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PHYSICAL MEASURES FOR MOSTLY SECTIONALLY EXPANDING FLOWS

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Abstract We prove that a partially hyperbolic attracting set for a C^2 vector field, having slow recurrence to equilibria, supports an ergodic physical/SRB measure if, and only if, the trapping region admits non-uniform sectional expansion on a positive Lebesgue measure subset. Moreover, in this case, the attracting set supports at most finitely many ergodic physical/SRB measures, which are also Gibbs states along the central-unstable direction. This extends to continuous time systems a similar well-known result obtained for diffeomorphisms, encompassing the presence of equilibria accumulated by regular orbits within the attracting set. In codimension two the same result holds, assuming only the trajectories on the trapping region admit a sequence of times with asymptotical sectional expansion, on a positive volume subset. We present several examples of application, including the existence of physical measures for asymptotically sectional hyperbolic attracting sets, and obtain physical measures in an alternative unified way for many known examples: Lorenz-like and Rovella attractors, and sectional-hyperbolic attracting sets (including the multidimensional Lorenz attractor).

Keywords: physical/SRB measure; positive Lyapunov exponent; non-uniform sectional expanding flow; hyperbolic times; *cu*-Gibbs state

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1. Introduction and statement of results

Much of the recent progress in Dynamics is a consequence of a probabilistic approach to the understanding of complicated dynamical systems, where one focuses on the statistical properties of "typical orbits", in the sense of large volume in the ambient space. We deal here with flows $\phi_t : M \to M$ on compact manifolds. The most basic statistical data are the time averages $T^{-1}\int_0^T \delta_{\phi_t(z)}\,dt$, where δ_w represents the Dirac measure at a point w. Birkhoff's Ergodic Theorem asserts that time averages admit asymptotic limits in the weak* topology at almost every point z with respect to any invariant probability μ.

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That is, for every continuous observable $\psi : M \to \mathbb{R}$ there exists a subset $E \subset M$ of full measure $\mu(E) = 1$ so that $\lim_{T\nearrow\infty} \frac{1}{T}\int_0^T \psi(\phi_t(z))\,dt = \widetilde{\psi}(z)$ is well-defined for each $z \in E$. Moreover, if the measure is ergodic, then the time average coincides with the space average, that is, $\widetilde{\psi}(z) = \int \psi\,d\mu$ for $z \in E$. However, many invariant measures are singular with respect to volume in general, and so the Ergodic Theorem is not enough to understand the behaviour of positive volume (Lebesgue measure) sets of orbits.

A *physical measure* is an invariant probability measure for which time averages exist and coincide with the space average, for a set of initial conditions with positive Lebesgue measure, i.e. in the weak* topology of convergence of probability measures we have

$$B(\mu) := \left\{ z \in M : \lim_{T\nearrow\infty} \frac{1}{T}\int_0^T \delta_{\phi_t(z)}\,dt = \mu \right\} \text{ with } \mathrm{Leb}(B(\mu)) > 0.$$

This set is the *basin* of the measure. Sinai, Ruelle and Bowen introduced this notion about fifty years ago, and proved that, for uniformly hyperbolic (Axiom A) diffeomorphisms and flows, time averages exist for Lebesgue almost every point and coincide with one of finitely many physical measures; see [27, 61].

The problem of existence (and finiteness) of physical measures, beyond the Axiom A setting, remains a main goal of Dynamics. The construction of the so called Gibbs u-states, by Pesin and Sinai in [53], was the beginning of the extension of the Sinai, Ruelle and Bowen ideas to partially hyperbolic systems, a fruitful generalization of the notion of uniform hyperbolicity, which more recently was shown to encompass Lorenz-like or singular-hyperbolic flows [48] and to be a consequence of robust transitivity: see the surveys [16, 23] for much of the progress obtained so far and the recent extensions to higher-dimensional flows [9, 41].

The papers of Alves, Bonatti and Viana [4, 25], and Dolgopyat [34] are of special interest to us here since they prove existence and finiteness of physical measures for partially hyperbolic diffeomorphisms, which are also u-Gibbs states, under the assumption that the central direction is either "mostly contracting" [25, 34] or "non-uniformly expanding" [4]. For local diffeomorphisms with a non-flat critical or singular set, the authors in [4] show that a slow recurrence condition is sufficient to obtain an absolutely continuous invariant probability measure when the system is non-uniformly expanding.

Here, we extend the results of [4] to attracting sets for smooth vector fields with a dominated splitting and a sequence of times with asymptotical sectional expansion on a positive volume subset. That is, every transversal direction to the vector field along the center subbundle has a positive Lyapunov exponent in a weak sense. In this setting partial hyperbolicity is natural. We additionally obtain finitely many physical/SRB measures for the flow, which are also cu-Gibbs states, whose ergodic basins cover the set of non-uniform sectional expanding orbits (except for a subset of volume zero), if we also have slow recurrence to the equilibria of the vector field.

The properties of continuous time dynamics enable us to show that weak asymptotical sectional expansion on a positive volume subset is a necessary and sufficient condition for existence of ergodic physical/SRB measures for partially hyperbolic attracting sets with central direction of codimension two. We present several examples of application,

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including the existence of physical measures for weak asymptotically sectional hyperbolic attracting sets.

Known examples satisfying these conditions are, besides hyperbolic (Axiom A) flows [27], all singular-hyperbolic attracting sets for C^2 smooth flows [16] (including the Lorenz attractor), the contracting Lorenz attractor (also known as the Rovella attractor [60]) and all sectional-hyperbolic attracting sets [9] (including the multidimensional Lorenz attractor [24]). Our results provide an alternative unified way to obtain physical/SRB measures for these known systems and for many others.

1.1. Statements of the results

Let M be a compact connected manifold with dimension $\dim M = m$, endowed with a Riemannian metric, induced distance d and volume form Leb. Let $\mathfrak{X}^r(M)$, $r \geq 1$, be the set of C^r vector fields on M endowed with the C^r topology and denote by ϕ_t the flow generated by $G \in \mathfrak{X}^r(M)$.

1.1.1. Preliminary definitions

We say that $\sigma \in M$ with $G(\sigma) = 0$ is an *equilibrium* or *singularity*. In what follows we denote by $\mathrm{Sing}(G)$ the family of all such points. We say that an equilibrium $\sigma \in \mathrm{Sing}(G)$ is *hyperbolic* if all the eigenvalues of $DG(\sigma)$ have non-zero real part.

An *invariant set* Λ for the flow ϕ_t, generated by the vector field G, is a subset of M which satisfies $\phi_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. A point $p \in M$ is *periodic* for the flow ϕ_t generated by G if $G(p) \neq 0$ and there exists $\tau > 0$ so that $\phi_\tau(p) = p$; its orbit $\mathcal{O}_G(p) = \phi_{\mathbb{R}}(p) = \phi_{[0,\tau]}(p) = \{\phi_t p : t \in [0, \tau]\}$ is a *periodic orbit*, an invariant simple closed curve for the flow. An invariant set is *nontrivial* if it is not a finite collection of periodic orbits and equilibria.

Given a compact invariant set Λ for $G \in \mathfrak{X}^r(M)$, we say that Λ is *isolated* if there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t\in\mathbb{R}} \mathrm{Closure}\, \phi_t(U)$. If U can be chosen so that $\mathrm{Closure}\, \phi_t(U) \subset U$ for all $t > 0$, then we say that Λ is an *attracting set* and U a *trapping region* (or *isolating neighbourhood*) for $\Lambda = \Lambda_G(U) = \cap_{t>0} \mathrm{Closure}\, \phi_t(U)$.

An *attractor* is a transitive attracting set, that is, an attracting set Λ with a point $z \in \Lambda$ so that its ω-limit $\omega(z) := \left\{ y \in M : \exists t_n \nearrow \infty \text{ s.t. } \phi_{t_n} z \xrightarrow[n\to\infty]{} y \right\}$ coincides with Λ.

1.1.2. Partial hyperbolic attracting sets for vector fields

Let Λ be a compact invariant set for $G \in \mathfrak{X}^r(M)$. We say that Λ is *partially hyperbolic* if the tangent bundle over Λ can be written as a continuous $D\phi_t$-invariant Whitney sum $T_\Lambda M = E^s \oplus E^{cu}$, where $d_s = \dim(E^s_x) \geq 1$ and $d_{cu} = \dim(E^{cu}_x) \geq 2$ for $x \in \Lambda$, and there exists a constant $\lambda > 0$ such that for all $x \in \Lambda$, $t \geq 0$, we have[1]

- domination of the splitting: $\|D\phi_t|E^s_x\| \cdot \|D\phi_{-t}|E^{cu}_{\phi_t x}\| \leq e^{-\lambda t}$;

[1] For some choice of the Riemannian metric on the manifold, see e.g. [36]. Changing the metric does not change the rate λ but might introduce the multiplication by a constant.

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- uniform contraction along E^s: $\|D\phi_t|E^s_x\| \le e^{-\lambda t}$.

We refer to E^s as the stable bundle and to E^{cu} as the centre-unstable bundle.

Lemma 1.1. [18, Lemma 3.2] *Let Λ be a compact invariant set for G.*

(1) *Given a $D\phi_t$-invariant and continuous splitting $T_\Lambda M = E \oplus F$ such that E is uniformly contracted, then $G(x) \in F_x$ for all $x \in \Lambda$.*
(2) *Assuming that Λ is non-trivial and has a continuous $D\phi_t$-invariant* and dominated *splitting $T_\Lambda M = E \oplus F$ such that $G(x) \in F_x$ for all $x \in \Lambda$, then E is a uniformly contracted subbundle.*

A *partially hyperbolic attracting set* is a partially hyperbolic set that is also an attracting set.

Remark 1.2. In the flow setting, a dominated splitting becomes partially hyperbolic whenever the flow direction is contained in the central-unstable bundle $G \in E^{cu}$, from Lemma 1.1. Thus, this inclusion is equivalent to partial hyperbolicity. Since the flow direction is invariant, then partial hyperbolicity is the natural setting to study invariant sets (which are not composed only of equilibria) for flows with a dominated splitting.

1.1.3. Singular/sectional-hyperbolicity

The centre-unstable bundle E^{cu} is *volume expanding* if there exists $K, \theta > 0$ such that $|\det(D\phi_t|E^{cu}_x)| \ge Ke^{\theta t}$ for all $x \in \Lambda$, $t \ge 0$.

We say that a compact nontrivial invariant set Λ is a *singular hyperbolic set* if all equilibria in Λ are hyperbolic, and Λ is partially hyperbolic with volume expanding centre-unstable bundle. A singular hyperbolic set which is also an attracting set is called a *singular hyperbolic attracting set*.

We say that E^{cu} is *(2-)sectionally expanding* if there are positive constants K, θ such that for every $x \in \Lambda$ and every 2-dimensional linear subspace $L_x \subset F_x$ one has $|\det(D\phi_t|L_x)| \ge Ke^{\theta t}$ for all $t \ge 0$. A *sectional-hyperbolic (attracting) set* is a partially hyperbolic (attracting) set whose central subbundle is sectionally expanding.

1.1.4. Asymptotical sectional-hyperbolicity

A compact invariant partially hyperbolic set Λ of a vector field G whose equilibria are hyperbolic, is *asymptotically sectional-hyperbolic* (ASH) if the center-unstable subbundle is eventually asymptotically sectional expanding outside the stable manifold of the equilibria. That is, there exists $c_* > 0$ so that

$$\limsup_{T \nearrow \infty} \frac{1}{T} \log |\det(D\phi_T\,|_{F_x})| \ge c_*, \tag{1}$$

for every $x \in \Lambda \setminus \cup\{W^s_\sigma : \sigma \in \mathrm{Sing}_\Lambda(G)\}$ and each 2-dimensional linear subspace F_x of E^{cu}_x, where we write $\mathrm{Sing}_\Lambda(G) = \mathrm{Sing}(G) \cap \Lambda$ and $W^s_\sigma = \{x \in M : \lim_{t\to+\infty} \phi_t x = \sigma\}$ is the *stable manifold* of the hyperbolic equilibrium σ. It is well-known that W^s_σ is a

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immersed submanifold of M; see e.g. [51]. This implies that all transverse directions to the vector field along the center-unstable subbundle have positive Lyapunov exponent; that is, if $v \in E^{cu}_x \setminus (\mathbb{R}\cdot G)$, then $\chi(x,v) := \limsup_{t\to+\infty} \log \|D\phi_t(x)v\|^{1/t} \geq c_* > 0$; see e.g. Theorem 1.6; Remark 1.7 and Conjecture 3.

Lemma 1.3 (Hyperbolic Lemma). *Every compact invariant subset Γ without equilibria contained in a asymptotically sectional-hyperbolic set is uniformly hyperbolic.*

We say that an invariant compact subset Γ is *(uniformly) hyperbolic* if Γ is partially hyperbolic and the central-unstable bundle admits a continuous splitting $E^{cu} = (\mathbb{R}\cdot G)\oplus E^u$, with $\mathbb{R}\cdot G$ the one-dimensional invariant flow direction and E^u a uniformly expanding subbundle. That is, we get the following dominated splitting $T_\Gamma M = E^s \oplus (\mathbb{R}\cdot G)\oplus E^u$ into three-subbundles; see e.g. [35].

Proof of Lemma 1.3. See e.g. [49, Proposition 1.8] for sectional-hyperbolic sets; and [43, Theorem 2.2] for the asymptotically sectional-hyperbolic case. □

1.2. Non-uniform sectional expansion

Let us fix $G \in \mathfrak{X}^2(M)$ endowed with a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ with a trapping region U. Then we can take a continuous extension $T_U M = \widetilde{E^s} \oplus \widetilde{E^{cu}}$ of $T_\Lambda M = E^s \oplus E^{cu}$ and for small $a > 0$ find center unstable and stable cones

$$C^{cu}_a(x) = \{v = v^s + v^c : v^s \in \widetilde{E^s}_x, v^c \in \widetilde{E^{cu}}_x, x \in U, \|v^s\| \leq a\|v^c\|\}, \quad \text{and} \tag{2}$$

$$C^{s}_a(x) = \{v = v^s + v^c : v^s \in \widetilde{E^s}_x, v^c \in \widetilde{E^{cu}}_x, x \in U, \|v^c\| \leq a\|v^s\|\},$$

which are invariant in the following sense

$$D\phi_t(x)\cdot C^{cu}_a(x) \subset C^{cu}_a(\phi_t(x)) \quad \text{and} \quad D\phi_{-t}\cdot C^{s}_a(x) \supset C^{s}_a(\phi_{-t}(x)), \tag{3}$$

for all $x \in \Lambda$ and $t > 0$ so that $\phi_{-s}(x) \in U$ for all $0 < s \leq t$; see § 2.1.1. We can assume that $\widetilde{E^{cu}_x} \subset C^{cu}_a(x)$ still contains the flow direction $G(x)$ for each $x \in U$. Otherwise, since $G(x) \in E^{cu}_x$ for $x \in \Lambda$ by Lemma 1.1(1), then $G(x) \in C^{cu}_a(x)$ and we can set $N^{cu}_x := \widetilde{E^{cu}} \cap G(x)^\perp$ and $\widehat{E^{cu}} := G(x) \oplus N^{cu}_x$ for all $x \in U$. Then $\widehat{E^{cu}}$ contains the flow direction by construction, is continuous and $\widehat{E^{cu}_x} \subset C^{cu}_a(x), x \in U$. We can also assume, without loss of generality according to [13], that the continuous extension of the stable direction E^s of the splitting is still $D\phi_t$-invariant and $\widetilde{E^s_x} \subset C^s_a(x), x \in U$. In what follows, we keep the notation $T_U M = E^s \oplus E^{cu}$ and write $N^{cu}_x = E^{cu}_x \cap G(x)^\perp, x \in U$.

In what follows we write $f := \phi_1$ for the time-1 diffeomorphism induced by the flow. We say that the attracting set Λ is *non-uniform 2-sectionally expanding* (NU2SE) if there exists $c_0 > 0$ so that

$$\Omega = \left\{ x \in U : \limsup_{n\nearrow\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \left\| \wedge^2 (Df\,|_{E^{cu}_{f^i x}})^{-1} \right\| \leq -c_0 \right\} \quad \text{and} \quad \mathrm{Leb}(\Omega) > 0. \tag{4}$$

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Non-uniform sectional expansion ensures the existence of at most finitely many physical/SRB measures whose ergodic basins cover the (NU2SE) subset Leb mod0.

Theorem A. *Let $G \in \mathfrak{X}^2(M)$ be a vector field with a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ with no equilibria $\mathrm{Sing}_\Lambda(G) = \emptyset$. Then we have (NU2SE) on $\Omega \subset U$ if, and only if, there are finitely many ergodic hyperbolic physical/SRB measures whose basins cover* Leb*-a.e. point of Ω:* $\mathrm{Leb}\left(\Omega \setminus \left(B(\mu_1) \cup \cdots \cup B(\mu_p)\right)\right) = 0$.

Here, hyperbolicity of an invariant probability measure means *non-uniform hyperbolicity*. More precisely, we set the forward Lyapunov exponent of $v \in T_xM \setminus \{\vec{0}\}$ as

$$\chi^+(x,v) := \limsup_{t\nearrow\infty} \log \|D\phi_t(x)v\|^{1/t}.$$

The function $\chi(x,\cdot)$ admits finitely many values only $\chi_1^+(x) > ... > \chi_{p(x)}^+(x)$ on $T_xM \setminus \{0\}$ (the *Lyapunov Spectrum* of μ) and generates a filtration $0 \subsetneq V_{p(x)}(x) \subsetneq \cdots \subsetneq V_1(x) = T_xM$ with $V_i(x) = \{v \in T_xM,\ \chi^+(x,v) \le \chi_i^+(x)\}$. The functions $p(x), \chi_i^+(x)$ are ϕ_t-invariant; the vector subspaces $V_i(x)$, $i = 1, ..., p(x)$ are $D\phi_t$-invariant; and each of them depend Borel measurably on x.

An invariant ergodic probability measure μ supported in Λ is *hyperbolic*, if the tangent bundle over Λ splits into a sum $T_zM = E_z^s \oplus (\mathbb{R}\cdot G(z)) \oplus F_z$ of invariant subspaces defined for μ-a.e. $z \in \Lambda$ and depending measurably on the base point z; $\mathbb{R}\cdot G(z)$ is the flow direction (with zero Lyapunov exponent); $\mathbb{R}\cdot G(z) \oplus F_z = E_z^{cu}$ and F_z is the direction with positive Lyapunov exponents, that is, it satisfies $\lim_{t\to+\infty} \log \left\|(D\phi_t \ |_{F_z})^{-1}\right\|^{-1/t} > 0$. Naturally, all directions along E_z^s have negative Lyapunov exponents $\lim_{t\to+\infty} \log \left\|D\phi_t \ |_{E_z^s})\right\|^{1/t} < 0$. In terms of the Lyapunov Spectrum this means that $\chi_{d_{cu}+1}^+(z) < 0 = \chi_{d_{cu}}^+(z) < \chi_{d_{cu}-1}^+(z)$.

1.2.1. Physical/SRB measures and cu-Gibbs states

In the uniformly hyperbolic setting, it is well known that physical measures μ, for hyperbolic attractors of C^2 diffeomorphisms g, admit a disintegration into conditional measures along the unstable manifolds of almost every point which are absolutely continuous with respect to the induced Lebesgue measure on these submanifolds, see [26, 27, 53]. By Leddrappier-Young characterization of measures satisfying (Pesin's) Entropy Formula [40], this is equivalent to

$$h_\mu(g) = \int \Sigma^+ \, d\mu = \int \log |\det Dg \ |_{E^u}| \, d\mu > 0, \tag{5}$$

where E^u is the unstable invariant subbundle over the hyperbolic attractor, and $\Sigma^+(z) = \sum_{i\le \dim(E^u)} \chi_i^+(z)$ is the sum of positive Lyapunov exponent (with multiplicities). In the hyperbolic setting for diffeomorphisms, condition (5) means that μ is a u-Gibbs state. These measures are known as *Sinai-Ruelle-Bowen (SRB)* measures.

In our setting, existence of unstable manifolds is guaranteed by the hyperbolicity of physical measures: the *strong-unstable manifolds* W_z^{uu} are the "integral manifolds"

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tangent to F_z defined by $W_z^{uu} = \{y \in M : \lim_{t\to-\infty} d(\phi_t(y), \phi_t(z)) = 0\}$ and exist for μ-a.e. z with respect to the measures obtained in Theorem A.

The weak-unstable manifolds W_z^{cu} are the saturation $\phi_{(-1,1)}(W_z^{uu})$ of W_z^{uu} by the flow, and are tangent to E_z^{cu}, μ-a.e. z. The sets W_z^{cu} are embedded sub-manifolds in a neighbourhood of z which, in general, depend only measurably (including its size) on the base point $z \in \Lambda$. We note that, since Λ is an attracting set, then $W_z^{cu} \subset \Lambda$ where defined[2].

The arguments of our proofs, adapted from [4], enable us to obtain not only ergodic hyperbolic physical invariant probability measures, but also the condition corresponding to (5) in the flow setting

$$h_\mu(\phi_1) = \int \Sigma^+ \, d\mu = \int \log |\det Df \,|_{E^{cu}}| \, d\mu > 0, \tag{6}$$

that is, the physical measures are *cu-Gibbs states*. The results from Leddrappier-Young [40] are still valid in our setting, that is, ergodic physical measures satisfying (6), supported on partially hyperbolic attractors in our setting, admit a disintegration into conditional measures along the unstable manifolds of almost every point, which are absolutely continuous with respect to the induced Lebesgue measure on these submanifolds – see e.g. [16] for the three-dimensional case, and [9] (and references therein) for the general case.

Example 1 (Hyperbolic examples). Anosov flows and hyperbolic attractors (attracting basic pieces of the spectral decomposition of Smale) for smooth vector fields admit a physical/SRB probability measure whose basin covers the trapping region except a zero volume subset [35, 38], and are asymptotically sectional hyperbolic, with no equilibria.

In fact, the central-unstable bundle splits $E^{cu}_\Lambda = \mathbb{R} \cdot G \oplus E^u$ into a pair of continuous subbundles: the direction of the flow and an unstable bundle E^u (uniformly contracting in negative time).

Using an adapted metric, we can assume without loss of generality that $T_\Lambda M = E^s \oplus \mathbb{R} \cdot G \oplus E^u$ is an *orthogonal* invariant splitting ; see e.g. [36]. Then, for all $x \in \Lambda$ and also in U, the backward contraction on E^u ensures that there exists $\lambda > 0$ so that, for any bivector $u \wedge v$ with $u, v \in E^u_x$, we may assume without loss of generality that $\langle u, v\rangle = 0$ and obtain $\| \wedge^2 D\phi_{-t} \cdot (u \wedge v)\| \le \|D\phi_{-t}u\| \cdot \|D\phi_{-t}v\| \le e^{-2\lambda t}\|u\| \cdot \|v\| = e^{-2\lambda t}\|u \wedge v\|$. If we instead consider a bivector $G(x) \wedge v$ for $v \in E^u_x$ we obtain

$$\| \wedge^2 D\phi_{-t} \cdot (G(x) \wedge v)\| \le e^{-\lambda t}\|v\| \cdot \|G(\phi_{-t}x)\|/\|G(x)\| \le Ke^{-\lambda t}\|G(x) \wedge v\|$$

since $\|G(x)\|$ is bounded above and also bounded away from zero on M. We thus obtain (ASH) for all $x \in M$. Moreover, we also obtain (NU2SE) for $f = \phi_n$, where $n \ge 1$ is such that $Ke^{-\lambda n} < 1$, that is, we get (4) with $\Omega = M$ and $f = \phi_n$.

[2] For if $y \in W^{uu}_z \cap U$ and $z \in \Lambda$, then $d(\phi_{-t}y, \phi_{-t}z) \to 0$ for $t \nearrow \infty$. Thus $\phi_{-t}y \subset U$ for all $t \ge 0$, that is, $y \in \cap_{t\ge0}\phi_t(U) = \Lambda$.

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1.2.2. Recurrence control to equilibria

To construct the physical probability measure in the presence of equilibria for a partially hyperbolic attracting set, we need to control the recurrence near the equilibria.

Definition 1 (Slow recurrence, continuous and discrete). *Let $U \subset M$ be a forward invariant set of M for the flow of a C^1 vector field G, where all equilibria are hyperbolic. We say that G has*

(CSR) continuous slow recurrence to equilibria *if, on the positive Lebesgue measure subset $\Omega \subset U$, for every $\varepsilon > 0$, we can find $\delta > 0$ so that*

$$\limsup_{T\nearrow\infty} \frac{1}{T}\int_0^T -\log d_\delta\big(\phi_t(x), \mathrm{Sing}_\Lambda(G)\big)\, dt < \varepsilon, \quad x \in \Omega, \tag{7}$$

where $d_\delta(x,S)$ δ-truncated distance from $x \in M$ to a subset S, that is

$$d_\delta(x,S) = \begin{cases} d(x,S) & \text{if } 0 < d(x,S) \le \delta; \\ \left(\frac{1-\delta}{\delta}\right) d(x,S) + 2\delta - 1 & \text{if } \delta < d(x,S) < 2\delta; \\ 1 & \text{if } d(x,S) \ge 2\delta. \end{cases}$$

(SR) slow recurrence to equilibria *if, on the positive Lebesgue measure subset $\Omega \subset U$, for every $\varepsilon > 0$, we can find $\delta > 0$ so that*

$$\limsup_{n\nearrow\infty} \frac{1}{n}\sum_{i=0}^{n-1} -\log d_\delta\big(f^i(x), \mathrm{Sing}_\Lambda(G)\big) < \varepsilon, \quad x \in \Omega, \tag{8}$$

Remark 1.4. If the function $|\log d_\delta(x, \mathrm{Sing}_\Lambda(G))|$ is μ-integrable, whenever μ is a physical measure, the slow recurrence to equilibria is a consequence of Birkhoff's Ergodic Theorem; see e.g. the comments in [5]. However, either this integrability property, or conditions (SR) or (CSR), are hard to obtain; see e.g. [19] for a setting where this was deduced from global properties of the transformation.

Clearly, the slow recurrence condition is vacuous (and automatically valid) if Λ contains no equilibria.

In the presence of equilibria, we obtain the same conclusion of Theorem A assuming non-uniform sectional expansion together with slow recurrence.

Theorem B. *Let $G \in \mathfrak{X}^2(M)$ be a vector field with a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ satisfying (SR) on $\Omega \subset U$, with $\mathrm{Leb}(\Omega) > 0$. Then we have (NU2SE) on Ω if, and only if, there are finitely many ergodic physical/SRB measures, which are cu-Gibbs states, and whose basins cover* Leb*-a.e. point of Ω:* $\mathrm{Leb}\Big(\Omega \setminus \big(B(\mu_1) \cup \cdots \cup B(\mu_p)\big)\Big) = 0$.

The proof of Theorem B relies on carefully constructing hyperbolic times and pre-disks[3] needed to apply the main technical results from [4].

[3] See Sections 3 and 4 together with e.g. [2, Chap. 7] and compare [4] for many more details.

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We adapt the main ideas from [4] to allow for the neutral (non-expanding nor contracting) flow direction along the center-unstable subbundle – we stress that this invariant direction $\mathbb{R} \cdot G$ (away from singularities) inside E^{cu} is *not part of a continuous splitting of* E^{cu} *in general*, as in the uniformly hyperbolic case; see e.g. [35]. The goal is to build, for each hyperbolic time of a given trajectory, a centre-unstable disk of uniform inner radius which is uniformly backward contracted along the sectional centre-unstable directions, and deduce a distortion bound to control the density of push-forwards of Lebesgue measure along these disks.

Example 2 (Singular-hyperbolic attracting sets). We recall that all singular-hyperbolic attracting sets, as the (geometric) Lorenz attractor [16], for C^2 smooth flows with codimension two partially hyperbolic attracting sets, admit finitely many (one only if transitive) physical/SRB probability measures, which are *cu*-Gibbs states and whose basins cover Leb-a.e. point of the trapping region of these attracting sets; see e.g. [9]. These are partially hyperbolic and sectional expanding attracting sets. But they also exhibit slow recurrence to equilibria. Indeed, it is even possible to obtain *exponentially* slow recurrence; see e.g. [19].

