$ for some k-algebra R that is finite dimensional as a k-vector space, and the *order* of G is $|G| = \dim_k R$; this recovers the usual notion when G is the constant group scheme associated to a finite abelian group. If $G = \{G[p^n]\}_n$ is a p-divisible group, then there is an integer h so that $|G[p^n]| = p^{nh}$ for all n, and we say that G is of *height* h, and write $\operatorname{ht}(G) := h$.

The *Dieudonné ring* relative to k is the (associative but noncommutative if $k \neq \mathbf{F}_p$) W-algebra $\mathcal{D}_k := W[F, V]$ generated by variables F (Frobenius) and V (Verschiebung) subject to the following relations:

- $FV = VF = p$;
- $F\lambda = \sigma(\lambda)F$; and
- $V\lambda = \sigma^{-1}(\lambda)V$.

for all $\lambda \in W$. By definition, a *Dieudonné module* is a (left) $\mathcal{D}_k$-module that is finitely generated as a W-module; equivalently, it is a finite type W-module M with additive maps $F, V : M \to M$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

satisfying the three conditions above. These form an abelian category, with morphisms $\mathcal{D}_k$-module homomorphisms, i.e. W-linear maps that are compatible with F and V. This category is equipped with an involutive duality functor: if M is a Dieudonné module of finite W-length, the *dual* of M is the Pontryagin dual $M^\vee := \mathrm{Hom}_W(M, W[1/p]/W)$ as a W-module, with $F_{M^\vee}(f)(x) := \sigma f(Vx)$ and $V_{M^\vee}(f)(x) := \sigma^{-1} f(Fx)$ for $f \in M^\vee$ and $x \in M$. If M is finite and free as a W-module, we set $M^\vee := \mathrm{Hom}_W(M, W)$ with Frobenius and Verschiebung defined in the same way.

Theorem 2.19 (Main Theorem of Dieudonné Theory). *There is an exact anti-equivalence of categories $G \rightsquigarrow \mathbf{D}_k(G)$ from the category of finite commutative k-group schemes of p-power order to the category of Dieudonné modules of finite W-length. If $G = \{G[p^n]\}_n$ is a p-divisible group, defining $\mathbf{D}_k(G) := \varprojlim_n \mathbf{D}_k(G[p^n])$ induces an exact anti-equivalence between the category of p-divisible groups and the category of Dieudonné modules that are free of finite rank over W. Furthermore, we have the following.*

(i) *Compatibility with duality: there is a natural isomorphism of covariant functors*
$$\mathbf{D}_k((\cdot)^\vee) \simeq \mathbf{D}_k(\cdot)^\vee.$$

(ii) *Compatibility with base change: if $k \to k'$ is any extension of perfect fields, there is a natural isomorphism of contravariant functors*
$$\mathbf{D}_k(\cdot) \otimes_W W(k') \simeq \mathbf{D}_{k'}((\cdot) \times_k k').$$

(iii) *For any G, the linearizations of F and V on $\mathbf{D}(G)$ correspond via functoriality to the relative Frobenius $G \to G^{(p)}$ and Verschiebung $V : G^{(p)} \to G$ morphisms via compatibility of $\mathbf{D}(\cdot)$ with the base change $\sigma : k \to k$.*

(iv) *For any G:*
- *$G = G^{\text{ét}}$ if and only if F is bijective and V is topologically nilpotent on $\mathbf{D}(G)$;*
- *$G = G^{\text{m}}$ if and only if F is topologically nilpotent and V is bijective on $\mathbf{D}(G)$; and*
- *$G = G^{\text{ll}}$ if and only if F and V are both topologically nilpotent on $\mathbf{D}(G)$.*

(v) *Let $e \in G(k)$ be the identity section. There is a functorial isomorphism of k-vector spaces*
$$\mathbf{D}(G)/F\mathbf{D}(G) \simeq \omega_G,$$
where $\omega_G := H^0(\mathrm{Spec}\, k, e^*\Omega^1_{G/k})$ is the cotangent space of G at the identity.

(vi) *If G is finite, $\log_p |G| = \mathrm{length}_W \mathbf{D}(G)$, and if G is p-divisible, $\mathrm{ht}(G) = \mathrm{rank}_W \mathbf{D}(G)$.*

Remark 2.20. When k is clear from context, we will simply write $\mathbf{D}$ in place of $\mathbf{D}_k$.

A crucial ingredient in the proofs of our main theorems is *Oda's theorem* [Oda69, Corollary 5.11], which provides an explicit description of Dieudonné modules of p-torsion group schemes of abelian varieties in terms of de Rham cohomology. We will only need this result for Jacobians of curves, where it can be stated in terms of the de Rham cohomology of the curves themselves.

Associated to any smooth and proper curve X over k is the short exact 'Hodge filtration' sequence of its de Rham cohomology:
$$0 \longrightarrow H^0(X, \Omega^1_{X/k}) \longrightarrow H^1_{\mathrm{dR}}(X/k) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0. \tag{2.11}$$
This sequence is functorial in finite morphisms of smooth and proper curves $\pi : Y \to X$ in *two ways*: we write π^* for the contravariant *pullback* map associated to π, and π_* for the covariant *trace* map; we have $\pi_*\pi^* = \deg(\pi)$ and when $Y \to X$ is generically Galois with group G, we moreover have $\pi^*\pi_* = \sum_{g \in G} g^*$. Cup-product pairing induces a canonical autoduality pairing $\langle \cdot, \cdot \rangle_X$ on (2.11), via which π_* and π^* are adjoint, for any finite $\pi : Y \to X$. In particular, as

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

the absolute Frobenius morphism $\mathrm{F}_X : X \to X$ is finite, the exact sequence (2.11) is equipped with semilinear endomorphisms $F := \mathrm{F}_X^*$ and $V := (\mathrm{F}_X)_*$, which are (semilinearly) adjoint under $\langle \cdot, \cdot \rangle_X$, and satisfy $VF = \deg \mathrm{F}_X = 0$. The map V is called the *Cartier operator*, and admits a more explicit description, as in [Oda69, Definition 5.5] (for a proof that our definition of V coincides with that of Oda, see [Cai18a, § 2.1]; also cf. [Ser58, § 10, Proposition 9]). In this way, (2.11) becomes a short exact sequence of Dieudonné modules: note that $F = 0$ on $H^0(X, \Omega^1_{X/k})$ as pullback of any differential form by the p-power map in characteristic p is zero, and by duality $V = 0$ in $H^1(X, \mathcal{O}_X)$.

On the other hand, writing J_X for the Jacobian of X, the relative Verschiebung of $J_X[p]^{(p^{-1})}$ sits in a short exact sequence of finite group schemes

$$0 \longrightarrow J_X[V] \longrightarrow J_X[p] \xrightarrow{V} J_X[F]^{(p^{-1})} \longrightarrow 0. \tag{2.12}$$

This short exact sequence is contravariantly functorial in finite morphisms of smooth and proper curves $\pi : Y \to X$ via Picard functoriality $\pi^* := \mathrm{Pic}^0(\pi)$ (pullback of line bundles) and covariantly functorial via Albanese functoriality $\pi_* := \mathrm{Alb}(\pi)$ (norm of line bundles). Applying the Dieudonné functor to (2.12) yields a short exact sequence of Dieudonné modules,

$$0 \longrightarrow k \underset{k,\sigma^{-1}}{\otimes} \mathbf{D}(J_X[F]) \longrightarrow \mathbf{D}(J_X[p]) \longrightarrow \mathbf{D}(J_X[F]) \longrightarrow 0, \tag{2.13}$$

that is *covariantly* functorial in finite morphisms $\pi : Y \to X$ via $\mathbf{D}(\pi^*)$ and *contravariantly* functorial via $\mathbf{D}(\pi_*)$. The principal polarization of J_X yields a canonical isomorphism $J_X[p]^\vee \simeq J_X[p]$ intertwining $(\pi^*)^\vee$ with π_* and $(\pi_*)^\vee$ with π^*; via the compatibility of $\mathbf{D}(\cdot)$ with duality, the dual of (2.13) is then functorially identified with (2.13).

THEOREM 2.21 (Oda). *Let X be a smooth and proper curve over k with Jacobian J_X. There is a canonical isomorphism of short exact sequences of Dieudonné modules*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k \underset{k,\sigma^{-1}}{\otimes} \mathbf{D}(J_X[F]) & \longrightarrow & \mathbf{D}(J_X[p]) & \longrightarrow & \mathbf{D}(J_X[F]) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \longrightarrow & H^0(X, \Omega^1_{X/k}) & \longrightarrow & H^1_{\mathrm{dR}}(X/k) & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & 0
\end{array}$$

wherein the top row is (2.13) and the bottom row is (2.11). This isomorphism is compatible with the autodualities of each row, and for any morphism $\pi : Y \to X$ of smooth and proper curves, intertwines $\mathbf{D}(\pi_*)$ on Dieudonné modules with π^* on de Rham cohomology, and $\mathbf{D}(\pi^*)$ with π_*.

Proof. This follows from [Oda69, Corollary 5.11] and [Cai10, Proposition 5.4]. □

We will also need the following variant of Oda's theorem for reduced but possibly non-smooth curves. If X is any reduced proper curve over k, its Jacobian $J_X := \mathrm{Pic}^0_{X/k}$ is a smooth commutative group scheme over k. While the kernel of multiplication by p on J_X may *not* be finite, the kernel of relative Frobenius $J_X \to J_X^{(p)}$ *is* finite, so we may consider its Dieudonné module. On the other hand, writing $\omega_{X/k}$ for the relative dualizing sheaf of X as in [Con00, § 5.2], Grothendieck's theory of the trace map attached to any finite map of proper reduced curves $\pi : Y \to X$ yields a map on dualizing sheaves $\pi_* : \pi_* \omega_{X/k} \to \omega_{Y/k}$ that is dual to the pullback map $\pi^* : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ via Grothendieck duality (see [Cai18a, Appendix A]). If π is in addition *flat*, $\pi_* \mathcal{O}_Y$ is a finite locally free $\mathcal{O}_X$-module, so one has a trace mapping $\pi_* : \pi_* \mathcal{O}_Y \to \mathcal{O}_X$ on functions, whose dual is a pullback map $\pi^* : \omega_{X/k} \to \pi_* \omega_{Y/k}$ on dualizing sheaves that coincides with pullback of differential forms when X and Y are smooth. In particular, the finite absolute Frobenius map induces a

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

semilinear mapping

$$V := (\mathrm{F}_X)_* : H^0(X, \omega_{X/k}) \longrightarrow H^0(X, \omega_{X/k}),$$

which recovers the previously defined Cartier operator when X is smooth. In this way, $H^0(X, \omega_{X/k})$ is naturally a Dieudonné module with $F = 0$.

Lemma 2.22. *Let X be a proper curve over k, and $J_X := \mathrm{Pic}^0_{X/k}$ its Jacobian. There is a natural isomorphism of Dieudonné modules*

$$\mathbf{D}(J_X[F]) \simeq H^0(X, \omega_{X/k}). \tag{2.14}$$

If $\pi : Y \to X$ is a finite map of reduced curves, then (2.14) intertwines $\mathbf{D}(\pi^*)$ with π_*, and if π is in addition flat, it intertwines $\mathbf{D}(\pi_*)$ with π^*.

Proof. Let $G := J_X[F]$. By Theorem 2.19(v), we have a natural isomorphism of Dieudonné modules $\mathbf{D}(G) \simeq \omega_G$. Now the contravariant cotangent space functor is right exact, so as $F : J_X \to J_X^{(p)}$ induces the zero map on cotangent spaces, we conclude that the closed immersion $G \hookrightarrow J_X$ induces a functorial isomorphism $\omega_G \simeq \omega_{J_X}$. It therefore suffices to prove that there is a canonical isomorphism $H^0(X, \omega_{X/k}) \simeq \omega_{J_X}$ intertwining π_* with $\mathrm{Pic}(\pi)^*$ and π_* with $\mathrm{Alb}(\pi)^*$ when π is flat. To verify this, we may dualize, where by Grothendieck duality we seek an isomorphism $H^1(X, \mathcal{O}_X) \simeq \mathrm{Lie}(J_X)$ intertwining π^* with $\mathrm{Lie}(\mathrm{Pic}(\pi))$ and, when π is flat, π_* with $\mathrm{Lie}(\mathrm{Alb}(\pi))$. The required isomorphism is provided by [LLR04, Proposition 1.3]. □

Remark 2.23. In fact, using Lemma 2.22 as a starting point, it is possible to generalize Oda's theorem 2.21 to the case of *arbitrary* proper curves.

Besides Oda's theorem, there is one other crucial tool in our proofs, Nakajima's *equivariant Deuring–Shafarevich formula* [Nak85]. To state it in the form that we will need, we first note that any Dieudonné module M admits a functorial decomposition

$$M = M^{V\text{-bij}} \oplus M^{F\text{-bij}} \oplus M^{F,V\text{-nil}},$$

where $M^{V\text{-bij}}$ (respectively, $M^{F\text{-bij}}$) is the maximal $\mathcal{D}_k$-stable submodule on which V (respectively, F) is bijective and F (respectively, V) is topologically nilpotent and $M^{F,V\text{-nil}}$ is the maximal submodule on which both F and V are topologically nilpotent. While the existence of this decomposition is a fact of 'pure' semilinear algebra, it corresponds to (2.10) via the equivalence of Theorem 2.19.

