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A dichotomy for a two-parameter piecewise rotation

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Abstract. We investigate a basic system T of two rotations with a common angle of rotation. Using a new idea, we prove that, except for an invertible T, the system T must have a global attractor and a global repeller. We give a specific condition for both cases.

1. *Introduction*

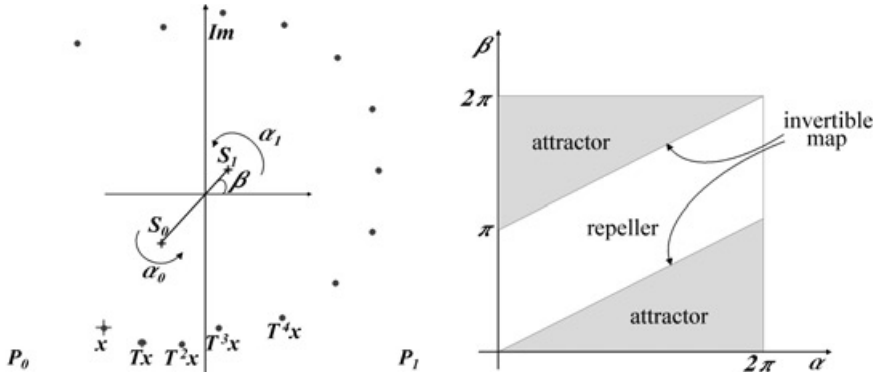
In this paper we study a very basic system of Euclidean rotations. We identify when a two-parameter piecewise rotation attracts and when it repels. This classification strengthens the theory about piecewise rotations with two atoms [**13, 14, 16**].

Piecewise rotations are examples of Euclidean piecewise isometries. These systems generalize well-known and well-studied interval exchanges to a class of piecewise isometries [**8, 15**]. Piecewise isometries appear in a variety of contexts and have been recently extensively studied as interval exchanges (see references in [**15**]), interval translations [**7, 9, 26**], rectangular exchanges [**19**], polygonal and polyhedron exchanges [**17**] and invertible two-dimensional piecewise rotations [**1, 3, 4, 17, 19, 20, 24**]. Piecewise isometric maps appear naturally in dual billiards [**18**], the theory of foliations, tilings and Hamiltonian systems. Piecewise rotations also appear in studies of the dynamics of electronic components known as digital filters (see, for example, [**1, 2, 5, 10, 11, 23**]).

The structure of the paper is as follows. In the next section, we set up the system, define attractors and repellers and briefly mention some known partial results about a piecewise rotation with two atoms. In §3, we state and briefly discuss the main result.

We conclude the article with various remarks. In particular (Remark 5.1), interval translation maps $g(\theta)$ can be associated with the considered transformations T and the analysis used in the paper applies to a more general setting when g is uniquely ergodic and of finite type [**7**].

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(a) (b)

FIGURE 1. (a) The action of the piecewise rotation T. (b) The parameter space in Theorem 1.

2. *Preliminaries*

We fix the angles of rotation $\alpha_0, \alpha_1 \in (0, 2\pi)$. Let T_0 be the rotation by α_0 about some point S_0 and let T_1 be the rotation by α_1 about another point S_1 (Figure 1).

Definition 1. (Piecewise rotation) We define a family $\mathcal{T}$ of piecewise rotations $T : \mathbb{C} \to \mathbb{C}$ such that

$$Tx = \begin{cases} T_0 x & \text{if } x \in P_0 = \{x : \operatorname{Re} x < 0\} \\ T_1 x & \text{if } x \in P_1 = \{x : \operatorname{Re} x \geq 0\}. \end{cases} \tag{1}$$

Let $\beta = \arg(S_1 - S_0) \in [0, 2\pi)$. The first basic case of a piecewise rotation for which $\beta = 0$, $S_0 \in P_0$ and $S_1 \in P_1$, was studied in [**13**]. The main result is that all orbits accumulate in a bounded set which is the union of two discs: $B(S_0, |S_0|) \cup B(S_1, |S_1|)$. However, if $\max(\alpha_0, \alpha_1) < \beta$, Goetz [**14**] showed that all orbits initiated outside the two fixed discs, $B(S_0, |S_0|)$ and $B(S_1, |S_1|)$, diverge to infinity.