1.3. Continuous time versus discrete time

A property similar to ASH on a positive volume subset of the trapping neighbourhood U of a partially hyperbolic attracting set Λ is the following. We say that Λ is *mostly asymptotically sectional expanding* (MASE) if there exists $c_0 > 0$ such that

$$\Omega = \left\{ x \in U : \limsup_{T \nearrow \infty} \frac{1}{T} \log \| \wedge^2 (D\phi_T \mid_{E^{cu}_x})^{-1} \| \leq -c_0 \right\} \quad \text{and} \quad \mathrm{Leb}(\Omega) > 0. \tag{9}$$

We can interrelate the previous notions (NU2SE), (MASE) and (SR),(CSR) as follows.

Theorem C (Equivalence between discrete and continuous time versions). *Let $G \in \mathfrak{X}^2(M)$ be given admitting a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$. Then*

(1) *the slow recurrence (SR) condition (8) holds for $x \in U$ if, and only if, continuous slow recurrence (CSR) condition (7) holds for x.*
(2) *if the subset $\Omega = \{x \in U : (8) \text{ holds for } x\}$ has positive volume, then the following pair of conditions are equivalent:*
   (A) there exists a hyperbolic physical/SRB invariant probability measure for the flow, which is a cu-Gibbs state with $\operatorname{supp} \mu \subset \Lambda$;
   (B) there exists $T > 0$ and $c_0 > 0$ and a positive volume subset $E \subset \Omega$ where we have (NU2SE) for ϕ_T, that is

$$\limsup_{n \nearrow \infty} \frac{1}{n} \sum\nolimits_{i=0}^{n-1} \log \left\| \wedge^2 \left(D\phi_T \mid_{E^{cu}_{\phi_{iT}(x)}} \right)^{-1} \right\|^{1/T} < -c_0 < 0, \quad x \in E. \tag{10}$$

(3) *if either condition of item (2) is met, then*

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(a) *the (MASE) condition* (9) *holds* Leb-*a.e. in* E*; and*
(b) *if, additionally,* Λ *is transitive, then there exists one ergodic physical/SRB measure such that* $\mathrm{Leb}(B(\mu) \setminus \Omega) = 0$.

Remark 1.5 ((NU2SE) implies (MASE)). This means that, under the assumption of slow recurrence, (NU2SE) implies (MASE) whenever we have hyperbolic physical/SRB measures, by either reparametrizing the flow $(\phi_t)_{t\in\mathbb{R}}$ to $(\phi_{tT})_{t\in\mathbb{R}}$ or, equivalently, replacing G by a multiple $T \cdot G$.

1.4. Non-uniformly sectional hyperbolic variants

We explore several possible definitions of non-uniform hyperbolicity for partially hyperbolic attracting set and relations between them, in order to exhibit the equivalence of some properties, and present some results where slow recurrence is not needed to prove the existence of physical measures.

Definition 2 (Linear Poincaré Flow). *If* x *is a regular point of the vector field* G *(i.e.* $G(x) \neq \vec{0}$*), denote by* $N_x = \{v \in T_xM : \langle v, G(x)\rangle = 0\}$ *the orthogonal complement of* $G(x)$ *in* T_xM*. Denote by* $O_x : T_xM \to N_x$ *the orthogonal projection of* T_xM *onto* N_x*. For every* $t \in \mathbb{R}$ *define, see Figure* 1

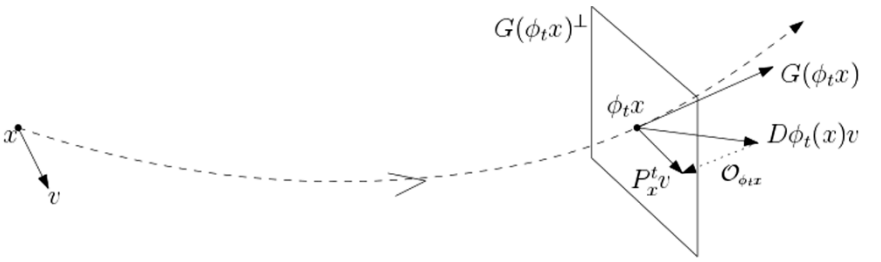


Figure 1. Sketch of the Linear Poincaré flow P_x^t *of a vector* $v \in T_xM$ *with* $x \in M \setminus \mathrm{Sing}(G)$ *and the orthogonal projection* $O_{\phi_t x} : T_{\phi_t x}M \to N_{\phi_t x}$*, with* $N_{\phi_t x} = G(\phi_t x)^\perp$.

$$P_x^t : N_x \to N_{\phi_t x} \quad \text{by} \quad P_x^t = O_{\phi_t x} \circ D\phi_t(x).$$

It is easy to see that $P = \{P_x^t : t \in \mathbb{R}, G(x) \neq 0\}$ *satisfies the cocycle relation* $P_x^{s+t} = P_{\phi_s x}^t \circ P_x^s$ *for every* $t, s \in \mathbb{R}$*. The family* $P = P_G$ *is called the* Linear Poincaré Flow *of* G.

Theorem 1.6 (Pointwise non-uniform sectional expanding variants). *Let* G *be a* C^1 *vector field generating the flow* $\phi_t : M \to M$ *on the compact connected finite dimensional manifold* M *admitting a compact invariant attracting subset* $\Lambda = \Lambda_G(U)$*, with a dominated splitting* $T_\Lambda M = E^{cs} \oplus E^{cu}$ *on the trapping region* U*, continuously extended to* U *with* $G(x) \in E_x^{cu}, x \in U$*. Let* $c_0 > 0$ *be given together with* $x \in U$ *so that its future trajectory* $\big(\phi_t(x)\big)_{t\geq 0}$ *does not converge to any equilibrium point. Then the following condition (A) implies condition (B):*

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(A) weak non-uniformly 2-sectional expanding **(wNU2SE)***:*

$$\liminf_{n\nearrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|\wedge^2(Df\mid_{E^{cu}_{f^ix}})^{-1}\|\le -c_0; \tag{11}$$

(B) weak non-uniformly sectional expanding **(wNUSE)***:*

$$\liminf_{n\nearrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|(P^1\mid_{N^{cu}_{f^ix}})^{-1}\|\le -c_0. \tag{12}$$

*Moreover, the former condition (B) on the time-*1 *map* f *implies the following conditions*

(a) weak mostly asymptotically sectional expanding **(wMASE)***:*

$$\liminf_{T\nearrow\infty}\frac{1}{T}\log\|\wedge^2(D\phi_T\mid_{E^{cu}_x})^{-1}\|\le -c_0; \tag{13}$$

(b) weak asymptotic sectional expansion for the Linear Poincaré Flow **(wASEP)***:*

$$\liminf_{T\nearrow\infty}\frac{1}{T}\log\|(P^T\mid_{N^{cu}_x})^{-1}\|\le -c_0; \tag{14}$$

and (a) *implies* (b)*. Again, the previous conditions imply the following conditions*

(i) weak asymptotically sectional expanding **(wASE)***: for each* 2*-dimensional linear subspace* F_x *of* E^{cu}_x *we have*

$$\limsup_{T\nearrow\infty}\frac{1}{T}\log|\det(D\phi_T\mid_{F_x})|\ge c_0. \tag{15}$$

(ii) weak positive sectional Lyapunov exponents **(wPSL)***:*

$$\limsup_{T\nearrow\infty}\frac{1}{T}\log\|D\phi_T(x)v\|\ge c_0, \forall v\in E^{cu}_x\setminus\mathbb{R}\cdot G_x; \tag{16}$$

and (i) *implies* (ii)*.*

Reciprocally, if $d_{cu}=\dim E^{cu}=2$*, then the latter condition (i) implies (A).*

In addition, the statements above are also true after exchanging $\liminf$ *and* $\limsup$*. In particular, the following is implied by (NU2SE):*

- non-uniformly sectional expanding **(NUSE)***:*

$$\limsup_{n\nearrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left\|\left(P^1\mid_{N^{cu}_{f^ix}}\right)^{-1}\right\|\le -c_0; \tag{17}$$

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Remark 1.7. In shorthand, we obtain the following relations

$$[(NU2SE) \implies (NUSE)] \quad \& \quad (NU2SE) \implies (MASE) \implies (ASEP) \implies (ASE)$$

together with the weak forms

$$[(11) \implies (12)] \quad \& \quad (11) \implies (13) \implies (14) \implies (15)$$

for trajectories not converging to any equilibrium in the future. Theorem 1.6 shows that the (wMASE) condition implies positive central-unstable Lyapunov exponents along directions transversal to the flow direction: positive sectional Lyapunov exponents, i.e. the condition (wPSL). Moreover, this condition is a consequence of asymptotical sectional expansion (wASE) as claimed when presenting the (ASH) condition (1).

It is easy to see that all asymptotic properties in the statement of Theorem 1.8 do not depend on the particular continuous extension of E^{cu}_Λ to U chosen before, due to the domination of the splitting; see Proposition 2.3.

The assumption (4) of (NU2SE) is close to the notion of non-uniform sectional hyperbolicity on critical elements (equilibria and periodic orbits) defined by Arbieto-S. [20, Definition 2.5 & Remark 2.6] to obtain sectional hyperbolicity for the non-wandering set on a C^1 residual subset of vector fields, among those with non-uniform sectional hyperbolic critical elements.

More precisely, let us assume that the attracting set Λ admits a *continuous invariant splitting* $E^{cs} \oplus E^{cu}$ for the flow of G. We say that the positive trajectory $(\phi_t(x))_{t\geq 0}$ of x is *weakly non-uniformly sectional hyperbolic* (wNUSH) if there exists $\omega > 0$ so that

$$\liminf_{T\nearrow\infty} \frac{1}{T} \int_0^T \log \|Df\,|_{E^{cs}_{\phi_t x}}\|\, dt < -\omega, \quad \text{and} \tag{18}$$

$$\liminf_{T\nearrow\infty} \frac{1}{T} \int_0^T \log \|\wedge^2 (Df\,|_{E^{cu}_{\phi_t x}})^{-1}\|\, dt < -\omega. \tag{19}$$

These conditions on a *total probability set*[4] ensure sectional-hyperbolicity. Moreover, a strong form of the property (19) is enough to get (NU2SE), as follows.

Theorem 1.8. *Let G be a C^1 vector field generating the flow $\phi_t : M \to M$ on the compact connected finite dimensional manifold M admitting a compact invariant attracting subset $\Lambda = \Lambda_G(U)$, with a dominated splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$ on the trapping region U, continuously extended to U with $G(x) \in E^{cu}_x, x \in U$. If there exists a total probability subset E of U satisfying (wNUSH), then Λ is sectional-hyperbolic.*

Moreover, if $\Lambda = \Lambda_G(U)$ is a partially hyperbolic attracting set which admits a positive volume subset of points in U satisfying (19), then we obtain a positive volume subset of

[4] That is, every $x \in E$ satisfy both (18) and (19) with the same c_0 and $\mu(E) = 1$ for each invariant probability measure μ supported in Λ.

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points of U satisfying (wNU2SE) for some $c_0 > 0$. In addition, if we replace (19) by

$$\limsup_{T\nearrow\infty} \frac{1}{T}\int_0^T \log \| \wedge^2 (Df\,|_{E^{cu}_{\phi_t x}})^{-1}\|\, dt < -\omega, \tag{20}$$

then we obtain (NU2SE), that is, condition (4) for some $c_0 > 0$.

This, together with Theorem C, enables us to restate Theorem B with the assumption (20) in the place of (NU2SE).

1.5. Weak non-uniform sectional expansion and physical measure

Combining the proof of the following Theorem E and of Theorem B, enable us to obtain the following result on existence of physical/SRB measures with weak non-uniform sectional expansion.

Corollary D. *Let $G \in \mathfrak{X}^2(M)$ be a vector field with a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ admitting no negative exponents along E^{cu}_Λ: for all $x \in \Lambda$ we have*

$$\liminf_{T\nearrow\infty} \frac{1}{T}\log \|D\phi_T(x)v\| \geq 0, \forall v \in E^{cu}_x \setminus \mathbb{R}\cdot G_x; \tag{21}$$

If the (wNU2SE) condition holds, then there exists an ergodic physical/SRB probability measure μ supported on Λ.

Moreover, if condition (SR) holds on $\Omega \subset U$, with Leb$(\Omega) > 0$*, then: we have (wNU2SE) on Ω if, and only if, there are finitely many ergodic physical/SRB measures, which are cu-Gibbs states, and whose basins cover* Leb*-a.e. point of Ω.*

Remark 1.9. *In Theorems A, B, C and Corollary D, there are no restrictions on the dimension of the central-unstable subbundle.*

Example 3 (Sectional-hyperbolic attracting sets). The multidimensional Lorenz attractor [24] is a sectional-hyperbolic attractor with a generalized Lorenz-like equilibrium. Sectional-expansion on an attracting set naturally implies sectional expansion along every orbit in a trapping neighbourhood U which, in turn, clearly ensures (MASE). In particular, it ensures the non-existence of negative Lyapunov exponents along the central-unstable direction: we get (21) with positive lim inf. Then Corollary D provides an alternative proof of existence of a unique ergodic physical/SRB measure for this family of attractors, complementing the arguments given in [24] and the proof presented in [41] for the general smooth sectional-hyperbolic attractor (which was extended in [9] for smooth sectional-hyperbolic attracting sets).

1.5.1. Codimension two partially hyperbolic attractors

In low codimension, we show that the (wMASE) condition on a positive volume subset is enough to ensure the existence of a physical/SRB measure.

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Theorem E. *Let a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ for a vector field $G \in \mathfrak{X}^2(M)$ be given, with $d_{cu} = \dim E^{cu} = 2$. Then Λ satisfies (wASE) on a positive volume subset*

$$\mathrm{Leb}\left(\left\{x \in U : \liminf_{T\nearrow\infty} \frac{1}{T}\log|\det(D\phi_T\,|_{E^{cu}_x})| > 0\right\}\right) > 0$$

if, and only if, *there exists a physical/SRB ergodic hyperbolic measure μ. If Λ is transitive, then μ is unique and* $\mathrm{Leb}(\Omega \setminus B(\mu)) = 0$.

Reciprocally, without restriction on d_{cu}, the existence of an invariant ergodic hyperbolic physical/SRB measure implies that (wMASE) holds on a positive volume subset of U.

Hence, to obtain a physical measure, it is enough to obtain a sequence of times with asymptotic sectional expansion, along the trajectories on a positive volume subset.

1.5.2. Codimension two partially hyperbolic attractors with slow recurrence

Restricting to codimension two with slow recurrence, we gain finiteness of physical measures.

Corollary F. *Let a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ for a vector field $G \in \mathfrak{X}^2(M)$ be given, with $d_{cu} = 2$ and assume that there exists $c_0 > 0$ so that (wASE) holds on a positive volume subset $\Omega := \left\{x \in U : \liminf_{T\nearrow\infty} \frac{1}{T}\log|\det(D\phi_T\,|_{E^{cu}_x})| > 0\right\}$ and points in Ω satisfy (SR). Then, there are finitely many ergodic physical/SRB measures $\mu_1, \ldots, \mu_p$, which are cu-Gibbs states, and whose basins cover* Leb*-a.e. point of Ω.*

Using the proof of the previous main results, we can deduce the following extension of the main statement from [44] on existence of physical measure for (ASH) attractors.

Corollary G. *Let a C^2 vector field G on M and a trapping region U be given containing a attracting set $\Lambda = \Lambda_G(U)$ with $d_{cu} = 2$ so that every $x \in \Lambda$ not converging to any equilibrium satisfies (wASE).*

If Λ contains only saddle-type hyperbolic equilibria, then there exists a physical/SRB probability measure supported on Λ. If Λ is transitive, then Λ supports a unique physical/SRB probability measure whose basin covers a neighbourhood of Λ.

This corollary shows that it is enough to obtain just a sequence of times with asymptotic sectional expansion, along the trajectories not converging to a singularity, to conclude the existence of a physical measure.

Example 4 (Rovella attractors). Another class of partially hyperbolic and mostly asymptotically sectional expanding attracting sets are the Rovella attractors [60] (also known as contracting Lorenz attractors) which are (weak) asymptotically singular-hyperbolic; see [43]. These attractors admit a physical/SRB probability measure whose basin covers the trapping region except a zero volume subset and, moreover, exhibit slow recurrence to the equilibrium at the origin[5]; see [45].

[5] The known proof of asymptotical sectional expansion for this family of attractors intertwines with the control of recurrence to the equilibrium point.

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1.6. Comments

Recently, Crovisier et al. [33] obtained physical measures for C^1-generic C^∞ multisingular vector fields. However this class of results fails to take into account non-transitive attracting singular sets as well as Rovella-like attractors, which are encompassed by our main statements.

1.6.1. On the proof of Theorems A and B

The reliance on discrete dynamics, through the reduction to asymptotic properties of iterations of the time-1 map $f = \phi_1$, suggests the classical alternative of reducing the flow dynamics to a global Poincaré return map to a well-chosen finite collection of cross-sections, as done in [7, 17, 19, 45] to obtain physical/SRB measures for contracting Lorenz attractors and sectional-hyperbolic attractors. This strategy however is technically challenging:

(1) non-uniform sectional expanding partial hyperbolic attractors *may admit different hyperbolic non-Lorenz-like equilibria*, especially with $d_{cu} > 2$, while sectional-hyperbolic or contracting Lorenz attractors admit only a well-controlled family of (generalized) Lorenz-like equilibria;
(2) continuous slow recurrence or slow recurrence *for the time*-1 *map* do not play a role in the construction of physical measures for sectional-hyperbolic or contracting Lorenz attractors because, in this special partial hyperbolic setting, the dynamics of the global Poincaré map can be further reduced to a one-dimensional quotient map over the stable foliation, which demands that
   - $d_{cu} = 2$ in order to obtain the one-dimensional quotient;
   - holonomies along the stable foliation exhibit some smoothness on the trapping region, to ensure that the quotient transformation is at least piecewise Hölder-C^1.

Circumventing the problems posed by the previous items in the (NUSE) setting, especially with $d_{cu} > 2$, with similar strategies to e.g. [7, 45] may be impossible, since higher dimensional invariant foliations are in general only Hölder-continuous [59]. However, we can adapt the construction from [4] to the vector field setting, as presented in this text, and also show that slow recurrence is automatic for trajectories not converging to equilibria.

1.6.2. Idea of the proof of Theorem E

The proof relies on reducing mostly asymptotical sectional expansion to a discrete version applied to the time-1 map of the flow, or some other fixed time after a time reparametrization, taking care to allow for the neutral (non-expanding nor contracting) flow direction along the centre-unstable subbundle; and applying the following useful extension of Pesin's Formula [52] obtained by Catsigeras-Cerminara-Enrich [32].

Theorem 1.10 (Generalized Pesin's Inequality [32]). *For any C^1 diffeomorphism f, if Λ is an invariant compact set with a dominated splitting $T_\Lambda M = E \oplus F$, then for Lebesgue almost every point x satisfying $\omega(x) \subset \Lambda$, the entropy of any weak* limit measure μ of the sequence $\left(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)}\right)_{n\geq 1}$ is bounded from below:*

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$$h_\mu(f) \geq \int \log |\det Df\,|_F\,|\,d\mu. \tag{22}$$

We use this for x satisfying the (wMASE) condition (13) to find an ergodic hyperbolic physical/SRB measure as an ergodic component of a limit measure μ as above. It is well known from the work of Ledrappier-Young [40] that for C^2 systems such measures are physical measures.

1.6.3. Conjectures and extensions

On the one hand, using the same techniques from [47], we may replace the domination condition by Hölder continuity of the splitting over Λ, keeping the conclusions of the main theorems and extending the thermodynamical methods from [32] to our vector field setting.

On the other hand, from Remark 1.2, we can replace the partial hyperbolic assumption on Λ by the assumption that Λ be a non-trivial attracting set with a dominated splitting $T_\Lambda M = E^s \oplus E^{cu}$, such that the vector field G is contained in E^{cu}, and keep the same conclusions of the main results. More precisely, we obtain the following statements with small adaptations of our arguments.

Theorem 1.11. *Let $\Lambda = \Lambda_G(U)$ be an attracting set for a vector field $G \in \mathfrak{X}^2(M)$ admitting an invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ such that*

- *either the splitting is Hölder-continuous and E^s is uniformly contracted;*
- *or the splitting is dominated and the flow direction is contained in E^{cu}.*

If either (SR) holds on $\Omega_0 \subset U$, or $d_{cu} = 2$, then Λ satisfies (NU2SE) on a positive volume subset $\Omega \subset \Omega_0$ if, and only if*, there exists an ergodic hyperbolic physical/SRB measure μ, which is a cu-Gibbs state with* $\mathrm{Leb}(B(\mu) \cap \Omega) > 0$.

Example 5 (Non-dominated continuous splitting). We consider the suspension flow ϕ_t, with constant roof of height 1, of the diffeomorphisms $f : \mathbb{T}^4 \to \mathbb{T}^4$ of the 4-torus studied by Tahzibi [62]. The diffeomorphism admits a Df-invariant dominated splitting $TM = E^{cs} \oplus E^{cu}$ together with a hyperbolic periodic point with expanding direction contained in E^{cs}, and another hyperbolic periodic point with contracting direction contained in E^{cu}. Hence, the naturally inherited $D\phi_t$-invariant splitting of the suspension $T\widetilde{M} = E^{cs} \oplus (E^{cu} \oplus (\mathbb{R} \cdot G))$ is *continuous but cannot be dominated*, where $\mathbb{R} \cdot G$ is the flow direction on the special manifold $\widetilde{M} := M/\sim$ defined by the natural dynamical identification.

Moreover, since the time-1 map $f = \phi_1$ satisfies $P^1 = Df$, then $E^{cu} = N^{cu}$ and, by the constructions presented in [62], we obtain condition (NUSE). It also satisfies average asymptotic contraction along E^{cs}. In addition, since f is transitive and admits a unique ergodic physical/SRB invariant measure, then the suspension of this measure is also physical/SRB with respect to the suspension flow ϕ_t; see e.g. § 4.3.

Recently, Cao-Mi-Zou [30] propose a method to construct Pesin unstable manifolds for hyperbolic measures admitting a continuous splitting of the tangent bundle. Example 5 indicates that the following stronger result should be attainable.

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Conjecture 1. *The same conclusion of Theorem 1.11 holds assuming a continuous splitting* $T_\Lambda M = E^{cs} \oplus E^{cu}$ *with* E^{cs} *having only negative Lyapunov exponents Lebesgue almost everywhere.*

Naturally we expect the dimensional restriction $\dim E^{cu} = 2$ on the statement of Theorem 1.6 to be just a technical shortcoming of our proof.

Conjecture 2. *We have (wASE)* $\Longrightarrow$ *(wNU2SE) for any value of* $\dim E^{cu} > 2$.

Example 10, namely inequalities (47), show that (wPSL) $\Longrightarrow$ (wASE) is subtle when we are dealing with trajectories which might not be Oseledets regular. On forward Oseledets regular points, conditions (A)-(B); (a)-(b) and (i)-(ii) are equivalent. From [37] our main results ensure that the ergodic basins of the physical measures are positive volume subsets of forward Oseledets regular points. It is thus natural to conjecture that this might be a priori true in more generality.

Conjecture 3. *The implications (wNUSE)* $\Longrightarrow$ *(wNU2SE); (wPSL)* $\Longrightarrow$ *(wASE) and (wNU2SE)* $\Longrightarrow$ *(19) hold on a full volume subset of points of the trapping region of a generic smooth partially hyperbolic attractor.*

Analogously to Theorem 1.8 if, in the setting of Theorem E, we have (wMASE) *on total probability*, that is, the relation (13) for all x on a total probability subset $\Omega \subset \Lambda$, then Λ becomes a sectional-hyperbolic attracting set and the conclusion of Theorem E still holds true by the result from [9] or Corollary D without any dimensional restriction.

The analogous condition for (local) diffeomorphisms was shown by Alves, Dias, Luzzatto and Pinheiro [1, 55] to be enough to obtain the same conclusions of the main results from [4]. Recently, a different approach to the proof of this result was presented by Burguet-Yang [29] and Theorem E can be seen as an extension of this result to the continuous time setting. Since (SR) was not assumed, it is natural to present the following.

Conjecture 4. *In the same setting of Theorem C, either the (wNUSE) condition (12), or the (wMASE) condition (13) is enough to obtain the same conclusion on existence and finiteness of physical/SRB measures, without assuming the slow recurrence (SR) condition.*

As for the number of distinct ergodic physical/SRB measures supported in the attracting set, motivated by the recent result [11] we propose the following.

Conjecture 5. *In the setting of the main results, the number* s *of ergodic physical/SRB measures supported in the attracting set satisfies* $s \leq 2 \cdot s_L$, *where* s_L *is the number of generalized Lorenz-like equilibria contained in* Λ.

The natural path after obtaining a physical/SRB measure is to study its statistical properties. Motivated by what has already been achieved for singular-hyperbolic attracting sets [7, 12, 14, 15]; for sectional-hyperbolic attracting sets [9]; for contracting Lorenz attractors [46]; and also in the discrete time case [2, 5, 6, 56], we propose the following.

Conjecture 6. *Given a non-uniformly sectional expanding partial hyperbolic attracting set* $\Lambda_G(U)$ *with hyperbolic singularities for a* C^2 *vector field* G, *then*

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- *modulo an arbitrary small perturbation of the norm of G, the field is topologically equivalent to a C^2 nearby vector field so that each ergodic physical/SRB is exponential mixing with respect to smooth observables; and*
- *both the flow ϕ_t of G and its time-1 map $f = \phi_1$ satisfy the Central Limit Theorem, the Law of the Iterated Logarithm and the Almost Sure Invariance Principle (for a comprehensive list, see e.g [54]) with respect to μ.*

Moreover, motivated by [10, 46], the physical/SRB measures supported on Λ should be statistically stable and also stochastically stable.

Remark 1.12. *Recently, Bruin-Farias [28] (see Example 9 in § 6), polynomial speed of mixing was proved for a neutral geometrical Lorenz-like attractor, so the assumption of hyperbolic equilibria should be necessary in Conjecture 6.*

1.7. Organization of the text

In § 2, we present preliminary results that are needed as tools for the overall construction, including the proof of Theorems 1.6 and 1.8. In § 3, we start the proof of Theorem B, using the domination of the splitting to obtain bounded distortion on *cu*-disks at hyperbolic times. Using this tool, in § 4, we study the push-forward of Lebesgue measure at hyperbolic times along *cu*-disks, obtaining the main tool to construct physical/SRB measures which are *cu*-Gibbs states. We then provide an overview of the construction of physical/SRB measures, citing the relevant results from Alves, Bonatti and Viana [4], and complete the proof of Theorem B.

In § 5, we prove Theorem C first. Then, using the Generalized Pesin's Inequality Theorem 1.10, we obtain Theorem E and deduce Corollaries D, F and G in this section.

Finally, in § 6, we present mostly asymptotically sectional expanding examples which are either non-sectional hyperbolic or non-singular hyperbolic, with or without hyperbolic equilibria, as well as counter-examples failing some of our assumptions and having no physical measure.

2. Auxiliary results

The following results will be used in our arguments.

2.1. Partial hyperbolic attracting sets

The following properties of partial hyperbolic attracting sets will be used as tools in our arguments.