Theorem 2.24 (Nakajima). *Let $\pi : Y \to X$ be a finite and generically Galois map of smooth proper and geometrically connected curves over an algebraically closed field k of characteristic $p > 0$, with group G that is a p-group. Let $\Sigma \subset X(k)$ be the set of branch points of π, and S any finite and nonempty set of points of X containing Σ, and write g_X and γ_X for the genus and p-rank of X, respectively.*

(i) *The $k[G]$-modules $H^0(Y, \Omega^1_{Y/k}(\pi^{-1}S))^{V\text{-bij}}$ and $H^1(X, \mathcal{O}_X(-\pi^{-1}S))^{F\text{-bij}}$ are each free of rank $\gamma_X + |S| - 1$.*

(ii) *If $\Sigma = \emptyset$, so $\pi : Y \to X$ is étale, then $H^0(Y, \Omega^1_{Y/k})^{F,V\text{-nil}}$ and $H^1(Y, \mathcal{O}_Y)^{F,V\text{-nil}}$ are each free $k[G]$-modules of rank $g - \gamma$.*

Proof. Serre duality restricts to a functorial and perfect duality pairing between the two spaces in each of (i) and (ii), so it suffices to prove that one of them is free of the asserted rank. Working with spaces of differentials, (i) then follows immediately from [Nak85, Theorem 1], while (ii) follows from [Nak85, Theorem 3]. □

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

In the remainder of this section, we record some additional results on the behavior of the trace mapping attached to finite maps of curves.

PROPOSITION 2.25. *Let $\pi : Y \to X$ be a finite, flat, and generically étale map of proper and geometrically irreducible curves over a field k. If there is a geometric point of X over which π is totally ramified, then $\pi^* : \mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{Y/k}$ has trivial scheme-theoretic kernel.*

Proof. The proof of [Cai18a, Lemma 2.2.2] goes through verbatim in the present (more general) situation, noting that the hypothesis that X and Y be smooth in *loc. cit.* is used *only* to ensure that $\pi : Y \to X$ is flat. □

PROPOSITION 2.26. *Let $\pi : Y \to X$ be a finite étale map of proper, smooth, and geometrically connected curves over k. Let $S \subseteq X(\overline{k})$ be a finite and nonempty set of geometric points of X and $T := \pi^{-1}S$, and consider S and T as (reduced and effective) divisors on X and Y, respectively. The trace mapping*

$$\pi_* : H^0(\Omega^1_{Y/k}(T)) \longrightarrow H^0(\Omega^1_{X/k}(S)) \tag{2.15}$$

is surjective.

Proof. As the formation of (2.15) is compatible with base change, by passing to a finite extension of k if need be, we may assume that $S \subseteq X(k)$, and we will induct on $|S|$. First, suppose that $S = \{s\}$ is a single point, and denote by Y_T the singular curve associated to the modulus T. Since T is reduced and $\pi(T) = s \in X(k)$, the restriction of $\pi : Y \to X$ to T scheme-theoretically factors through $X(k)$ [Sta22, Tag 0356]. By the universal property of $Y \to Y_T$, we obtain a unique map $\rho : Y_T \to X$ through which π factors. Now the curve Y_T is by construction geometrically integral, and hence (geometrically) reduced and therefore Cohen–Macaulay, and X is by hypothesis smooth. It follows from [Mat89, Corollary to Theorem 23.1] that ρ is flat, and necessarily totally ramified over s. By Proposition 2.25, the map $\rho^* : \mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{Y_T/k}$ has trivial scheme-theoretic kernel. Using the fact that $\mathrm{Lie}(\cdot)$ is left exact, together with [LLR04, Proposition 1.3], we deduce that $\rho^* : H^1(X, \mathcal{O}_X) \to H^1(Y_T, \mathcal{O}_{Y_T})$ is injective, so by Grothendieck duality the trace map

$$\rho_* : H^0(Y_T, \omega_{Y_T}) \to H^0(X, \Omega^1_X) = H^0(X, \Omega^1_X(S))$$

is surjective; the final equality uses the fact that the sum of the residues at all singular points of a meromorphic differential on a proper curve is zero. On the other hand, by Lemma 2.13, the normalization map $\nu : Y \to Y_T$ induces an identification $\nu_* : H^0(Y, \Omega^1_Y(T)) \simeq H^0(Y_T, \omega_{Y_T/k})$. As $\pi = \rho \circ \nu$, we conclude that π_* is surjective when $S = \{s\}$.

Now supposing that the result holds for a given finite and nonempty set $S \subseteq X(k)$, let $s \in X(k)$ be a point not in S, and set $S' := S \cup \{s\}$ and $T' := T \cup \pi^{-1}s$. We then have a commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\Omega^1_Y(T)) & \longrightarrow & H^0(\Omega^1_Y(T')) & \xrightarrow{\eta \mapsto (\mathrm{res}_t\, \eta)_t} & \bigoplus_{t \in \pi^{-1}(s)} k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \sum_{t \in \pi^{-1}(s)}} & & \\
0 & \longrightarrow & H^0(\Omega^1_Y(S)) & \longrightarrow & H^0(\Omega^1_X(S')) & \xrightarrow[\eta \mapsto \mathrm{res}_s\, \eta]{} & k & \longrightarrow & 0
\end{array} \tag{2.16}$$

Indeed, exactness of the rows is a straightforward consequence of the Riemann–Roch theorem, and the (right square of the) diagram commutes due to [Cai18a, Remark 2.2]. The left vertical arrow of (2.16) is now surjective by our inductive hypothesis, while surjectivity of the right vertical arrow is clear. We conclude that the middle vertical arrow is surjective as well. □

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

3. Proofs

In this section, we prove our main results stated in § 1, whose setting and notation we maintain, and now briefly recall. Throughout, we will fix a function field K in one variable over k and a Γ-extension L/K with k algebraically closed in L and Γ an infinite pro-p group equipped with a countable basis of the identity $\Gamma = \Gamma_0 \supsetneq \Gamma_1 \supsetneq \cdots$ consisting of open, normal subgroups. We write $K_n := L^{\Gamma_n}$ for the fixed field of Γ_n, and X_n for the unique smooth, projective, and geometrically connected curve over k with function field K_n, and for all $n \geqslant m$, we denote by $\pi_{n,m} : X_n \to X_m$ the map of curves corresponding to the inclusion of function fields $K_m \hookrightarrow K_n$. For all n, we assume that $\pi_{n,0} : X_n \to X_0$ is étale outside a finite (possibly empty) set Σ of closed points of X_0 that is independent of n, and totally ramified at every point of Σ.

Remark 3.1. Let U be the complement of Σ in X_0, and let γ be the p-rank of X_0. By [Sha47], the maximal pro-p quotient $\pi_1^{\text{ét}}(U_{\overline{k}})^{(p)}$ of the geometric étale fundamental group of U is a *free* pro-p group on γ generators when $\Sigma = \emptyset$, and on as many generators as the cardinality of $\overline{k}$ when $\Sigma \neq \emptyset$ (for a modern proof of this fact via cohomological dimension and étale cohomology, see [Gil00, § 1]). It follows from this and [RZ10, Corollary 2.6.6], that $\pi_1^{\text{ét}}(U)^{(p)}$ admits a countable basis of the identity consisting of open normal subgroups if and only if k is countable; in particular, our assumption that Γ admit a countable basis of the identity is automatically verified whenever k itself is countable.

When $\Sigma \neq \emptyset$, we set $S := \Sigma$, and when $\Sigma = \emptyset$, we fix a closed point x_0 of X_0 and put $S := \{x_0\}$. For each n, let $S_n := (\pi_{n,0}^{-1} S)_{\text{red}}$ be the reduced closed subscheme underlying the scheme-theoretic fiber of S in X_n, so $S_0 = S$. In the étale case $\Sigma = \emptyset$, note that S_n is just the (scheme-theoretic) fiber over x_0, while in the ramified case $\Sigma \neq \emptyset$, it is the reduced closed subscheme of X_n at which $\pi_{n,0}$ is non-étale. For each n, we write $\mathscr{G}_n := J_{X_n}[p^\infty]$ for the p-divisible group of the Jacobian of X_n, and $\mathscr{G}_{n,S_n} := J_{X_n,S_n}[p^n]$ for the inductive system of p^n-torsion group schemes on the generalized Jacobian with modulus S_n (Definition 2.17). By Corollary 2.18, each $\mathscr{G}_{n,S_n}$ is a p-divisible group, and there is a canonical exact sequence of p-divisible groups,

$$0 \longrightarrow \mathscr{T}_n \longrightarrow \mathscr{G}_{n,S} \longrightarrow \mathscr{G}_n \longrightarrow 0, \tag{3.1}$$

with $\mathscr{T}_n$ of multiplicative type. We write $\mathscr{Q}_n := \mathscr{T}_n^\vee$ for the dual p-divisible group, which is étale as $\mathscr{T}_n$ is multiplicative. It follows from Proposition 2.16 that Γ acts on J_{X_n,S_n} in two ways, with $\gamma \in \Gamma$ acting either as γ^*, or as γ_*. We will refer to the first of these actions as the Picard, or $(\cdot)^*$-action, and the second as the Albanese or $(\cdot)_*$-action. Note that as $\gamma_*\gamma^* = \deg(\gamma) = 1$, it follows that $\gamma_* = (\gamma^*)^{-1}$ for all $\gamma \in \Gamma$. These actions induce actions of Γ on (3.1), with Γ_n acting trivially. For each pair of integers $n > m$, the degeneracy maps $\pi_{n,m} : X_n \to X_m$ likewise induce *two* morphisms of the exact sequence (3.1) by Proposition 2.16, and we will frequently abbreviate $\pi^* := \pi_{n,m}^*$ for the morphism induced by Picard functoriality, and $\pi_* := (\pi_{n,m})_*$ for that induced by Albanese functoriality: note that π^* is equivariant with respect to the $(\cdot)^*$-action of Γ, while π_* is equivariant with respect to the $(\cdot)_*$-action. Furthermore, the autoduality of Jacobians induces a canonical identification $\mathscr{G}_n^\vee \simeq \mathscr{G}_n$ for all n, via which one has $(\pi_*)^\vee = \pi^*$ as morphisms $\mathscr{G}_m \to \mathscr{G}_n$.

Definition 3.2. Let $\star \in \{\text{ét}, \text{m}, \text{ll}, \text{null}\}$. For each n, set $\mathbf{D}_n^\star := \mathbf{D}(\mathscr{G}_n^\star)$ and write $\rho := \mathbf{D}(\pi^*)$ for the induced transition maps. When $\star = \text{m}, \text{ll}$, we likewise put $\mathbf{D}_{n,S_n}^\star := \mathbf{D}(\mathscr{G}_{n,S_n}^\star)$, equipped with transition maps $\rho = \mathbf{D}(\pi^*)$. For $\star = \text{ét}$, we instead define

$$\mathbf{D}_{n,S_n}^{\text{ét}} := \mathbf{D}((\mathscr{G}_{n,S_n}^{\text{m}})^\vee) = \mathbf{D}((\mathscr{G}_{n,S_n}^\vee)^{\text{ét}}),$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

which we make into an inverse system via the transition maps $\rho' := \mathbf{D}((\pi_*)^\vee)$. We equip $\mathbf{D}^\star_n$ and $\mathbf{D}^{\mathrm{m}}_{n,S_n}$ with the *left* action of Γ in which $\gamma \in \Gamma$ acts as $\mathbf{D}(\gamma^*)$, while we equip $\mathbf{D}^{\text{ét}}_{n,S_n}$ with the *left* Γ-action wherein γ acts as $\mathbf{D}((\gamma_*)^\vee)$. These actions commute with F and V, and Γ_n acts trivially.

For any $\star$, we then define the Iwasawa–Dieudonné modules

$$\mathbf{D}^\star := \varprojlim_{n,\rho} \mathbf{D}^\star_n,$$

with projective limits taken with respect to the indicated transition maps. Similarly, we define

$$\mathbf{D}^\star_S := \varprojlim_{n,\rho} \mathbf{D}^\star_{n,S_n} \text{ for } \star \in \{\mathrm{m}, \mathrm{ll}\}, \quad \text{and} \quad \mathbf{D}^{\text{ét}}_S := \varprojlim_{n,\rho'} \mathbf{D}^{\text{ét}}_{n,S_n}.$$

In each case, the given transition maps are equivariant with respect to the specified (left) actions of Γ, so $\mathbf{D}^\star$ and $\mathbf{D}^\star_S$ are naturally (left) Λ-modules with additive maps F, V that are semilinear over σ and σ^{-1}, respectively, and satisfy $FV = VF = p$.