These two results motivated us to investigate the intermediate cases and to identify when all orbits are bounded and when orbits diverge to infinity. These intermediate cases and, in particular, the structure of the attractors appear to be extremely complex [**12**]. For instance, in [**14**], an example of the piecewise rotation T was constructed for which the accumulation set of all orbits contained an infinite number of disjoint discs located in a bounded region around the origin (this construction was achieved when β is close to zero). In [**16**] an example of an attracting system consisting of a Sierpiński gasket was constructed. Finally, [**22**] constructed an example of the piecewise rotation T, for which the attractor consists of a convex hull of the two maximal polygons that are fixed by T.

In this article, we succeed in classifying a subclass of piecewise rotations for which the angles of rotation are the same. However, before we state the main result, we need the following two definitions.

Definition 2. (Globally attracting map) A map T is *globally attracting* if there is a number $M > 0$ such that for all $x \in \mathbb{C}$,

$$\limsup_{n \to \infty} |T^n x| < M.$$

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Definition 3. (Globally repelling map) A map T is *globally repelling* if there is a number $M > 0$ such that for all $x \in \mathbb{C}$, $|x| > M$,

$$\lim_{n\to\infty} |T^n x| = \infty.$$

3. *Main results*

THEOREM 1. *Let $T \in \mathcal{T}$ be a piecewise rotation with a common angle of rotation $\alpha_0 = \alpha_1 = \alpha \in (0, 2\pi) - \{\pi\}$. Let $\beta = \arg(S_1 - S_0)$. Then the following hold.*

(1) *If $\sin(\beta - \frac{1}{2}\alpha) < 0$ (Figure 1(b)), then the piecewise rotation T is globally attracting.*

(2) *If $\sin(\beta - \frac{1}{2}\alpha) > 0$, then T is globally repelling. Moreover, if α is not commensurate with π, all orbits are repelled with the linear rate:*

$$\lim_{n\to\infty} \frac{|T^n x|}{n} = 4(|S_1 - S_0|) \sin\frac{1}{2}\alpha \sin\left(\beta - \frac{1}{2}\alpha\right). \tag{2}$$

It is remarkable to note that the remaining case, $\sin(\beta - \frac{1}{2}\alpha) = 0$, is precisely the condition for T to be invertible. The reader may check this by noting that T is invertible if and only if $T_0 I = T_1 I$, where I is the imaginary line. The invertible case appears to be very complex. We briefly comment on it in the last section.

Here we point out that the following measure-theoretic argument shows that if T is invertible, then T cannot be globally attracting. Otherwise, for some $M > 0$ we have the pointwise convergence of the sets

$$B_m = \{x : |x| < 2M, |T^n x| < M \text{ for } n \geq m\} \xrightarrow{m\to\infty} \{x : |x| < 2M\},$$

which implies that their Lebesgue measure converges:

$$\mu(B_m) \xrightarrow{m\to\infty} 4\pi M^2.$$

However, since T preserves the Lebesgue measure, for all m, $\mu(B_m) = \mu(T^m B_m) \leq \mu(\{x : |x| < M\}) = \pi M^2$, a contradiction.

4. *Proofs*

Proof of Theorem 1. The proof consist of two parts. In the first part we show that the result holds for α not commensurate with π. The remaining case when α is a rational multiple of π uses a different argument and it is given later in the second part. □

4.1. *Part 1: The angle α is not commensurate with π.* First, we consider the case when the centers $S_0 \in P_0$ and $S_1 \in P_1$ (i.e. when S_0 and S_1 are fixed by T). Without loss of generality we may assume that $0 \in \overline{S_0 S_1}$. Thus $S_0 = -|S_0|e^{\beta i}$, $S_1 = |S_1|e^{\beta i}$.

The proof is based on a series of lemmas. Lemma 1 is an application of the uniform distribution of the irrational rotation on the circle.

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LEMMA 1. *Let $f(\theta)$ be a 2π-periodic real function which is Riemann integrable on $[0, 2\pi]$, and let α be an angle non-commensurate with π. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(\theta_0 + k\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \tag{3}$$

uniformly in $\theta_0 \in \mathbb{R}$.

The proof of Lemma 1 is routine (see, for example, [**21**, Ch. 1, Corollary 1.1]).