2.1.1. Extension of the stable bundle and center-unstable cone fields

Let $\mathcal{D}^k$ denote the k-dimensional open unit disk and let $\mathrm{Emb}^r(\mathcal{D}^k, M)$ denote the set of C^r embeddings $\psi : \mathcal{D}^k \to M$ endowed with the C^r distance. We say that *the image of any such embedding is a C^r k-dimensional disk.*

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Proposition 2.1 ([13, Proposition 3.2, Theorem 4.2 and Lemma 4.8]). *Let Λ be a partially hyperbolic attracting set.*

(1) *The stable bundle E^s over Λ extends to a continuous uniformly contracting $D\phi_t$-invariant bundle E^s on an open positively invariant neighbourhood U of Λ.*
(2) *There exists a constant $\lambda \in (0,1)$, such that*
   (a) *for every point $x \in U$ there is a C^r embedded d_s-dimensional disk $W^s_x \subset M$, with $x \in W^s_x$, such that $T_x W^s_x = E^s_x$; $\phi_t(W^s_x) \subset W^s_{\phi_t x}$ and $d(\phi_t x, \phi_t y) \le \lambda^t d(x,y)$ for all $y \in W^s_x$, $t \ge 0$ and $n \ge 1$.*
   (b) *the disks W^s_x depend continuously on x in the C^0 topology: there is a continuous map $\gamma : U \to \mathrm{Emb}^0(\mathcal{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathcal{D}^{d_s}) = W^s_x$. Moreover, there exists $L > 0$ such that $\mathrm{Lip}\,\gamma(x) \le L$ for all $x \in U$.*
   (c) *the family of disks $\mathcal{F}^s = \{W^s_x : x \in U\}$ defines a topological foliation $\mathcal{W}^s$ of U: every $x_0 \in U$ admits a neighbourhood $V \subset U$ and a homeomorphism $\psi : V \to \mathbb{R}^{d_s} \times \mathbb{R}^{d_{cu}}$ so that $\psi(W^s_x) = \pi_s^{-1}\{\pi_s(\psi(x))\}$ where $\pi^s : \mathbb{R}^{d_s} \times \mathbb{R}^{d_{cu}} \to \mathbb{R}^{d_s}$ is the canonical projection.*

Remark 2.2. For any two close enough d_{cu}-disks D_1, D_2 contained in U and transverse to $\mathcal{F}^s$ there exists an open subset $\hat{D}_1$ of D_1 so that $W^s_x \cap D_2$ is a singleton for each $x \in \hat{D}_1$. This defines the *holonomy map* $h : \hat{D}_1 \to D_2, \hat{D}_1 \ni x \mapsto W^s_x \cap D_2$ and Proposition 2.1 ensures that h is continuous.

The splitting $T_\Lambda M = E^s \oplus E^{cu}$ extends continuously to a splitting $T_U M = E^s \oplus E^{cu}$ where E^s is the invariant uniformly contracting bundle in Proposition 2.1 – however E^{cu} is not invariant in general, but the center-unstable cone field satisfies the following.

Proposition 2.3. *Let Λ be an attracting set with a dominated splitting so that the flow direction is contained in the centre-unstable bundle $G \in E^{cu}$. Then, for any $a > 0$, after possibly shrinking U, we can find $\kappa > 0$ so that $D\phi_t \cdot \mathcal{C}^{cu}_a(x) \subset \mathcal{C}^{cu}_{\kappa e^{-\lambda^t} a}(\phi_t x)$ for all $t > 0$, $x \in U$.*

Proof. See [13, Proposition 3.1] considering the choice of adapted Riemannian metric as defined in § 1.1.2: we estimate for $v \in C^{cu}_a(x)$ (using only the domination of the splitting)

$$\frac{\|D\phi_t(x)\cdot v^s\|}{\|D\phi_t(x)\cdot v^c\|} \le \frac{\|D\phi_t\,|_{E^s_x}\| \cdot \|v^s\|}{\|(D\phi_t\,|_{E^{cu}_x})^{-1}\|^{-1} \cdot \|v^c\|} \le e^{-\lambda^t} a.$$

However, since E^{cu} extended to U is not necessarily $D\phi_t$-invariant, we need to project $D\phi_t(x)\cdot v^c$ to $E^{cu}_{\phi_t x}$ parallel to $E^s_{\phi_t x}$ to decompose $D\phi_t(x)\cdot v$ into stable/centre-unstable components. Because both $E^{cu}_{\phi_t x}$ and $D\phi_t \cdot E^{cu}_x$ are contained in $C^{cu}_a(\phi_t x)$, then we can find $\kappa = \kappa(a) > 0$ so that $\kappa\|\pi^{cu}\cdot D\phi_t(x)\cdot v^c\| \ge \|D\phi_t(x)\cdot v^c\|$, and so $\frac{\|D\phi_t(x)\cdot v^s\|}{\|\pi^{cu}\cdot D\phi_t(x)\cdot v^c\|} \le \kappa e^{-\lambda^t} a$, which completes the proof of the statement. □

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2.1.2. Partial hyperbolicity of Poincaré maps

Let Σ, Σ' be a small cross-sections to G contained in U and let $R : \mathrm{dom}(R) \to \Sigma'$ be a Poincaré map[6] $R(y) = \phi_{t(y)}(y)$ from an open subset $\mathrm{dom}(R)$ of Σ to Σ' (possibly $\Sigma = \Sigma'$). The splitting $E^s \oplus E^{cu}$ over U induces a continuous splitting $E^s_\Sigma \oplus E^{cu}_\Sigma$ of the tangent bundle $T\Sigma$ to Σ (analogously for Σ') as

$$E^s_\Sigma(y) = E^s_y \cap T_y\Sigma \quad \text{and} \quad E^{cu}_\Sigma(y) = E^{cu}_y \cap T_y\Sigma. \tag{23}$$

The splitting (23) is partially hyperbolic for R, as follows.

Proposition 2.4. ([14, Proposition 4.1]& [16, Lemma 8.25]) *Let $R : \Sigma \to \Sigma'$ be a Poincaré map with Poincaré time $t(\cdot)$. Then $DR \cdot E^s_\Sigma(x) = E^s_\Sigma(R(x))$ at every $x \in \Sigma$ and $DR \cdot E^{cu}_\Sigma(x) = E^{cu}_\Sigma(R(x))$ at every $x \in \Lambda \cap \Sigma$. Moreover, for $x \in \Sigma$ we have $\|DR\,|_{E^s_\Sigma(x)}\| < \lambda^{t(x)}$, and $\|DR\,|_{E^s_\Sigma(x)}\| \cdot \|\big(DR\,|_{E^{cu}_\Sigma(x)}\big)^{-1}\| < \lambda^{t(x)}$.*

Given a cross-section Σ, $b > 0$ and $x \in \Sigma$, the unstable cone of width b at x is

$$C^u_b(\Sigma, x) = \{v = v^s + v^u : v^s \in E^s_\Sigma(x),\, v^u \in E^{cu}_\Sigma(x) \text{and } \|v^s\| \le b\|v^u\|\}. \tag{24}$$

Corollary 2.5. *There exists $b > 0$ small enough so that, for each $R : \Sigma \to \Sigma'$ as in Proposition 2.4, we have $DR(x) \cdot C^u_b(\Sigma, x)) \subset C^u_{b\lambda^{t(x)}}(\Sigma', R(x))$ for all $x \in \Sigma$.*

Proof. Cf. proof of [13, Proposition 3.1] which is similar to the proof of Proposition 2.1. □

2.2. Hölder control of the tangent bundle in the centre-unstable direction

We recall that we have continuous extensions of the two subbundles E^s and E^{cu} defined on an isolating neighbourhood U of Λ, and the respective cone fields $C^s_a(x), C^u_a(x), x \in U$ for a small $0 < a < 1$ which are invariant in the sense of (3).

We may assume without loss of generality that, up to increasing the value of $\lambda > 0$ by a small amount and reducing the neighbourhood U of Λ, a "bunched domination condition" holds true for vectors in these cone fields: there exists $\zeta \in (0, 1)$ so that

$$\|D\phi_t \cdot u\| \cdot \|D\phi_{-t} \cdot v\|^{1+\zeta} \le e^{-\lambda^t} \cdot \|u\| \cdot \|v\|, \quad \text{for } t > 0, x \in U, u \in C^s_a(x) \;\&\; v \in C^{cu}_a(\phi_t x).$$

A C^1 disk D on M, that is, the image of a C^1 embedding $\psi : B(0,1) \subset \mathbb{R}^{d_{cu}} \to M$ defined on the unit ball of an Euclidean space, is a *cu-disk* if $T_yD \subset C^{cu}_a(y), y \in D$.

We fix $\rho_0 > 0$ so that the inverse of the exponential map $\exp_x$ is defined on the ρ_0 neighbourhood of each point $x \in U$, which we identify with the corresponding neighbourhood V_x of the origin 0 in T_xM, and x with 0.

We may assume without loss of generality that $E^s_x \subset C^s_a(y)$ for all $y \in V_x$ so that, in particular, $E^s_x \cap C^{cu}_a(x) = \{0\}$. If $x \in D$ then T_yD is given by the graph of the linear map $A_x(y) : T_xD \to E^s_x$ for each $y \in V_x \cap D$.

[6] Note that R needs not correspond to the first time the orbits of Σ encounter Σ' nor it is defined everywhere in Σ.

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We say that *the tangent bundle TD is (C,ζ)-Hölder* if there are constants $C,\zeta>0$ such that $\|A_x(y)\| \le C d_x(y)^\zeta$ for $y \in D \cap V_x$, where $d_x(y)$ is the intrinsic distance from x to y within $D \cap V_x$.[7] Given a *cu*-disk D we write $\kappa(D)$ for the least $C>0$ so that the tangent bundle of D is (C,ζ)-Hölder.

We recall the notation $f=\phi_1$ for the time-1 map of the flow of G. Then we can prove the following, since Λ is also a partially hyperbolic attracting set for f.

Proposition 2.6. ([4, Proposition 2.2 & Corollary 2.4]) *There exists $C_1>0$ so that each cu-disk $D \subset U$ satisfies*

(a) *there exists $n_0 \ge 1$ such that $\kappa(f^n(D)) \le C_1$ for every $n \ge n_0$ such that $f^k(D) \subset U$ for all $0 \le k \le n$; and, if $\kappa(D) \le C_1$, then $n_0 = 1$;*
(b) *$J_k : f^k(D) \ni x \mapsto \log|\det\left(Df\mid_{T_x f^k(D)}\right)|$ are (L_1,ζ)-Hölder continuous for $0 \le k \le n$ whenever D and n are as above, where $L_1 = L_1(f,C_1) > 0$ depends only on C_1 and G.*

Remark 2.7. For any small $\varepsilon>0$, the family $\{\phi_{(-\varepsilon,\varepsilon)}\left[D \cap \exp_z\left(N^u_z \cap B(0,\rho_0)\right)\right] : z \in U\}$ of *cu*-disks is flow invariant. Then all of them have curvature bounded above by C_1.

2.3. Equivalences and implications between non-uniform sectional expansion versions

Here we prove Theorem 1.6 first and then Theorem 1.8.

Proof of Theorem 1.6. We recall that the extension of E^{cu}_Λ to U is not necessarily $D\phi_t$-invariant and note that, given any 2-subspace F_x of E^{cu}_x, the map $t \mapsto |\det(D\phi_t\mid_{F_x})|$ is a multiplicative function $|\det(D\phi_{t+s}\mid_{F_x})| = |\det(D\phi_s\mid_{D\phi_t\cdot F_x})|\cdot|\det(D\phi_t\mid_{F_x})|$, for all $t,s \ge 0, x \in U$.

In addition, by Proposition 2.3, we get $D\phi_t \cdot E^{cu}_x \subset C^{cu}_{\kappa\lambda^t a}(\phi_t x)$ for constants $\kappa, a>0$. Then the stable direction $E^s_{\phi_t x}$ is complementary to both the $D\phi_t\cdot E^{cu}_x$ and $E^{cu}_{\phi_t x}$ directions at $\phi_t x$. Therefore there exists a natural isomorphism $\pi^s : (D\phi_t \cdot E^{cu}_x) \to E^{cu}_{\phi_t x}$ given by the projection parallel to $E^s_{\phi_t x}$. Hence, $\pi^s(D\phi_t \cdot F_x) = \widehat{F}_{\phi_t x} \subset E^{cu}_{\phi_t x}$ and since the width $\kappa\lambda^t a$ of the center-unstable cone around $E^{cu}_{\phi_t x}$ is small for large $t>0$, then we obtain $\xi_t \to 1$ when $t \nearrow \infty$ such that for any fixed $s \ge 0$

$$\xi_t^{-1}|\det(D\phi_s\mid_{\widehat{F}_{\phi_t x}})| \le \frac{|\det(D\phi_{t+s}\mid_{F_x})|}{|\det(D\phi_t\mid_{F_x})|} \le \xi_t|\det(D\phi_s\mid_{\widehat{F}_{\phi_t x}})|. \tag{25}$$

We first show that **(wASE)**, given by (15), implies the **(wNU2SE)** condition (11), i.e., (i) $\Longrightarrow$ (A), when $d_{cu}=2$. By assumption, given any 2-subspace $F_x \subset E^{cu}_x$ and sufficiently small $\varepsilon>0$, there exists a strictly increasing sequence t_k such that $t_k \nearrow \infty$ and $\log|\det(D\phi_{t_k}\mid_{F_x})| \ge (c_0-\varepsilon)t_k$. Since $|\det(D\phi_{[t_k]}\mid_{F_x})| = \prod_{i=0}^{[t_k]-1}\frac{|\det(D\phi_{i+1}|_{F_x})|}{|\det(D\phi_i|_{F_x})|}$ we

[7] The length of the shortest curve connecting x to y inside $D \cap V_x$.

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get[8] from (25)

$$\left|\log|\det(D\phi_{t_k}\,|_{F_x})^{-1}| + \sum_{i=0}^{[t_k]-1}\log|\det(Df\,|_{\widehat{F}_{f^ix}})|\right| \le \sum_{i=0}^{[t_k]}|\log\xi_i|.$$

Hence, for all k and each 2-subspace $F_x \subset E^{cu}_x$, we can write

$$\sum_{i=0}^{[t_k]-1}\log|\det(Df\,|_{F_{f^ix}})^{-1}| \le (-c_0+\varepsilon)t_k + \sum_{i=0}^{[t_k]-1}|\log\xi_i|.$$

Since $\varepsilon>0$ was arbitrary, we obtain $\liminf \frac{1}{n}\sum_{i=0}^{n-1}\log\|\wedge^2(Df\,|_{E^{cu}_{f^ix}})^{-1}\| \le -c_0$, that is, (wNU2SE) if $\dim E^{cu}=2$. Analogously, we deduce that **(ASE)** $\Longrightarrow$ **(NU2SE)**, i.e., (i) $\Longrightarrow$ (A) after exchanging $\liminf$ with $\limsup$, and note that no assumption on non-convergence to equilibria was needed here.

We now easily show both **(NU2SE)** $\Longrightarrow$ **(NUSE)** and **(wNU2SE)** $\Longrightarrow$ **(wNUSE)**, i.e., (A) $\Longrightarrow$ (B), without restrictions on d_{cu}. For a regular point $x\in U$, any 2-subspace $F_x\subset E^{cu}_x$ containing $G(x)$ admits an orthonormal basis $\{G(x)/\|G(x)\|, n(x)\}$[9] and $\widehat{F}_{f^ix}=\pi^s(Df^i_x\cdot F_x)$ admits an orthonormal basis $\{G(f^ix)/\|G(f^ix)\|, n(f^ix)\}$ for $i\ge 1$, where $n(f^ix)$ is a unit vector on the line $\widehat{F}_{f^ix}\cap G(f^ix)^{\perp}$[10]. These choices ensure that

$$|\det(D\phi_1\,|_{\widehat{F}_{f^ix}})| = \det\begin{pmatrix}\|G(f^{i+1}x)\|/\|G(f^ix)\| & \star\\ 0 & \|P^1\cdot n(f^ix)\|\end{pmatrix}. \tag{26}$$

Let $\mathcal{G}_2(f^ix)$ be the compact family of all 2-subspaces of $E^{cu}_{f^ix}$ which contain $G(f^ix)$. From (26) we get

$$\sup_{F\in\mathcal{G}_2(f^ix)}|\det(Df\,|_F)^{-1}| = \big(\|G(f^ix)\|/\|G(f^{i+1}x)\|\big)\cdot\sup_{n\in N^{cu1}_{f^ix}}\|P^1\cdot n\|^{-1},$$

where $N^{cu1}_{f^ix}$ is the unit sphere on the normal bundle $N^{cu}_{f^ix}$ and $i\ge 1$. Since this is a subset of all 2-subspaces of E^{cu}_x, we get

$$\frac{\|G(f^ix)\|}{\|G(f^{i+1}x)\|}\cdot\|(P^1\,|_{N^{cu}_{f^ix}})^{-1}\| \le \|\wedge^2(Df\,|_{E^{cu}_{f^ix}})^{-1}\|. \tag{27}$$

Let $\Gamma(x) := \log\|\wedge^2(Df\,|_{E^{cu}_x})^{-1}\|$ and $\psi(x) := \log\|(P^1\,|_{N^{cu}_x})^{-1}\|$ in what follows. Therefore

$$\sum_{i=0}^{n-1}\Gamma(f^ix)+\sum_{i=0}^{n-1}\log\frac{\|G(f^{i+1}x)\|}{\|G(f^ix)\|}\ge\sum_{i=0}^{n-1}\psi(f^ix),\quad n\ge 1.$$

[8] We write $[t]=\sup\{\ell\le t:\ell\in\mathbb{Z}^+\}$ for the integer part of $t\in\mathbb{R}$.
[9] Recall that $E^{cu}_x\ni G(x)$ for $x\in U$ by construction; see § 1.2.
[10] In general, we do not know if E^{cu} is orientable, however the choice $\pm n(f^ix)$ is immaterial since we consider the modulus of the Jacobian in what follows

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Since we assume that the future trajectory of x does not converge to any equilibrium, we can use the following.

Lemma 2.8. *We have* $\limsup \frac{1}{n}\sum_{i=0}^{n-1}\log\frac{\|G(f^{i+1}x)\|}{\|G(f^ix)\|} = 0 = \limsup \frac{1}{n}\log\|G(f^nx)\|$ *when* $n \nearrow \infty$*; and also* $\limsup \frac{1}{t}\log\|G(\phi_t x)\| = 0$ *when* $t \nearrow \infty$*, for every* $x \in U$ *whose future trajectory does not converge to any equilibrium. In particular, it holds for* Leb*-a.e* $x \in U$*.*

Hence, dividing by n and taking lim sup when $n \nearrow \infty$, Lemma 2.8 together with (27) ensures that **(NU2SE)** $\Longrightarrow$ **(NUSE)**. Analogously, using lim inf provides **(wNU2SE)** $\Longrightarrow$ **(wNUSE)** since, for all $\varepsilon > 0$ there exists $N > 1$ so that $\frac{1}{n}\sum_{i=0}^{n-1}\log\frac{\|G(g^{i+1}x)\|}{\|G(g^ix)\|} < \varepsilon$ for all $n \geq N$, by Lemma 2.8. Thus, for any $\varepsilon > 0$ we get

$$\liminf_{n\nearrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}\psi(f^ix) \leq \varepsilon + \liminf_{n\nearrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}\Gamma(f^ix) \leq \varepsilon - c_0.$$

Both **(wMASE)** $\Longrightarrow$ **(wASEP)** and **(MASE)** $\Longrightarrow$ **(ASEP)**, i.e., (a) $\Longrightarrow$ (b), follow from the analogous to (26) and (27): we get for the family of 2-subspaces F of E^{cu}_x with orthonormal basis $\{G(x), n(x)\}$ for some $n(x) \in N^{cu1}_x$

$$|\det(D\phi_t\,|_F)| = \det\begin{pmatrix}\|G(\phi_t x)\|/\|G(x)\| & \star\\ 0 & \|P^t\cdot n(x)\|\end{pmatrix} \tag{28}$$

and it follows, as before, that $\frac{\|G(x)\|}{\|G(\phi_t x)\|}\cdot\|(P^t\,|_{N^{cu}_x})^{-1}\| \leq \|\wedge^2(D\phi_t\,|_{E^{cu}_x})^{-1}\|$. Then, for each $t > 0$, we get $\frac{1}{t}\left(\log\|\wedge^2(D\phi_t\,|_{E^{cu}_x})^{-1}\| + \log\frac{\|G(\phi_t x)\|}{\|G(x)\|}\right) \geq \frac{1}{t}\log\|(P^t\,|_{N^{cu}_x})^{-1}\|$, and the claimed relations follow just like above using Lemma 2.8.

Analogously, we obtain (b) $\Longrightarrow$ (i), that is, **(wASEP)** $\Longrightarrow$ **(wASE)** and **(ASEP)** $\Longrightarrow$ **(ASE)**. For any 2-subspace F of E^{cu}_x containing $G(x)$ and each fixed $t > 0$, we have (28). For F not containing $G(x)$, we choose a basis $\{v_i + \omega_i G(x), i = 1, 2\}$ of F, with $\omega_1\omega_2 \neq 0$; and $\{v^t_i + \omega^t_i G(\phi_t x), i = 1, 2\}$ of $F_t := D\phi_t(x)F$, such that $\{v^t_1, v^t_2\}$ is orthonormal. Thus, $P^{-t}\big(v^t_i + \omega^t_i G(\phi_t x)\big) = v_i, i = 1, 2$ and $|\det(D\phi_t\,|_F)^{-1}|$ is given by

$$\frac{\|\big(v_1 + \omega_1 G(x)\big)\wedge\big(v_2 + \omega_2 G(x)\big)\|}{\|\big(v^t_1 + \omega^t_1 G(\phi_t x)\big)\wedge\big(v^t_2 + \omega^t_2 G(\phi_t x)\big)\|} = \|(P^{-t}v^t_1)\wedge(P^{-t}v^t_2)\| \leq \|(P^t\,|_{N^{cu}_x})^{-1}\|^2.$$

Then $|\det(D\phi_t\,|_F)^{-1}| \leq \max\left\{\frac{\|G(x)\|}{\|G(\phi_t x)\|}\|(P^t\,|_{N^{cu}_x})^{-1}\|, \|(P^t\,|_{N^{cu}_x})^{-1}\|^2\right\} = \max\{a_t, b_t\}$. From (wASEP) and Lemma 2.8, for any $\varepsilon > 0$ we have $T > 0$ so that $\|G(\phi_t x)\| > e^{-\varepsilon t}$ for all $t > T$, and then $\liminf \frac{1}{t}\log a_t$ satisfies

$$\liminf_{t\nearrow\infty}\frac{1}{t}\log\left(\frac{\|G(x)\|}{\|G(\phi_t x)\|}\|(P^t\,|_{N^{cu}_x})^{-1}\|\right) \leq \varepsilon + \liminf_{t\nearrow\infty}\frac{1}{t}\log\|(P^t\,|_{N^{cu}_x})^{-1}\| \leq -c_0 + \varepsilon.$$

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Since $\varepsilon > 0$ is arbitrary, we see that for any sequence $t_n \nearrow \infty$ there exists $N > 1$ so that for all $n > N$ we have $|\det(D\phi_{t_n}\,|_F)^{-1}| \le a_{t_n}$ and so

$$\limsup_{n\nearrow\infty} \frac{1}{n} \log|\det(D\phi_t\,|_F)| = \liminf_{n\nearrow\infty} \frac{1}{n} \log|\det(D\phi_t\,|_F)^{-1}| \le -c_0.$$

Since F is an arbitrary 2-subspace of E^{cu}_x, we have obtained (wASE), as claimed. If we assume (ASEP), then from Lemma 2.8 we have

$$\limsup_{t\nearrow\infty} \frac{1}{t} \log a_t \le \liminf_{t\nearrow\infty} \frac{1}{t} \log \frac{\|G(\phi_t x)\|}{\|G(x)\|} + \limsup_{t\nearrow\infty} \frac{1}{t} \log\|(P^t\,|_{N^{cu}_x})^{-1}\| \le -c_0.$$

Hence, by the same reasoning as above, we conclude (ASE)

$$\liminf_{n\nearrow\infty} \frac{1}{n} \log|\det(D\phi_t\,|_F)| = \limsup_{n\nearrow\infty} \frac{1}{n} \log|\det(D\phi_t\,|_F)^{-1}| \le -c_0.$$

We can now show that (i) $\Longrightarrow$ (ii), that is, **(wASE)** $\Longrightarrow$ **(wPSL)** and **(ASE)** $\Longrightarrow$ **(PSL)**. Indeed, given any $v \in E^{cu}_x \setminus \mathbb{R}\cdot G(x)$, we let F be the 2-subspace of E^{cu}_x generated by $\{G(x), n(x)\}$, where $n(x) \in N^{cu}_x$ is the orthogonal projection of v. Then, assuming (wASE), given $\varepsilon > 0$, there exists $t_n \nearrow \infty$ so that

$$e^{\varepsilon t_n}\|P^{t_n} n(x)\| \ge \frac{\|G(\phi_{t_n}x)\|}{\|G(x)\|}\|P^{t_n} n(x)\| = |\det(D\phi_{t_n}\,|_F)| \ge e^{(c_0-\varepsilon)t_n}$$

from (28) and Lemma 2.8. By definition of the Linear Poincaré Flow, since $v = n(x) + \omega G(x)$ with $\omega \in \mathbb{R}$, then there exits $\eta > 0$ so that

$$\|D\phi_{t_n}(x)v\|^2 = \|P^{t_n} n(x) + \eta G(\phi_{t_n}x)\|^2 = \|P^{t_n} n(x)\|^2 + \|\eta G(\phi_{t_n}x)\|^2 \ge \|P^{t_n} n(x)\|^2.$$

Altogether we obtain $\|D\phi_{t_n}(x)v\| \ge e^{(c_0-2\varepsilon)t_n}$. Since $\varepsilon > 0$ is arbitrary, we get **(wPSL)**. Assuming now **(ASE)**, that is $\liminf \frac{1}{T}\log|\det(D\phi_T \mid F)| \ge c_0$, then we obtain the same estimates above for all times t larger than some $T > 0$ in the place of t_n. We thus conclude that $\|D\phi_t(x)v\| \ge e^{(c_0-2\varepsilon)t}$ for all $t > T$. Hence, we get $\liminf \frac{1}{t}\log\|D\phi_t(x)v\| \ge c_0$, i.e, (PSL).

Finally, for (A) $\Longrightarrow$ (a), that is, **(wNU2SE)** $\Longrightarrow$ **(wMASE)** and **(NU2SE)** $\Longrightarrow$ **(MASE)**: these follow from the subadditivity of the continuous map $(t,x) \in \mathbb{R}^+ \times U \mapsto \psi_t(x) = \log\|\wedge^2 (D\phi_t\,|_{E^{cu}_x})^{-1}\|$ together with the relative compactness of U. Indeed, for any real $T > 0$ we have

$$\begin{aligned}\frac{\log\|\wedge^2 (D\phi_T\,|_{E^{cu}_x})^{-1}\|}{T} &\le \frac{\log\|\wedge^2 (D\phi_{T-[T]}\,|_{E^{cu}_{f^{[T]}x}})^{-1}\|}{T} + \sum_{i=0}^{[T]-1} \frac{\log\|\wedge^2 (Df\,|_{E^{cu}_{f^i x}})^{-1}\|}{T} \\ &\le \frac{[T]}{T}\cdot\frac{1}{[T]}\sum\nolimits_{i=0}^{[T]-1} \Gamma(f^i x) + \frac{2}{T}\log L,\end{aligned}$$

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where $L := \sup\{\log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1}\| : x \in U, 0 \le t \le 1\}$. So letting $T \nearrow \infty$ shows that the discrete time average of Γ is larger than the asymptotic average, as claimed.

This completes the proof of Theorem 1.6, except for the proof Lemma 2.8, which we present next. □

Proof of Lemma 2.8. The sum in the statement is telescopic, so it becomes

$$\limsup_{n\nearrow\infty}(\log \|G(f^n x)\| - \log \|G(x)\|)/n = \limsup_{n\nearrow\infty} \log \|G(f^n x)\|^{1/n}. \tag{29}$$

Since $\|G\|$ is bounded above, we get (29)$\le \limsup_{t\nearrow\infty} \frac{1}{t} \log \|G(\phi_t x)\| \le 0$. Hence, it is enough to find the same lower bound.

Let us assume that the future trajectory of x does not converge to an equilibrium. We argue by contradiction: if there exists $\varepsilon_0 > 0$ so that $\limsup_{t\nearrow\infty} \log \|G(\phi_t x)\|^{1/t} < -\varepsilon_0$, then we can find $T = T(x, \varepsilon_0) > 1$ so that for all $t > T$ we get $\|G(\phi_t x)\| \le e^{-\varepsilon_0 t}$. This implies that $\phi_t x$ is converging to the finite subset $\mathrm{Sing}_\Lambda(G)$, and so converges to some equilibrium of this set. This contradiction shows that $\limsup \frac{1}{t} \log \|G(\phi_t x)\| = 0$. The same argument replacing a real $t \nearrow \infty$ with an integer $n \nearrow \infty$ completes the proof of the first statement of the lemma.