Remark 3.3. The definition of $\mathbf{D}^{\text{ét}}_{n,S_n} = \mathbf{D}((\mathscr{G}^{\mathrm{m}}_{n,S_n})^\vee)$ may seem strange at first, but note that as $\mathscr{G}_{n,S_n}$ is an extension of $\mathscr{G}_n$ by a p-divisible group of *multiplicative* type, we in fact have $\mathscr{G}^{\text{ét}}_{n,S_n} = \mathscr{G}^{\text{ét}}_n$, so via the autoduality identification $\mathscr{G}^{\text{ét}}_n \simeq (\mathscr{G}^{\mathrm{m}}_n)^\vee$ we see that $\mathscr{G}^{\text{ét}}_n$ is a sub-p-divisible group of $(\mathscr{G}^{\mathrm{m}}_{n,S_n})^\vee$, with étale quotient $\mathscr{Q}_n := \mathscr{T}^\vee_n$. Note that $\mathscr{G}_{n,S_n}$ is *not* autodual; consequently, its étale part is too small to effectively study $\mathscr{G}^{\text{ét}}_n$. Via the identifications $\mathscr{G}^\vee_n \simeq \mathscr{G}_n$, the transition maps $\rho := \mathbf{D}(\pi^*)$ and $\rho' := \mathbf{D}((\pi_*)^\vee)$ *coincide* as maps $\mathbf{D}(\mathscr{G}_n) \to \mathbf{D}(\mathscr{G}_m)$, so that the projective limit $\mathbf{D}^{\text{ét}}$ can be formed via either.

Remark 3.4. As in Remark 3.3, the kernel of $\mathscr{G}_{n,S_n} \twoheadrightarrow \mathscr{G}_n$ is of multiplicative type, whence we have an isomorphism $\mathscr{G}^{\mathrm{ll}}_{n,S} \simeq \mathscr{G}^{\mathrm{ll}}_n$ for all n, so in particular $\mathbf{D}^{\mathrm{ll}} \simeq \mathbf{D}^{\mathrm{ll}}_S$.

For $\star \in \{\mathrm{m}, \mathrm{ll}\}$, the compatibility of $\mathbf{D}(\cdot)$ with duality gives a natural isomorphism of Dieudonné modules

$$\mathbf{D}((\mathscr{G}^\star_{n,S_n})^\vee) \simeq \mathrm{Hom}_W(\mathbf{D}^\star_{n,S_n}, W), \tag{3.2}$$

intertwining $\mathbf{D}((\gamma_*^{-1})^\vee)$ with $\mathbf{D}(\gamma_*^{-1})^\vee = \mathbf{D}(\gamma^*)^\vee$. It follows that the corresponding W-bilinear perfect duality pairing

$$\langle \cdot, \cdot \rangle : \mathbf{D}^\star_{n,S_n} \times \mathbf{D}((\mathscr{G}^\star_{n,S_n})^\vee) \longrightarrow W \tag{3.3}$$

satisfies

$$\langle \gamma \cdot x, y \rangle = \langle \mathbf{D}(\gamma^*)(x), y \rangle = \langle x, \mathbf{D}((\gamma_*^{-1})^\vee)(y) \rangle.$$

For any $n \geqslant m$, we likewise have

$$\begin{aligned} \langle \mathbf{D}(\pi^*)x, \mathbf{D}((\pi_*)^\vee)y \rangle = \langle x, \mathbf{D}((\pi_*)^\vee) \circ \mathbf{D}((\pi^*)^\vee)(y) \rangle &= \langle x, \mathbf{D}((\pi^*)^\vee \circ (\pi_*)^\vee)(y) \rangle \\ &= \langle x, \mathbf{D}((\pi_*\pi^*)^\vee)(y) \rangle. \end{aligned}$$

From Proposition 2.16, we have the identity

$$\pi_*\pi^* = \sum_{\gamma \in \Gamma_m/\Gamma_n} \gamma^* = \sum_{\gamma \in \Gamma_m/\Gamma_n} (\gamma_*)^{-1},$$

Using additivity of the functors $\mathbf{D}(\cdot)$ and $(\cdot)^\vee$ and the fact that the identification $(\mathscr{G}^{\mathrm{ll}}_n)^\vee \simeq \mathscr{G}^{\mathrm{ll}}_n$ intertwines $(\gamma_*)^\vee$ with γ^*, we deduce that the duality pairings between $\mathbf{D}^{\mathrm{m}}_{n,S_n}$ and $\mathbf{D}^{\text{ét}}_{n,S_n}$ and between $\mathbf{D}^{\mathrm{ll}}_{n,S_n} \simeq \mathbf{D}^{\mathrm{ll}}_n$ and itself induced by (3.3) verify the hypotheses of Proposition 2.10.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

3.1 The ramified case

By hypothesis, $\pi_{n,0}: X_n \to X_0$ is totally ramified at every point of S_n, and so induces an identification of étale k-schemes $S_n \simeq S_0 = S$ for all n. We will henceforth make these identifications, which induce identifications $\mathscr{T}_n \simeq \mathscr{T}_0$ and $\mathscr{Q}_n \simeq \mathscr{Q}_0$ for all n, which we likewise make, denoting these common p-divisible groups simply by $\mathscr{T}_S$ and $\mathscr{Q}_S$, respectively. Similarly, we often write $\mathscr{G}_{n,S}$ for $\mathscr{G}_{n,S_n}$, and $\mathbf{D}^{\star}_{n,S}$ in place of $\mathbf{D}^{\star}_{n,S_n}$.

In order to prove Theorem C, we apply Propositions 2.9 and 2.10 to the Γ-towers of W-modules $\{\mathbf{D}^{\mathrm{m}}_{n,S}, \rho\}$ and $\{\mathbf{D}^{\text{ét}}_{n,S}, \rho'\}$. Recalling the notation of Proposition 2.9(i), it suffices to prove the following.

Lemma 3.5. *For each pair of positive integers $m \leqslant n$, the maps*

$$\overline{\rho}: \overline{\mathbf{D}}^{\mathrm{m}}_{n,S} \longrightarrow \overline{\mathbf{D}}^{\mathrm{m}}_{m,S} \quad \text{and} \quad \overline{\rho}': \overline{\mathbf{D}}^{\text{ét}}_{n,S} \longrightarrow \overline{\mathbf{D}}^{\text{ét}}_{m,S}$$

are surjective. Moreover, $\overline{\mathbf{D}}^{\mathrm{m}}_{n,S}$ and $\overline{\mathbf{D}}^{\text{ét}}_{n,S}$ are free Ω_n-modules of rank $d := \gamma + \deg S - 1$ for all n.

Proof. We may assume that k is algebraically closed.[6] On the one hand, Theorem 2.19 provides identifications of Dieudonné modules

$$\mathbf{D}(\mathscr{G}^{\mathrm{m}}_{n,S}) \otimes_W k \simeq \mathbf{D}(J_{X_n,S}[p]^{\mathrm{m}}) \simeq \mathbf{D}(J_{X_n,S}[F])^{V\text{-bij}}.$$

For the final isomorphism, note that if a finite commutative k-group scheme G is killed by p, then one has $FV = 0$ on $\mathbf{D}(G)$, so that G is of multiplicative type if and only if V is bijective and $F = 0$ on $\mathbf{D}(G)$ due to Theorem 2.19(iv). On the other hand, Lemmas 2.13 and 2.22 provide a natural identification

$$\mathbf{D}(J_{X_n,S}[F])^{V\text{-bij}} \simeq H^0(X_n, \Omega^1_{X_n/k}(S))^{V\text{-bij}}.$$

These identifications and the duality isomorphism of (3.2) yield natural isomorphisms of Ω_n-modules

$$\overline{\mathbf{D}}^{\mathrm{m}}_{n,S} := \mathbf{D}(\mathscr{G}^{\mathrm{m}}_{n,S}) \otimes_W k \simeq \mathbf{D}(J_{X_n,S}[p]^{\mathrm{m}}) \simeq H^0(X_n, \Omega^1_{X_n/k}(S))^{V\text{-bij}}, \tag{3.4a}$$

$$\overline{\mathbf{D}}^{\text{ét}}_{n,S} := \mathbf{D}((\mathscr{G}^{\mathrm{m}}_{n,S})^{\vee}) \otimes_W k \simeq \mathbf{D}(J_{X_n,S}[p]^{\mathrm{m}})^{\vee} \simeq (H^0(X_n, \Omega^1_{X_n/k}(S))^{V\text{-bij}})^{\vee}. \tag{3.4b}$$

It follows from this and Nakajima's theorem 2.24(i) that $\overline{\mathbf{D}}^{\mathrm{m}}_{n,S}$ and $\overline{\mathbf{D}}^{\text{ét}}_{n,S}$ are each free Ω_n-modules of rank d. Now, (3.4a) and (3.4b) are compatible with change in n, in the sense that for all $m \leqslant n$,

[6] Indeed, whether or not $\overline{\rho}$ is surjective may be checked after extension of scalars along the faithfully flat map $k \to \overline{k}$, and the Dieudonné module functor commutes with any base change of perfect fields by Theorem 2.19(ii). Moreover, whether or not a finitely generated Ω_n-module is free may be checked after extension of scalars along $k \to \overline{k}$. More generally, if $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ is any flat, local map of local rings, then a finitely generated left R-module M is free of rank r if and only if $M' := R' \otimes_R M$ is free of rank r. To see this, suppose that M' is R'-free of rank r and note that

$$k' \otimes_{R'} M' = k' \otimes_{R'} (R' \otimes_R M) = k' \otimes_k (k \otimes_R M) = k' \otimes_k (M/\mathfrak{m}M),$$

from which it follows that $M/\mathfrak{m}M$ has dimension r as a k-vector space. Choosing $m_1, \ldots, m_r \in M$ whose images in $M/\mathfrak{m}M$ form a k-basis, Nakayama's lemma [Lam01, (4.22) (3)] implies that $m_1, \ldots, m_r$ generate M as an R-module, so the induced map $\varphi: R^r \twoheadrightarrow M$ is surjective. The scalar extension of φ along $R \to R'$ is then a surjective map of free modules of the same rank r, and therefore an isomorphism (see footnote 5). As $R \to R'$ is flat, it follows that $R' \otimes_R (\ker \varphi) = 0$, whence $\ker \varphi = 0$ as $R \to R'$ is faithfully flat [Lam99, Theorem 4.74].

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

each of the two diagrams

$$\begin{array}{ccc} \overline{\mathbf{D}}^{\mathrm{m}}_{n,S} & \xrightarrow{\simeq} & H^0(X_n, \Omega^1_{X_n/k}(S))^{V\text{-bij}} \\ \downarrow{\scriptstyle \bar{\rho}} & & \downarrow{\scriptstyle \pi_*} \\ \overline{\mathbf{D}}^{\mathrm{m}}_{m,S} & \xrightarrow{\simeq} & H^0(X_m, \Omega^1_{X_m/k}(S))^{V\text{-bij}} \end{array} \qquad \begin{array}{ccc} \overline{\mathbf{D}}^{\text{ét}}_{n,S} & \xleftarrow{\simeq} & (H^0(X_n, \Omega^1_{X_n/k}(S))^{V\text{-bij}})^\vee \\ \downarrow{\scriptstyle \bar{\rho}'} & & \downarrow{\scriptstyle (\pi^*)^\vee} \\ \overline{\mathbf{D}}^{\text{ét}}_{m,S} & \xleftarrow{\simeq} & (H^0(X_m, \Omega^1_{X_m/k}(S))^{V\text{-bij}})^\vee \end{array}$$

commutes. We are thereby reduced to the claim that the map π^* (respectively, π_*) is injective (respectively, surjective) on differential forms. Pullback of differential forms is injective as $\pi_{n,m}$ is generically étale, and to see that trace is surjective, we may dualize, apply Serre duality, and argue that the pullback map

$$\pi^* : H^1(X_m, \mathcal{O}_{X_m}(-S)) \longrightarrow H^1(X_n, \mathcal{O}_{X_n}(-S)) \tag{3.5}$$

is injective. This pullback map fits into the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X_m, \mathcal{O}_{X_m}) & \longrightarrow & H^0(X_m, \mathcal{O}_S) & \longrightarrow & H^1(X_m, \mathcal{O}_{X_m}(-S)) & \longrightarrow & H^1(X_m, \mathcal{O}_{X_m}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ 0 & \longrightarrow & H^0(X_n, \mathcal{O}_{X_n}) & \longrightarrow & H^0(X_n, \mathcal{O}_S) & \longrightarrow & H^1(X_n, \mathcal{O}_{X_n}(-S)) & \longrightarrow & H^1(X_n, \mathcal{O}_{X_n}) \longrightarrow 0 \end{array}$$

Since X_n is (geometrically) connected for all n, we have $H^0(X_n, \mathcal{O}_{X_n}) \simeq k$ and the first vertical map is an isomorphism. As S is reduced, we have $H^0(X_n, \mathcal{O}_S) \simeq k^{\deg S}$ for all n, and the second vertical map is likewise an isomorphism. The injectivity of the final vertical map follows from Proposition 2.25, arguing as in the proof of Proposition 2.26. A diagram chase then implies that (3.5) is injective as well. □

Proof of Theorem C. Due to Lemma 3.5 and our calculations immediately below (3.3), the Γ-towers of W-modules $\{\mathbf{D}^{\mathrm{m}}_{n,S}, \rho\}$ and $\{\mathbf{D}^{\text{ét}}_{n,S}, \rho'\}$ satisfy the hypotheses of Propositions 2.9 and 2.10, whose conclusions immediately give Theorem C. □

Lemma 3.6. *For each pair of positive integers $m \leqslant n$, the map $\pi_{n,m} : X_n \to X_m$ induces commutative diagrams of p-divisible groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T}_S & \longrightarrow & \mathscr{G}_{n,S} & \longrightarrow & \mathscr{G}_n & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \\ 0 & \longrightarrow & \mathscr{T}_S & \longrightarrow & \mathscr{G}_{m,S} & \longrightarrow & \mathscr{G}_m & \longrightarrow & 0 \end{array} \tag{3.6a}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T}_S & \longrightarrow & \mathscr{G}_{n,S} & \longrightarrow & \mathscr{G}_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [\Gamma_m : \Gamma_n]} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\ 0 & \longrightarrow & \mathscr{T}_S & \longrightarrow & \mathscr{G}_{m,S} & \longrightarrow & \mathscr{G}_m & \longrightarrow & 0 \end{array} \tag{3.6b}$$

compatibly with change in m, n. Moreover, (3.6a) and (3.6b) are equivariant for the $(\cdot)^*$-action and the $(\cdot)_*$-action of Γ, respectively.