LEMMA 2. *Let $x = \mathrm{Re}^{i\theta}$, $\theta \in [\frac{1}{2}\pi, \frac{3}{2}\pi)$. Then*

$$\lim_{R\to\infty} (|Tx| - |x| - f(\theta)) = 0, \tag{4}$$

where

$$f(\theta) = \begin{cases} 2|S_0| \sin(\beta - \frac{1}{2}\alpha - \theta) \sin\frac{1}{2}\alpha & \text{if } \theta \in [\frac{1}{2}\pi, \frac{3}{2}\pi) \\ 2|S_1| \sin(\beta - \frac{1}{2}\alpha - \theta) \sin\frac{1}{2}\alpha & \text{if } \theta \in [-\frac{1}{2}\pi, \frac{3}{2}\pi). \end{cases}$$

Proof of Lemma 2. We check only the case when $x \in P_0$ (i.e. $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$). The case $x \in P_1$ is similar. We need the following computation.

$$\begin{aligned} |Tx| &= |e^{i\alpha}(x - S_0) + S_0| = |e^{i\alpha}\mathrm{Re}^{i\theta} + S_0(1 - e^{i\alpha})| = |R + S_0(1 - e^{i\alpha})e^{-i(\alpha+\theta)}| \\ &= \left|R - |S_0|e^{\beta i} e^{i\alpha/2}(e^{-i\alpha/2} - e^{i\alpha/2})e^{-i(\alpha+\theta)}\right| = \left|R + |S_0|e^{(\beta-\alpha/2-\theta)i} 2i \sin\tfrac{1}{2}\alpha\right|. \end{aligned} \tag{5}$$

Next we use the observation that for a fixed $z \in \mathbb{C}$,

$$\lim_{R\to\infty} (|R + z| - R - \mathrm{Re}(z)) = 0. \tag{6}$$

Substitute $z = |S_0|e^{(\beta-\alpha/2-\theta)i} 2i \sin\frac{1}{2}\alpha$ in (6). Then (4) follows from (5). The details are straightforward and left to the reader. □

LEMMA 3. *Let $f(\theta)$ be defined as in Lemma 2. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = 4(|S_0| + |S_1|) \sin\frac{1}{2}\alpha \sin\left(\beta - \frac{1}{2}\alpha\right).$$

The proof of the lemma is straightforward and is left to the reader.

Assume that either one of the hypotheses of Theorem 1 is satisfied, i.e. $\sin(\beta - \frac{1}{2}\alpha) \neq 0$. Denote

$$I = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

Then $I \neq 0$.

Fix $\epsilon > 0$ and such that $\epsilon < |I|/2$. Then, by Lemma 1, there exists an N_0 such that

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(\theta_0 + k\alpha) - I \right| < \frac{\epsilon}{4} \tag{7}$$

for all $\theta_0 \in \mathbb{R}$ and all $N \geq N_0$. Fix $N > N_0$ such that $\sup|f|/N < \frac{1}{4}\epsilon$.

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Consider a partial T-orbit of x, $\{x, Tx, \dots, T^{N-1}x\}$. Denote $\theta_k = \arg T^k x$ for $0 \le k \le N-1$. By Lemma 2,

$$\lim_{|x|\to\infty} (|T^{k+1}x| - |T^k x| - f(\theta_k)) = 0 \quad \text{for all } 0 \le k \le N-1, \tag{8}$$

because $\lim_{|x|\to\infty} |T^k x| = \infty$ for any $0 \le k \le N-1$.

Adding (8) for $k = 0, \dots, N-1$, we obtain

$$\lim_{|x|\to\infty} \left(|T^N x| - |x| - \sum_{k=0}^{N} f(\theta_k) \right) = 0. \tag{9}$$

Since $\lim_{|x|\to\infty} e^{(\theta_{k+1}-\theta_k)i} = e^{\alpha i}$ for $0 \le k \le N-1$,

$$\lim_{|x|\to\infty} e^{(\theta_k-\theta_0)i} = e^{\alpha k i}. \tag{10}$$

The function $f(\theta)$ is Lipshitz on each of the intervals $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and $(\frac{1}{2}\pi, \frac{3}{2}\pi)$, with constant $c = 2\max(|S_0|, |S_1|) \sin \frac{1}{2}\alpha$. Therefore, for sufficiently large $|x|$,

$$\begin{aligned} \left| \sum_{k=0}^{N-1} f(\theta_k) - \sum_{k=0}^{N-1} f(\theta_0 + k\alpha) \right| &\le \sum_{k=0}^{N-1} |f(\theta_k) - f(\theta_0 + k\alpha)| \\ &\le \sum_{k=0}^{N-1} c|e^{\theta_k i} - e^{(\theta_0+k\alpha)i}| + 2\sup|f| \end{aligned} \tag{11}$$

because, since α is non-commensurate with π, at most one of the numbers $\theta_0 + k\alpha$, $0 \le k \le N-1$ may turn out to be close to either $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ modulo 2π. For all other numbers $\theta_0 + k\alpha$, the Lipshitz condition applies.