Let $S(\Omega) \subset \mathrm{Sing}_\Lambda(G)$ be the collection of equilibria accumulated by the future trajectories of points of Ω. Then $\sigma \in S(\Omega)$ cannot be a sink (otherwise trajectories converging to σ cannot have sectional expansion) nor a source. Since $S(G)$ consists of finitely many hyperbolic fixed points of saddle-type, then every approach of the trajectory to any element $\sigma \in S(G)$ is followed by a departure from σ, except for the points of the stable manifold of σ. But, in this setting, the union $\cup_{\sigma\in S(G)} W^s_\sigma$ of the local stable manifolds has zero volume. Indeed, the stable manifold of a hyperbolic critical element in an immersed submanifold [51], so it has zero volume as a subset of the ambient manifold. Hence, $U \setminus \cup\{W^s_\sigma : \sigma \in \mathrm{Sing}_\Lambda(G)\}$ has full volume in U.

Thus, the previous convergence of $f^n x$ or $\phi_t x$ to $\mathrm{Sing}_\Lambda(G) \supset S(\Omega)$ is impossible on a positive Lebesgue measure subset. This completes the proof of the lemma. □

Proof of Theorem 1.8. Let $\Gamma(x) := \log \| \wedge^2 (Df \mid_{E^{cu}_x})^{-1}\|$ and set $E := \{x \in U : (19)$ holds for $x\}$. If $\mu(E) = 1$ for some ergodic invariant measure μ for the flow, we obtain $\mu(\Gamma) = \lim_{t\nearrow\infty} \frac{1}{t} \int \Gamma(\phi_t x)\, d\mu(x) = \int \lim_{t\nearrow\infty} \frac{1}{t} \Gamma(\phi_t x)\, d\mu(x) < -\omega$ from Kingman's Subadditive Ergodic Theorem, since the limit exists and coincides with the limit inferior for μ-a.e. x. But by subadditivity $\int \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1}\|\, d\mu(x)$ is bounded above by

$$\int \log \| \wedge^2 (D\phi_{t-[t]} \mid_{E^{cu}_{\phi_{[t]}x}})^{-1}\|\, d\mu(x) + \int \log \| \wedge^2 (D\phi_{[t]} \mid_{E^{cu}_x})^{-1}\|\, d\mu(x)$$
$$\le \sup_{x\in U} \sup_{s\in[0,1]} \log \| \wedge^2 (D\phi_s \mid_{E^{cu}_x})^{-1}\| + \sum\nolimits_{i=0}^{[t]-1} \int \Gamma(f^i x)\, d\mu(x) = C + [t]\mu(\Gamma)$$

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where $f := \phi_1$, C is a constant depending on the flow, and we used that μ is f-invariant. Therefore, again by the Subadditive Ergodic Theorem

$$\int \lim_{t\nearrow\infty} \frac{1}{t} \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \| \, d\mu(x) = \lim_{t\nearrow\infty} \frac{1}{t} \int \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \| \, d\mu(x)$$
$$\leq \limsup_{t\nearrow\infty} \frac{[t]}{t} \cdot \mu(\Gamma) < -\omega.$$

Since we assume that the above holds for any invariant ergodic probability measure μ for the flow, we are in the conditions of the following result.

Proposition 2.9. ([20, Corollary 4.2]) *Let $\{t \mapsto f_t : \Lambda \to \mathbb{R}\}_{t\in\mathbb{R}}$ be a continuous family of continuous function which is subadditive and suppose that, for every invariant probability measure μ for the flow, the limit $\widetilde{f}(x) := \lim_{t\nearrow\infty} \frac{1}{t} f_t(x)$ defined for μ-a.e. x satisfies $\int \widetilde{f}\, d\mu < 0$. Then there exist constants $T > 0$ and $\lambda > 0$ such that for every $x \in \Lambda$ and every $t > T$ we have $f_t(x) \leq -\lambda \cdot t$.*

If we set $f_t(x) := \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \|$ we obtain $T_u, \lambda_u > 0$ such that

$$\| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \| \leq e^{-\lambda_u t} \quad \text{for } t \geq T_u \text{ and all } x \in \Lambda.$$

Analogously, if we instead set $f_t(x) = \log \|D\phi_t \mid_{E^{cs}_x} \|$ and reapply the same reasoning, we conclude the existence of $T_s, \lambda_s > 0$ so that $\|D\phi_t \mid_{E^{cs}_x} \| \leq e^{-\lambda_s t}$ for all $x \in \Lambda$ and $t \geq T_s$.

This ensures that the compact set Λ with the invariant continuous splitting $E^{cs} \oplus E^{cu}$ is a sectional-hyperbolic set, and concludes the proof of the first statement of the theorem.

For the second statement, we write $\psi_\delta(x) := \int_0^\delta \Gamma(\phi_t x)\, dt$ where $\delta = 1/m > 0$ for some $m > 1$ so large that for $0 < \zeta < \omega/4$ we get $\exp\big(\Gamma(\phi_\delta x) - \Gamma(x)\big) \leq 1 + \zeta$ and so $|\Gamma(\phi_\delta x) - \Gamma(x)| \leq \log(1 + \zeta) < \omega/4$ for all $x \in U$. Then $\psi_\delta(x) = \Gamma(x) + \xi(x)$ where $\xi : U \to (-\zeta, \zeta)$ is a continuous function.

Lemma 2.10. *Let us consider a point $x \in U$ and a sequence $T_n \nearrow \infty$ satisfying*

$$\int_0^{T_n} \Gamma(\phi_t x)\, dt < -\omega T_n \tag{30}$$

for all large $n > 1$. Then there exists $0 \leq \widehat{j} < m$ so that for n large enough, with $\ell = \ell_n = [T_n/\delta]/m$ and $g := \phi_\delta$, we have $\sum_{i=0}^{\ell-1} \Gamma(f^i g^{\widehat{j}} x) < -\ell\omega/4m$.

Proof. We rewrite the integral in (30) as follows

$$\int_0^{T_n} \Gamma(\phi_t x)\, dt = \int_{[T_n]}^{T_n - \delta[T_n/\delta]} \Gamma(\phi_t x)\, dt + \sum\nolimits_{i=0}^{[T_n/\delta]-1} \psi_\delta(g^i_\delta x). \tag{31}$$

Since the integrand Γ is uniformly bounded by a constant C and $T_n - \delta[T_n/\delta] < \delta$, can rewrite (30) as $\sum_{i=0}^{[T_n/\delta]-1} \psi_\delta(g^i x) < -\omega T_n + C\delta < (-\omega + C\delta/T_n)T_n < -[T_n]\omega/2$ for

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all n large enough. We group iterates that are time-1 apart rewriting the summation as follows: we let $[T_n/\delta] = \ell m$ for some $\ell = \ell_n > 1$ and

$$\sum_{j=0}^{m-1} \sum_{i=0}^{\ell-1} \psi_\delta(f^i g^j x) < -[T_n/\delta]\delta\omega/2 = -\ell\omega/2.$$

So, one of the inner sums is negative. More precisely, there exists $\widehat{j} = \widehat{j}(x) \in \{0, \dots, m-1\}$ so that $\sum_{i=0}^{\ell-1} \psi_\delta(f^i g^{\widehat{j}} x) < -\ell\omega/2m$. By the choice of δ we obtain

$$\begin{aligned} &\sum_{i=0}^{\ell-1} \big(\Gamma(f^i g^{\widehat{j}} x) \pm \xi(f^i g^{\widehat{j}} x)\big) < -\ell\omega/2m \\ &\quad \Longrightarrow \sum_{i=0}^{\ell-1} \Gamma(f^i g^{\widehat{j}} x) < -\ell\omega/2m + \zeta < -\ell\omega/4m \end{aligned}$$

for n large enough, with $\ell = \ell_n = [T_n/\delta]/m$, as in the statement of the lemma. □

We define $E_j := \left\{ x \in U : \liminf_{\ell\nearrow\infty} \sum_{i=0}^{\ell-1} \Gamma(f^i g^j x)^{1/\ell} < -\omega/4m \right\}$ for each $0 \le j < m$. If we assume that $\mathrm{Leb}(E) > 0$, then from Lemma 2.10 we have that $E \subset \cup_{j=0}^{m-1} E_j$ and so there exists $\widehat{j} \in \{0, \dots, m-1\}$ such that $\mathrm{Leb}(E_{\widehat{j}}) > 0$. By smoothness of f, the set $\Omega = g^{-\widehat{j}} E_{\widehat{j}}$ satisfies $\mathrm{Leb}(\Omega) > 0$ and every $x \in \Omega$ is such that

$$\liminf_{n\nearrow\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Gamma(f^i x) \le -\omega/4m.$$

This is (wNU2SE) with $c_0 = \omega/4m$, completing the proof of the first part of the second statement of Theorem 1.8.

Exchanging lim inf by lim sup in E and E_j enables us to follow the same reasoning to show that (20), on a positive volume subset of U, implies (NU2SE) on a positive volume subset of U with a slightly smaller rate $c_0 = \omega/m$. This completes the proof of Theorem 1.8. □

3. Hyperbolic times and center-unstable disks

In this section, we start the proof of Theorem B. We only use the domination of the splitting and hyperbolic times along the sectional centre-unstable direction. We assume that (NUSE) hold for a positive Lebesgue measure subset Ω as in (17); see Theorem 1.6.

3.1. Hyperbolic (Pliss) times for the Linear Poincaré Flow

The following is a very useful tool which enables us to use *hyperbolic times*.

Lemma 3.1. ([4, Lemma 3.1]) *Let $A \ge c_2 > c_1$ be real numbers and $\zeta = (c_2 - c_1)/(A - c_1)$. Given real numbers $a_1, \dots, a_N$ satisfying*

$$\sum_{j=1}^{N} a_j \ge c_2 N \quad \text{and} \quad a_j \le A \ \text{ for all } \ 1 \le j \le N,$$

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there are $\ell > \zeta N$ *and* $1 < n_1 < \ldots < n_\ell \le N$ *such that*

$$\sum_{j=n+1}^{n_i} a_j \ge c_1 \cdot (n_i - n) \text{ for each } 0 \le n < n_i,\ i = 1, \ldots, \ell.$$

Let us fix $x \in \Omega$ satisfying (17). Since $(P^1 \mid_{N_x^{cu}})^{-1} = \mathcal{O}_x \cdot Df^{-1} \mid_{P^1(N_x^{cu})}$, then $\|(P^1 \mid_{N_x^{cu}})^{-1}\| \le \|Df^{-1}\| \le e^L$ with $L = \sup_{x \in U} \|DG(x)\|$, which is finite because U is relatively compact. We also have $\|(P^1 \mid_{N_x^{cu}})^{-1}\| \ge \|P^1 \mid_{N_x^{cu}}\|^{-1} \ge e^{-L}$, and thus $A_0 = \sup_{x \in U} \left|\log \|(P^1 \mid_{N_x^{cu}})^{-1}\|\right| \le e^L$. Then we apply Lemma 3.1 to $a_i = -\log \|(P^1 \mid_{N^{cu}_{f^{i-i_x}}})^{-1}\|$ for $i = 1, \ldots, N$ so that $\sum_{i=1}^N a_i \ge c_0 N/2$ – this inequality holds for all large enough $N = N(x) > 1$.

We obtain $\ell > \zeta_1 N$ with $\zeta_1 = (c_0/2 - c_0/4)/(A_0 - c_0/4) = c_0/(4A_0 - c_0) > 0$ and times $1 < n_1 < \ldots < n_\ell \le N$ such that

$$\prod_{j=n}^{n_i - 1} \|(P^1 \mid_{N^{cu}_{fjx}})^{-1}\| \le e^{-c_0(n_i - n)/4}, \quad 0 \le n < n_i, \quad i = 1, \ldots, \ell. \tag{32}$$

We say that n_i is a *hyperbolic time* for x if $\mathrm{Sing}_\Lambda(G) = \emptyset$. In the presence of equilibria, we need to control the visits of the future orbit of x near these fixed points where the Linear Poincaré flow is not defined. For that, we reapply Lemma 3.1 to (8), as follows.

For $\varepsilon_0 \in (0, \zeta_1 c_0/32)$ we take $\delta_0 > 0$ satisfying (8) for all $x \in \Omega$, so that for some $N(x) > 1$ we have

$$n \ge N(x) \implies \sum_{i=0}^{n-1} \log d_{\delta_0}\big(f^i(x), \mathrm{Sing}_\Lambda(G)\big) \ge -2\varepsilon_0 \cdot n.$$

Since the summands are non-positive, we can take $A = 0$, $c_2 = -2\varepsilon_0$ and $c_1 = -c_0/16$ to obtain $\zeta_2 = (c_2 - c_1)/(A - c_1) = 1 - c_2/c_1 > 1 - \zeta_1$. Hence $\zeta_1 + \zeta_2 > 1$ and for $\zeta = \zeta_1 + \zeta_2 - 1 > 0$ we have $\ell \ge \zeta N$ and times $1 < n_1 < \ldots < n_\ell \le N$ simultaneously satisfying (32) and the conclusion of Pliss' Lemma for the last summation. We have proved the following.

Proposition 3.2. *For each sufficiently small* $\varepsilon_0 > 0$*, we can find a small enough* $\delta_0 > 0$ *such that there are* θ, ε_0 *and for* $x \in \Omega$ *we can find* $N = N(x) \in \mathbb{Z}^+$ *so that for any given integer* $T \ge N$*, there exists* $\ell \ge \theta T$ *and times* $1 < n_1 < \cdots < n_\ell \le T$ *satisfying* (32) *and*

$$d_{\delta_0}\big(f^j x, \mathrm{Sing}_\Lambda(G)\big) > e^{-c_0(n_i - j)/16}, \quad 0 \le j < n_i, \quad i = 1, \ldots, \ell. \tag{33}$$

The times n_i satisfying the conclusion of Proposition 3.2 will be referred to as *hyperbolic times* for x when $\mathrm{Sing}_\Lambda(G) \ne \emptyset$.

Remark 3.3 (uniform positive asymptotical density). This shows that for every $x \in \Omega$ satisfying (17) there exists a subset $HT(x) \subset \mathbb{Z}^+$ formed of hyperbolic times so that $\limsup_{n \nearrow \infty} \#(HT(x) \cap [1, n])/n \ge \theta > 0$.

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3.2. Estimates for nearby points and roughness of hyperbolic times

We observe that the map $x \in U \mapsto E^{cu}_x$ is Hölder-continuous, by the domination of the splitting, see e.g. [13, Subsection 4.2]. In addition, both $x \mapsto Df(x)$ and $x \in U^* \mapsto \mathcal{O}_x$ are Lipschitz, where $U^* = U \setminus \mathrm{Sing}(G)$, because G is of class C^2 and the unit vector field $\hat{G} := G/\|G\|$ defined in U^* has derivative $\hat{G}(x)' = \mathcal{O}_x \circ DG(x) \cdot \hat{G}_x$ whose norm is uniformly bounded from above. Hence $\Psi : U^* \times U^* \to \mathbb{R}, (x,y) \mapsto \log \frac{\|(P^1|_{N^u_y})^{-1}\|}{\|(P^1|_{N^u_x})^{-1}\|}$ is Hölder-continuous and $\Psi(x,x) = 0$ for all $x \in U^*$. Therefore, there exists a constant $C_2 > 0$ and an exponent $\omega \in (0,1)$ so that $\Psi(x,y) \le C_2 \cdot d(x,y)^\omega$.

We recall that $\rho_0 > 0$ is such that $\left(\exp_x |_{B(0,\rho_0)}\right)^{-1}$ is well-defined at every $x \in U$.

For $z \in M$ with $G(z) \neq \vec{0}$ we define the cone

$$C^\perp_a(z) = \{v + \lambda G(z) : v \in G(z)^\perp,\ \lambda \in \mathbb{R}\ \&\ \|v\| \le a\|\lambda G(z)\|\}.$$

We let $a > 0$ be small enough so that for $y \in U^*$

$$\|\mathcal{O}_y \cdot Df^{-1}(f(y))v\| \le e^{c_0/16}\left\|\left(P^1|_{N^{cu}_y}\right)^{-1}\right\| \cdot \|v\|, \quad v \in C^{cu}_a(f(y)) \cap C^\perp_a(f(y)).$$

We choose $0 < \rho_1 \le \min\{\delta_0, \rho_0, 1\}$ such that $C_2\rho_1^\omega < c_0/16$ and, for each $x, y \in U^*$ with both $d(x,y) < \rho_1$ and $d(x,y) < d(x, \mathrm{Sing}_\Lambda(G))/2$, then together with the Hölder condition on Ψ we get for $v \in C^{cu}_a(f(y)) \cap C^\perp_a(f(y))$

$$\|\mathcal{O}_y \cdot Df^{-1}(f(y))v\| \le e^{c_0/8} \cdot \left\|\left(P^1|_{N^{cu}_x}\right)^{-1}\right\| \cdot \|v\|. \tag{34}$$

We recall that δ_0 is small enough satisfying Proposition 3.2.

Lemma 3.4. *There exists $b_* > 0$ so that for each $x \in U^*$, if $d(x, \mathrm{Sing}_\Lambda(G)) < \delta_0$, then*

$$b_* L \le \frac{\|G(x)\|}{d(x,\sigma)} \le L \quad \textit{and} \quad 2 \cdot d(y,x) < d(x, \mathrm{Sing}_\Lambda(G)) \implies \|G(x)\| \ge b_*\|G(x)\|. \tag{35}$$

Proof. Since all equilibria in $\mathrm{Sing}_\Lambda(G)$ are hyperbolic, there are at most finitely many and those accumulated by the orbit of $x \in \Omega$ are of saddle type. Thus, there exists $b > 0$ so that $\|(DG(\sigma))^{-1}\| < 1/b$ for all $\sigma \in \omega_G(x) \cap \mathrm{Sing}_\Lambda(G)$.

Moreover, we have $\|DG(y) - DG(\sigma)\| \le \kappa_0 d(y,\sigma)^\beta$ for all $y \in B(\sigma, 2\delta_0)$ and some constants $\kappa_0 > 0$ and $0 < \beta \le 1$ since G is of class C^{1+} — here and in the following estimates, we identify $B(\sigma,\rho_0)$ with the ρ_0-ball on $T_\sigma M$. Hence, if $2\kappa_0\delta_0^\beta < b$, then by the Mean Value Inequality

$$\|G(x) - G(\sigma) - DG(\sigma)\cdot(x-\sigma)\| \le \kappa d(x,\sigma)^\beta \|x - \sigma\| \le \kappa_0 \delta_0^\beta d(x,\sigma)$$

and so $\|G(x)\| \ge \|DG(\sigma)\cdot(x-\sigma)\| - \kappa_0\delta_0^\beta d(x,\sigma) \ge (b - \kappa_0\delta_0^\beta) d(x,\sigma)$. On the other hand, by the smoothness of G we have that $\|G(x)\| = \|G(x) - G(\sigma)\| \le L \cdot d(x,\sigma)$. Finally, if $2d(y,x) < d(x,\sigma)$, then

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$$\|G(y)\| \geq (b - \kappa_0 \delta_0^\beta) d(y, \sigma) \geq \frac{b - \kappa_0 \delta_0^\beta}{2} d(x, \sigma) \geq \frac{b - \kappa_0 \delta_0^\beta}{L} \|G(x)\|$$

which completes the proof after setting $b_* = (b - \kappa_0 \delta_0^\beta)/L$. □

Now we show that hyperbolic times are rough along a trajectory, in the following sense.

Proposition 3.5. *There exists $s_0 > 0$ small so that, if $n > 1$ is a hyperbolic time for $x \in U^*$, then n is also a hyperbolic time for $\phi_s x$, for all $|s| < s_0$, but with a contracting rate of $e^{-c_0/8}$.*

Proof. First choose $s_0 > 0$ small enough so that $d(z, \phi_s z) < \rho_1$ for all $z \in U$ and $|s| < s_0$, and use (34) to obtain

$$\prod_{i=n-k}^{n-1} \|\big(P^1 \mid_{N^{cu}_{\phi_s f^i x}}\big)^{-1}\| = \prod_{i=n-k}^{n-1} \left(\frac{\|\big(P^1 \mid_{N^{cu}_{\phi_s f^i x}}\big)^{-1}\|}{\|\big(P^1 \mid_{N^{cu}_{f^i x}}\big)^{-1}\|} \cdot \|\big(P^1 \mid_{N^{cu}_{f^i x}}\big)^{-1}\| \right)$$
$$\leq e^{kc_0/8} \cdot e^{-kc_0/4} = e^{-kc_0/8}; \quad k = 1, \ldots, n; |s| < s_0.$$

Then, if $d(f^i x, \mathrm{Sing}_\Lambda(G)) < \delta$, we note that $d(\phi_s f^i x, \mathrm{Sing}_\Lambda(G))$ is bounded from below by

$$d(f^i x, \mathrm{Sing}_\Lambda(G)) - d(\phi_s f^i x, f^i x) \geq d(f^i x, \mathrm{Sing}_\Lambda(G)) - |s| \sup_{|t|<s} \|G(\phi_t f^i x)\|.$$

Lemma 3.4 provides $\|G(\phi_t f^i x)\| \geq b_* \|G(x)\|$ whenever $2d(\phi_t f^i x, f^i x) < d(f^i x, \mathrm{Sing}_\Lambda(G))$, which holds for $|t| < d(f^i x, \mathrm{Sing}_\Lambda(G))/(2\|G(x)\|) \leq (2b_* L)^{-1}$. In this case, we obtain

$$d(\phi_s f^i x, \mathrm{Sing}_\Lambda(G)) \geq d(f^i x, \mathrm{Sing}_\Lambda(G)) - |s| \cdot b_* \|G(x)\|$$
$$\geq (1 - b_* L \cdot |s|) \cdot d(f^i x, \mathrm{Sing}_\Lambda(G)).$$

Hence, choosing $s_0 \in (0, (2b_* L)^{-1})$ small enough so that $d(z, \phi_s z) < \rho_1$ for all $z \in U$ we can assume without loss of generality that

$$d_\delta(\phi_s f^i x, \mathrm{Sing}_\Lambda(G)) \geq e^{-L} d_\delta(f^i x, \mathrm{Sing}_\Lambda(G)) \geq e^{-L} e^{-(n-i)c_0/16}; \quad i = 0, \ldots, n; |s| < s_0.$$

The above conclusions show that, modulo a small change of rates, n is still a hyperbolic time for $\phi_s x$ for each $|s| < s_0$. □

3.3. Distortion bounds at hyperbolic times along the sectional center-unstable direction

We fix a *cu*-disk $D \subset U$ so that $x \in D$ admits $n > 1$ as a hyperbolic time with a choice of $\varepsilon_0, \delta_0 > 0$ satisfying Proposition 3.2.

We set $\Sigma_z = \exp_z(B(z, \rho_1) \cap G(z)^\perp)$ as a cross-section to G through $z \in U^*$; $\Sigma_z^i = \exp_z(B(z, (\rho_1/2)e^{-ic_0/16 - L}) \cap G(z)^\perp)$ a scaled cross-section; $D_n = f^n(D)$ and $D_n^\perp(z) =$

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$D_n \cap \Sigma_z$ for each $z \in D_n$, that is, a section of D_n through z in the direction orthogonal to the vector field.

We then consider the Poincaré first hitting maps $R_i : \mathrm{dom}(R_i) \subset \Sigma^{n-i}_{f^i(x)} \to \Sigma^{n-i-1}_{f^{i+1}(x)}$ and note that $DR_i(f^i(x)) = P^1_{f^i x} : N_{f^i x} \to N_{f^{i+1}x}$, for $i = 0, \dots, n-1$; see Figure 3.

We note that *these maps are well-defined even if U contains equilibria*, by the distance bound provided by the condition (33).

Lemma 3.6 (Local sectional continuity). *For each $i = 0, \dots, n-1$, let $D_i^\perp = f^i(D) \cap \Sigma^{n-i-1}_{f^{i+1}x}$. Then the Poincaré maps satisfy $\|DR_i(R_i y)^{-1} \,|_{T_{R_i y} D_i^\perp}\| \le e^{c_0/8} \cdot \|(P^1 \,|^{-1}_{N^u_{f^i x}})\|$, for $y \in \mathrm{dom}(R_i)$.*

Proof. We have $y \in \Sigma^{n-i}_{f^i x}$ by definition of R_i and so $2 \cdot d(y, f^i x) \le \rho_1 e^{-(n-i)c_0/16 - L}$. Moreover, $d_{\delta_0}(f^i x, \mathrm{Sing}_\Lambda(G)) > e^{-(n-i)c_0/16}$ with $\delta_0 < \rho_1 \le 1$. Hence $2 \cdot d(y, f^i x) \le d_{\delta_0}(f^i x, \mathrm{Sing}_\Lambda(G))$ and thus $2 \cdot d(y, \mathrm{Sing}_\Lambda(G)) > d_{\delta_0}(f^i x, \mathrm{Sing}_\Lambda(G))$. At this point, we divide the argument into two cases.

Away from equilibria: If $d\big(f^i x, \mathrm{Sing}_\Lambda(G)\big) \ge \delta_0 \ge \rho_1$, then $\Sigma^{n-i}_{f^i x}$ is away from $\mathrm{Sing}(G)$ and the Poincaré time from $\mathrm{dom}(\Sigma^{n-i}_{f^i x})$ to $\Sigma^{n-i-1}_{f^{i+1}x}$ is between $1-\xi$ and $1+\xi$ for some uniform small $\xi > 0$ depending on ρ_1.

This ensures that $R_i y = (\phi_s \circ f) y$ with $s = s(y)$ such that $|s-1| < \xi$ and so for $y \in \mathrm{dom}(R_i)$ and $v \in T_{R_i y} D^\perp_{i+1}$

$$\begin{aligned} \|DR_i(R_i y)^{-1} v\| &= \|\mathcal{O}_y \cdot [D(\phi_s \circ f)(R_i y)]^{-1} v\| \\ &= \|\mathcal{O}_y \cdot Df^{-1}(\phi_{-s} R_i y) \cdot D(\phi_s)^{-1}(R_i y) v\| \\ &= \|\mathcal{O}_y \cdot Df^{-1}(fy) \cdot D(\phi_s)^{-1}(R_i y) v\|. \end{aligned} \tag{36}$$

The time $s = s(y)$ can be seen as the Poincaré first visit time from the cross-section $S = f(\Sigma^{n-i}_{f^i x})$ to $\Sigma^{n-i-1}_{f^{i+1}x}$, and so $D(\phi_s)^{-1}(R_i y) v \in T_{fy}(S \cap f^{i+1}(D)) \subset C^{cu}_a(fy) \cap C^\perp_a(fy)$ by the proximity between $R_i y, fy$ and $f^{i+1}x$. Then the statement of the lemma follows from (34).

Close to equilibria: Otherwise, $d\big(f^i x, \mathrm{Sing}_\Lambda(G)\big) < \delta_0$ and $\Sigma^{n-i}_{f^i x}$ is close to an equilibrium $\sigma \in \mathrm{Sing}_\Lambda(G)$.

We show that we can repeat the above argument by obtaining a flow box from $\mathrm{dom}(R_i)$ to $\Sigma^{n-i-1}_{f^{i+1}x}$ with flight time bounded from above.

Reducing δ_0 if necessary, we may assume, without loss of generality, that the flow on $B(\sigma, 2\delta_0)$ is topologically conjugated to the flow of $\dot{X} = DG(\sigma) \cdot X$, *because σ is hyperbolic*. That is, there exists a bi-Hölder homeomorphism $h : B(\sigma, 2\delta_0) \to \mathbb{R}^m$ so that $(h \circ \phi_t)(z) = e^{t \cdot DG(\sigma)} h(z)$ for $t > 0$ such that $\phi_{[0,t]} z \subset B(\sigma, 2\delta_0)$; see e.g. [22]. We arrange so that $\mathbb{R}^m = \mathbb{R}^u \times \mathbb{R}^s$ is the decomposition into stable and unstable subspaces of $DG(\sigma)$, which decomposes in block form as $\mathrm{diag}\{A_u, A_s\}$. We identify $f^i x$ with $(v, w) \in \mathbb{R}^u \times \mathbb{R}^s$ and then $f^{i+1}x$ becomes $(v_1, w_1) = (e^{A_u} v, e^{A_s} w)$.