Proof. This follows immediately from Proposition 2.16 and Corollary 2.18. □

Passing to multiplicative parts on (3.6a) and applying the (contravariant) Dieudonné module functor yields, for each $m \leqslant n$, the following commutative diagram of Dieudonné modules, whose

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

rows are short exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D}_n^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{n,S}^{\mathrm{m}} & \longrightarrow & \mathbf{D}(\mathscr{T}_S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbf{D}_m^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{m,S}^{\mathrm{m}} & \longrightarrow & \mathbf{D}(\mathscr{T}_S) & \longrightarrow & 0 \end{array} \tag{3.7}$$

Similarly, passing to multiplicative parts on (3.6*b*), dualizing, and applying $\mathbf{D}(\cdot)$ yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D}(\mathscr{Q}_S) & \longrightarrow & \mathbf{D}_{n,S}^{\text{ét}} & \longrightarrow & \mathbf{D}_n^{\text{ét}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [\Gamma_m:\Gamma_n]} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho'=\rho} & & \\ 0 & \longrightarrow & \mathbf{D}(\mathscr{Q}_S) & \longrightarrow & \mathbf{D}_{m,S}^{\text{ét}} & \longrightarrow & \mathbf{D}_m^{\text{ét}} & \longrightarrow & 0 \end{array} \tag{3.8}$$

where we have used the canonical identifications $\mathscr{G}_n^\vee \simeq \mathscr{G}_n$ coming from principal polarizations on Jacobians. As duality of Jacobians interchanges Picard and Albanese functoriality, these identifications are equivariant for the action of $(\gamma_*)^\vee$ on $\mathscr{G}_n^\vee$ and γ^* on $\mathscr{G}_n$, and identify the transition maps ρ' and ρ on $\mathbf{D}_n^{\text{ét}}$ as in Remark 3.3. These diagrams give short exact sequences of projective systems.

To prove Theorem D, we will need to apply the functor $\varprojlim$ to the exact sequences of inverse systems given by (3.7) and (3.8). To analyze the result, we will need to know that certain derived limits vanish, which is the content of the next lemma.

Lemma 3.7. *For each pair of positive integers $m \leqslant n$, the map*

$$\rho : \mathbf{D}_n \longrightarrow \mathbf{D}_m$$

is surjective. In particular, the map $\rho : \mathbf{D}_n^\star \to \mathbf{D}_m^\star$ is surjective for any $\star \in \{\text{ét}, \mathrm{m}, \mathrm{ll}, \mathrm{null}\}$.

Proof. As $\mathbf{D}_n^\star$ is a functorial direct summand of $\mathbf{D}_n$, the second assertion follows from the first. To prove the first, it suffices to do so after applying $(\cdot) \otimes_W k$, and, as in the proof of Lemma 3.5, we may assume that k is algebraically closed. Arguing as in the proof of Lemma 3.5, by Oda's theorem 2.21, for all $m \leqslant n$ we have the following commutative diagram with indicated isomorphisms.

$$\begin{array}{ccc} \overline{\mathbf{D}}_n \simeq \mathbf{D}(J_{X_n}[p]) & \xrightarrow{\simeq} & H^1_{\mathrm{dR}}(X_n/k) \\ \downarrow{\scriptstyle \bar{\rho}} & & \downarrow{\scriptstyle \pi_*} \\ \overline{\mathbf{D}}_m \simeq \mathbf{D}(J_{X_m}[p]) & \xrightarrow{\simeq} & H^1_{\mathrm{dR}}(X_m/k) \end{array}$$

Therefore, it suffices to prove that the trace map on de Rham cohomology is surjective. This map fits into the following commutative diagram with short exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X_n, \Omega^1_{X_n/k}) & \longrightarrow & H^1_{\mathrm{dR}}(X_n/k) & \longrightarrow & H^1(X_n, \mathcal{O}_{X_n}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\ 0 & \longrightarrow & H^0(X_m, \Omega^1_{X_m/k}) & \longrightarrow & H^1_{\mathrm{dR}}(X_m/k) & \longrightarrow & H^1(X_m, \mathcal{O}_{X_m}) & \longrightarrow & 0 \end{array} \tag{3.9}$$

Therefore, it is enough to prove that the flanking vertical maps are surjective. Dualizing and using Serre duality, we equivalently wish to show that the pullback maps

$$\pi^* : H^1(X_m, \mathcal{O}_{X_m}) \longrightarrow H^1(X_n, \mathcal{O}_{X_n}) \quad \text{and} \quad \pi^* : H^0(X_m, \Omega^1_{X_m/k}) \longrightarrow H^0(X_n, \Omega^1_{X_n/k}) \tag{3.10}$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

are injective. As in the proof of Lemma 3.5, the second map of (3.10) is injective as $\pi_{n,m}$ is generically étale, while the first map is injective due to Proposition 2.25, as in the proof of Proposition 2.26. □

Proof of Theorem D. We apply $\varprojlim$ to the exact sequences of inverse systems given by (3.7) and (3.8). By Lemma 3.7, the inverse system $\{\mathbf{D}_n^{\mathrm{m}}, \rho\}$ is Mittag–Leffler, so the derived limit $\varprojlim^1_{n,\rho} \mathbf{D}_n^{\mathrm{m}}$ vanishes, which yields the short exact sequence of Theorem D(i). Likewise, $\mathbf{D}(\mathscr{Q}_S)$ is a finite free W-module, so it follows from Lemma 2.11 that applying $\varprojlim$ to (3.8) results in an isomorphism of Λ-modules $\mathbf{D}_S^{\text{ét}} \simeq \mathbf{D}^{\text{ét}}$, as in Theorem D(i). This isomorphism and the control aspect of Theorem C yield canonical isomorphisms of Λ_n-modules

$$\Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\text{ét}} \simeq \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^{\text{ét}} \simeq \mathbf{D}_{n,S}^{\text{ét}}$$

and the first short exact sequence of Theorem D(ii) follows from (3.8). To obtain the second exact sequence, we apply $\Lambda_n \otimes_\Lambda (\cdot)$ to the short exact sequence of (i) to get a long exact sequence that fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}^1_\Lambda(\Lambda_n, \mathbf{D}(\mathscr{T}_S)) & \longrightarrow & \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\mathrm{m}} & \longrightarrow & \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^{\mathrm{m}} & \longrightarrow & \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}(\mathscr{T}_S) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & & & \mathbf{D}_n^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{n,S}^{\mathrm{m}} & \longrightarrow & \mathbf{D}(\mathscr{T}_S) \longrightarrow 0 \end{array}$$

whose bottom row is (3.7) and whose vertical maps are the canonical projections. The rightmost vertical map is identified with the multiplication map which, as the Λ-module structure map of $\mathbf{D}(\mathscr{T}_S)$ factors through the augmentation map $\Lambda \twoheadrightarrow \Lambda_n \twoheadrightarrow W$, is easily seen to be an isomorphism. The middle vertical map is an isomorphism due to Theorem C(ii), so a diagram chase yields the short exact sequence

$$0 \longrightarrow \mathrm{Tor}^1_\Lambda(\Lambda_n, \mathbf{D}(\mathscr{T}_S)) \longrightarrow \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\mathrm{m}} \longrightarrow \mathbf{D}_n^{\mathrm{m}} \longrightarrow 0. \tag{3.11}$$

On the other hand, applying $(\cdot) \otimes_\Lambda \mathbf{D}(\mathscr{T}_S)$ to the tautological exact sequence

$$0 \longrightarrow I_n \longrightarrow \Lambda \longrightarrow \Lambda_n \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow \mathrm{Tor}^\Lambda_1(\Lambda_n, \mathbf{D}(\mathscr{T}_S)) \longrightarrow I_n \underset{\Lambda}{\otimes} \mathbf{D}(\mathscr{T}_S) \longrightarrow \Lambda \underset{\Lambda}{\otimes} \mathbf{D}(\mathscr{T}_S) \longrightarrow \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}(\mathscr{T}_S) \longrightarrow 0.$$

Again, as the Λ-module structure on $\mathbf{D}(\mathscr{T}_S)$ factors through $\Lambda \to \Lambda_n \to W$, the final map above is an isomorphism, which yields

$$\mathrm{Tor}^\Lambda_1(\Lambda_n, \mathbf{D}(\mathscr{T}_S)) \simeq I_n \underset{\Lambda}{\otimes} \mathbf{D}(\mathscr{T}_S) \simeq (I_n \underset{\Lambda}{\otimes} W) \underset{W}{\otimes} \mathbf{D}(\mathscr{T}_S) \simeq \frac{I_n}{I \cdot I_n} \underset{W}{\otimes} \mathbf{D}(\mathscr{T}_S), \tag{3.12}$$

where the final isomorphism results from the canonical identification $W \simeq \Lambda/I$ via the augmentation map. Putting (3.11) and (3.12) together gives the second exact sequence in Theorem D(ii).

It remains to establish the explicit descriptions of $\mathbf{D}(\mathscr{T}_S)$ and $\mathbf{D}(\mathscr{Q}_S)$ given in Theorem D(iii). By Corollary 2.18 and the very definition of $\mathscr{T}_S$, we have a short exact sequence of p-divisible groups

$$0 \longrightarrow \mu_{p^\infty} \xrightarrow{\Delta} \mathrm{Res}_{S/k}\, \mu_{p^\infty} \longrightarrow \mathscr{T}_S \longrightarrow 0.$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Applying $\mathbf{D}(\cdot)$ to this gives the claimed description of $\mathbf{D}(\mathscr{T}_S)$, and the description of $\mathbf{D}(\mathscr{Q}_S)$ then follows from the very definition $\mathscr{Q}_S := \mathscr{T}_S^\vee$ and the compatibility of $\mathbf{D}(\cdot)$ with duality. □

Proof of Theorem E. Let d be the dimension of the p-adic Lie group Γ, and without loss of generality assume that $\{\Gamma_n\}_n$ is the lower central p-series of Γ, so that Γ/Γ_n is a finite p-group of order p^{dn}.

The functorial decompositions $\mathscr{G}_n = \mathscr{G}_n^{\text{ét}} \times \mathscr{G}_n^{\text{m}} \times \mathscr{G}_n^{\text{ll}}$ yield a canonical isomorphism of Λ-modules

$$\mathbf{D} \simeq \mathbf{D}^{\text{ét}} \times \mathbf{D}^{\text{m}} \times \mathbf{D}^{\text{ll}},$$

so as $\mathbf{D}^\star$ is finitely generated (even as a Λ-module) for $\star = \text{ét}, \text{m}$ due to Theorems C and D, it suffices to prove that $\mathbf{D}$ is not a finitely generated $\Lambda[\![F, V]\!]$-module. Assume to the contrary that it is, and choose $\Lambda[\![F, V]\!]$-module generators $\delta_1, \ldots, \delta_r$. Since the transition maps $\rho : \mathbf{D}_n \to \mathbf{D}_m$ are surjective due to Lemma 3.7, the canonical projection $\mathbf{D} \to \mathbf{D}_n$ is surjective for every n, whence the images of $\delta_1, \ldots, \delta_r$ generate $\mathbf{D}_n$ as a $\Lambda_n[F, V]$-module. It follows that $\mathbf{D}_n/(F, V)$ is generated by at most r elements as a Λ_n-module, for all n. The canonical isomorphisms $\mathbf{D}_n/p\mathbf{D}_n \simeq H^1_{dR}(X_n/k)$ of Oda's theorem induce isomorphisms

$$D_n/(F, V) \simeq \operatorname{coker}(V : H^0(\Omega^1_{X_n}) \to H^0(\Omega^1_{X_n}))$$

for each n so that the cokernel of V is generated as an Ω_n-module by at most r generators. It follows that the cokernel, and hence the kernel, of V on $H^0(\Omega^1_{X_n})$ has k-dimension at most $r \cdot |\Gamma/\Gamma_n| = rp^{dn}$. Writing a_n for this dimension, we will prove that for *any* $D > 0$, there exists N with $a_n > Dp^{dn}$ for all $n > N$, contradicting the above bound coming from our finite generation assumption. To do so, we may, and henceforth do, assume that k is algebraically closed.

