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On 2-dimensional aspherical complexes and a problem of J. H. C. Whitehead

By E. LUFT

Mathematics Department, University of British Columbia, 121-1984 *Mathematics Road, Vancouver, B.C., Canada, V6T 1Z2*

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1. *Introduction*

In [**W**] J. H. C. Whitehead posed the following question: 'Is every subcomplex K of a 2-dimensional aspherical complex L itself aspherical?'

This problem is usually referred to as the 'Whitehead Conjecture' though it was only stated in the form of a question. For convenience we treat it also as a conjecture.

The Whitehead Conjecture has been proved in special cases: if the subcomplex K has only one 2-cell, and also in the case where $\pi_1(K)$ is either finite, abelian, of free [**C**]. For more partial results see, for example, the introduction of [**H1**].

In [**H2**] J. Howie proved the following remarkable theorem:

If the Whitehead Conjecture is false then there exists a counterexample $K \subset L$ *satisfying one of the following sets of conditions*:

(*a*) L *is finite and contractible and* $K = L - e$ *for some 2-cell* e *of* L;

(*b*) L *is the union of an infinite ascending chain of finite non-aspherical subcomplexes* $K = K_0 \subset K_1 \subset \ldots$ *such that each inclusion map* $K_i \to K_{i+1}$ *is nullhomotopic.*

His proof is based on tower constructions in the category of transverse CW-complexes.

We show that if the Whitehead Conjecture is false then there must exist a counterexample of type (*b*).

We note first

Theorem 1. *If there exists a finite contractible counterexample to the Whitehead Conjecture then there exists a counterexample of type* (*b*).

Thus in the theorem of J. Howie above the existence of a counterexample of type (*a*) actually implies the existence of a counterexample of type (*b*).

Applying Theorem 1 we obtain now a rather elementary proof of

Theorem 2. *If the Whitehead conjecture is false then there is a counterexample of type* (*b*).

The existence of a counterexample of type (*a*) remains an important unsolved problem.

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2. *Preliminaries*

By a complex we mean a CW complex. If L is a complex, $H_i(L)$ denotes the ith integral cellular homology group and $Z_i(L)$ the ith group of integral cellular cycles. A topological space X is aspherical (or an Eilenberg–MacLane space) if it is path connected and if $\pi_i(X) = 0$, $i = 2, 3, \ldots$.

Considering the universal covering space and applying the Hurewicz Isomorphism Theorem it follows that a connected 2-dimensional complex L is aspherical if and only if $\pi_2(L) = 0$.

We will apply the following well-known results.

LEMMA 1. *Let L be a contractible complex and let $L_0 \subset L$ be a finite subcomplex. Then there is a finite subcomplex $L_1 \subset L$, containing L_0, such that the inclusion $L_0 \to L_1$ is nullhomotopic.*

LEMMA 2. *Let L be a path connected not simply connected finite complex and let $p: \tilde{L} \to L$ be the universal covering projection. If $\tilde{L}$ is contractible then $\pi_1(L)$ is infinite.*

3. *Proofs of the theorems*

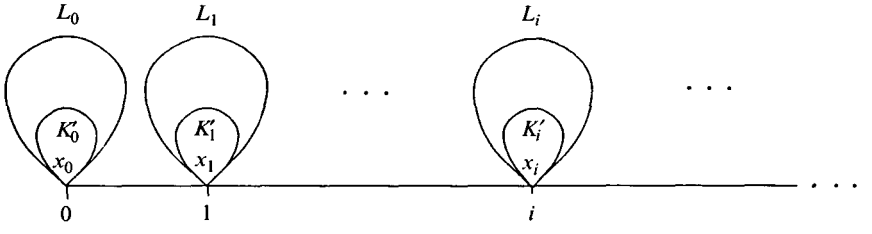
Proof of Theorem 1. Let L_0 be a finite contractible aspherical 2-dimensional complex and let $K_0' \subset L_0$ be a non-aspherical subcomplex. Let $(K_i', L_i), i = 1, 2, \ldots$, be infinitely many copies of (K_0', L_0). Select a point $x_i \in K_i'$, $i = 0, 1, 2, \ldots$. Denote $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geqslant 0\}$ and define

$$L = \mathbb{R}_+ \cup \bigcup_{i=0}^{\infty} L_i$$

where x_i is identified with $i \in \mathbb{R}_+$, $i = 0, 1, 2, \ldots$, see the Figure.

Let $K_i = [0, i] \cup \bigcup_{k=0}^{i-1} L_k \cup K_i'$, $i = 1, 2, \ldots$.

Then K_i is non-aspherical and the inclusion map $K_i \to K_{i+1}$ is nullhomotopic, $i = 1, 2, \ldots$.