For a point in $\Sigma^{n-i}_{f^i x}$ to arrive at $\Sigma^{n-i-1}_{f^{i+1}x}$, the flow time is close to the time τ it takes a point $(v', w) \in W_0 = \mathbb{R}^u \times \{w\}$, with $\|v\| - \|v'\|$ small but positive, to arrive at the segment $W_1 = \{e^{A_u} v\} \times \mathbb{R}^s$; see Figure 2. For each $y \in \Sigma^{n-i}_{f^i x}$ we have $h(y) = (v', w)$ and

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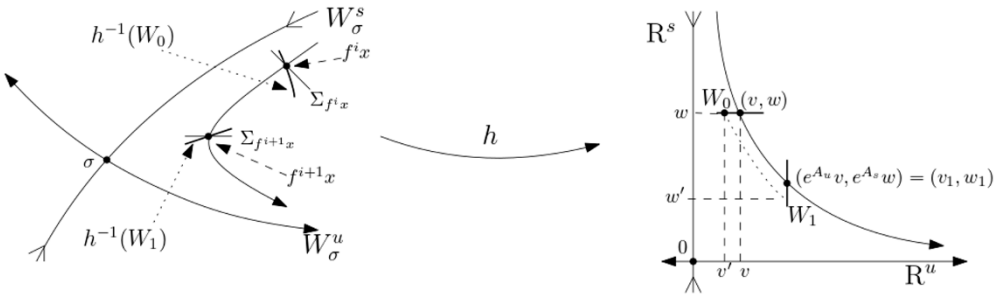


Figure 2. The estimation of the flight time in the linearized setting.

for some $\kappa > 0$ and $\beta \in (0,1)$ we get $\|h(y) - h(f^i x)\| \le \kappa d(y, f^i x)^\beta \le \frac{\kappa}{2^\beta} d(f^i x, \sigma)^\beta$ and $d(f^i x, \sigma) \le \kappa d(h(f^i x), h(\sigma))^\beta = \kappa \|h(f^i x)\|^\beta$, then

$$\begin{aligned}\|h(y)\| &\ge \|h(f^i x)\| - \frac{\kappa}{2^\beta} d(f^i x, \sigma)^\beta = \|h(f^i x)\| \left(1 - \frac{\kappa}{2^\beta} \frac{d(f^i x, \sigma)^\beta}{\|h(f^i x)\|}\right) \\ &\ge \|h(f^i x)\| \left(1 - \frac{\kappa^{1+\beta}}{2^\beta} d(f^i x, \sigma)^{\beta+\beta^{-1}}\right) \\ &\ge \|h(f^i x)\| \left(1 - \frac{\kappa^{1+\beta}}{2^\beta} \delta_0^{\beta+\beta^{-1}}\right) \ge \frac{1}{2} \|h(f^i x)\|,\end{aligned}$$

if $\delta_0 > 0$ is small enough, depending only on G. Using the previous bound, we estimate $\|e^{\tau A_u} v'\| = \|v_1\| \ge e^{\tau \|A_u^{-1}\|^{-1}} \|v'\|$ since $\|v\| \le 2\|v'\|$ and $e^{\tau A} \cdot h(y) = (v_1, w')$ for some w', and so we obtain

$$\begin{aligned}\tau &\le \|A_u^{-1}\| \log \frac{\|v_1\|}{\|v'\|} = \|A_u^{-1}\| \log \left(\frac{\|v_1\|}{\|v\|} \cdot \frac{\|v\|}{\|v'\|}\right) \le \|A_u^{-1}\| \log \left(2 \frac{\|v_1\|}{\|v\|}\right) \\ &= \|A_u^{-1}\| \log(2e^{\|A_u\|})\end{aligned}$$

because $v_1 = e^{A_u} v'$ satisfies $\|v_1\| = \|e^{A_u} v'\| \le e^{\|A_u\|} \|v'\|$. This shows that τ is bounded depending only on $DG(\sigma)$.

Going back to the original coordinates, the cross-sections $h^{-1}(W_i), i = 0, 1$ touch $\Sigma^{n-1}_{f^i x}$ and $\Sigma^{n-i-1}_{f^{i+1}x}$ at $f^i x, f^{i+1} x$, respectively; see Figure 2. The vector field in between these sections has norm uniformly bounded away from zero and close to $G(f^i x)$, by the estimate (35). Hence, the flight time is also bounded above depending only on G in a neighbourhood of σ.

We have recovered a flow box with bounded flight time from $\mathrm{dom}(R_i)$ to $\Sigma^{n-i-1}_{f^{i+1}x}$. We can thus finish repeating the argument as before, using (36) and (34). □

We write $g_k = R_{n-1} \circ \cdots \circ R_{n-k}, 1 \le k \le n$ in what follows, and $\mathrm{dist}_D(x, y)$ for the distance between two points x, y in the disk D, measured along D as the least length of smooth curves from x to y within D.

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Lemma 3.7. (Local sectional backward contraction) *Given any cu-disk $D \subset U$ tangent to the centre-unstable cone field, $x \in D$ and $n \geq 1$ a hyperbolic time for x*

$$\operatorname{dist}_{D^{\perp}_{n-k}(x)}(f^{n-k}(x), g_k(y)) \leq e^{-kc_0/8} \cdot \operatorname{dist}_{D^{\perp}_n(x)}(f^n(x), g_n(y)), \quad k = 1, \dots, n;$$

for each $y \in D^{\perp}_0(x)$ satisfying $\operatorname{dist}_{D^{\perp}_n(x)}(f^n x, g_n y) \leq \rho_1$. Moreover, there exists $\tau_* > 0$ such that, defining t_k as the least positive real so that $g_k(y) = \phi_{t_{n-k}}(y)$, then $|t_{n-k} - (n-k)| \leq \tau_*$.

Proof. We follow the proof of [4, Lemma 2.7]. Let γ_0 be a curve of minimal length in $D^{\perp}_n(x)$ from $f^n x$ to $g_n y$ and let $\gamma_k = (g_k)^{-1}(\gamma_0), k = 1, \dots, n$. Arguing by induction, let us assume that for some $k = 1, \dots, n$ we have the following bound for the length: $\ell(\gamma_k) \leq e^{-kc_0/8}\ell(\gamma_0)$. The choice of ρ_1 in (34) together with the definition of hyperbolic times and Lemma 3.6 ensure that

$$\begin{aligned} \left\| D(R_{n-1} \circ \cdots \circ R_{n-k-1})^{-1} \gamma_0'(z) \right\| &\leq e^{(k+1)c_0/8} \|\gamma_0'(z)\| \prod_{j=n-k}^{n} \|(P^1 \mid_{N^u_{fjx}})^{-1}\| \\ &\leq e^{-(k+1)c_0/8} \|\gamma_0'(z)\| \end{aligned}$$

where $\gamma_0'(z)$ denotes the tangent vector to γ_0 at z. Therefore

$$\ell(\gamma_{k+1}) \leq e^{-(k+1)c_0/8}\ell(\gamma_0) = e^{-(k+1)c_0/8} \operatorname{dist}_{D^{\perp}_n(x)}(f^n x, g_n y) \leq \rho_1 e^{-(k+1)c_0/8} \leq \rho_1,$$

which shows that the maps are well-defined on their domains and completes the induction. Finally, as a standard consequence of Gronwall's Inequality, if $y \in D^{\perp}_0(x)$ is such that $\operatorname{dist}_{D^{\perp}_n(x)}(f^n x, g_n y) \leq \rho_1$ and $k = 1, \dots, n$, then

$$d(f^{n-k+1}x, f g_k y) \leq e^L d(f^{n-k}x, g_k y) \leq e^L \operatorname{dist}_{D^{\perp}_{n-k}(x)}(f^{n-k}x, g_k y) \leq \rho_1 e^L e^{-kc_0/8}.$$

Hence, the time τ_k it takes for $f(g_k y)$ to arrive at $\phi_{t_{n-k+1}} y = g_{k-1} y \in D^{\perp}_{n-k+1}(x)$ is bounded from above by the above distance divided by the speed of flow. Thus, since for z in a $\rho_1 e^{-(n-k+1)c_0/8}$-neighbourhood of $f^{n-k+1}x$, we have either $\|G(z)\| \geq \gamma_0$, or $\|G(z)\| \geq b_* \|G(f^{n-k+1}x)\|$, we get from Lemma 3.4 in the worst case

$$\begin{aligned} |\tau_k| &\leq \frac{\rho_1 e^{L-kc_0/8}}{b_* \|G(f^{n-k+1}x)\|} \leq \frac{\rho_1 e^{L-kc_0/8}}{b_*^2 L d(f^{n-k+1}x, \sigma)} \leq \frac{\rho_1 e^{L-kc_0/8}}{b_*^2 L e^{-(k-1)c_0/16}} \\ &= \frac{\rho_1 e^{L+c_0/16}}{b_*^2 L} e^{-(k-1)c_0/16}. \end{aligned}$$

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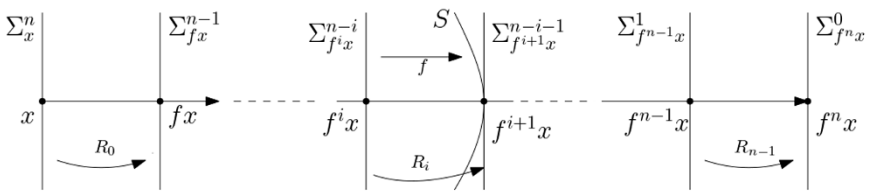


Figure 3. The Poincaré first hitting time maps R_i.

This shows that $t_{n-k+1} - t_{n-k} = 1 + \tau_k$. Since $t_0 = 0$, then

$$|t_{n-k} - (n-k)| = \Big|\sum_{i=k}^{n-1}(t_{n-i+1} - t_{n-i}) - (n-k)\Big| \le \sum_{i=k}^{n-1}|\tau_i|$$
$$\le \frac{\rho_1 e^{L+c_0/16}}{b_*^2 L}\sum_{i=k}^{n-1} e^{-(k-1)c_0/16}.$$

which completes the proof after setting $\tau_* = \frac{\rho_1 e^{L+c_0/16}}{b_*^2 L}(1 - e^{-c_0/16})^{-1}$. □

Proposition 3.8 (Sectional bounded distortion). *There exists $C_2 > 1$ so that, given a cu-disk D tangent to the centre-unstable cone field with $\kappa(D) \le C_1$, and given $x \in D$ and $n \ge 1$ a hyperbolic time for x, then*

$$\frac{1}{C_2} \le \frac{|\det Dg_n\,|_{T_yD_0^\perp(x)}\,|}{|\det Df^n\,|_{T_xD_0^\perp(x)}\,|} \le C_2$$

in the notation of Lemma 3.7, for every $y \in D_0^\perp(x)$ such that $\operatorname{dist}_{D_n^\perp(x)}(g_n y, f^n x) \le \rho_1$.

Proof. Follow [4, Proposition 2.8] we write $J_i(y) = |\det DR_i\,|_{T_{g_iy}D_i^\perp(x)}\,|$ and so by the Chain Rule

$$\log\frac{|\det Dg_n\,|_{T_yD_0^\perp(x)}\,|}{|\det Df^n\,|_{T_xD_0^\perp(x)}\,|} = \sum_{i=0}^{n-1}\big(\log J_i(y) - \log J_i(x)\big).$$

We recall that $R_i = \hat{R}_i \circ f$, where $\hat{R}_i : S = f(\Sigma^{n-i}_{f^ix}) \to \Sigma^{n-i-1}_{f^{i+1}x}$ is the Poincaré first visit map; see the proof of Lemma 3.7 and Figure 3. Since $\Sigma^{n-i}_{f^ix}$ is a restriction of a *cu*-disk with curvature bounded by C_1, and contained in the *cu*-disk $W_i = \phi_{(-\varepsilon,\varepsilon)}\Sigma^{n-i}_{f^ix}$ with $\kappa(W_i) \le C_1$ by Remark 2.7, then S has bounded curvature by Proposition 2.6. By construction, S is also tangent to $\Sigma^{n-i-1}_{f^{i+1}x}$ at $f^{i+1}x$. Hence, we can see S as a graph of a (L_1, ζ)-Hölder continuous smooth map $h : \Sigma^{n-i-1}_{f^{i+1}x} \to \mathbb{R}\cdot G(f^{i+1}x)$ and $\hat{R}_i$ as the projection from this graph to its domain. Moreover, $f : W_i \to \phi_{(-\varepsilon,\varepsilon)}S$ is a C^2 diffeomorphism from a flat submanifold to a manifold whose curvature is bounded by C_1. Thus

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$$\log J_i(y) = \log |\det D\hat{R}_i \mid_{T_{f_{g_i y}}S}| + \log |\det Df \mid_{T_{g_i y}D_0^{\perp}(x)}|$$

and both summands are restrictions of (L_1, ζ)-Hölder continuous maps.

Therefore, the sum is bounded above by $\sum_{i=0}^{n-1} 2L_1\big(e^{-ic_0/8}\rho_1\big)^{\zeta} \le \frac{2L_1\rho_1^{\zeta}}{1-e^{-\zeta c_0/8}}$. The proof is complete after setting $C_2 = \exp(2L_1\rho_1^{\zeta}/(1-e^{-\zeta c_0/8}))$. □

4. Lebesgue measure and hyperbolic times

We extend the construction of backward contraction to a full neighbourhood of points in a *cu*-disk at hyperbolic times in § 4.1. This provides the tools needed to construct the physical/SRB measure, outlined in § 4.2, leading to the proof of Theorem B in § 4.3.

4.1. Distortion bounds and central-unstable disks at hyperbolic times

In what follows, we set $\operatorname{dist}_D(x, \partial D) = \inf_{y\in\partial D} \operatorname{dist}_D(x, y)$ for the distance from a point $x \in D$ to the boundary of D. We assume without loss of generality that U contains a ρ_1-neighbourhood of Λ.

Let $z \in U^*$, $N_z^u := E_X^{cu} \cap G(z)^{\perp}$ and $W = \exp_z\big(N_z^u \cap B(0, \rho_0)\big)$ be such that the *cu*-disk $D = \phi_{(-\rho,\rho)}W$ for some $\rho = \rho(z) > 0$ satisfies $\ell(\phi_{(-\rho,\rho)}z) > 2\rho_1$ and $\operatorname{Leb}_D(\Omega) > 0$, where Leb_D is the volume measure induced in the embedded disk D by the Riemannian metric of M. Remark 2.7 ensures that $\kappa(D) \le C_1$.

We note that this disk is an union of segments of trajectories of the flow – we say that this is a *cu-disk*. Moreover, since there exists $\gamma_* > 0$ such that $\|G(x)\| \le \gamma_*$ for all $x \in U$, we necessarily have that $2\rho_1 < \ell\big(\phi_{(-\rho,\rho)}z\big) \le 2\rho\gamma_*$ and so $\rho > \rho_1\gamma_*^{-1}$. We set

$$A = A(D, \rho_1) = \{x \in D \cap \Omega : \operatorname{dist}_D(x, \partial D) \ge \rho_1\}$$

so that $\operatorname{Leb}_D(A) > 0$, reducing ρ_1 if necessary.

Let $\gamma_0 > 0$ be such that $\|G(x)\| \ge \gamma_0$ for all $x \in U^*$ with $d(x, \operatorname{Sing}_\Lambda(G)) > \delta_0$.

Next result states robust local sectional backward contraction and bounded distortion, together with the consequence for the push-forward of Lebesgue measure along *cu*-disks.

Proposition 4.1. *Let $x \in A$ and $n > 1$ be a hyperbolic time for x. Then there exists an open neighbourhood $V_n(x)$ of x in D, a δ_0-ball $W_n(x)$ inside $f^n(D)$ centered at $f^n x$ such that*

(1) *$f^n \mid_{V_n(x)} : V_n(x) \to W_n(x)$ is a diffeomorphism, and*
(2) *there exists $s_0 > 0$ such that $\phi_{(-s_0,s_0)}x \subset V_n(x)$ and $f^n(\phi_{(-s_0,s_0)}x)$ has length at least ρ_1, and for all $-s_0 < s < s_0$*
   (a) *n is a hyperbolic time for $\phi_s x$, $D_0^{\perp}(\phi_s x) \subset V_n(x)$, and*
   (b) *the translated Poincaré maps $R_i^s : \operatorname{dom}(R_i^s) \subset \Sigma^{n-i}_{\phi_s f^i(x)} \to \Sigma^{n-i-1}_{\phi_s f^{i+1}(x)}, i = 0, \dots, n-1$ composed to form $g_n^s = R_{n-1}^s \circ \cdots \circ R_0^s$ satisfy:*
      (i) *$\big(g_n^s \mid_{D_0^{\perp}(\phi_s x)}\big)^{-1} : D_n^{\perp}(\phi_s f^n) \to D_0^{\perp}(\phi_s x)$ is a $e^{-nc_0/8}$-contraction, and*

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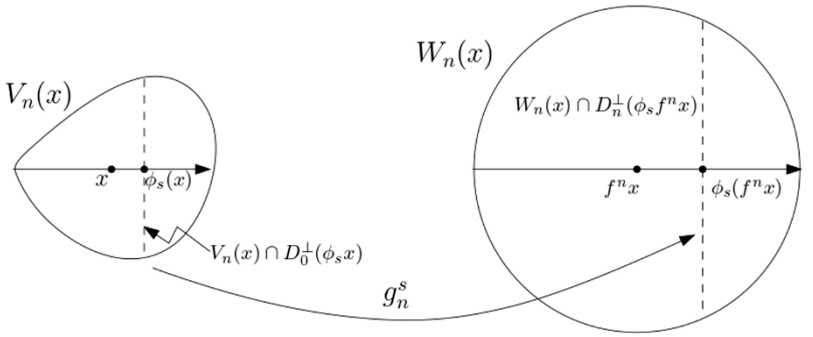


Figure 4. Sectioning $V_n(x)$ and $W_n(x)$ through normal cross-sections to the trajectory of x, to then apply Fubini's Theorem.

(ii) *for every* $y \in D_0^\perp(\phi_s x)$ *such that* $\operatorname{dist}_{D_n^\perp(\phi_s x)}(g_n^s y, \phi_s f^n x) \le \rho_1$ *we get*

$$C_2^{-1} \le \frac{|\det Dg_n^s\,|_{T_y D_0^\perp}(\phi_s x)|}{|\det Df^n\,|_{T_{\phi_s x} D_0^\perp}(\psi_s x)|} \le C_2.$$

(3) *there exists* $C_3 > 0$ *so that* $f_*^n\big(\operatorname{Leb}|_{V_n(x)}\big) \le C_3 \cdot \big(\operatorname{Leb}|_{W_n(x)}\big)$.

Proof. Fixing $x \in A$ and n a hyperbolic time for x, then n is also a hyperbolic time for $\phi_s(x) \in D$ for $-s_0 < s < s_0$ with s_0 given by Proposition 3.5. Moreover, $D_0^\perp(\phi_s x) \subset D$ and we obtain item (2a).

Hence, $d(\phi_s f^n x, \operatorname{Sing}_\Lambda(G)) > \delta_0$, which implies that $\|G(\phi_s f^n x)\| \ge \gamma_0$ for all $|s| < s_0$. Thus $\ell(\phi_{(-s_0,s_0)} f^n x) \ge 2 s_0 \gamma_0$.

In addition, since $\|G(y) - G(x)\| \le L d(x,y)$, if $d(x, \operatorname{Sing}_\Lambda(G)) < \delta_0$, then $s_0 b_* \|G(x)\| \le \ell(\phi_{(-s_0,s_0)} x) \le s_0 \cdot 2Ld(x, \operatorname{Sing}_\Lambda(G)) < \frac{1}{2} d_{\delta_0}(x, \operatorname{Sing}_\Lambda(G))$, by Lemma 3.4.

To obtain item (2b), we apply Lemma 3.7 together with Proposition 3.8 to each $\phi_s x$ with $|s| < s_0$, and we also get that $\operatorname{dist}_{D_n}(f^n x, \partial D^n) > \min\{\rho_1, 2 s_0 \gamma_0\} = \rho_1$ (reducing ρ_1 if needed).

To obtain item (1), we consider the open set $W_n(x) = \cup_{|s|<\rho_x} D_n^\perp(\phi_s f^n x)$ together with $V_n(x) = f^{-n} W_x(x)$, and note that $W_n(x)$ contains a ρ_1-ball around $f^n x$ by construction.

Finally, since $\{D_n^\perp(\phi_s f^n x) = g_n^s\big(D_0^\perp(\phi_s x) \cap V_n(x)\big) : |s| < s_0\}$ is a measurable partition of $W_n(x)$, we use Lemma 3.7 together with Proposition 3.8 as follows; see Figure 4. We write

$$v(\phi_s x) = \operatorname{Leb}^\perp|_{V_n(x) \cap D_0^\perp(\phi_s x)} \quad \& \quad w(\phi_s x) = \operatorname{Leb}^\perp|_{W_n(x) \cap D_n^\perp(\psi_s x)}, \quad |s| < s_0;$$

for the normalized volume measures induced on $D_0^\perp(\psi_s x) \cap V_n(x)$ and $W_n(x) \cap D_n^\perp(\psi_s x)\big)$, respectively. Since $\big(\operatorname{Leb}|_{V_x(x)}\big)(E) = \int_{-s_0}^{s_0} v(\phi_s x)(E)\, \frac{ds}{\|G(\phi_s x)\|}$ for any measurable subset E, then we can apply Fubini's Theorem.

We have that $g_n^s(y) = \phi_{\tau(s,y)} \circ f^n$ for each $y \in D_0^\perp(\psi_s x)$ with $|\tau(s,y)| \le \tau_*$, so $f^n = h \circ g_n^s$ and $h : V_n(x) \to W_n(x)$ is a diffeomorphism with bounded derivatives. Thus,

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item (2b)(ii) ensures that $(g_n^s)_* v(\phi_s x) \le C_2 w(\phi_s x)$, and there exists a constant $K > 0$ so that

$$\begin{aligned} f_*^n\big(\operatorname{Leb}|_{V_n(x)}\big)(E) &= f_*^n\left(\int_{-s_0}^{s_0} v(\phi_s x)(E)\,\frac{ds}{\|G(\phi_s x)\|}\right) \\ &= \int_{-s_0}^{s_0} [(h\circ g_n^s)_* v(\phi_s x)](E)\,\frac{ds}{\|G(\phi_s x)\|} \\ &\le C_2 \int_{-s_0}^{s_0} [(h)_* w(\phi_s x)](E)\,\frac{ds}{\|G(\phi_s x)\|} \\ &\le C_2 K \int_{-s_0}^{s_0} w(\phi_s x)(E)\,\frac{ds}{\|G(\phi_s x)\|} \\ &\le C_2 K \int_{\phi_{(-s_0,s_0)} f^n x} \big(\operatorname{Leb}|_{W_n(x)\cap D_n^{\perp}(z)}\big)(E)\,dz \\ &= C_2 K\big(\operatorname{Leb}|_{W_n(x)}\big)(E). \end{aligned}$$

This completes the proof after setting $C_3 = KC_2$. □

4.2. Construction of a physical probability measure

We now have all the basic tools needed to follow the construction presented in [4, Sections 3 & 4] to obtain a physical/SRB probability measure for the flow. We present a step by step overview in what follows.

For each $n > 1$ we set $H_n = \{x \in A(D,\rho_1) : n \text{ is a hyperbolic time for } x\}$. From Proposition 4.1, if $x \in H_n$, then $f^n x$ is ρ_1-away from the boundary of $f^n D$. For $\delta > 0$, we denote $\Delta_n(x,\delta)$ the δ-neighbourhood of $f^n x$ inside $f^n(D)$. If Leb_D is the probability measure $\operatorname{Leb}_D(E) = \operatorname{Leb}(D\cap E)/\operatorname{Leb}(D)$ for every Borel subset $E \subset D$, obtained by normalizing the Riemannian induced volume measure on D, then $(f_*^n \operatorname{Leb}_D)\,|_{\Delta_n(x,\delta_1)} \le C_3 \operatorname{Leb}_{f^n(D)}\,|_{\Delta_n(x,\delta_1)}$, again from Proposition 4.1.

The following is a geometrical consequence of the finite dimensionality and bounded curvature of *cu*-disks.

Proposition 4.2. *There exists $\tau > 0$ so that for $n > T_2$ there exists a finite subset $\widehat{H}_n$ of H_n such that the balls $\Delta_n(z,\rho_1/4)$ in $f^n(D)$ centered at $z \in f^n(\widehat{H}_n)$ are pairwise disjoint, and their union Δ_n satisfies $(f_*^n \operatorname{Leb}_D)(\Delta_n \cap \Omega) \ge (f_*^n \operatorname{Leb}_D)(\Delta_n \cap f^n(H_n)) \ge \tau \operatorname{Leb}_D(H_n)$.*

Proof. See [4, Proposition 3.3 & Lemma 3.4]. □

Let $\mathcal{D}_n = \{\Delta_n(z,\rho_1/4) : z \in f^n(\widehat{H}_n)\}$ be the collection of balls that form Δ_n. We note that all these balls are δ_0-away from $\operatorname{Sing}_\Lambda(G)$, and we define

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\operatorname{Leb}_D); \quad \nu_n := \frac{1}{n}\sum_{j=0}^{n-1}(f_*^j \operatorname{Leb}_D)\,|_{\Delta_j} \quad \text{and} \quad \eta_n := \mu_n - \nu_n. \tag{37}$$

Proposition 4.3. *There is $\alpha > 0$ so that both $\nu_n(\Omega) \ge \alpha$ and $\nu_n\big(\cup_{i=0}^{n-1} f^i(D\cap\Omega)\big) \ge \alpha$ for all sufficiently large $n > T_2$.*

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Proof. Just follow [4, Proposition 3.5], using Proposition 4.2 together with the positive asymptotic frequency of hyperbolic times for each $x \in \Omega$, given by Proposition 3.2, and Remark 3.3. □

We consider weak* accumulation points μ and ν of $(\mu_n), (\nu_n)$ respectively, along some subsequence $(n_k)_k$. It is standard that μ is a f-invariant probability measure and that $\widetilde{\mu} = \int_0^1 (\phi_t)_* \mu \, dt$ is a flow invariant probability measure; see e.g. [64]. In addition, $\nu(U) \geq \limsup_k \nu_k(U) \geq \alpha > 0$.

We claim that ν has a property of absolute continuity along certain disks contained in its support. We define the collection of these disks in what follows.

Note that ν_n is supported on the union $\cup_{j=0}^{n-1} \Delta_j$ of disks with uniform size and δ_0-away from $\mathrm{Sing}_\Lambda(G)$. Then $\operatorname{supp} \nu$ is contained in $\Delta_\infty = \cap_{n=1}^{\infty} \operatorname{Closure}\big(\cup_{j \geq n} \Delta_j\big)$, the family of accumulation points of such disks. That is, for $y \in \Delta_\infty$ there are $(j_i)_i \to \infty$, disks $\widetilde{\Delta}_i = \Delta_{j_i}(x_i, \delta_1/4) \subset \Delta_{j_i}$, and points $y_i \in \widetilde{\Delta}_i$ so that $y_i \to y$ when $i \nearrow \infty$.

We may assume, without loss of generality, taking subsequences if necessary, that x_i converges to some point x. By uniform size and bounded curvature, we can use the Ascoli-Arzela Theorem to conclude that $\widetilde{\Delta}_i$ converge to a *cu*-disk $\widetilde{\Delta}(x)$ with radius $\rho_1/4$ centered at x. Then $y \in \operatorname{Closure} \widetilde{\Delta}(x) \subset \Delta_\infty$.

Lemma 4.4. *Every $y \in \widetilde{\Delta}(x)$ is such that N_y^u is uniformly expanding: $\|\big(P^k \mid_{N_y^u}\big)^{-1}\| \leq e^{-kc_0/8}$ for all $k \geq 1$. The disk $\widetilde{\Delta}(x)$ is contained in Λ and also in the unique[11] center-unstable manifold $W_x^{cu}(\rho_1)$ tangent to E_x^{cu} containing a ρ_1-ball around x.*

Proof. Let $j_i \nearrow \infty$, $x_i \to x$, and $\widetilde{\Delta}_i \to \widetilde{\Delta}(x)$ be as in the construction described previously. We have that $\widetilde{\Delta}_i$ is contained in the j_ith-iterate of D, which is a *cu*-disk. The domination of the splitting on U ensures that $\angle\big(\widetilde{\Delta}_i, E^{cu}\big) \to 0$ as $i \nearrow \infty$, uniformly on Λ; this is a consequence of Proposition 2.3.