Since Γ/Γ_n is a p-group, it is solvable, and we may find a subgroup $H_n \leqslant \Gamma/\Gamma_n$ of order p. Let $L_n \subset K_n$ be the fixed field of H_n, and so K_n/L_n a degree-p Galois extension, with Galois group $H_n \simeq \mathbf{Z}/p\mathbf{Z}$, totally ramified over the finite set of places of L_n lying over $S \subseteq K_0$. For $Q \in S$, let $d_{Q,n}$ be the unique break in the lower numbering ramification filtration of H_n above Q. As the lower numbering of the ramification groups is inherited by subgroups, $d_{Q,n}$ is the largest lower break in the ramification filtration of Γ/Γ_n above Q. Working locally at Q, let $K_{n,Q}$ be the localization of K_n at Q, and L_Q the compositum of all $K_{n,Q}$. Then L_Q/K_Q is *strictly APF* [Win79, Théorème 1.2], so the Herbrand functions $\psi(x) := \int_0^x [\Gamma : \Gamma^t] dt$ and $\phi := \psi^{-1}$ are well-defined and continuous, piecewise linear increasing bijections on $[0, \infty)$. Let $u_n := \phi(d_n)$ be the last upper break in the ramification filtration of Γ/Γ_n at Q, so that $d_n = \psi(u_n)$. By [Win79, Théorème 1.2], the ratio $\psi(x)/[\Gamma : \Gamma^x]$ tends to infinity with x. It follows that for any $D > 0$, we have $\psi(x) > D[\Gamma : \Gamma^x]$ for all x sufficiently large. Evaluating on $x = u_n + \varepsilon$, and noting that by definition of u_n we have $\Gamma^x \subseteq \Gamma_n$ for all $x > u_n$, we find that for any D, there exists N so that, for all $n > N$ and all $\varepsilon > 0$,

$$\psi(u_n + \varepsilon) > D[\Gamma : \Gamma_n] \cdot [\Gamma_n : \Gamma^{u_n + \varepsilon}]. \tag{3.13}$$

As ψ is continuous, we conclude that $d_{Q,n} = \psi(u_n) \geqslant D[\Gamma : \Gamma_n]$ for all $n > N$. We may clearly arrange for this to hold for all $Q \in S$ simultaneously.

On the other hand, by [BC20, Theorem 1.1] we have the lower bound

$$a_n > \left(1 - \frac{1}{p}\right) \left\lfloor \frac{p}{2} \right\rfloor \sum_{Q \in S} \left(\left\lceil \frac{p}{2} \right\rceil \frac{d_{Q,n}}{p} - 1 \right)$$

and we conclude that for any $D > 0$, there exists N so that for all $n > N$ we have $a_n > D|\Gamma/\Gamma_n|$. □

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

To prove Theorem F, we will need the following lemma.

LEMMA 3.8. *Let A be an abelian variety over a finite field k of cardinality p^r, let $\mathscr{G} := A[p^\infty]$ be the p-divisible group of A, and let $\wp := 1 - F^r$ be the Lang isogeny. There is a functorial exact sequence*

$$0 \longrightarrow \mathbf{D}(\mathscr{G}^{\text{ét}}) \xrightarrow{\wp} \mathbf{D}(\mathscr{G}^{\text{ét}}) \longrightarrow \mathscr{G}(k)^* \otimes_{\mathbf{Z}_p} W \longrightarrow 0,$$

where $\mathscr{G}(k)^* := \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{G}(k), \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin dual of $\mathscr{G}(k)$.

Proof. As A is smooth and geometrically connected, $\wp : A \to A$ is surjective, and yields a short exact sequence of abelian sheaves

$$0 \longrightarrow A(k) \longrightarrow A \xrightarrow{\wp} A \longrightarrow 0,$$

from which we deduce the exact sequence

$$0 \longrightarrow \mathscr{G}(k) \longrightarrow \mathscr{G}^{\text{ét}} \xrightarrow{\wp} \mathscr{G}^{\text{ét}} \longrightarrow 0. \tag{3.14}$$

Indeed, for each n, multiplication by p^n and the snake lemma give exact sequences

$$0 \longrightarrow A[p^n](k) \longrightarrow A[p^n] \xrightarrow{\wp} A[p^n] \longrightarrow A(k)/p^n A(k) \longrightarrow 0, \tag{3.15}$$

which, for variable n, form an inductive system in which the transition maps $A(k)/p^n A(k) \to A(k)/p^{n+1}A(k)$ are multiplication by p. Passing to étale parts (which is exact) and applying the exact functor $\varinjlim$ to (3.15) therefore yields (3.14). Now, the Dieudonné module functor for a constant group scheme $\underline{H}$ is particularly simple:

$$\mathbf{D}(\underline{H}) = \varinjlim_n \mathrm{Hom}_{k\text{-gp}}(\underline{H}, W_n) = (\mathrm{Hom}(H, \mathbf{Q}_p/\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} W(\overline{k}))^{\mathrm{Gal}(\overline{k}/k)} = \mathrm{Hom}(H, \mathbf{Q}_p/\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} W,$$

as the Galois action on $\underline{H}(\overline{k}) = H$ is trivial. Applying $\mathbf{D}(\cdot)$ to (3.14) therefore gives the exact sequence of Lemma 3.8. □

Remark 3.9. If G is *any* p-divisible group or finite k-group, one always has an exact sequence of étale abelian sheaves

$$0 \longrightarrow G(k) \longrightarrow G^{\text{ét}} \xrightarrow{\wp} G^{\text{ét}},$$

which yields the isomorphism of finite length W-modules

$$\mathbf{D}(G^{\text{ét}})/\wp\mathbf{D}(G^{\text{ét}}) \simeq G(k)^* \otimes_{\mathbf{Z}_p} W.$$

Proof of Theorem F. By Lemma 3.8, for each n we have exact sequences of Λ_n-modules,

$$0 \longrightarrow \mathbf{D}(\mathscr{G}_n^{\text{ét}}) \xrightarrow{\wp} \mathbf{D}(\mathscr{G}_n^{\text{ét}}) \longrightarrow \mathscr{G}_n(k)^* \otimes_{\mathbf{Z}_p} W \longrightarrow 0, \tag{3.16}$$

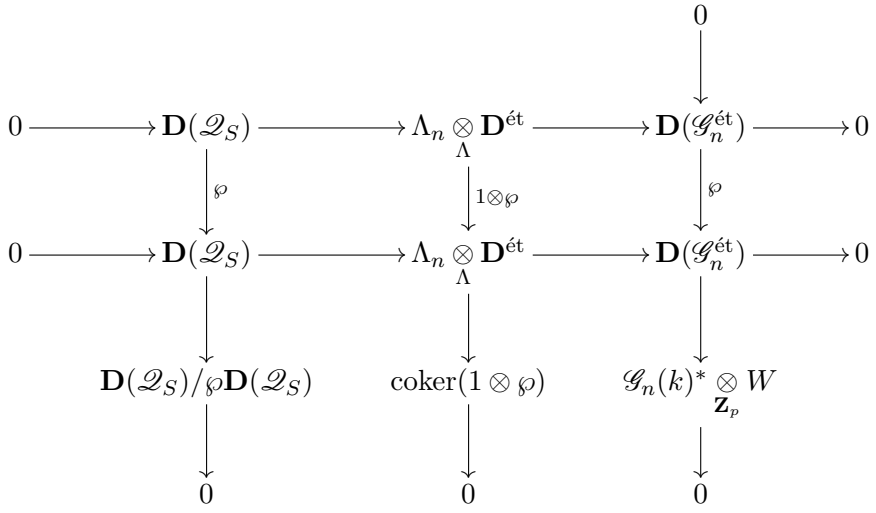
which are compatible with change in n using the maps induced by Picard functoriality. By Lemma 3.7, applying $\varprojlim$ results in the first short exact sequence of Theorem F once we observe that the canonical map

$$M_W := (\varprojlim_n \mathscr{G}_n(k)^*) \otimes_{\mathbf{Z}_p} W \longrightarrow \varprojlim_n (\mathscr{G}_n(k)^* \otimes_{\mathbf{Z}_p} W)$$

is an isomorphism, due to the fact that $\mathscr{G}_n(k)$ is *finite* (and hence of finite length as a $\mathbf{Z}_p$-module) and W is finitely presented as a $\mathbf{Z}_p$-module. Theorem D(ii) and the exact sequence (3.16) provide

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

the following commutative diagram with exact rows and columns.



From this, we deduce (via the snake lemma) the short exact sequence of Λ_n-modules

$$0 \longrightarrow \mathbf{D}(\mathscr{Q}_S)/\wp\mathbf{D}(\mathscr{Q}_S) \longrightarrow \operatorname{coker}(1 \otimes \wp) \longrightarrow \mathscr{G}_n(k)^* \underset{\mathbf{Z}_p}{\otimes} W \longrightarrow 0.$$

As the formation of cokernels commutes with tensor product, the already established first exact sequence of Theorem F identifies the middle term above with $\Lambda_n \otimes_\Lambda M_W$, while Remark 3.9 provides the identification $\mathbf{D}(\mathscr{Q}_S)/\wp\mathbf{D}(\mathscr{Q}_S) \simeq \mathscr{Q}_S(k)^* \otimes_{\mathbf{Z}_p} W$; this yields the second exact sequence of Theorem F. □

Remark 3.10. If every point of S is k-rational, then F^r acts trivially on $\mathbf{D}(\mathscr{Q}_S)$ due to the explicit description given in Theorem D(iii), so that $\wp = 0$ on $\mathbf{D}(\mathscr{Q}_S)$ in this case.

Proof of Corollary 1.5. Applying $(\cdot) \otimes_W k$ to the second exact sequence of Theorem F gives an exact sequence

$$\mathscr{Q}_S(k)^* \underset{\mathbf{Z}_p}{\otimes} k \longrightarrow (\Lambda_n \underset{\Lambda}{\otimes} M_W) \underset{W}{\otimes} k \longrightarrow \mathscr{G}_n(k)^* \underset{\mathbf{Z}_p}{\otimes} k \longrightarrow 0.$$

On the other hand, for each n we have canonical isomorphisms

$$(\Lambda_n \underset{\Lambda}{\otimes} M_W) \underset{W}{\otimes} k \simeq \Omega_n \underset{\Omega}{\otimes} (M_W \underset{W}{\otimes} k)$$

and for any abelian group G, we have $G^*/pG^* = (G[p])^*$, whence the exact sequence

$$\mathscr{Q}_S[p](k)^* \underset{\mathbf{F}_p}{\otimes} k \longrightarrow \Omega_n \underset{\Omega}{\otimes} (M_W \underset{W}{\otimes} k) \longrightarrow \mathscr{G}_n[p](k)^* \underset{\mathbf{F}_p}{\otimes} k \longrightarrow 0,$$

and we deduce that for all n,

$$\log_p |\mathscr{G}_n[p](k)| = \dim_k(\mathscr{G}_n[p](k) \underset{\mathbf{F}_p}{\otimes} k) = \dim_k(\Omega_n \underset{\Omega}{\otimes} (M_W \otimes_W k)) + O(1).$$

On the other hand, applying $(\cdot) \otimes_W k$ to the first exact sequence of Theorem F shows that $M_W \otimes_W k$ is a finitely generated Ω-module, and the result follows from [EP20, Proposition 2.18] for general Γ and from [Mon83a, Theorem 1.8] when $\Gamma = \mathbf{Z}_p^d$ is abelian. □

Remark 3.11. As noted in [EP20, Remark 2.19], a stronger version of [EP20, Proposition 2.18], which would imply that we may *always* take $\mu = \nu$ in Corollary 1.5, is stated in [CE11, Theorem 2.3], which refers to [AB06, § 5] for a proof. We have unfortunately not been able to deduce a proof

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

of [CE11, Theorem 2.3] from the provided reference. According to [EP20, Remark 2.20], a weaker version of [EP20, Proposition 2.18] follows from (the proof of) [Har00, Theorem 1.10], though we have not checked the details (see also [Har79]). In [Har00], Harris writes 'I understand that Y. Ochi, in his 1998 Cambridge PhD thesis, has also found [a] new [proof] of Theorem 1.10 ...'. However, in the introduction to his 1999 Cambridge PhD thesis, Ochi writes '...the asymptotic formula [Theorem 1.10] of Harris will not be found in this paper' [Och99, p. 14].

3.2 The étale case

We now suppose that $\Sigma = \emptyset$ – i.e. that $\pi_{n,0} : X_n \to X_0$ is étale for all n – and we recall that $S = \{x_0\}$ for a fixed closed point x_0 of X_0, and $S_n = \pi_{n,0}^{-1}(S_0)$ is the fiber of $\pi_{n,0}$ over S. The proofs of Theorems A and B proceed along much the same lines as those of Theorems C and D, though there are some key differences.