LEMMA 3. *Let L' be a not simply connected aspherical 2-dimensional complex. Suppose it contains a non-aspherical finite subcomplex K' such that the inclusion $K' \to L'$ is nullhomotopic. Then there exists a counter-example of type (b).*

Proof. Let $p: \tilde{L}' \to L'$ be the universal covering projection. Then $\pi_n(\tilde{L}') = 0, n = 2, 3, \ldots$ by the Hurewicz Isomorphism Theorem. Therefore $\tilde{L}'$ is contractible. By Lemma 2, $\pi_1(L')$ is infinite. Let $p^{-1}(K') = \bigcup \tilde{K}_k, \tilde{K}_k$ a component of $p^{-1}(K')$, $k \in \pi_1(L')$.

Note that $p|: \tilde{K}_k \to K'$ is a homeomorphism and that $\tilde{K}_k \cap \tilde{K}_l = \varnothing$ if $k \neq l$.

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Select a component K_1 of $p^{-1}(K')$. Then K_1 is a finite non-aspherical subcomplex of $\tilde{L}'$. We proceed inductively. Suppose $K_i \subset \tilde{L}'$ is a finite non-aspherical subcomplex. By Lemma 1 there is a finite subcomplex $K_i \subset L_i \subset \tilde{L}'$ such that the inclusion map $K_i \to L_i$ is nullhomotopic. Select a component $\tilde{K}_{k_i} \subset p^{-1}(K')$ such that $\tilde{K}_{k_i} \cap L_i = \varnothing$. Let $I_i \subset \tilde{L}'$ be an arc connecting L_i and $\tilde{K}_{k_i}$ with

$$I_i \cap (L_i \cup \tilde{K}_{k_i}) = \partial I_i.$$

Define $K_{i+1} = L_i \cup I_i \cup \tilde{K}_{k_i}$.

Then the inclusion $K_i \to K_{i+1}$ is nullhomotopic.

Claim. K_{i+1} is not aspherical.

Proof of claim. Let $q: K^*_{i+1} \to K_{i+1}$ be the universal covering projection. Let $K^* \subset q^{-1}(\tilde{K}_{k_i})$ be a component. Then K^* is simply connected and

$$Z_2(K^*) = H_2(K^*) \cong \pi_2(K^*) \cong \pi_2(\tilde{K}_{k_i}) \cong \pi_2(K') \neq 0.$$

The inclusion $K^* \to K^*_{i+1}$ induces an injective homomorphism $Z_2(K^*) \to Z_2(K^*_{i+1})$. Therefore $\pi_2(K_{i+1}) \cong \pi_2(\tilde{K}_{i+1}) \cong H_2(\tilde{K}_{i+1}) = Z_2(\tilde{K}_{i+1}) \neq 0$.

Proof of the Theorem 2. Suppose L' is an aspherical 2-dimensional complex and $K' \subset L'$ is a non-aspherical sub-complex. By Theorem 1 we may assume that if L' is finite then it is not simply connected. Let $p: \tilde{L}' \to L'$ be the universal covering projection. (If L' is simply connected $\tilde{L}' = L'$). Then $\tilde{L}'$ is contractible. Select a component $\tilde{K}' \subset p^{-1}(K')$. Then $\pi_2(K') \neq 0$. Let $f: S^2 \to \tilde{K}'$ be a map with $[f] \neq 0$ in $\pi_2(\tilde{K}')$ and let $K_1 \subset \tilde{K}'$ be a finite subcomplex with $f(S^2) \subset K_1$. Again $\pi_2(K_1) \neq 0$.

By Lemma 1 there is a finite subcomplex $K_2 \subset \tilde{L}'$ containing K_1 such that the inclusion map $K_1 \to K_2$ is nullhomotopic. If $\pi_2(K_2) \neq 0$ we continue this construction. Proceeding inductively there are two possibilities.

Case 1. We obtain an ascending chain of finite non-aspherical subcomplexes of $\tilde{L}'$

$$K_1 \subset K_2 \subset \ldots \subset K_i \subset \ldots$$

such that the inclusion maps $K_i \to K_{i+1}$ are nullhomotopic. In this case let $L = \bigcup_{i=1}^{\infty} K_i$.

Case 2. The process terminates. We obtain a finite sequence of finite subcomplexes

$$K_1 \subset K_2 \subset \text{———} \subset K_n$$

with $\pi_2(K_n) = 0$. Then $K_1 \subset K_n$ is a finite non-aspherical subcomplex of the finite aspherical complex K_n, and the inclusion $K_1 \to K_n$ is nullhomotopic. If K_n is simply connected then K_n is contractible and we apply Theorem 1. If K_n is not simply connected we apply Lemma 3.

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