By Proposition 4.1, f^{-k} is a $e^{-kc_0/8}$-contraction on $\widetilde{\Delta}_i \cap \exp_{\phi_s f^{j_i} x}\big(N^u_{\phi_s f^{j_i} x}\big)$ for every large i and any given fixed $k \geq 1$ and $|s| \leq s_0$. Passing to the limit in i, we get that f^{-k} is a $e^{-kc_0/8}$-contraction on $\widetilde{\Delta}(x) \cap \exp_{\phi_s x}\big(N^u_{\phi_s x}\big)$, and $\widetilde{\Delta}(x)$ is a *cu*-disk in Λ by continuity of the splitting on U.

We have shown that the subspace E^{cu} is uniformly sectionally expanding for Df on $\widetilde{\Delta}(x)$. Since $Df \mid_{E^s}$ is uniformly contracted, we are in the setting of [21, Section 3 of Chapter 7] with respect to f^{-1}. Then there exists a unique *center-unstable* manifold $W_x^{cu}(\rho_1)$ tangent to E_x^{cu} containing a ρ_1-ball around x. □

4.2.1. Absolute continuity

The same arguments in [4, Section 4.1] imply the following result, where we write *cylinder* for any diffeomorphic image of $\mathcal{D}^{d_{cu}} \times \mathcal{D}^{d_s}$ into U.

Proposition 4.5. *There exist $C_3 > 1$ and a cylinder $\mathcal{C} \subset M$, with a family $\mathcal{K}_\infty$ of disjoint disks, contained in $\mathcal{C} \cap \Delta_\infty$, which are graphs over $\mathcal{D}^{d_{cu}}$, such that*

[11] The center-unstable manifold might depend on the radius, but it is uniquely defined given the radius.

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(1) *the union* K_∞ *of all disks in* $\mathcal{K}_\infty$ *has positive* ν*-measure;*
(2) *the restriction* $\nu\,|_{K_\infty}$ *has absolutely continuous conditional measures* ν_γ *with respect the induced volume* Leb_γ *along the disks* $\gamma \in \mathcal{K}_\infty$*, whose density is bounded:* $C_3^{-1} \leq d\nu_\gamma / d\,\mathrm{Leb}_\gamma \leq C_3$.

Proof. See [4, Proposition 4.1 & Lemma 4.4], whose proof uses the properties of Δ_∞ already obtained. □

Remark 4.6. (Uniformly sized cylinder) The cylinder $\mathcal{C}$ is based on a centre-unstable manifold $W^{cu}_x(\rho_1)$ with uniform inner radius obtained in Lemma 4.4 and contained in the ergodic basin of μ_*. Since the size of the stable leaves is also uniform by partial hyperbolicity, then the volume of the ergodic basin is uniformly bounded from below away from zero, with a bound depending only on ρ_1, which depends on the map f.

4.2.2. Ergodicity and ergodic basin.

Following [4, Section 4.2] we obtain the next result.

Lemma 4.7. *The* f*-invariant probability measure* $\mu = \nu + \eta$ *has an ergodic component* μ_* *whose Lyapunov exponents are all non-zero, except along the direction of the vector field, and whose conditional measures along local unstable manifolds are absolutely continuous with respect to Lebesgue measure. Moreover,* $\operatorname{supp}\mu_* \subset \Lambda$ *and* $\mathrm{Leb}_D(B(\mu_*) \cap H) > 0$.

Proof. This is [4, Lemma 4.5], whose proof uses the properties of Δ_∞ already obtained in the previous arguments. □

4.3. Finitely many physical/SRB measures for the flow

The following completes the proof of the existence statement of Theorem B. The reciprocal is contained in the proof of item (2) of Theorem C in § 5.2.1.

Corollary 4.8. *There exist finitely many ergodic hyperbolic physical/SRB invariant probability measures* $\eta_1, \ldots, \eta_k$ *for* f *and* $\mu_1, \ldots, \mu_k$ *for the flow* ϕ_t *of* G*, supported on* Λ*, such that* $\mathrm{Leb}\left(\Omega \setminus \cup_{i=1}^k B(\mu_i)\right) = 0$ *and* $\mathrm{Leb}\left(\Omega \cap (B(\eta_i) \triangle B(\mu_i))\right) = 0, \forall i$.

Proof. The existence of finitely many ergodic hyperbolic physical/SRB measures $\eta_1, \ldots, \eta_k$ with respect to f supported in Λ and satisfying $\mathrm{Leb}\left(\Omega \setminus \cup_{i=1}^k B(\eta_i)\right) = 0$ follows by [4, Corollary 4.6] using the properties already obtained.

We are left to obtain the G-invariant ergodic physical probability measures. The probability measures $\mu_i = \int_0^1 (\phi_t)_* \eta_i \, dt$ are ϕ_t-invariant for every $t > 0$ and μ_i are ergodic for the flow, $i = 1, \ldots, k$.

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Moreover, if $\varphi : M \to \mathbb{R}$ is continuous and $x \in B(\eta_i)$, then $\psi = \int_0^1 \varphi \circ \phi_s \, ds$ is also continuous, and since ϕ_t and f commute

$$\int \varphi \, d\mu_i = \int \psi \, d\eta_i = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j x)$$
$$= \lim_{n\to+\infty} \frac{1}{n} \int_0^n \varphi(\phi_s x) \, ds = \lim_{T\nearrow\infty, T\in\mathbb{R}} \frac{1}{T} \int_0^T \varphi(\phi_s x) \, ds,$$

where the last equality follows from boundedness of φ. This shows that $B(\eta_i) \subset B(\mu_i)$ and so μ_i becomes a physical measure and also $\mathrm{Leb}\left(\Omega \cap (B(\eta_i) \triangle B(\mu_i))\right) = 0$ for $i = 1, \ldots, k$.

Hyperbolicity of μ_i follows from partial hyperbolicity together with the implication (NU2SE) $\Longrightarrow$ (MASE) from Theorem 1.6, so that we have (MASE) for each $x \in B(\mu_i)$. From the results of [37] the Lyapunov spectrum of $x \in B(\mu_i)$ and of μ_i coincide — thus, μ_i has positive Lyapunov exponents along all the directions in E^{cu}_z, except $G(z)$, for μ_i-a.e. $z \in \Lambda$.

By smoothness of the flow, the absolute continuity of conditional measures of η_i along unstable manifolds implies absolute continuity of conditional measures of μ_i along weak-unstable (or center-unstable) manifolds, so that each μ_i is also a *cu*-Gibbs state. That is, each μ_i is an ergodic hyperbolic physical/SRB measure for the flow, completing the proof. □

5. Discrete versus continuous: the main theorems and corollaries

To easily deduce Theorem C from Theorem B, we recall some general properties of Gibbs *cu*-states first. Then we prove Theorem C. Afterwards, we obtain Theorem E and Corollaries D, F & G.

5.1. Properties of Gibbs *cu*-states

We collect some useful results here.

Theorem 5.1. *Let $\Lambda = \Lambda_G(U)$ be a partially hyperbolic attracting set for a C^2 vector field G which is NU2SE on $\Omega \subset U$ with $\mathrm{Leb}(\Omega) > 0$. Then there exists $k \in \mathbb{Z}^+$, $\varrho > 0$ and k distinct f-invariant ergodic probability measures $\mu_1, \ldots, \mu_k$ satisfying*

(1) *the family $\mathbb{E}$ of all G-invariant physical probability measures μ such that $\mathrm{Leb}(\Omega \cap B(\mu)) > 0$ is the convex hull $\mathbb{E} = \{\sum_{i=1}^k t_i\mu_i : \sum_i t_i = 1, t_i \geq 0, i = 1, \ldots, k\}$. The same holds replacing G-invariance by ϕ_t-invariance, for some fixed value of $t > 0$.*
(2) *for a G-invariant (or f-invariant)* hyperbolic *probability measure μ supported in Λ, with $\mu(\Omega) > 0$, the following are equivalent*
   (a) *the Entropy Formula: $h_\mu(f) = \int \log |\det Df|_{E^{cu}}| \, d\mu$;*
   (b) *μ is a cu-Gibbs state, that is, admits an absolutely continuous disintegration along center-unstable manifolds;*
   (c) *μ is a physical measure, i.e., its basin $B(\mu)$ has positive Lebesgue measure.*

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(3) *the basin* $B(\mu)$ *of a physical measure* μ *supported in* Λ*, with* $\mathrm{Leb}(B(\mu)\cap\Omega)>0$*, admits an open subset* V*, containing a* ϱ*-ball, which intersects* Λ *and is contained in the ergodic basin except a zero volume subset, that is,* $\mathrm{Leb}(V\setminus B(\mu))=0$ *and* $V\cap\Lambda\neq\emptyset$*.*

(4) *if* Λ *is transitive then* $k=1$*, i.e., there exists only one physical probability measure* $\mu=\mu_1$*, which is also a Gibbs-cu-state, such that* $m(B(\mu)\setminus\Omega)=0$*.*

Proof. For item (1), Theorem B (cf. Corollary 4.8) ensures the existence of finitely many ergodic hyperbolic physical/SRB measures $\mu_1,\dots,\mu_k$ such that the union of their ergodic basins covers Ω Lebesgue almost everywhere: $\mathrm{Leb}\left(\Omega\setminus\left(\cup_{i=1}^k B(\mu_i)\right)\right)=0$. We note that if there are no equilibria, then we can take $\Omega=U$. The measures considered can either be invariant for the flow, or f-invariant, or even ϕ_t-invariant for any fixed $t>0$.

Since $\mathrm{Leb}(B(\mu)\cap\Omega)>0$, it follows that $B(\mu)\cap\Omega=\Omega\cap\left(\sum_{i=1}^k B(\mu)\cap B(\mu_i)\right)$ Lebesgue modulo zero. By definition of ergodic basin, for each continuous observable $\varphi:U\to\mathbb{R}$ we get

$$\begin{aligned}\int\varphi\,d\mu&=\frac{1}{\mathrm{Leb}(\Omega\cap B(\mu))}\int_{\Omega\cap B(\mu)}\int\varphi\,d\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^jx}\right)\,d\,\mathrm{Leb}(x)\\&=\sum_{i=1}^k\frac{\mathrm{Leb}(B(\mu)\cap B(\mu_i)\cap\Omega)}{\mathrm{Leb}(B(\mu)\cap\Omega)}\int\varphi\,d\mu_i,\end{aligned}$$

where the limit is in the weak* topology of the probability measures of the ambient space M. Hence, we deduce $\mu=\sum_{i=1}^k\frac{\mathrm{Leb}(B(\mu)\cap B(\mu_i)\cap\Omega)}{\mathrm{Leb}(B(\mu)\cap\Omega)}\mu_i$ and μ as a convex linear combination of the ergodic physical/SRB measures provided by Theorem B.

For item (2), since G is contained is E^{cu} and has zero Lyapunov exponent, then domination of the splitting $E^s\oplus E^{cu}$ ensures that all Lyapunov exponents along E^s are strictly negative and so $\int\log|\det Df\mid_{E^{cu}}|\,d\mu=\int\Sigma^+\,d\mu$ by Oseledets' Multiplicative Ergodic Theorem. This holds either for G-invariant of f-invariant probability measures, or even ϕ_t-invariant for a fixed value of $t>0$.

Then, assumption (2a) means $h_\mu(f)=\int\Sigma^+\,d\mu>0$. In particular, μ is non-atomic[12] and this becomes the necessary and sufficient condition for absolutely continuous disintegration along unstable manifolds W^{uu}_x for μ-a.e. x, by the characterization of measures satisfying the Entropy Formula [40] for C^2 smooth dynamics. This means, more precisely, that for μ-a.e. $x\in\Lambda$ there exists $\rho=\rho(x)>0$ so that

$$\Pi_x=\{W^{uu}_y:y\in B(x,\rho)\,\&\,W^{uu}_y\text{crosses }B(x,\rho)\}$$

[12] For otherwise by Ergodic Decomposition, Jacob's Theorem [64, Chpt. 9, Sec. 6] and Margulis-Ruelle Inequality [64, Theorem 9.4.4] we would obtain the Entropy Formula for each ergodic component ν of μ. In particular, if ν is supported on a critical element of Λ, either an equilibrium or a periodic orbit, then $0=h_\nu(f)=\int\log|\det f\mid_{E^{cu}}|\,d\nu=\int\Sigma^+\,d\nu$, contradicting the hyperbolicity assumption on μ.

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and[13] the normalized restriction $\tilde{\mu} = \mu\,|_{\cup\Pi_x}$ disintegrates along the leaves of Π_x as $\tilde{\mu} = \int \mu_y\, d\hat{\mu}(y)$. Here μ_y is a probability measure supported on γ_y equivalent to the restriction Leb_y of Lebesgue measure on this submanifold and $\hat{\mu} = \pi_*\tilde{\mu}$, where $\pi : \cup\Pi_x \to \Pi_x$ is the quotient map associating a point of $\cup\Pi_x$ to the corresponding leaf of Π_x.

Each manifold W^{uu}_x is contained in Λ with dimension $\dim E^{cu} - 1$ and tangent bundle in E^{cu}. The center-unstable (or weak-unstable) manifolds $W^{cu}_x = \phi_{(-1,1)}(W^u_x)$ are tangent to the centre-unstable bundle E^{cu} at each point and also the disjoint union of strong-unstable leaves transported by the flow. By smoothness of the flow, the disintegration of μ along the centre-unstable leaves is also absolutely continuous.

Indeed, for small enough ρ, $W^{uu}_y \in \Pi_x$ if, and only if, W^{cu}_y crosses $B(x,\rho)$. Considering $\Pi^c_x = \{W^{cu}_y : y \in B(x,\rho)\,\&\, W^{cu}_y \text{ crosses } B(x,\rho)\}$, then $\cup\Pi_x = \cup\Pi^c_x$ and $\tilde{\mu} = \int \nu_z\, d\hat{\nu}(z)$, with $\hat{\nu} = \hat{\pi}_*\tilde{\mu}$ where $\hat{\pi} : \cup\Pi^c_x \to \Pi^c_x$ is the corresponding quotient map, and $\nu_z = \int \mu_y\, d(\pi_*\nu_z)$ is equivalent to Leb induced on the connected component γ^{cu}_z of $W^{cu}_z \cap B(x,\rho)$ containing z. This is the property stated in item (2b).

Assuming condition (2b), the Ergodic Theorem provides a full μ-measure subset B of Birkhoff generic points for μ which is also a full $\tilde{\mu}$-measure subset. Hence, B has full ν_z-measure for $\hat{\nu}$-a.e. z. If we fix a center unstable disk γ^{cu}_z for a $\hat{\nu}$-generic z, then $\nu_z(B) = 1$ and B is also a full Leb_z-measure subset of γ^{cu}_z. Since the stable foliation is defined at all points of Λ, tangent to the stable bundle E^s which makes an angle with the center-unstable bundle uniformly bounded away from zero, then the subset $V = \{W^s_w(\varepsilon) : w \in \gamma^{cu}_z\}$ is an open neighbourhood of z for small enough $\varepsilon > 0$, where $W^s_w(\varepsilon)$ is the connected component of $W^s_w \cap B(w,\varepsilon)$ containing w. Moreover, the stable foliation is absolutely continuous [14, Section 6], and so the subset $W = \{W^s_w(\varepsilon) : w \in B \cap \gamma^{cu}_z\}$ has full Leb-measure in V: $\mathrm{Leb}(W \setminus V) = 0$. In addition, each $y \in W$ is such that $d(\phi_t y, \phi_t w) \to 0$ when $t \nearrow \infty$ for some $w \in B \cap \gamma^{cu}_z$. Hence, for any given continuous observable $\varphi : U \to \mathbb{R}$ we obtain

$$\lim_{T\nearrow\infty} \frac{1}{T}\int_0^T \varphi(\phi_t y)\, dt = \lim_{T\nearrow\infty} \frac{1}{T}\int_0^T \varphi(\phi_t w)\, dt = \int \varphi\, d\mu \tag{38}$$

and thus $W \subset B(\mu)$ with $\mathrm{Leb}(W) > 0$ and μ becomes a physical measure, as stated in item (2c). For an f-invariant measure, we replace (38) by $\lim n^{-1}\sum_{i=0}^{n-1} \varphi(f^i x)$ an argue in the same way.

We note that we immediately obtain item (3) from the previous construction, since $\mathrm{Leb}(V \setminus W) = 0$, once we show that (2c) implies (2a). Indeed, the uniformed sized ϱ-ball inside V is given by the cylinder $\mathcal{C}$ obtained in the proof of Theorem B; see Remark 4.6.

Moreover, from item (3) we easily obtain item (4). Indeed, if there are two physical measures μ_1, μ_2, then from item (3) there exist open subsets V_i such that $\mathrm{Leb}(V_i \setminus B(\mu_i)) = 0$ and $V_i \cap \Lambda \neq \emptyset, i = 1, 2$. Transitivity ensures that there exist $x_1 \in V_1$ and $t > 0$ so that $\phi_t x_1 \in V_2$. Smoothness of the flow ensures that $\phi_{-t}B(\mu_2)$ has positive volume in V_1, thus by flow invariance of the ergodic basin we find $y \in B(\mu_1) \cap B(\mu_2)$, which implies that $\mu_1 = \mu_2$.

[13] We say that W^{uu}_y *crosses* $B(x,\rho)$ if the connected component γ_y of $W^{uu}_y \cap B(x,\rho)$ containing y projects into the corresponding connected component γ_x of $W^{uu}_y \cap B(x,\rho)$ containing x, thought the stable holonomy map π^s in a one-to-one way, i.e., $\pi^s\,|_{\gamma_y} : \gamma_y \to \gamma_x$ is injective.

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We are left with showing that condition (2c) implies (2a). But this is an easy consequence of item (1), since a physical probability measure μ is a linear convex combination of the finitely many ergodic physical/SRB measures provided by Theorem B which are *cu*-Gibbs states, that is, satisfy (2a). The proof is complete. □

5.2. Proof of the main theorems

We are now ready to deduce Theorem C first, assuming the statement of Theorem B. Afterwards, we prove Theorem E and then Corollaries D, F and G.

5.2.1. Discrete (NU2SE) versus continuous (MASE)

Proof of Theorem C. For item (1), to obtain slow recurrence (SR) from continuous slow recurrence (CSR), we note that, since G is L-Lipschitz, where $L = \sup_{x\in U} \|DG(x)\|$, we have for $\varphi(x) = d(x, \mathrm{Sing}(G))$

$$\left|\frac{d}{dt}\varphi(t)\right| \le \|G(\phi_t x)\| = \|G(\phi_t x) - G(\sigma)\| \le L \cdot d(x,\sigma) = L \cdot \varphi(t), \tag{39}$$

whenever x is near $\sigma \in \mathrm{Sing}(G)$. Hence, $e^{-Ls} \le \varphi(\phi_s x)/\varphi(x) \le e^{Ls}$ for $|s|$ small enough so that $\varphi(\phi_s x) = d(\phi_s x, \sigma)$. Therefore, setting $\varphi_\delta(x) = d_\delta(x, \mathrm{Sing}_\Lambda(G))$, given $\delta > 0$ we can take $s > 0$ so that $Ls < -\log \delta^{1/2}$ and if $d(x,\sigma) < \delta$, then for $0 \le t \le s$

$$-\log \varphi_\delta(\phi_t x) = -\log \varphi(\phi_t x) \ge -\log \varphi(x) - Lt \ge -(1/2)\log \varphi_\delta(x). \tag{40}$$

Thus, from (7): for any $\varepsilon > 0$ we can find δ and $k \ge 2, k \in \mathbb{Z}^+$ so that $L/k < -\log \delta$ and for all $x \in \Omega$ there exists $N = N(x) > 1$ so that for each $n \in \mathbb{Z}^+$ satisfying $n \ge N$ we have

$$\varepsilon n \ge \int_0^n \varphi_\delta(\phi_s x)\, ds = \sum\nolimits_{i=0}^{nk-1} \int_0^{1/k} \varphi_\delta(\phi_{s+i/k}(x))\, ds \ge \frac{1}{2}\sum\nolimits_{i=0}^{nk-1} \varphi_\delta(\phi^i_{1/k}(x)).$$

Setting $g := \phi_{1/k}$, this ensures that for $m \in \mathbb{Z}^+$, if $n = [m/k] + 1$, then

$$\sum\nolimits_{i=0}^{m-1} -\log \varphi_\delta(g^i x) \le \sum\nolimits_{i=0}^{nk-1} -\log \varphi_\delta(g^i x) \le 2\int_0^n -\log \varphi_\delta(\phi_s x)\, ds < 2n\varepsilon$$

if $x \in \Omega$ and $m > k \cdot N(x)$. Thus we obtain the next time reparametrization of (8):

$$(1/m)\sum\nolimits_{i=0}^{m-1} -\log \varphi_\delta(g^i x) < 2\big(1/k + 1/m\big)\varepsilon. \tag{41}$$

Noting that from (39) we may likewise deduce the reverse inequality to (40), then a similar argument to the previous one enables us to reciprocally obtain continuous slow recurrence (11) from slow recurrence (8). This completes the proof of the statement of item (1).

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For item (2): Theorem B shows that (B) implies (A). For the reciprocal, let μ be a hyperbolic physical/SRB measure for the flow, which is also a *cu*-Gibbs state supported on the partial hyperbolic attracting set $\Lambda = \Lambda_G(U)$.

To obtain the (NU2SE) condition (4), we note that by hyperbolicity of μ, all central-unstable directions transversal to the vector field have positive Lyapunov exponents, thus $\chi(x) := \lim_{t\to+\infty} \log \left\| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \right\|^{1/t} \leq -c_0 < 0$ for μ-a.e. x and some constant $c_0 > 0$, by Oseledets' Theorem. By Fatou's Lemma for bounded sequences of functions we get $-c_0 > \int \chi \, d\mu \geq \limsup_{T\nearrow\infty} \int \log \| \wedge^2 (D\phi_T \mid_{E^{cu}_x})^{-1} \|^{1/T} \, d\mu(x)$. Thus, we find $T > 0$ so that $\int \log \| \wedge^2 (D\phi_t \mid_{E^{cu}})^{-1} \|^{1/t} \, d\mu \leq -c_0/2$ for all $t \geq T$. Hence, there exists an ergodic component ν of μ so that $\int \log \| \wedge^2 (D\phi_t \mid_{E^{cu}})^{-1} \|^{1/t} \, d\nu \leq -c_0/2$.

Moreover, ν is also a physical/SRB measure for the flow; see Theorem 5.1. Since ν is flow invariant, we can assume without loss of generality that ν is ergodic with respect to $g = \phi_T$, using the flow invariance and ergodicity of ν.

Remark 5.2. (Ergodicity for continuous time ensures dicrete time ergodicity) *The ergodicity of a flow invariant probability measure μ implies that, for a co-countable set of times $t_* \in \mathbb{R}$, we have that μ is ϕ_{t_*}-ergodic; see e.g. [58]. That is, if a measurable set A is ϕ_{t_*}-invariant $\phi_{-t_*}(A) = A$ for this fixed value of t_*, then $\mu(A) \cdot \mu(M \setminus A) = 0$.*

This property is not true for transformations, i.e., if mu is g-ergodic, then not necessarily μ is g^k-ergodic for some $k > 1$.

Hence, because $U \ni x \mapsto \Gamma(x) := \log \|(Dg \mid_{E^{cu}_x})^{-1}\|$ is continuous and ν is also physical/SRB with respect to g, we get $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Gamma(g^i x) = \int \Gamma \, d\nu \leq -c_0/2$, for all $x \in B_g(\mu)$, where $B_g(\mu)$ is the ergodic basin of μ as a g-invariant probability measure.

From Theorem 5.1 this shows that (4) holds on the positive Lebesgue measure subset $B_g(\nu)$. We also have $B_g(\nu) \subset B(\nu)$ and so $\mathrm{Leb}(B_g(\nu) \cap \Omega) > 0$, completing the proof that (A) implies (B) with $E = B_g(\nu)$. This finishes the proof of item (2).

Item (3b) is a straightforward consequence of item (4) of Theorem 5.1.

For item (3a), we assume from now on that we are in the setting of the statement of Theorem B and its conclusion, i.e., we have both properties (A) and (B) and take μ an ergodic hyperbolic physical/SRB measure for the flow which is also f-ergodic (reparametrizing the flow if needed) and satisfying $\mathrm{Leb}(E \cap B(\mu)) > 0$; recall Remark 5.2. Since, from Theorem 1.6, all trajectories in U satisfying (NU2SE) automatically satisfy (MASE), then we obtain a full volume measure subset of E satisfying (MASE).

Since $E = \cup_{i=1}^{k} (E \cap B(\mu_i))$ except perhaps a subset of zero Lebesgue measure, we can repeat this argument for each μ_i to cover Leb-a.e. point of E. This completes the proof of item (3a) and of Theorem C. □

5.2.2. Weak asymptotic sectional expansion: proof of the main corollaries

Proof of Theorem E. We recall that $(wASE) \implies (wNU2SE)$ from Theorem 1.6 if $\dim E^{cu} = d_{cu} = 2$. Hence we have (wNU2SE) on a positive volume subset Ω of U.

To obtain the physical/SRB measure, we set $\Gamma(x) := \log \| \wedge^2 (Df \mid_{E^{cu}_x})^{-1} \|$ and $J^{cu}(x) := \log |\det(Df \mid_{E^{cu}_x})|$ in what follows. We note that for $x \in \Omega$ we have a subsequence of the integers n_k so that $\lim \frac{1}{n_k} \sum_{i=0}^{n_k-1} \Gamma(f^i(x)) < 0$ and the sequence

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$\left(\frac{1}{n_k}\sum_{i=0}^{n_k-1}\delta_{f^i x}\right)_{k\geq 1}$ weak* accumulates on a f-invariant probability measure μ which satisfies (22) (after Theorem 1.10) and, for some subsequence m_i of n_k

$$\mu(\Gamma) = \lim_{i\to+\infty}\frac{1}{m_i}\sum_{j=0}^{m_i-1}\Gamma(f^j(x)) = \lim_{k\to+\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\Gamma(f^i(x)) < 0. \tag{42}$$

Since μ is not necessarily ergodic, we consider its ergodic decomposition to write

$$\begin{aligned} 0 &\leq \int \left(h_{\mu_y}(f) - \mu_y(J^{cu})\right) d\mu(y) \\ &= \int_{y\in\Upsilon}\left(h_{\mu_y}(f) - \mu_y(J^{cu})\right) d\mu(y) + \int_{y\in\operatorname{supp}\mu\setminus\Upsilon}\left(h_{\mu_y}(f) - \mu_y(J^{cu})\right) d\mu(y) \end{aligned} \tag{43}$$

where $\Upsilon := \{y : \mu_y(\Gamma) < 0\}$; $\mu(\Gamma) < 0$ by (42); and $\mu_y := \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(y)}$ (in the weak* topology) is a f-invariant ergodic probability measure so that $y \in \operatorname{supp}\mu_y$ for μ-a.e y; see e.g. [42, Chapter II.6].

Claim 5.3. *There exists $y \in \Upsilon$ so that $h_{\mu_y}(f) = \mu_y(J^{cu})$.*

Assuming the claim, since for $y \in \Upsilon$ all the Lyapunov exponents along E^{cu} are either positive or zero along the flow direction, and negative along E^s by partial hyperbolicity, then we have that $\nu := \mu_y$ is a hyperbolic ergodic measure; and $\nu(J^{cu})$ coincides with the sum Σ^+ of all the positive Lyapunov exponents of ν. Hence we arrive at Pesin's Formula $h_\nu(f) = \Sigma^+$. From the characterization of measures satisfying this formula [40] we obtain that ν is SRB; and ergodic hyperbolic SRB measures for C^2 diffeomorphisms are physical; see e.g. [23]. We have obtained an ergodic physical/SRB measure for the time-1 map f of the flow of G. The same argument of the proof of Corollary 4.8 ensures that $\eta := \int_0^1 (\phi_t)_*\nu\, dt$ is flow invariant, ergodic and also a hyperbolic cu-Gibbs state, and a physical/SRB measure for the flow of G.