Lemma 3.12. *For $\star \in \{\text{ét}, \text{m}, \text{ll}\}$ and each pair of integers $n \geqslant m \geqslant 0$, the maps $\overline{\rho} : \overline{\mathbf{D}}^{\star}_{n,S} \to \overline{\mathbf{D}}^{\star}_{n,S}$ are surjective. Moreover, $\overline{\mathbf{D}}^{\star}_{n,S}$ is a free Ω_n-module of rank γ when $\star = \text{m}, \text{ét}$, and is free of rank $2(g-\gamma)$ when $\star = \text{ll}$.*

Proof. As noted in Remark 3.4, we have $\mathbf{D}^{\text{ll}}_{n,S} \simeq \mathbf{D}^{\text{ll}}_{n}$ for all n, so that $\mathbf{D}^{\text{ll}}_{S} \simeq \mathbf{D}^{\text{ll}}$. We first show that for all integers $m \leqslant n$, the map $\overline{\rho} : \overline{\mathbf{D}}^{\text{ll}}_{n} \to \overline{\mathbf{D}}^{\text{ll}}_{m}$ is surjective, and $\overline{\mathbf{D}}^{\text{ll}}$ is a free Ω_n-module of rank $2(g-\gamma)$. To do so, as in the proof of Lemma 3.5, we may assume that k is algebraically closed. Oda's theorem gives a commutative diagram with horizontal isomorphisms

$$\begin{array}{ccc} \overline{\mathbf{D}}^{\text{ll}}_{n} & \xrightarrow{\simeq} & H^1_{\text{dR}}(X_n/k)^{F,V\text{-nil}} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi_*} \\ \overline{\mathbf{D}}^{\text{ll}}_{m} & \xrightarrow{\simeq} & H^1_{\text{dR}}(X_m/k)^{F,V\text{-nil}} \end{array}$$

wherein $M^{F,V\text{-nil}}$ denotes the maximal submodule of M on which F and V are nilpotent. The trace map on de Rham cohomology fits into the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X_n, \Omega^1_{X_n/k})^{F,V\text{-nil}} & \longrightarrow & H^1_{\text{dR}}(X_n/k)^{F,V\text{-nil}} & \longrightarrow & H^1(X_n, \mathcal{O}_{X_n})^{F,V\text{-nil}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\ 0 & \longrightarrow & H^0(X_m, \Omega^1_{X_m/k})^{F,V\text{-nil}} & \longrightarrow & H^1_{\text{dR}}(X_m/k)^{F,V\text{-nil}} & \longrightarrow & H^1(X_m, \mathcal{O}_{X_m})^{F,V\text{-nil}} & \longrightarrow & 0 \end{array} \tag{3.17}$$

Therefore, it suffices to prove that the flanking vertical maps are surjective and that each of the flanking terms (say, in the top row) is free over Ω_n of rank $g-\gamma$. As π is separable, pullback of differential forms is injective, whence the dual map $\pi_* : H^1(X_n, \mathcal{O}_{X_n}) \to H^1(X_m, \mathcal{O}_{X_m})$ is surjective: as the right vertical map in (3.17) is a direct summand of this map, it is surjective as well. On the other hand, for *any* smooth and proper curve X/k with *reduced* divisor S, the canonical inclusion

$$H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^1_{X/k}(S))$$

induces an isomorphism on V-nilpotent subspaces, due to the formula $\text{res}_x(V\eta)^p = \text{res}_x(\eta)$. Thus, the surjectivity of the left vertical map in (3.17) follows from Proposition 2.25.

By Nakajima's theorem 2.24(ii), the Ω_n-module $H^0(X_n, \Omega^1_{X_n/k})^{F,V\text{-nil}}$ is free of rank $g-\gamma$. Since $H^1(X, \mathcal{O}_X)^{F,V\text{-nil}}$ is functorially dual to this by Serre duality, it is isomorphic to the contragredient Ω_n-module and hence free of rank $g-\gamma$ as well.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

The argument that $\overline{\mathbf{D}}^{\mathrm{m}}_S$ and $\overline{\mathbf{D}}^{\text{ét}}_S$ are each free Ω_n-modules of rank γ and that the mod p transition maps are surjective is essentially the same as in the proof of Lemma 3.5: by Oda's theorem and duality, it is enough to prove that $H^0(X_n, \Omega^1_{S_n}(S_n))^{V\text{-bij}}$ is free as a Ω_n-module, and the pullback (respectively, trace) map

$$H^0(X_m, \Omega^1_{X_m/k}(S_m)) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} H^0(X_n, \Omega^1_{X_n/k}(S_n))$$

is *injective* (respectively, surjective). The desired freeness follows from Nakajima's theorem, and pullback of differentials along a generically étale (even étale!) map is injective. The surjectivity of π_* is Corollary 2.26. □

Proof of Theorem A. Lemma 3.12 and the discussion below (3.3) show that the Γ-towers of W-modules $\{\mathbf{D}^{\mathrm{m}}_{n,S_n}, \rho\}$ and $\{\mathbf{D}^{\text{ét}}_{n,S_n}, \rho'\}$ satisfy the hypotheses of Propositions 2.9 and 2.10, as does $\{\mathbf{D}^{\mathrm{ll}}_{n,S_n}, \rho\}$. The conclusions of these propositions immediately give Theorem A. □

For a finite group G, write BT_G for the category of p-divisible groups equipped with an action of G by automorphisms; for simplicity, we put $\mathrm{BT} := \mathrm{BT}_1$. If H is a subgroup of G, we write Ind^G_H (respectively, CoInd^G_H) for the functor which is left (respectively, right) adjoint to the forgetful functor $\mathrm{BT}_G \to \mathrm{BT}_H$. Explicitly, if $g_1, \ldots, g_n$ are coset representatives for H in G, then $\mathrm{Ind}^G_H \mathscr{G}$ (respectively, CoInd^G_H) is the coproduct (respectively, product) of n copies of $\mathscr{G}$, indexed by $\{g_i\}$, with obvious G-action. As finite products and coproducts coincide, these functors are naturally isomorphic. We write $\mathrm{Nm} : \mathrm{Ind}^G_H \mathscr{G} \to \mathscr{G}$ (respectively, $\Delta : \mathscr{G} \to \mathrm{CoInd}^G_H \mathscr{G}$) for the unique morphism corresponding to the identity map on $\mathscr{G}$ via the left (respectively, right) adjointness of each functor. If $H = 1$, we simply write Ind^G and CoInd^G in place of Ind^G_1 and CoInd^G_1, respectively. Note that $\mathrm{Ind}^G_H \circ \mathrm{Ind}^H = \mathrm{Ind}^G$, and when H is normal in G and acts trivially on $\mathscr{G}$, we may view $\mathscr{G}$ in BT or in BT_H, and have the formula $\mathrm{Ind}^G_H \mathscr{G} = \mathrm{Ind}^{G/H} \mathscr{G}$ in BT_G; analogous formulae hold for CoInd. In this way, if H is normal in G and acts trivially on $\mathscr{G}$, we obtain maps in BT_G,

$$\mathrm{CoInd}^{G/H} \mathscr{G} \simeq \mathrm{CoInd}^G_H \mathscr{G} \xrightarrow{\mathrm{CoInd}^G_H(\Delta)} \mathrm{CoInd}^G_H \mathrm{CoInd}^H \mathscr{G} \simeq \mathrm{CoInd}^G \mathscr{G}$$

and

$$\mathrm{Ind}^G \mathscr{G} \simeq \mathrm{Ind}^G_H \mathrm{Ind}^H \mathscr{G} \xrightarrow{\mathrm{Ind}^G_H(\mathrm{Nm})} \mathrm{Ind}^G_H \mathscr{G} \simeq \mathrm{Ind}^{G/H} \mathscr{G},$$

which, by a slight abuse of notation, we again denote simply by Δ and Nm, respectively.

Lemma 3.13. *Assume that the chosen point x_0 is k-rational, and that the étale k-scheme $S_n = \pi_n^{-1} x_0$ splits for all n. For each pair of positive integers $m \leqslant n$, the map $\pi_{n,m} : X_n \to X_m$ induces commutative diagrams of p-divisible groups with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_{p^\infty} & \xrightarrow{\Delta} & \mathrm{CoInd}^{\Gamma/\Gamma_n} \mu_{p^\infty} & \longrightarrow & \mathscr{G}_{n,S_n} & \longrightarrow & \mathscr{G}_n \longrightarrow 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\
0 & \longrightarrow & \mu_{p^\infty} & \xrightarrow{\Delta} & \mathrm{CoInd}^{\Gamma/\Gamma_m} \mu_{p^\infty} & \longrightarrow & \mathscr{G}_{m,S_m} & \longrightarrow & \mathscr{G}_m \longrightarrow 0
\end{array} \tag{3.18a}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_{p^\infty} & \xrightarrow{\Delta} & \mathrm{Ind}^{\Gamma/\Gamma_n} \mu_{p^\infty} & \longrightarrow & \mathscr{G}_{n,S_n} & \longrightarrow & \mathscr{G}_n \longrightarrow 0 \\
 & & \downarrow{\scriptstyle [\Gamma_m:\Gamma_n]} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
0 & \longrightarrow & \mu_{p^\infty} & \xrightarrow{\Delta} & \mathrm{Ind}^{\Gamma/\Gamma_m} \mu_{p^\infty} & \longrightarrow & \mathscr{G}_{m,S_m} & \longrightarrow & \mathscr{G}_m \longrightarrow 0
\end{array} \tag{3.18b}$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

compatibly with change in m, n. Moreover, (3.18a) and (3.18b) are equivariant for the $(\cdot)^*$-action and the $(\cdot)_*$-action of Γ, respectively.

Proof. The hypotheses that x_0 is k-rational and that S_n splits for all n ensure that one has natural isomorphisms of functors $\mathrm{BT} \to \mathrm{BT}_{\Gamma/\Gamma_n}$:

$$\mathrm{Ind}^{\Gamma/\Gamma_n} \simeq \mathrm{Res}_{S_n/k} \simeq \mathrm{CoInd}^{\Gamma/\Gamma_n}.$$

This result then follows from Proposition 2.16 and Corollary 2.18. □

Next, we apply $\mathbf{D}(\cdot)$ to the multiplicative part of (3.18a), which yields the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{D}_n^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{n,S_n}^{\mathrm{m}} & \longrightarrow & \Lambda_n & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbf{D}_m^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{m,S_m}^{\mathrm{m}} & \longrightarrow & \Lambda_m & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

In this, pr is the canonical quotient map, and we deduce the following commutative diagram with short exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{D}_n^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{n,S_n}^{\mathrm{m}} & \longrightarrow & I_{\Gamma/\Gamma_n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{pr}} & & \\
0 & \longrightarrow & \mathbf{D}_m^{\mathrm{m}} & \longrightarrow & \mathbf{D}_{m,S_m}^{\mathrm{m}} & \longrightarrow & I_{\Gamma/\Gamma_m} & \longrightarrow & 0
\end{array} \tag{3.19}$$

Proof of Theorem B. As in Remark 3.4, we have $\mathbf{D}_{n,S_n}^{\mathrm{ll}} \simeq \mathbf{D}_n^{\mathrm{ll}}$ for all n so that (i) of Theorem B follows from Theorem A(i) and (ii). Passing to projective limits on (3.19) and using Lemma 3.12 yields the second exact sequence in Theorem B(ii). To obtain the first, we pass to multiplicative parts on (3.18b), dualize, apply $\mathbf{D}(\cdot)$, and invoke the canonical identifications $\mathscr{G}_n^\vee \simeq \mathscr{G}$ to obtain the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{\Delta_n} & \Lambda_n & \longrightarrow & \mathbf{D}_{n,S_n}^{\text{ét}} & \longrightarrow & \mathbf{D}_n^{\text{ét}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [\Gamma_m:\Gamma_n]} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & W & \xrightarrow[\Delta_m]{} & \Lambda_m & \longrightarrow & \mathbf{D}_{m,S_m}^{\text{ét}} & \longrightarrow & \mathbf{D}_m^{\text{ét}} & \longrightarrow & 0
\end{array} \tag{3.20}$$

In this, pr is the canonical projection and $\Delta_n : W \to \Lambda_n$ is the 'diagonal map' taking $\alpha \in W$ to $\alpha \sum_{\gamma \in \Gamma/\Gamma_n} \gamma$. We break this diagram up into two diagrams with short exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{\Delta_n} & \Lambda_n & \longrightarrow & \operatorname{coker} \Delta_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [\Gamma_m:\Gamma_n]} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow & & \\
0 & \longrightarrow & W & \xrightarrow[\Delta_m]{} & \Lambda_m & \longrightarrow & \operatorname{coker} \Delta_m & \longrightarrow & 0
\end{array} \tag{3.21}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker} \Delta_n & \longrightarrow & \mathbf{D}_{n,S_n}^{\text{ét}} & \longrightarrow & \mathbf{D}_n^{\text{ét}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \operatorname{coker} \Delta_m & \longrightarrow & \mathbf{D}_{m,S_m}^{\text{ét}} & \longrightarrow & \mathbf{D}_m^{\text{ét}} & \longrightarrow & 0
\end{array} \tag{3.22}$$

As the projection map $\mathrm{pr} : \Lambda_n \to \Lambda_m$ is surjective for all $n \geqslant m$, so too is the map $\operatorname{coker} \Delta_n \to \operatorname{coker} \Delta_m$ due to the snake lemma applied to (3.21). It follows that the inverse system $\operatorname{coker} \Delta_n$ is

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Mittag–Leffler, so passing to inverse limits on (3.22) yields a short exact sequence of Λ-modules

$$0 \longrightarrow \varprojlim_n \operatorname{coker} \Delta_n \longrightarrow \mathbf{D}^{\text{ét}}(S) \longrightarrow \mathbf{D}^{\text{ét}} \longrightarrow 0. \tag{3.23}$$

On the other hand, passing to inverse limits on (3.21) gives, due to Lemma 2.11 and our assumption that Γ is infinite, an isomorphism of Λ-modules

$$\Lambda = \varprojlim_n \Lambda_n \simeq \varprojlim_n \operatorname{coker} \Delta_n. \tag{3.24}$$

Together with (3.23), this establishes the first exact sequence of Theorem B(ii).