Remark 5.4. There exists a neighbourhood W of x where Leb-a.e. points are in $B(\eta)$. Indeed, by construction, we have $B(\eta) \subset \Omega$. Then, on $\Omega_0 := B(\eta)$ we have (NU2SE) and from the proof of item (3) of Theorem 5.1, we obtain the existence of an open subset V such that $V \cap \Lambda \neq \emptyset$ and $\operatorname{Leb}(V \setminus B(\eta)) = 0$. Hence $\nu(V) > 0$ and also $\mu(V) > 0$. We can take a non-negative continuous bump function $\varphi : M \to [0,1]$ supported on V so that $0 < \mu(\varphi) = \lim\frac{1}{n_k}\sum_{i=0}^{n_k-1}\varphi(f^i(x))$, ensuring that $f^k(x) \in V$ for some $k > 1$. Hence, we may take $W = f^{-k}(V)$.

If Λ is an attractor, we conclude that η is the unique physical/SRB measure in the same way as in the proof of item (4) of Theorem 5.1. This proves the existence and uniqueness statements of Theorem E.

It remains to prove the claim.

Proof of Claim 5.3. Arguing by contradiction, since $\mu(\Upsilon) > 0$ and for $x \in \Upsilon$ we have $h_{\mu_x}(f) - \mu_x(J^{cu}) = h_{\mu_x}(f) - \mu_x(\Sigma^+) \leq 0$ by Ruelle's Inequality, we assume that $h_{\mu_x}(f) - \mu_x(J^{cu}) < 0$ for all $x \in \Upsilon$. Then the left hand side of (43) is negative, thus $\mu(\operatorname{supp}(\mu) \setminus \Upsilon) > 0$ and there must be $z \in \operatorname{supp}(\mu) \setminus \Upsilon$ so that $h_{\mu_z}(f) - \mu_z(J^{cu}) > 0$.

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Since $\mu_z(\Sigma^+) \geq h_{\mu_z}(f) \geq \mu_z(J^{cu}) = \mu_z\big(\sum_{i\leq d_{cu}} \chi_i^+\big)$, then such z admits a negative sectional Lyapunov exponent on E^{cu}. *Because we are assuming that* $d_{cu} = \dim(E^{cu}) = 2$, we thus obtain an ergodic measure $\nu = \int_0^1 (\phi_t)_*\mu_z\, dt$ for the flow of G, whose Lyapunov exponents are all negative with the exception of the flow direction. Hence, this measure is supported on a hyperbolic periodic attracting orbit $\mathcal{O}$ — a sink; see e.g. [8]. In particular, μ_z is an atom of μ, so $\mu(B_\varepsilon(\mathcal{O})) > 0$ for all $\varepsilon > 0$. Since μ is a finite measure, we may find $\varepsilon_0 > 0$ arbitrarily small such that the ε_0-neighbourhood of the periodic orbit has null boundary: $\mu(\partial B_{\varepsilon_0}(\mathcal{O})) = 0$. By Portmanteau Theorem on weak* convergence, this ensures that $0 < \mu(B_{\varepsilon_0}(\mathcal{O})) = \lim \frac{1}{n_k} \sum_{i=0}^{n_k-1} 1_{B_{\varepsilon_0}(\mathcal{O})}(f^i(x))$. and so $f^k(x) \in B_{\varepsilon_0}(\mathcal{O})$ for some $k > 1$. But this implies that x belongs to the basin of a sink, contradicting $\mu(\Gamma) < 0$. This contradiction proves the claim. □

Remark 5.5. (Other setting where the claim holds) *The same proof of the claim follows, with arbitrary* $d_{cu} \geq 2$, *if we know that no negative Lyapunov exponents exist along* E^{cu} *on the attracting set. For instance, under condition (21) of Corollary D, or for sectional hyperbolic attractors as in Example 3; see also Example 7.*

For the reciprocal statement in Theorem E, let μ be an hyperbolic ergodic physical/SRB measure for the partial hyperbolic attracting set $\Lambda = \Lambda_G(U)$ with $\mathrm{Leb}(B(\mu) \cap \Omega) > 0$, and let us deduce mostly asymptotic sectional expansion (MASE), without any restriction on $\dim E^{cu}$.

We start by noting that G-invariance, ergodicity and hyperbolicity of μ, together with Kingman's Subadditive Ergodic Theorem, ensure that there exists $c_0 > 0$ so that

$$\inf_{t>0} \int \log \| \wedge^2 (D\phi_t \mid_{E^{cu}})^{-1} \|^{1/t} \, d\mu = \lim_{t\nearrow\infty} \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \|^{1/t} < -c_0, \quad \mu - \text{a.e. x.}$$

For a given small $\varepsilon > 0$ let us fix $g = \phi_T$ with $\int \log \| \wedge^2 (Dg \mid_{E^{cu}})^{-1} \|^{1/T} \, d\mu \leq -c_0 + \varepsilon$ for some $T > 0$ so that μ is g-ergodic, from Remark 5.2. Note that g is a partially hyperbolic diffeomorphism with respect to the same splitting $E^s \oplus E^{cu}$ over Λ. Since μ is a hyperbolic cu-Gibbs state even for the dynamics of g, then μ is a physical measure for g also; see the proof that (2b) implies (2c) in Theorem 5.1.

Using that μ is g-ergodic and physical, together with the subadditive property of the continuous function $(x,t) \mapsto \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \|$, we obtain for Leb-a.e. $x \in B_g(\mu)$, since $g^n x = \phi_{nT}(x)$ for $n \geq 0$

$$\begin{aligned} -c_0 + \varepsilon &\geq \int \frac{\log \| \wedge^2 (Dg \mid_{E^{cu}})^{-1} \|}{T} \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \wedge^2 (D\phi_T \mid_{E^{cu}_{g^i x}})^{-1} \|^{1/T} \\ &\geq \limsup_{n\to\infty} \log \| \wedge^2 (D\phi_{nT} \mid_{E^{cu}_x})^{-1} \|^{1/nT} \\ &\geq \liminf_{t\to\infty} \log \| \wedge^2 (D\phi_t \mid_{E^{cu}_x})^{-1} \|^{1/t}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that we have (wMASE) as in (13) on the positive volume subset $B_g(\mu) \subset B(\mu)$, which satisfies $\mathrm{Leb}(B_g(\mu) \cap \Omega) > 0$, completing the proof of Theorem E, since (wMASE) $\Longrightarrow$ (wASE) on a full Lebesgue measure subset, from Theorem 1.6. □

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Proof of Corollary D. We start by assuming (wNU2SE) which, in the proof of Theorem E was deduced as the first step. Thus, from the proof of Theorem E, we obtain a probability measure μ supported in Λ so that $h_\mu(f) = \mu(J^{cu})$. Since we additionally assume (21), which ensures the non-existence of negative Lyapunov exponents along the central-unstable direction on Λ, we can use Claim 5.3 by Remark 5.5. This enables us to obtain a physical/SRB measure μ supported in Λ. This proves the first statement of Corollary D.

Moreover, such measure μ is such that its basin $B(\mu)$ is contained in Ω, from Remark 5.4.

If we additionally assume that points in Ω satisfy the slow recurrence condition (SR), then all points of $B(\mu)$ satisfy both (SR) and (NU2SE) conditions, the latter one with a rate c_0. Reapplying the construction of *cu*-disks using hyperbolic times from the proof of Theorem B, we get that ergodic basin $B(\mu)$ of μ has volume bounded from below away from zero by a uniform constant, depending only on the map f; see Remark 4.6.

If $\mathrm{Leb}(\Omega \setminus B(\mu)) = 0$, then the proof is complete. Otherwise, we set $\mu_1 = \mu$ and reapply the proof of Theorem E starting with $x \in \Omega \setminus B(\mu)$, to obtain a new ergodic physical/SRB measure μ_2 with $B(\mu_2) \subset \Omega$. Proceeding by induction, we obtain a family of distinct ergodic physical/SRB probability measures $\mu_1, \mu_2, \ldots$ whose basins are contained in Ω with positive volume bounded away from zero. Since $\Omega \subset U$ with finite volume (U is a relatively compact neighbourhood of Λ in a finite dimensional manifold), after finitely many steps we get finitely distinct ergodic physical/SRB measures whose basis cover Ω except perhaps a zero volume subset. This completes the proof of the existence part of Corollary D. The reciprocal follows just as in the proof of Theorem E. □

Proof of Corollary F. As in the proof of Theorem E, we use (wASE) $\Longrightarrow$ (wNU2SE) with rate c_0 from Theorem 1.6; and the ergodic physical/SRB measure μ obtained is such that its basin $B(\mu)$ is contained in Ω, from Remark 5.4.

The rest of the proof of finiteness follows just like in the proof of Corollary D, since we are assuming also that (SR) condition holds. □

Proof of Corollary G. We show that a partially hyperbolic attracting set satisfying (wASE), when $\mathrm{Sing}_\Lambda(G)$ contains only saddle-type hyperbolic equilibria, is in the setting of Theorem E.

We start by observing that if Λ is transitive, then $\mathrm{Sing}_\Lambda(G)$ cannot contain sinks or sources. We are left to find a positive Lebesgue measure subset $\Omega \subset U$ whose trajectories are weak asymptotically sectional expanding (wASE).

In order to find Ω, we note that, from Theorem 1.6 (see e.g. the proof of Theorem E), any point of $\Lambda' := \Lambda \setminus \cup\{W^s_\sigma : x \in \mathrm{Sing}_\Lambda(G)\}$ satisfies (NU2SE).

Therefore, just like in the proof of Theorem E, for $x \in \Lambda'$ any f-invariant probability measure μ obtained as a weak* accumulation point of $\left(\frac{1}{n_k}\sum_{i=0}^{n_k-1}\delta_{f^i x}\right)_{k\geq 1}$ satisfies $\mu(\Gamma) \leq -c_u < 0$, where $\Gamma(x) = \log\left\| \wedge^2 (Df\mid_{E^{cu}_x})^{-1}\right\|$.

Thus, there exists an ergodic component ν of μ such that $\nu(\Gamma) \leq -c_u$. This ensures that $\eta = \int_0^1 (\phi_t)_*\nu\, dt$ is a hyperbolic measure for the flow. Hence, there exists a Pesin unstable-manifold $W^u_z(\varepsilon)$ contained in Λ (since Λ is an attracting subset) and a corresponding center-unstable manifold $W^{cu}_z(\varepsilon) = \cup_{|t|<\varepsilon}\phi_t\big(W^u_z(\varepsilon)\big)$, which is an embedded *cu*-disk inside Λ.

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Moreover, any compact connected part of the stable manifold W^s_σ of each saddle-type hyperbolic equilibrium $\sigma \in \mathrm{Sing}_\Lambda(G)$ is a compact submanifold transversal to $\Delta := W^{cu}_z(\varepsilon)$. Therefore, since $\mathrm{Sing}_\Lambda(G)$ is a finite set, the intersection $\Delta \cap \big(\cup\{W^s_\sigma : \sigma \in \mathrm{Sing}_\Lambda(G)\}\big)$ has zero measure with respect to the induced volume measure Leb_Δ on Δ.

Thus, $\Delta' := \Delta \cap \Lambda'$ has Leb_Δ-full measure. Since each $w \in \Delta'$ has a local stable manifold, we have that $y \in W^s_w$ also satisfies (wNU2SE) since $d(\phi_t y, \phi_t x) \to 0$ as $t \nearrow \infty$.

Therefore, we are left to check that the subset $\Omega := \cup\{W^s_w : w \in \Delta'\}$ has positive volume to complete the conditions of Theorem E and conclude the existence of an ergodic physical/SRB measure supported on Λ.

The flow is Hölder-C^1 (it is in fact of class C^2 as the vector field G) and so, from standard results, the stable leaves form an absolutely continuous lamination; see e.g. [14, Section 6] & [35, Appendix B.7] and references therein. This ensures that $\mathrm{Leb}(\Omega)$ is positive as a consequence of the positivity of $\mathrm{Leb}_\Delta(\Delta')$. This completes the proof of Corollary G. □

6. New examples of application

Here, we present mostly asymptotically sectional expanding examples which are either non-sectional hyperbolic or non-singular hyperbolic, with or without hyperbolic equilibria; as well as counter-examples failing some of our assumptions and having no physical measure.

6.1. Mostly asymptotically sectional expanding and non-sectional hyperbolic

Example 6 (Mostly asymptotically sectional expanding, singular-hyperbolic and not sectional-hyperbolic, with no equilibria). We consider the hyperbolic (Anosov) automorphism f_0 of the 3-torus $\mathbb{T} = (\mathbb{S}^1)^3$ induced by the linear map defined by

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \text{ with } \mathrm{sp}(A) = \{\lambda_3 \approx 0.198062 < 1 < \lambda_2 \approx 1.55496 < \lambda_1 \approx 3.24698\}.$$

Let p be the fixed point at the class of the origin $(0,0,0) \in \mathbb{R}^3$ and a small neighbourhood V of p with a choice of basis $\{v_1, v_2, v_3\}$ where v_i is a unit eigenvector corresponding to the eigenvalue $\lambda_i, i = 1,2,3$. In V the map f_0 has the expression $(x,y,z) \mapsto (\lambda_1 x, \lambda_2 y, \lambda_3 z)$. We consider the one-parameter family of maps of the real line

$$f_\mu(x) = \psi(x)\lambda_2 x + (1 - \psi(x))((1-\mu)\lambda_2 x + \mu \cdot h(x)), \quad 0 < \mu < 1;$$

where $h(x) = (1-b)x(1 + x^2(x^2 - a^2))$ for $0 < b < a < \xi < 1$ (with small b) has 3 fixed points at $0, \pm\xi$ satisfying $h'(0) = 1 - b < 1$ and $h'(\pm\xi) = (1-b)(1 \pm \xi(4\xi^2 - 2a^2)) > 1$; and $\psi : \mathbb{R} \to [0,1]$ is a C^∞ bump function so that $\mathrm{supp}\,\psi \subset \mathbb{R} \setminus [-1+b, 1-b]$ and $\psi(x) = 1$ whenever $|x| \geq 1 + b$; see the left hand side of Figure 5. Moreover, we can also assume that

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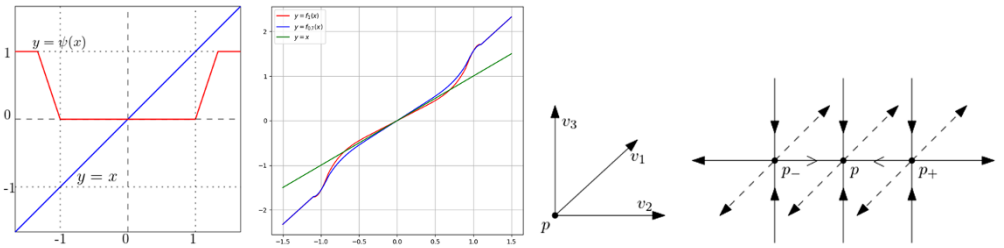


Figure 5. From left to right: the graph of a continuous bump function together with $y = x$; graphs of the identity, $y = f_1(x)$ and $y = f_{0.7}(x)$; depiction of the eigenvalue directions at the origin of T_pM; and the dynamical behaviour of the (un)stable directions in a neighbourhood of p for f_1.

$$\lambda_3 < f'_\mu(x) < \lambda_1, \quad x \in \mathbb{R} \quad \text{and} \quad 0 < \mu < 1. \tag{44}$$

In addition, since $f'_\mu(0) = \mu h'(0) = \mu(1-b)$ is the minimum of $f'_\mu(x)$, then

$$\lambda_1 + f'_\mu(x) > 1, \quad x \in \mathbb{R}. \tag{45}$$

We replace the second coordinate map $y \mapsto \lambda_2 y$ by the one-parameter family $y \mapsto \varepsilon_1 \cdot f_\mu(y/\varepsilon_1)$ for $\varepsilon_1 > 0$ small enough so that the ball of radius $\varepsilon_1(1+b)$ around p is contained in V, and the properties stated above are preserved at corresponding points after scaling. For $\mu = 0$ we have the original map f_0.

For $\mu = 1$ we have a map f_1 coinciding with f_0 outside of V and having inside V three fixed hyperbolic saddle points: p with index 2; and $p_\pm$ with index 1, symmetrically placed with respect to p along the line segment $[-\varepsilon_1(1+b), \varepsilon_1(1+b)]v_2$ inside V; see the right hand side of Figure 5.

We note that f_1 has a partially hyperbolic splitting $E^s \oplus E^{cu}$ defined on all of $\mathbb{T}$ which is volume hyperbolic:

- E^s coincides with the stable bundle of f_0 spanned everywhere by v_3 and is uniformly contracted $\|Df_1 \mid_{E^s}\| = \lambda_3$;
- E^{cu} coincides with the unstable bundle of f_0 spanned everywhere by $\{v_1, v_2\}$, domination of the splitting is a consequence of (44); and
- $\det(Df_1 \mid_{E^{cu}}) > 1$ as a consequence of (45).

The invariant bundle E^{cu} further decomposes into the continuous splitting $E^c \oplus E^u$, where E^u is spanned everywhere by v_1 and uniformly expanded: $\|Df \mid_{E^u}\| = \lambda_1$; and E^c is spanned everywhere by v_2 and dominated by E^u. We claim that f_1 is non-uniformly expanding, that is

$$\limsup_{n\nearrow\infty} \sum_{i=0}^{n-1} \log \|(Df_1 \mid_{E^{cu}_{f_1^i x}})^{-1}\|^{1/n} < 0, \quad \text{Leb}-\text{a.e.} x \in \mathbb{T}. \tag{46}$$

But $f = f_1$ is a perturbation of the Anosov automorphism f_0 on the 3-torus around the fixed point satisfying all the conditions stated in [4, Appendix], namely:

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(1) f admits invariant cone fields C^{cu} and C^{cs}, with small width containing, respectively, the unstable and stable bundle of the Anosov diffeomorphism f_0;
(2) there are $0 < \sigma_2 < 1 < \sigma_1$ and $\delta_0 > 0$ so that for disks D^{cu} and D^{cs} through x tangent, respectively, to the centre-unstable cone field C^{cu} and to centre-stable cone field C^{cs}, we have
   (a) $\min\{|\det(Df\,|_{T_xD^{cu}})|, |\det(Df\,|_{T_xD^{cs}})^{-1}|\} > \sigma_1$, for $x \in M$;
   (b) $\max\{\|(Df\,|_{T_xD^{cu}})^{-1}\|, \|(Df\,|_{T_xD^{cs}})\|\} < \sigma_2$, for $x \in M \setminus V$;
   (c) $\max\{\|(Df\,|_{T_xD^{cu}})^{-1}\|, \|(Df\,|_{T_xD^{cs}})\|\} < (1+\delta_0)$, for $x \in V$.

Then it follows that f_1 satisfies (46); see e.g. [4, Appendix] or [2, Section 7.6].

We now consider the suspension flow G on a 4-dimensional manifold $\widetilde{\mathbb{T}} = \mathbb{T} \times [0,1]/\sim$ of the diffeomorphism f_1, where the equivalence relation is given by $(x,1) \sim (f_1(x),0)$ for $x \in \mathbb{T}$; see e.g. [51, Proposition 3.7]. Since f_1 can be taken of class C^r for every $r \geq 1$, the same holds for G. Moreover, $\widetilde{\mathbb{T}}$ is also a parallelizable manifold as $\mathbb{T}$ is; thus we can consider v_1, v_2, v_3 as globally defined vector fields transverse to G.

We observe that the flow becomes singular-hyperbolic but not sectional-hyperbolic: the splitting $T\widetilde{\mathbb{T}} = F^s \oplus (F^c \oplus \mathbb{R}\cdot G \oplus F^u)$ where F^s, F^c, F^u are respectively spanned by v_1, v_2, v_3 everywhere on $\widetilde{\mathbb{T}}$ is such that $F^{cu} = F^c \oplus \mathbb{R}\cdot G \oplus F^u$ is volume expanding, since the action of the flow ϕ_t of G along G is a translation. However, at the point $p_0 = p \times \{0\}$ we have $\det(D\phi_1\,|_{F^c \oplus \mathbb{R}\cdot G}) = 1 - b < 1$, contradicting sectional-expansion, since p_0 belongs to a periodic orbit of G with period 1. We claim that this flow satisfies the (MASE) condition.

Indeed, we note that since each submanifold $\Sigma_s = \mathbb{T} \times \{s\}, 0 \leq s < 1$ is a global cross-section for the flow ϕ_t with constant return time equal to 1, then $\phi_1\,|_{\Sigma_s} \colon \Sigma_s \to \Sigma_s$ is the Poincaré First Return Map to Σ_s, and such return maps all coincide with f_1 by construction of the suspension flow as a translation on the last coordinate. In addition, we get $P^1 = D(\phi_1\,|_{\Sigma_s})$. Hence, since f_1 is a partially hyperbolic non-uniformly expanding diffeomorphism, we obtain (17) for Leb-a.e. $x \in \Sigma_s$ for each $0 \leq s < 1$. Thus by Fubini's Theorem, we get (17) for Leb-a.e. point of $\widetilde{\mathbb{T}}$, because $\{\Sigma_s : 0 \leq s < 1\}$ is a smooth foliation of $\widetilde{\mathbb{T}}$.

We have (NUSE) without singularities, so that (SR) holds automatically by vacuity. Moreover, since the flow direction makes an angle bounded away from zero from the global cross-sections Σ_s, we also obtain (NU2SE). We can then ensure the existence of finitely many ergodic physical/SRB measures for this class of systems from the proof of Theorem B (which just assumes (NUSE)). From this we obtain (MASE) from Theorem C.

Example 7 (Mostly asymptotically sectional expanding, with equilibria and not sectionally hyperbolic). We adapt the construction of the multidimensional Lorenz attractor, first presented by Bonatti, Pumariño and Viana in [24], to obtain an example of a mostly asymptotically sectional expanding attracting set with an equilibrium.

We consider a "solenoid" constructed over a uniformly expanding map $g : \mathbb{T} \to \mathbb{T}$ of the k-dimensional torus $\mathbb{T}$, for some $k \geq 2$. That is, let $\mathbb{D}$ be the unit disk on $\mathbb{R}^2$ and consider a smooth embedding $F_0 : N \circlearrowleft$ of $N = \mathbb{T} \times \mathbb{D}$ into itself, which preserves and contracts the foliation $\mathcal{F}^s = \{\{z\} \times \mathbb{D} : z \in \mathbb{T}\}$. The natural projection $\pi : N \to \mathbb{T}$ on the first factor conjugates F_0 to g: $\pi \circ F_0 = g \circ \pi$. We assume that the initial expanding

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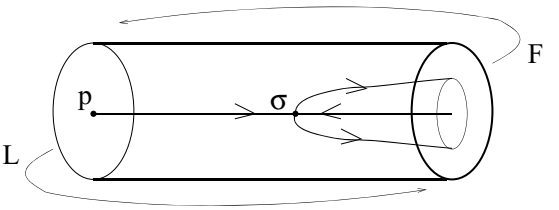


Figure 6. A sketch of the modification of the vector field leading to the multidimensional Lorenz attractor.

map g has simple spectrum $\{\lambda_1 > \lambda_2 > \ldots > \lambda_k\}$ and that F_0 admits two distinct fixed points p and q.

We have that $DF_0(q) : T_qN \to T_qN$ is hyperbolic with a 2-dimensional contracting invariant subspace, and a complementary k dimensional expanding invariant subspace. Let $\{v_1, \ldots, v_k, u_1, u_2\}$ be a basis of $TN = \mathbb{R}^k \times \mathbb{R}^2$ formed by unit vectors so that v_i is an eigenvector corresponding to $\lambda_i, i = 1, \ldots, k$. We choose coordinates on a neighbourhood V of q in N so that $F_0 \mid_V$ has the expression $(x, y_1, \ldots, y_k) \mapsto (Ax, \lambda_1 y_1, \ldots, \lambda_k y_k)$ with $x = x_1 u_1 + x_2 u_2$ and A a linear contraction on $\mathbb{R}^2$.

We perform the same perturbation as in Example 6 replacing the weakest expanding coordinate map $y_k \mapsto \lambda_k y_k$ by $y_k \mapsto \varepsilon_1 f_\mu(y_k/\varepsilon_1)$ around the fixed point q, obtaining a new base map $F : N \to N$. We choose $\mu \in (0, 1)$ so that the $DF(q)$ has eigenvalue 1 along the y_k direction and keeps the expanding/contracting eigenvalues along the other directions.

We note that F is a partially hyperbolic map with an invariant splitting $E^s \oplus E^c \oplus E^u$, where $E^s = \{0\} \times \mathbb{R}^2$, $E^c = \mathbb{R} \times v_k$ and E^u is everywhere spanned by $v_1, \ldots, v_{k-1}$, with $\| \wedge^2 \left(DF \mid_{E^{cu}_z}\right)^{-1} \| < 1$ for $z \in N \setminus \{q\}$ with respect to the standard product metric in N where $E^{cu} = E^c \oplus E^u$ is central-unstable subbundle; and $\| \wedge^2 \left(DF \mid_{E^{cu}_q}\right)^{-1} \| = 1$. That is, we have area expansion along any two-dimensional subspace of the central-unstable subbundle away from q. We also have an attracting subset $\Lambda_0 = \cap_{n \geq 0} F^n(N)$ with N as topological basin of attraction.

We further consider the constant vector field $X = (0, 1)$ on $M = N \times [0, 1]$ and modify this field on the cylinder $C = U \times \mathbb{D} \times [0, 1]$ around the periodic orbit of the point $p = (z, 0) \in N \times \{0\}$, where U is a neighbourhood of z in $\mathbb{T}$ such that $V \cap (U \times \mathbb{D}) = \emptyset$, in such a way as to create a hyperbolic (generalized Lorenz-like) equilibrium σ of saddle-type with k expanding and 3 contracting eigenvalues, as depicted in Figure 6. The eigenspace of one of the contracting eigenvalues lies along the direction of X, the other two-dimensional contracting directions still lie on the direction of $\mathbb{D}$, and the remaining expanding eigenspaces are transversal to the X direction.

This vector field Y defines a transition map from $\Sigma_\varepsilon = N \times \{\varepsilon\}$ to $\Sigma_{1-\varepsilon} = N \times \{1 - \varepsilon\}$ for some fixed small $\varepsilon > 0$, which is the identity in the first coordinate when restricted to $\Sigma_\varepsilon \setminus (U \times \mathbb{D} \times \{\varepsilon\})$.

We assume that the standard inner product satisfies $\langle Y, X \rangle > 0$ on $\Sigma_\varepsilon \cup \Sigma_{1-\varepsilon}$ and take a C^∞ bump function $\psi : [0, 1] \circlearrowleft$ so that $\psi \mid_{[\varepsilon/2, 1-\varepsilon/2]} \equiv 0$ and $\psi \mid_{[0, \varepsilon/3] \cup [1-\varepsilon/3, 1]} \equiv 1$. We define the vector field $G(x, t) = \psi(t) \cdot X + (1 - \psi(t)) \cdot Y(x, t), (x, t) \in M$ which generates a smooth transition map L from $\Sigma_0^* = (N \setminus \{p\}) \times \{0\}$ to $\Sigma_1 = N \times \{1\}$. Together with

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the identification $(x,0) \sim (F_0(x),1), x \in N$ we obtain a smooth parallelizable manifold $\widetilde{M} = M/\sim$ where G induces a C^∞ vector field which we denote by the same letter.

We may assume that the splitting $E^s \oplus E^{cu}$ is still preserved by $F \circ L$: this is clear outside of the cylinder C, inside C this is obtained by the choice of Y and, moreover, in C the E^{cu} bundle in uniformly expanded.

We may now induce invariant bundles for the flow ϕ_t of G on $\widetilde{M}$ by parallel transport: $F^{cu} = \mathbb{R}\cdot G \oplus E^{cu}$ and $F^s = E^s$ and consider the maximal invariant subset $\Lambda = \cap_{t>0}\phi_t(\widetilde{M})$ which is a attracting set with basin $\widetilde{M}$. Since $q \in N$ becomes a periodic point with period 1 for G and $p \in W^s_\sigma$, we still have area expansion along F^{cu}_z for $z \neq q$ and non-sectional-expansion along F^{cu}_q for ϕ_t. But, considering the cone fields C^{cs} and C^{cu} of small width around F^s, F^{cu} respectively, we obtain the sufficient conditions (1-2) presented in the previous Example 6 for non-uniform sectional expansion of $f = \phi_1$. Condition (NUSE) is obtained again as in Example 6, as well as condition (NU2SE).

Moreover, by construction, there are no asymptotic contracting directions along the F^{cu} subbundle: we have condition (21) from Corollary D. Hence there exists an ergodic physical/SRB probability measure. We can also obtain transitivity, ensuring the uniqueness of this measure and that $\widetilde{M} = B(\mu)$, Leb mod0.

6.2. Mostly asymptotically sectional expanding and not singular-hyperbolic

Example 8 (Geometric Lorenz-like attractor with non-hyperbolic periodic orbit). We start with a one-dimensional Lorenz-like transformation with two expanding fixed repellers at the boundary of the interval, which is an adaptation of the "intermittent" Manneville map [57] into a local homeomorphism of the circle; . We consider $I = [-1,1]$ and the map $f : I \to I$, $x \mapsto \begin{cases} 2\sqrt{x} - 1 & \text{if } x \geq 0, \\ 1 - 2\sqrt{|x|} & \text{otherwise.} \end{cases}$; see the left hand side of Figure 7.

Then we perform the geometric Lorenz construction in such a way to obtain this map as the quotient over the stable leaves of the Poincaré first return map to the global cross-section of a vector field G_0; see the right hand side of Figure 7.

As usual in the geometric Lorenz construction, we assume that in the cube I^3 the flow is linear $\dot{G}_0 = A \cdot G_0$ with $A = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ and a Lorenz-like equilibrium at the origin σ_0 satisfying $\lambda_1 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$; see e.g. the detailed description in [16, Chap. 3, Sec. 3].

The map f preserves Lebesgue measure λ on I which is f-ergodic; see [3, Sec. 5]. In particular, f is transitive (in fact, it is locally eventually onto, and so topologically mixing) and since $|log|x||$ is λ-integrable, then we have slow recurrence to the unique singularity at 0.

We thus obtain an attractor Λ for the flow of the vector field G depicted in the right hand side of Figure 7 which is partially hyperbolic (with $d_s = 1$ and $d_{cu} = 2$) and admits two periodic orbits $\mathcal{O}(p_\pm)$ corresponding to the indifferent fixed points of f which are not hyperbolic. Indeed, the Poincaré first return map $R : S^* \to S$ to the cross-section $S = I^2 \times \{1\}$, with domain $S^* = S \setminus \big(\{0\} \times I \times \{1\}\big)$ given by all the points of S away from the singular line, is a skew-product map $R(x,y) = (f(x), g(x,y))$, where g is a

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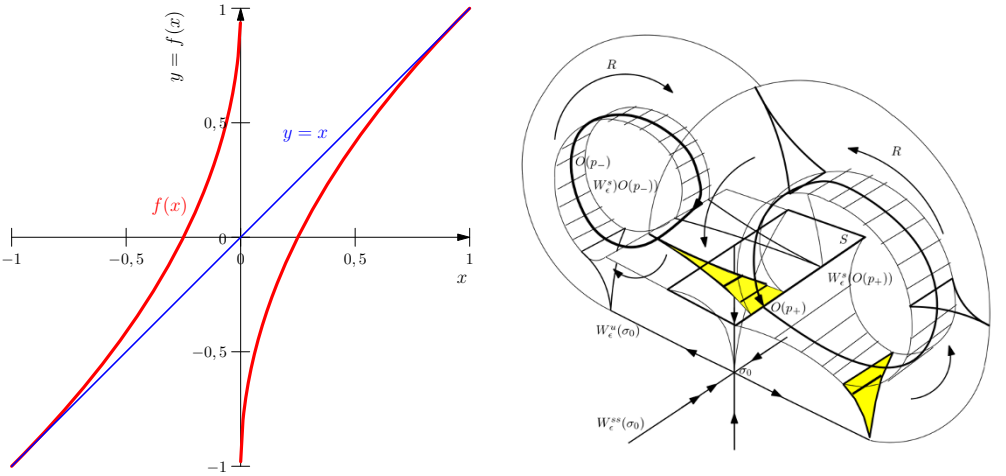


Figure 7. Lorenz one-dimensional transformation with repelling fixed points at the extremes of the interval on the left; and the geometric Lorenz construction with this map as the quotient over the contracting invariant foliation on the cross-section S, with two corresponding periodic saddle-type periodic orbits $\mathcal{O}(p_{\pm})$.

contraction on the second coordinate. The non-hyperbolicity of $\mathcal{O}(p_{\pm})$ ensures that *the attractor* Λ *of the* 3*-vector field* G *is not singular-hyperbolic.*

Following the standard construction described in [16, Chap. 7, Sec. 3.4], there exists an ergodic physical R-invariant probability measure ν on S whose marginal $\pi_*\nu$ is λ, where $\pi : S \simeq I^2 \to I$ is the natural projection on the first coordinate. Finally, we obtain a physical ergodic invariant probability measure μ for the flow of G by considering the suspension flow with base map R and roof function provided by the Poincaré first return time $\tau : S^* \to \mathbb{R}^+$ to S.

Moreover, $|f'(x)| > 1$ for all $x \in I \setminus \{0, \pm 1\}$ and so, if $(\phi_t)_{t\in\mathbb{R}}$ is the flow of G, then since τ is constant on the fibers of the skew-product and λ-integrable

$$\int \log |\det D\phi_1 \,|_{E^c}\,| \, d\mu \geq h_\mu(\phi_1) = \frac{h_\nu(R)}{\mu(\tau)} \geq \frac{h_\lambda(f)}{\mu(\tau)} = \frac{1}{\mu(\tau)} \int \log |f'| \, d\lambda > 0,$$

we conclude that Λ is mostly asymptotically sectional expanding (MASE) while not being singular-hyperbolic. From Theorem E, these attractors admit a unique physical/SRB measure due to transitivity and $\dim E^{cu} = 2$.

Example 9 (Geometric Lorenz-like attractor with non-hyperbolic equilibrium). In the recent work [28] Bruin-Farias construct (similarly to the previous example) and study a geometric Lorenz-like attractor with a neutral equilibrium replacing the hyperbolic Lorenz-like equilibrium from the classical (geometrical) Lorenz attractor. This neutral equilibrium is neither Lorenz-like nor Rovella-like.

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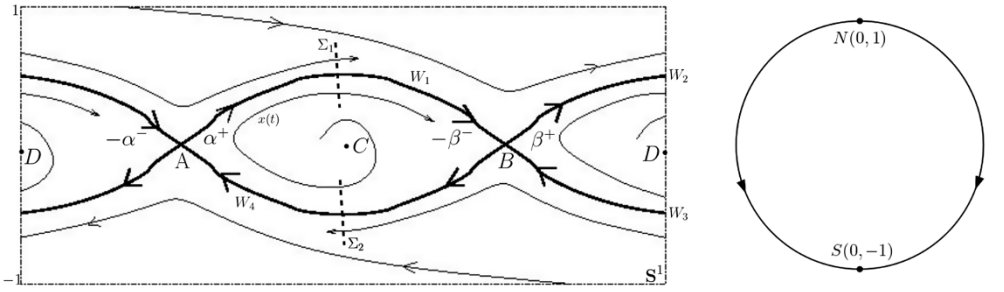


Figure 8. The double heteroclinic connection with non-existing time averages for a full Lebesgue measure subset; on the right hand side. On the left hand side, the North-South flow on the circle.

The authors show that there exists a unique physical/SRB measure and proceed to study its mixing rate (obtaining polynomial upper bounds). This implies mostly asymptotic sectional expansion without singular-hyperbolicity.

Remark 6.1. *Example 9 shows, in particular, that the assumption of hyperbolic equilibria is not necessary for the existence of a physical/SRB measure and so also not necessary to obtain asymptotical sectional expansion. Hence, this assumption should be regarded as a simplifying general assumption which is used in our line of proof.*

6.3. Non-uniform weak expansion without slow recurrence nor physical measure

Example 10 (Non-uniform expanding and no physical measure). We consider the well-known vector field X generating the flow $(\phi_t)_{t\geq 0}$ of the cylinder $N := \mathbb{S}^1 \times \mathbb{R}$ with a double heteroclinic connection (the "Bowen's eye" flow), e.g., from Takens work [63] showing that *Birkhoff averages may not exist almost everywhere*; see the left hand side of Figure 8. In this system time averages exist only for the sources C, D and for the set of separatrixes and saddle equilibria $W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup \{A, B\}$. Moreover the orbit $(\phi_t(x))_{t\geq 0}$ of each x not in W and different from C, D tends to W as $t \nearrow \infty$.

Letting $f := \phi_1$ denote the time 1 map of the flow, we see that W is a compact f-invariant attracting set, since $W = \cap_{n\geq 0} f^n(U)$ for all sufficiently small neighbourhoods U of W. Moreover, we may choose the saddles eigenvalues and adapted coordinates near A and B to obtain the following for every $x \in N \setminus \{C, D\}$

$$\limsup_{T\nearrow\infty} \log |\det D\phi_T(x)|^{1/T} < 0 < \limsup_{n\nearrow\infty} \sum_{i=0}^{n-1} \log \|Df(f^i x)\|^{1/n}. \quad (47)$$

This shows that this system, although with some average asymptotic expansion, is asymptotically sectional contracting on an open and full Lebesgue measure subset – which shows that these trajectories are not Oseledets regular; see e.g. [21] and [31]. Since physical measures cannot exist in this system due to the non-existence of Birkhoff time averages, then we obtain a weak counterexample to the following conjecture.

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Conjecture 7. ([23, Conjecture 12.37]) *If a smooth map f has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits some SRB measure.*

The proof of (47) is a consequence of the following (see also Kiriki et al. [39] and Ott-Yorke [50]).

Theorem 6.2 ([63, Theorem 1]). *If g is a continuous function on N with $g(A) > g(B)$ and the positive trajectory of x accumulates W, then*

$$\limsup_{T\nearrow\infty} \frac{1}{T}\int_0^T g(\phi_t x)\,dt = \frac{\sigma g(A) + g(B)}{1+\sigma} \quad \text{and} \quad \liminf_{T\nearrow\infty} \frac{1}{T}\int_0^T g(\phi_t x)\,dt = \frac{\lambda g(B) + g(A)}{1+\lambda},$$

where $\lambda := \alpha^-/\beta^+$ and $\sigma := \beta^-/\alpha^+$ from the spectra $\mathrm{sp}(DX(A)) = \{\alpha^+, -\alpha^-\}$ and $\mathrm{sp}(DX(B)) = \{\beta^+, -\beta^-\}$ with $\alpha^\pm, \beta^\pm > 0$.

Indeed, to ensure that W is attracting it is enough to have $\lambda\sigma > 1$ and we can set this together with $\delta_A := \alpha^+ - \alpha^- < 0$ and $\delta_B := \beta^+ - \beta^- < 0$. Since $|\det D\phi_t(x)| = \exp\int_0^t \mathrm{Tr}(DX(\phi_s x))\,ds$ we set $g(x) = \mathrm{Tr}(DX(x))$ to get $\log|\det D\phi_T(x)| = \int_0^T g(\phi_s x)\,ds$ and both $g(A) = \delta_A$ and $g(B) = \delta_B$ strictly negative. Thus, the left hand side inequality from (47) follows from Theorem 6.2.

For the right hand side inequality, we set $g(x) = \log\|Df(x)\|$ and note that $t \mapsto g(\phi_t x)$ is C^1. Hence, we can write $\int_0^n g(\phi_t x)\,dt = \sum_{i=0}^{n-1}\int_0^1 g(\phi_t f^i x)\,dt$ and $g(\phi_t f^i x) = g(f^i x) + t\cdot\partial_s(g\circ\phi_s f^i x)\,|_{s=s(t)}$ by the Mean Value Theorem for some $s(t)\in(0,t)$. Moreover,

$$\partial_s(g\circ\phi_s f^i x)\,|_{s=s(t)} = \nabla g(\phi_s f^i x)\cdot DX(\phi_s f^i x)X(\phi_s f^i x)$$

is uniformly bounded from above and below, so we can find $\bar{L}$ so that

$$\int_0^1 g(\phi_t f^i x)\,dt \le \int_0^1 \big(g(f^i x) + t\bar{L}\big)\,dt \le g(f^i x) + \frac{1}{2}\bar{L}.$$

This ensures that $(1/n)\sum_{i=0}^{n-1} g(f^i x) \ge (1/n)\int_0^n g(\phi_t x)\,dt - \bar{L}/2n$ and so the right hand inequality of (47) follows again from Theorem 6.2, since for our choice of g we have both $g(A) = \log\|e^{DX(A)}\| = \alpha^+$ and $g(B) = \log\|e^{DX(B)}\| = \beta^+$ strictly positive.

Example 11 (Partially hyperbolic nonuniform sectional expanding with no physical measure). Continuing from the previous example, we consider the compactification $\mathbb{S}^2$ of N with a source at infinity and the direct product $M = \mathbb{S}^2\times\mathbb{S}^1$ with the "North-South flow" on the circle; see right hand side of Figure 8.

We get a flow $(\psi_t : M \to M)_{t\in\mathbb{R}}$ with an attracting set $\mathcal{A} := \mathbb{S}^2\times\{S\}$ so that $d(\psi_t(z),\mathcal{A})\to 0$ when $t\nearrow\infty$ for all $z\in M\setminus\mathcal{A}$, where d is any Riemannian distance on M. If we let the contraction rate at the sink S of the North-South flow to be stronger than the contracting rates of the saddles A, B from $(\phi_t)_{t>0}$, then $\mathcal{A}$ becomes a partially hyperbolic attracting set with splitting $T_{\mathcal{A}}M = E^s\oplus E^c$ given by $E^s = \{0\}\times T_S\mathbb{S}^1$ and $E^c = T\mathbb{S}^2\times\{0\}$.

We note that the region between the saddle connections W_1 and W_4 containing C has a closure F which is invariant and $K := F\times V_S$ becomes also a partially hyperbolic

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forward invariant set, where V_S is any compact positively invariant neighbourhood of the sink S in $\mathbb{S}^1$ with respect to the North-South flow, with the same splitting as above since we have a direct product.

Moreover, because all future trajectories starting in K accumulate $W_1 \cup W_4 \cup \{A, B\}$, from (47) we obtain

$$\limsup_{T\nearrow\infty} \log |\det D\psi_T \,|_{E^c_x}\,|^{1/T} < 0 < \limsup_{n\nearrow\infty} \sum_{i=0}^{n-1} \log \|D\bar{f}\,|_{E^c_{\bar{f}^i x}}\,\|^{1/n} \tag{48}$$

for all $x \in K \setminus \{C\} \times V_S$ where $\bar{f} := \psi_1$. Thus, for an open and full Leb-measure subset of the partially hyperbolic forward invariant set K we have average asymptotic expansion along the central bundle together with asymptotic sectional contraction, and no physical/SRB measure.

Moreover, *we do not have slow recurrence*. Indeed, for any given $\delta, L > 0$ the continuous function $g(x) := \min\{L, -\log d_\delta(x, \{A, B, C\})\}$ is such that $\limsup_{T\nearrow\infty} \frac{1}{T}\int_0^T g(\phi_t x)\, dt = L$ since $g(A) = g(B) = L$ for all x whose future trajectory accumulates W, as a direct consequence of Theorem 6.2. Hence, for these trajectories we arrive at

$$\limsup_{T\nearrow\infty} \frac{1}{T}\int_0^T -\log d_\delta(\phi_t x, \{A, B, C\})\, dt = +\infty$$

for each small $\delta > 0$. Analogously, since $\|G(x)\|$ is comparable to $d_\delta(\phi_t x, \mathrm{Sing}(X))$ (see Lemma 3.4) we obtain the same results replacing the distance to the equilibria with the norm of the vector field.

Remark 6.3. The proof of the existence of a physical measure for asymptotic sectional hyperbolic attractors presented in [44] — in the case when the attractor contains non-Lorenz-like equilibria – is based on the assumption that *the right hand side inequality of (48) on a positive Lebesgue measure subset of points $x \in U$ implies the existence of some physical measure*. From Examples 10 and 11 we see that the proof in [44] is incomplete.

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References

1. J. Alves, C. L. Dias, S. Luzzatto and V. Pinheiro, SRB measures for partially hyperbolic systems whose central direction is weakly expanding, *J. Eur. Math. Soc.* **19**(10) (2017), 2911–2946.
2. J. F. Alves, *Nonuniformly hyperbolic attractors: geometric and probabilistic aspects.*, (Springer International Publishing, 2020).
3. J. F. Alves and V. Araujo, Hyperbolic times: frequency versus integrability, *Ergodic Theory and Dynamical Systems* **24**(2) (2004), 1–18.
4. J. F. Alves, C. Bonatti and M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly expanding, *Invent. Math.* **140**(2) (2000), 351–398.
5. J. F. Alves, S. Luzzatto and V. Pinheiro, Markov structures and decay of correlations for non-uniformly expanding dynamical systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22**(6) (2005), 817–839.
6. J. F. Alves and V. Pinheiro, Gibbs-markov structures and limit laws for partially hyperbolic attractors with mostly expanding central direction, *Advances in Mathematics* **223**(5) (2010), 1706–1730.
7. V. Araújo and E. Trindade, Robust exponential mixing and convergence to equilibrium for singular-hyperbolic attracting sets, *Journal of Dynamics and Differential Equations* **35**(3) (2021), 2487–2536. doi:10.1007/s10884-021-10100-7.
8. V. Araujo, Sinks and sources for C^1 dynamics whose Lyapunov exponents have constant sign, *Kyoto Journal of Mathematics* **57**(4) (2020), 751–788.
9. V. Araujo, Finitely many physical measures for sectional-hyperbolic attracting sets and statistical stability, *Ergodic Theory and Dynamical Systems* **41**(9) (2021), 2706–2733.
10. V. Araujo, On the statistical stability of families of attracting sets and the contracting Lorenz attractor, *Journal of Statistical Physics* **182**(3) (2021), 53–68 doi:10.1007/s10955-021-02729-x.
11. V. Araujo, On the number of ergodic physical/SRB measures of singular-hyperbolic attracting sets, *Journal of Differential Equations* **354** (may) (2023), 373–402. doi:10.1016/j.jde.2023.02.008.
12. V. Araujo and I Melbourne, Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor, *Annales Henri Poincaré* **17**(11) (2016), 2975–3004. doi:10.1007/s00023-016-0482-9.
13. V. Araujo and I Melbourne, Existence and smoothness of the stable foliation for sectional hyperbolic attractors, *Bulletin of the London Mathematical Society* **49**(2) (2017), 351–367.
14. V. Araujo and I Melbourne, Mixing properties and statistical limit theorems for singular hyperbolic flows without a smooth stable foliation, *Advances in Mathematics* **349** (2019), 212–245.

https://doi.org/10.1017/S0013091525101260 Published online by Cambridge University Press

15. V. Araujo, I Melbourne and P. Varandas, Rapid Mixing for the Lorenz Attractor and Statistical Limit Laws for Their Time-1 Maps, *Communications in Mathematical Physics* **340**(3) (2015), 901–938.
16. V. Araujo and M. J. Pacifico, *Three-dimensional flows*. Of *Results in Mathematics and Related Areas. 3rd Series. A Series Of Modern Surveys in Mathematics.*, Volume 53, (Springer, Heidelberg, 2010).
17. V. Araujo, M. J. Pacifico, E. R. Pujals and M. Viana, Singular-hyperbolic attractors are chaotic, *Transactions of the A.M.S.* **361**(5) (2009), 2431–2485. doi:10.1090/S0002-9947-08-04595-9.
18. V. Araujo and L. Salgado, Infinitesimal Lyapunov functions for singular flows, *Mathematische Zeitschrift* **275**(3-4) (2013), 863–897. doi:10.1007/s00209-013-1163-8.
19. V. Araujo, A. Souza and E. Trindade, Upper large deviations bound for singular-hyperbolic attracting sets, *Journal of Dynamics and Differential Equations* **31**(2) (2019), 601–652.
20. A. Arbieto and L. Salgado, On critical orbits and sectional hyperbolicity of the nonwandering set for flows, *Journal of Differential Equations* **250**(6) (2011), 2927–2939.
21. L. Barreira and Y. Pesin, Dynamics of systems with nonzero Lyapunov exponents. Of *Encyclopedia Of Mathematics and its Applications.*, Volume 115, (Cambridge University Press, Cambridge, 2007).
22. L. Barreira and C. Valls, Hölder Grobman-Hartman linearization, *Disc. Cont. Dyn. Syst. A* **18**(1) (2007), 187–197. doi:10.3934/dcds.2007.18.187.
23. C. Bonatti, L. J. Diaz and M. Viana, *Dynamics beyond uniform hyperbolicity*. Of *Encyclopaedia Of Mathematical Sciences.*, Volume 102, (Springer-Verlag, Mathematical Physics, III, 2005). A global geometric and probabilistic perspective.
24. C. Bonatti, A. Pumariño and M. Viana, Lorenz attractors with arbitrary expanding dimension, *C. R. Acad. Sci. Paris Sér. I Math.* **325**(8) (1997), 883–888.
25. C. Bonatti and M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, *Israel J. Math.* **115**(1) (2000), 157–193. doi:10.1007/BF02810585.
26. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Of *Lecture Notes in Mathematics*, Volume 470, (Berlin: Springer Verlag, 1975). doi:10.1007/978-3-540-77695-6.
27. R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, *Invent. Math.* **29**(3) (1975), 181–202. doi:10.1007/bf01389848.
28. H. Bruin and H. H. Canales Farías, Mixing rates of the geometrical neutral lorenz model, *Journal of Statistical Physics* **190**(12) (2023), 1572–9613. doi:10.1007/s10955-023-03212-5.
29. D. Burguet and D. Yang, Srb measures for mostly expanding partially hyperbolic diffeomorphisms via the variational approach, *International Mathematics Research Notices* **2025**(6) (2025), 1–24. doi:10.1093/imrn/rnaf053.
30. Y. Cao, Z. Mi and R. Zou, C^1 pesin (un)stable manifolds without domination, *International Mathematics Research Notices* **2025**(19) (2025), 1687–0247. doi:10.1093/imrn/rnaf298.
31. M. Carvalho, V. Coelho, L. Salgado and P. Varandas, Sensitivity and historic behavior for continuous maps on baire metric spaces, *Ergodic Theory and Dynamical Systems* **44**(1) (2023), 1–30. doi:10.1017/etds.2023.3.
32. E. Catsigeras, M. Cerminara and H. Enrich, The Pesin entropy formula for C^1 diffeomorphisms with dominated splitting, *Ergodic Theory Dynam. Systems* **35**(3) (2015), 737–761.

https://doi.org/10.1017/S0013091525101260 Published online by Cambridge University Press

33. S. Crovisier, X. Wang, D. Yang and J. Zhang, On physical measures of multi-singular hyperbolic vector fields, *Transactions of the AMS* **377**(10) (2024), 6937–6980. doi:10.1090/tran/9161.
34. D. Dolgopyat, On dynamics of mostly contracting diffeomorphisms, *Comm. Math. Phys.* **213**(1) (2000), 181–201.
35. T. Fisher and B. Hasselblatt, *Hyperbolic Flows.*, (European Mathematical Society Publishing House, Zuerich, Switzerland, 2019).
36. N. Gourmelon, Adapted metrics for dominated splittings, *Ergodic Theory Dynam. Systems* **27**(6) (2007), 1839–1849.
37. D. -Q. Jiang, M. Qian and M. -P. Qian, *Lyapunov exponents of hyperbolic attractors*, pp. 189–214, (Springer, Berlin Heidelberg, 2003).
38. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*. Of *Encyclopeadia Appl. Math.*, Volume 54, (Cambridge University Press, Cambridge, 1995). volume.
39. S. Kiriki, X. Li, Y. Nakano and T. Soma, Abundance of Observable Lyapunov Irregular Sets, *Communications in Mathematical Physics* **391**(3) (2022), 1241–1269.
40. F. Ledrappier and L. S. Young, The metric entropy of diffeomorphisms I. Characterization of measures satisfying Pesin's entropy formula, *Annals of Mathematics* **122**(3) (1985), 509–539. doi:10.2307/1971328.
41. R. Leplaideur and D. Yang, SRB measure for higher dimensional singular partially hyperbolic attractors, *Annales de l'Institut Fourier* **67**(2) (2017), 2703–2717.
42. R. Mañé, *Ergodic Theory and Differentiable dynamics.*, (Springer Verlag, New York, 1987).
43. B. S. Martín and K. Vivas, The Rovella attractor is asymptotically sectional-hyperbolic, *NonLinearity* **33**(6) (2020), 3036–3049.
44. B. S. Martín and K. Vivas, Physical measures for ASH attractors, *Journal of Differential Equations* **367**(sep) (2023), 366–381. doi:10.1016/j.jde.2023.05.003.
45. R. J. Metzger, Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17**(2) (2000), 247–276.
46. R. J. Metzger, Stochastic stability for contracting Lorenz maps and flows, *Comm. Math. Phys.* **212**(2) (2000), 277–296.
47. Z. Mi, Y. Cao and D. Yang, SRB measures for attractors with continuous invariant splittings, *Mathematische Zeitschrift* **288**(1-2) (2017), 135–165. doi:10.1007/s00209-017-1883-2.
48. C. A. Morales, M. J. Pacifico and E. R. Pujals, Singular hyperbolic systems, *Proc. Amer. Math. Soc.* **127**(11) (1999), 3393–3401.
49. C. A. Morales, M. J. Pacifico and E. R. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, *Ann. of Math* **160**(2) (2004), 375–432.
50. W. Ott and J. A. Yorke, When Lyapunov exponents fail to exist, *Physical Review E* **78**(5) (2008), 056203. doi:10.1103/physreve.78.056203.
51. J. Palis and W. de Melo, *Geometric theory of dynamical systems.*, (Springer Verlag, 1982).
52. Y. B. Pesin, Characteristic Lyapunov exponents and Smooth Ergodic Theory, *Russian Math. Surveys* **324**(4) (1977), 55–114. doi:10.1070/rm1977v032n04abeh001639.
53. Y. Pesin and Y. Sinai, Gibbs measures for partially hyperbolic attractors, *Ergod. Th. & Dynam. Sys.* **2**(3-4) (1982), 417–438. doi:10.1017/s014338570000170x.
54. W. Philipp and W. Stout, *Almost sure invariance principles for partial sums of weakly dependent random variables*. Memoirs of the AMS., (AMS, 1975).
55. V. Pinheiro, Sinai-Ruelle-Bowen measures for weakly expanding maps, *Nonlinearity* **19**(5) (2006), 1185–1200.

https://doi.org/10.1017/S0013091525101260 Published online by Cambridge University Press

56. V. Pinheiro, Expanding measures. *Annales de l Institut Henri Poincaré, Analyse non Linéaire*, **28**(dec) (2011), 899–939. doi:10.1016/j.anihpc.2011.07.001.
57. Y. Pomeau and P. Manneville, Intermittent Transition to Turbulence in Dissipative Dynamical Systems, *Communications in Mathematical Physics* **74**(2) (1980), 189–197.
58. C. Pugh and M. Shub, Ergodic elements of ergodic actions, *Compositio Math.* **23**(1) (1971), 115–122.
59. C. Pugh, M. Shub and A. Wilkinson, Hölder foliations, *Duke Math. J.* **86**(1) (1997), 517–546. doi:10.1215/S0012-7094-97-08616-6.
60. A. Rovella, The dynamics of perturbations of the contracting Lorenz attractor, *Bull. Braz. Math. Soc.* **24**(2) (1993), 233–259.
61. Y. Sinai, Gibbs measures in ergodic theory, *Russian Math. Surveys* **27**(4) (1972), 21–69. doi:10.1070/RM1972v027n04ABEH001383.
62. A. Tahzibi, Stably ergodic diffeomorphisms which are not partially hyperbolic, *Israel Journal of Mathematics* **142**(1) (2004), 315–344. doi:10.1007/BF02771539.
63. F. Takens, Heteroclinic attractors: time averages and moduli of topological conjugacy, *Bull. Braz. Math. Soc.* **25**(1) (1994), 107–120. doi:10.1007/bf01232938.
64. M. Viana and K. Oliveira, *Foundations of ergodic theory*. Of *Cambridge Studies in Advanced Mathematics.*, Volume 151, (Cambridge University Press, Cambridge, 2016).

https://doi.org/10.1017/S0013091525101260 Published online by Cambridge University Press