By the very construction of the first exact sequence of Theorem B(ii) as the projective limit of the short exact sequence of inverse systems (3.22), for each n we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Tor}_1^{\Lambda}(\Lambda_n, \mathbf{D}^{\text{ét}}) & \longrightarrow & \Lambda_n \underset{\Lambda}{\otimes} (\varprojlim_n \operatorname{coker} \Delta_n) & \longrightarrow & \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}_S^{\text{ét}} & \longrightarrow & \Lambda_n \underset{\Lambda}{\otimes} \mathbf{D}^{\text{ét}} \longrightarrow 0 \\
 & & & & \downarrow\!\!\downarrow & & \downarrow\!\simeq & & \downarrow \\
0 & \longrightarrow & & & \operatorname{coker} \Delta_n & \longrightarrow & \mathbf{D}_{n,S_n}^{\text{ét}} & \longrightarrow & \mathbf{D}_n^{\text{ét}} \longrightarrow 0
\end{array}$$

In this, the leftmost vertical map is surjective, as was observed above, and the middle vertical map is an isomorphism by Theorem A(ii). The snake lemma then implies that the rightmost vertical map is an isomorphism for all n, and a diagram chase coupled with (3.24) and (3.21) yields the claimed isomorphism $\operatorname{Tor}_1^{\Lambda}(\Lambda_n, \mathbf{D}^{\text{ét}}) \simeq W$. □

Remark 3.14. In the proofs of Theorems D and B, we must show that certain derived limits vanish. To do so, we have appealed to the Mittag–Leffler criterion [Sta22, Tag 0598] and to Lemma 2.11, which relies on the explicit description of $\varprojlim^1$ given in [Sta22, Tag 091D]. In order to use these results, it is essential that all of our inverse systems are indexed by *countable* directed sets (see [EH76, Proposition 2.3] and cf. [Bou70, III.94 Exercise 4d]). It is for this reason that we have assumed at the outset that the infinite pro-p group Γ admits a *countable* basis of the identity consisting of open normal subgroups, which is equivalent to Γ being first countable as a topological space. Equivalently [HR79, Theorem 8.3], we demand that Γ be metrizable as a topological space. As in Remark 3.1, this condition is automatic whenever k itself is countable.

Remark 3.15. Writing ρ_S for the middle vertical map of (3.19), we obtain from Theorem A the description $\ker \rho_S = I_{\Gamma_m/\Gamma_n} \mathbf{D}^{\mathrm{m}}_{n,S_n}$. Applying the snake lemma to (3.19), and noting that the surjective map $\mathbf{D}^{\mathrm{m}}_{n,S_n} \twoheadrightarrow I_{\Gamma/\Gamma_n}$ carries $I_{\Gamma_m/\Gamma_n}\mathbf{D}^{\mathrm{m}}_{n,S_n}$ onto $I_{\Gamma_m/\Gamma_n} \cdot I_{\Gamma/\Gamma_n}$, we thereby deduce isomorphisms

$$\operatorname{coker}(\rho : \mathbf{D}^{\mathrm{m}}_n \to \mathbf{D}^{\mathrm{m}}_m) \simeq I_m/(I_m \cdot I + I_n).$$

Assuming that $\Gamma_n = P_n$ is the lower central p-series of Γ as in Definition 2.2, it follows from the above and Lemma 2.3 that $\operatorname{coker}(\rho)$ is annihilated by $I + p^{n-m}\Lambda$, whence $\operatorname{im}(\rho)$ contains $(I + p^{n-m}\Lambda)\mathbf{D}^{\mathrm{m}}_m$. Thus, while ρ is not surjective when $n > m$, its image is quite large. In general, we do not know anything more about $\operatorname{im}(\rho)$ and, in particular, we do not know whether or not $\mathbf{D}^{\mathrm{m},1}$ vanishes.

Proof of Corollary 1.1. Let $K_0 := K$ and for $n \geqslant 1$ inductively define K_n to be the compositum of *all* unramified $\mathbf{Z}/p\mathbf{Z}$-extensions of K_{n-1}. Note that K_n/K is a finite Galois p-extension, and that the maximal unramified p-extension of K is the union of all K_n. Let X_n be the unique smooth proper curve with function field K_n, so that $\{X_n\}_n$ is a Γ-tower of curves, with Γ the maximal pro-p quotient of the étale fundamental group of X_0.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

For each n, we claim that the transition map $\rho : \mathbf{D}^{\mathrm{m}}_{n+1} \to \mathbf{D}^{\mathrm{m}}_{n}$ has image contained in $p\mathbf{D}^{\mathrm{m}}_{n}$. Granting this claim, Lemma 2.11 gives $\mathbf{D}^{\mathrm{m}} = 0 = \mathbf{D}^{\mathrm{m},1}$ as desired. To prove the claim, by Oda's theorem 2.21 we are reduced to proving that for all n, the trace mapping

$$\pi_* : H^0(X_{n+1}, \Omega^1_{X_{n+1}})^{V\text{-bij}} \longrightarrow H^0(X_n, \Omega^1_{X_n})^{V\text{-bij}}$$

is *zero*. To see this, let $\omega \in H^0(X_{n+1}, \Omega^1_{X_{n+1}})^{V\text{-bij}}$ be arbitrary, and suppose that $\pi_*\omega \neq 0$. We will derive a contradiction.

As $\pi_*\omega \neq 0$, there exists $u \in H^1(X_n, \mathcal{O}_{X_n})$ with $\langle \pi_*\omega, u\rangle \neq 0$, where

$$\langle \cdot, \cdot, \rangle : H^0(X_n, \Omega^1_{X_n/k}) \times H^1(X_n, \mathcal{O}_{X_n}) \longrightarrow k$$

is the canonical (perfect!) Serre duality pairing. We may uniquely decompose $u = u_0 + z$ with $u_0 \in H^1(X_n, \mathcal{O}_{X_n})^{F\text{-bij}}$ and $z \in H^1(X_n, \mathcal{O}_{X_n})^{F,V\text{-nil}}$. Choose N large enough so that $F^N z = 0$: as $\pi_*\omega$ lies in the V-bijective subspace, we may find ω_0 so that $V^N\omega_0 = \pi_*\omega$. Then

$$\langle \pi_*\omega, u\rangle = \langle \pi_*\omega, u_0\rangle + \langle \pi_*\omega, z\rangle = \langle \pi_*\omega, u_0\rangle + \langle V^N\omega_0, z\rangle = \langle \pi_*\omega, u_0\rangle + \sigma^{-N}\langle \omega_0, F^N z\rangle = \langle \pi_*\omega, u_0\rangle.$$

As k is algebraically closed, the canonical map

$$H^1(X_n, \mathcal{O}_{X_n})^{F=1} \underset{\mathbf{F}_p}{\otimes} k \longrightarrow H^1(X_n, \mathcal{O}_{X_n})^{F\text{-bij}}$$

is an isomorphism of k-vector spaces, so that $H^1(X_n, \mathcal{O}_{X_n})^{F\text{-bij}}$ has a k-basis consisting of F-fixed vectors. Since $\langle \pi_*\omega, u_0\rangle \neq 0$, there must exist some F-fixed basis vector e with $\langle \pi_*\omega, e\rangle \neq 0$. Let $Y \to X_n$ be the unramified $\mathbf{Z}/p\mathbf{Z}$-cover of X_n corresponding to e, and let E be its function field. By construction, E is an unramified $\mathbf{Z}/p\mathbf{Z}$-extension of K_n, whence a subfield of K_{n+1}, so the covering map $\pi : X_{n+1} \to X_n$ factors as

$$\pi : X_{n+1} \xrightarrow{\rho} Y \xrightarrow{\tau} X_n.$$

We then compute

$$\langle \pi_*\omega, e\rangle = \langle \tau_*\rho_*\omega, e\rangle = \langle \rho_*\omega, \tau^* e\rangle = 0,$$

since by construction, the pullback $\tau^* e$ of e to $H^1(Y, \mathcal{O}_Y)^{F=1}$ is zero. This is a contradiction, whence $\pi_*\omega = 0$ as claimed.

From Theorem B(ii), we deduce a canonical isomorphism $\mathbf{D}^{\mathrm{m}}_S \simeq I$, whence I is a free Λ-module of rank γ by Theorem A(i). Writing $\Lambda' := \mathbf{Z}_p[\![\Gamma]\!]$ and I' for its augmentation ideal, it follows that I' is a free Λ'-module of rank γ. One way to see this is to use the base change isomorphism

$$\Lambda \underset{\Lambda'}{\otimes} \mathrm{Tor}^1_{\Lambda'}(\mathbf{F}_p, I') \simeq \mathrm{Tor}^1_{\Lambda}(\Lambda \underset{\Lambda'}{\otimes} \mathbf{F}_p, \Lambda \underset{\Lambda'}{\otimes} I') \simeq \mathrm{Tor}^1_{\Lambda}(k, I) = 0$$

together with faithful flatness of $\Lambda' \to \Lambda$ to conclude that $\mathrm{Tor}^1_{\Lambda'}(\mathbf{F}_p, I') = 0$, and hence that I' is a projective module over the local ring Λ'. The isomorphism $\Lambda \otimes_{\Lambda'} I' \simeq I$ then shows that I' has rank γ as a Λ'-module. If M is any finite Γ-module of p-power order, applying $\mathrm{Hom}_{\Lambda'}(\cdot, M)$ to the tautological exact sequence

$$0 \longrightarrow I' \longrightarrow \Lambda' \xrightarrow{\varepsilon} \mathbf{Z}_p \longrightarrow 0$$

therefore yields

$$H^i(\Gamma, M) \simeq \mathrm{Ext}^i_{\Lambda'}(\mathbf{Z}_p, M) = 0$$

for $i > 1$ as well as the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\Lambda'}(\mathbf{Z}_p, M) \xrightarrow{\varepsilon^*} \mathrm{Hom}_{\Lambda'}(\Lambda', M) \longrightarrow \mathrm{Hom}_{\Lambda'}(I', M) \longrightarrow \mathrm{Ext}^1_{\Lambda'}(\mathbf{Z}_p, M) \longrightarrow 0.$$

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

When $M = \mathbf{F}_p$, the map ε^* is an isomorphism as any Λ'-module homomorphism $\Lambda' \to \mathbf{F}_p$ factors through $\varepsilon : \Lambda' \twoheadrightarrow \mathbf{Z}_p$, and we find that

$$H^1(\Gamma, \mathbf{F}_p) \simeq \mathrm{Ext}^1_{\Lambda'}(\mathbf{Z}_p, \mathbf{F}_p) \simeq \mathrm{Hom}_{\Lambda'}(I', \mathbf{F}_p) \simeq \mathbf{F}_p^{\gamma}.$$

By Propositions 21 and 24 of [Ser02, Chapter I, § 4], Γ is a free pro-p group on γ generators. □

Proof of Corollary 1.4. This proceeds along the same lines as the proof of Corollary 1.5: from the isomorphisms $\mathbf{D}^{\mathrm{ll}}_n/J\mathbf{D}^{\mathrm{ll}}_n \simeq \mathbf{D}(\mathscr{G}_n[J])$ and Theorem B(i) we deduce

$$\mathbf{D}(\mathscr{G}_n[J]) \simeq \Omega_n \otimes_{\Omega} \mathbf{D}^{\mathrm{ll}}/J\mathbf{D}^{\mathrm{ll}},$$

and the asymptotic formula then follows from Theorem 2.19(vi), [EP20, Proposition 2.18], and, when $\Gamma = \mathbf{Z}_p^d$ is abelian, [Mon83a, Theorem 1.8]. □

Remark 3.16. Let Γ be a torsion-free p-adic Lie group of dimension d, and $\{X_n\}$ a Γ-tower with X_n corresponding to the nth subgroup in the lower central p-series of Γ. Note that when $J = (p)$, we know that $\mathbf{D}^{\mathrm{ll}}/J\mathbf{D}^{\mathrm{ll}}$ is a *free* Ω-module by Theorem B(i), and it follows from [Ven02, Proposition 3.5(ii)] that $\mathbf{D}^{\mathrm{ll}}/p\mathbf{D}^{\mathrm{ll}}$ has dimension $\delta = d$. The asymptotic formula of Corollary 1.5 therefore yields constants $\nu \geqslant \mu \geqslant 1/d!$ with $\mu p^{nd} + O(p^{(d-1)n}) \leqslant \log_p |\mathscr{G}_n^{\mathrm{ll}}[p]| \leqslant \nu p^{nd} + O(p^{(d-1)n})$. On the other hand, by definition $\log_p |\mathscr{G}_n^{\mathrm{ll}}[p]| = h_n$, where h_n is the height of the p-divisible group $\mathscr{G}_n$, and, since $\mathscr{G}_n$ is the p-divisible group of the Jacobian of X_n, one *knows* that $h_n = 2(g_n - \gamma_n)$, where g_n and γ_n are the genus and p-rank of X_n, respectively. From the Riemann–Hurwitz and Deuring–Shafarevich formulae, we thus obtain the *exact* formula

$$\log_p |\mathscr{G}_n^{\mathrm{ll}}[p]| = 2(g_0 - \gamma_0) \cdot p^{dn}$$

so we may take $\mu = \nu = 2(g_0 - \gamma_0)$ in this case. One has analogous exact formulae when $J = (F)$ or $J = (V)$; in general, the order of $\mathscr{G}_n^{\mathrm{ll}}[J]$ can be *extremely* subtle, and we do not know of any exact formula beyond the three cases $J = (p)$, $J = (F)$, and $J = (V)$ that we have mentioned. In this way, we may view Corollary 1.4 as an asymptotic generalization of the Deuring–Shafarevich formula to local–local p-torsion subgroup schemes in unramified Γ-towers.

Conflicts of interest

None.

Financial support

This work was supported by National Science Foundation grant numbers DMS-1902005 and DMS-2302072.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

AB06 K. Ardakov and K. A. Brown, *Ring-theoretic properties of Iwasawa algebras: A survey*, Doc. Math. **Extra Volume: John H. Coates' Sixtieth Birthday** (2006), 7–33; MR 2290583.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

BC20 J. Booher and B. Cais, *a-numbers of curves in Artin–Schreier covers*, Algebra Number Theory **14** (2020), 587–641; MR 4113776.

BC23 J. Booher and B. Cais, *Iwasawa theory for p-torsion class group schemes in characteristic p*, Nagoya Math. J. **250** (2023), 298–351.

BCK$^+$24 J. Booher, B. Cais, J. Kramer-Miller and J. Upton, *Iwasawa theory of Frobenius-torsion class group schemes*, Preprint (2024), arXiv:2407.13966.

BLR90 S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21 (Springer-Verlag, Berlin, 1990); MR 1045822.

Bou70 N. Bourbaki, *Éléments de mathématique. Théorie des ensembles* (Hermann, Paris, 1970); MR 0276101.

Cai10 B. Cais, *Canonical extensions of Néron models of Jacobians*, Algebra Number Theory **4** (2010), 111–150; MR 2592016.

Cai18a B. Cais, *The geometry of Hida families I: Λ-adic de Rham cohomology*, Math. Ann. **372** (2018a), 781–844; MR 3856828.

Cai18b B. Cais, *The geometry of Hida families II: Λ-adic (φ, Γ)-modules and Λ-adic Hodge theory*, Compositio Math. **154** (2018b), 719–760; MR 3778192.

CE11 F. Calegari and M. Emerton, *Mod-p cohomology growth in p-adic analytic towers of 3-manifolds*, Groups Geom. Dyn. **5** (2011), 355–366; MR 2782177.

CCO14 C.-L. Chai, B. Conrad and F. Oort, *Complex multiplication and lifting problems*, Mathematical Surveys and Monographs, vol. 195 (American Mathematical Society, Providence, RI, 2014); MR 3137398.

Con00 B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. 1750 (Springer-Verlag, Berlin, 2000); MR 1804902.

Cre84 R. M. Crew, *Etale p-covers in characteristic p*, Compositio Math. **52** (1984), 31–45; MR 742696.

Cuo80 A. A. Cuoco, *The growth of Iwasawa invariants in a family*, Compositio Math. **41** (1980), 415–437; MR 589090.

CM81 A. A. Cuoco and P. Monsky, *Class numbers in $\mathbf{Z}_p^d$-extensions*, Math. Ann. **255** (1981), 235–258; MR 614400.

DWX16 C. Davis, D. Wan and L. Xiao, *Newton slopes for Artin–Schreier–Witt towers*, Math. Ann. **364** (2016), 1451–1468; MR 3466874.

DdSM$^+$99 J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, Cambridge Studies in Advanced Mathematics, vol. 61, second edition (Cambridge University Press, Cambridge, 1999); MR 1720368.

DF14 N. Dummigan and S. Farwa, *Exact holomorphic differentials on a quotient of the Ree curve*, J. Algebra **400** (2014), 249–272; MR 3147373.

EH76 D. A. Edwards and H. M. Hastings, *On topological methods in homological algebra*, Proc. Amer. Math. Soc. **59** (1976), 389–393; MR 424906.

Elk11 A. Elkin, *The rank of the Cartier operator on cyclic covers of the projective line*, J. Algebra **327** (2011), 1–12; MR 2746026.

EP07 A. Elkin and R. Pries, *Hyperelliptic curves with a-number 1 in small characteristic*, Albanian J. Math. **1** (2007), 245–252; MR 2367217.

EP20 M. Emerton and V. Paškūnas, *On the density of supercuspidal points of fixed regular weight in local deformation rings and global Hecke algebras*, J. Éc. polytech. Math. **7** (2020), 337–371; MR 4077579.

Fon77 J.-M. Fontaine, *Groupes p-divisibles sur les corps locaux*, Astérisque, vol. 47–48 (Société Mathématique de France, Paris, 1977); MR 0498610.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Fre18 S. Frei, *The a-number of hyperelliptic curves*, in *Women in numbers Europe II*, Cham, 2016, eds I. I. Bouw, E. Ozman, J. Johnson-Leung and R. Newton (Springer International, 2018), 107–116.

FGM+13 H. Friedlander, D. Garton, B. Malmskog, R. Pries and C. Weir, *The a-numbers of Jacobians of Suzuki curves*, Proc. Amer. Math. Soc. **141** (2013), 3019–3028; MR 3068955.

Gil00 P. Gille, *Le groupe fondamental sauvage d'une courbe affine en caractéristique $p>0$*, in *Courbes semi-stables et groupe fondamental en géométrie algébrique*, Luminy, 1998, Progress in Mathematics, vol. 187 (Birkhäuser, Basel, 2000), 217–231; MR 1768103.

GK88 R. Gold and H. Kisilevsky, *On geometric $\mathbf{Z}_p$-extensions of function fields*, Manuscripta Math. **62** (1988), 145–161; MR 963002.

Har79 M. Harris, *p-adic representations arising from descent on abelian varieties*, Compositio Math. **39** (1979), 177–245; MR 546966.

Har00 M. Harris, *Correction to: 'p-adic representations arising from descent on abelian varieties' [Compositio Math. **39** (1979), no. 2, 177–245; MR0546966 (80j:14035)]*, Compositio Math. **121** (2000), 105–108; MR 1753112.

HR79 E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups, integration theory, group representations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, second edition (Springer-Verlag, Berlin, 1979); MR 551496.

KL81 N. M. Katz and S. Lang, *Finiteness theorems in geometric classfield theory*, Enseign. Math. (2) **27** (1981), 285–319 (1982), with an appendix by Kenneth A. Ribet; MR 659153.

KW88 T. Kodama and T. Washio, *Hasse-Witt matrices of Fermat curves*, Manuscripta Math. **60** (1988), 185–195; MR 924086.

Lam99 T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189 (Springer, New York, 1999); MR 1653294.

Lam01 T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, vol. 131, second edition (Springer, New York, 2001); MR 1838439.

Laz65 M. Lazard, *Groupes analytiques p-adiques*, Publ. Math. Inst. Hautes Études Sci. **26** (1965), 5–219; MR 209286.

LZ97 C. Li and J. Zhao, *Iwasawa theory of $\mathbf{Z}_p^d$-extensions over global function fields*, Expo. Math. **15** (1997), 315–337; MR 1486400.

LLR04 Q. Liu, D. Lorenzini and M. Raynaud, *Néron models, Lie algebras, and reduction of curves of genus one*, Invent. Math. **157** (2004), 455–518; MR 2092767, see corrigendum in MR 3858404.

Mat89 H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, second edition (Cambridge University Press, Cambridge, 1989), translated from the Japanese by M. Reid; MR 1011461.

MM74 B. Mazur and W. Messing, *Universal extensions and one dimensional crystalline cohomology*, Lecture Notes in Mathematics, vol. 370 (Springer-Verlag, Berlin, 1974); MR 0374150.

MW83 B. Mazur and A. Wiles, *Analogies between function fields and number fields*, Amer. J. Math. **105** (1983), 507–521; MR 701567.

Mon83a P. Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983a), 43–49; MR 697329.

Mon83b P. Monsky, *p-ranks of class groups in $\mathbf{Z}_p^d$-extensions*, Math. Ann. **263** (1983b), 509–514; MR 707245.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Mon86a P. Monsky, *Class numbers in $\mathbf{Z}_p^d$-extensions. II*, Math. Z. **191** (1986a), 377–395; MR 824440.

Mon86b P. Monsky, *Class numbers in $\mathbf{Z}_p^d$-extensions. III*, Math. Z. **193** (1986b), 491–514; MR 867344.

Mon87 P. Monsky, *Class numbers in $\mathbf{Z}_p^d$-extensions. IV*, Math. Z. **196** (1987), 547–572; MR 917237.

Mon89 P. Monsky, *Fine estimates for the growth of e_n in $\mathbf{Z}_p^d$-extensions*, in *Algebraic number theory*, Advanced Studies in Pure Mathematics, vol. 17 (Academic Press, Boston, MA, 1989), 309–330; MR 1097621.

MS18 M. Montanucci and P. Speziali, *The a-numbers of Fermat and Hurwitz curves*, J. Pure Appl. Algebra **222** (2018), 477–488; MR 3694465.

Nak85 S. Nakajima, *Equivariant form of the Deuring–Šafarevič formula for Hasse–Witt invariants*, Math. Z. **190** (1985), 559–566; MR 808922.

NSW08 J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, second edition (Springer-Verlag, Berlin, 2008); MR 2392026.

Neu88 A. Neumann, *Completed group algebras without zero divisors*, Arch. Math. (Basel) **51** (1988), 496–499; MR 973723.

Och99 Y. Ochi, *Iwasawa modules via homotopy theory*, PhD thesis, University of Cambridge (1999).

Oda69 T. Oda, *The first de Rham cohomology group and Dieudonné modules*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 63–135; MR 0241435.

Re01 R. Re, *The rank of the Cartier operator and linear systems on curves*, J. Algebra **236** (2001), 80–92; MR 1808346.

Ren19 R. Ren, *Newton slopes for twisted Artin–Schreier–Witt towers*, Int. J. Number Theory **15** (2019), 2089–2105; MR 4033247.

RWX$^+$18 R. Ren, D. Wan, L. Xiao and M. Yu, *Slopes for higher rank Artin–Schreier–Witt towers*, Trans. Amer. Math. Soc. **370** (2018), 6411–6432; MR 3814335.

RZ10 L. Ribes and P. Zalesskii, *Profinite groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, second edition (Springer-Verlag, Berlin, 2010); MR 2599132.

Ser58 J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p*, in *Symp. int. de topología algebraica, Int. symp. on algebraic topology* (Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958), 24–53; MR 0098097.

Ser88 J.-P. Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics, vol. 117 (Springer, New York, 1988), translated from French; MR 918564.

Ser02 J.-P. Serre, *Galois cohomology*, Springer Monographs in Mathematics, English edition (Springer-Verlag, Berlin, 2002), translated from French by Patrick Ion and revised by the author; MR 1867431.

Sha47 I. Shafarevitch, *On p-extensions*, Rec. Math. [Mat. Sbornik] N.S. **20** (1947), 351–363; MR 0020546.

Sta22 The Stacks Project Authors, *The Stacks Project* (2022), https://stacks.math.columbia.edu.

Ven02 O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc. (JEMS) **4** (2002), 271–311; MR 1924402.

Vol88 J. Voloch, *A note on algebraic curves in characteristic 2*, Comm. Algebra **16** (1988), 869–875; MR 932639.

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press

Wan19 D. Wan, *Class numbers and p-ranks in $\mathbb{Z}_p^d$-towers*, J. Number Theory **203** (2019), 139–154; MR 3991398.

WK86 T. Washio and T. Kodama, *Hasse–Witt matrices of hyperelliptic function fields*, Sci. Bull. Fac. Ed. Nagasaki Univ. **37** (1986), 9–15; MR 853690.

Win79 J.-P. Wintenberger, *Extensions de Lie et groupes d'automorphismes des corps locaux de caractéristique p*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), A477–A479; MR 529480.

Xia18 L. Xiang, *The stable property of Newton slopes for general Witt towers*, J. Number Theory **185** (2018), 144–159; MR 3734346.

Bryden Cais cais@arizona.edu
Department of Mathematics, The University of Arizona, Tucson, AZ 85721, USA

https://doi.org/10.1017/S0010437X26103455 Published online by Cambridge University Press