In view of (10) and by the choice of N,

$$\limsup_{|x|\to\infty} \left| \frac{1}{N} \sum_{k=0}^{N-1} f(\theta_k) - \frac{1}{N} \sum_{k=0}^{N-1} f(\theta_0 + k\alpha) \right| \le \frac{2\sup|f(\theta)|}{N} < \frac{\epsilon}{2}.$$

It follows from (7) that for sufficiently large $|x|$,

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(\theta_k) - I \right| < \frac{3}{4}\epsilon.$$

By (9) and since $\epsilon < |I|/2$, we obtain

$$\left| \frac{|T^N x| - |x|}{N} - I \right| < \epsilon < \frac{|I|}{2}, \tag{12}$$

whence

$$|T^N x| > |x| + \frac{N|I|}{2} \quad \text{if } I > 0, \tag{13}$$

and

$$|T^N x| < |x| - \frac{N|I|}{2} \quad \text{if } I < 0 \tag{14}$$

for x sufficiently large, $|x| > M$ for some $M > 0$.

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In the case $I < 0$, (14) implies that $S = T^N$ (hence also T itself) is globally attracting, providing the first conclusion of Theorem 1. This is because every S-orbit eventually enters the ball $B = B(0, M)$ and then stays in the bounded set $B \cup S(B)$.

In the case $I > 0$, (13) implies that T is globally repelling.

In order to compute the repelling rate (2), for $|x| > M$, from (13), we have

$$M < |x| < |T^N x| < |T^{2N} x| < \cdots,$$

and hence by (12), for $k > 0$,

$$\left| \frac{|T^{(k+1)N} x| - |T^{kN} x|}{N} - I \right| < \epsilon. \tag{15}$$

Summing (15) for $k = 0, \ldots, K-1$, we obtain

$$\left| \frac{|T^{KN} x| - |x|}{N} - KI \right| < K\epsilon \tag{16}$$

for all $K \geq 1$. Thus

$$\limsup_{K \to \infty} \left| \frac{|T^{KN} x| - |x|}{KN} - I \right| \leq \epsilon.$$

Since $\lim_{K \to \infty} (|T^{K+1} x| / |T^K x|) = 1$, we conclude that

$$\limsup_{K \to \infty} \left| \frac{|T^K x| - |x|}{K} - I \right| \leq \epsilon.$$

Finally, since the choice of $\epsilon > 0$ is arbitrary ($< \frac{1}{2} I$), conclusion (2) follows. This completes the proof of Theorem 1 in the case when the centers $S_0 \in P_0$ and $S_1 \in P_1$.

For the remaining cases when $S_0 \in P_1$ and $S_1 \in P_0$, or if S_0, S_1 belong to one of the half-planes, the proof is analogous. (If $\overline{S_0 S_1}$ is not parallel to the imaginary axis, then as in the proof, assume without loss of generality that the line $S_0 S_1$ passes through the origin. The case when $\overline{S_0 S_1}$ is parallel to the imaginary axis can be reduced to the previous case by an approximation argument or by a direct computation.)

This concludes the proof of Theorem 1 in the case when the rotation angle α is non-commensurate with π.

4.2. *Part 2: The angle α is commensurate with π.* We now prove Theorem 1 when the angle of rotation α satisfies

$$\alpha = \frac{k}{s} \pi,$$

where k are co-prime and $s > 1$. The key idea of the proof is the computation of the directions of the translation vectors for a power of T (Lemma 4). Partition the plane into $2s$ cones as follows:

$$C_j = \left\{ z : \frac{\pi}{2} + j\frac{\pi}{s} < \arg z \leq \frac{\pi}{2} + (j+1)\frac{\pi}{s} \right\} \quad \text{for } j = 0, \ldots, 2s-1.$$

For each j, define a cone $E_j \subset C_j$ (Figure 2) to be the translation of C_j along the symmetry line of C_j such that $\operatorname{dist}(\partial C_j, E_j) = d$, where $d = 4s \max(|S_0|, |S_1|)$. First, we show the following.

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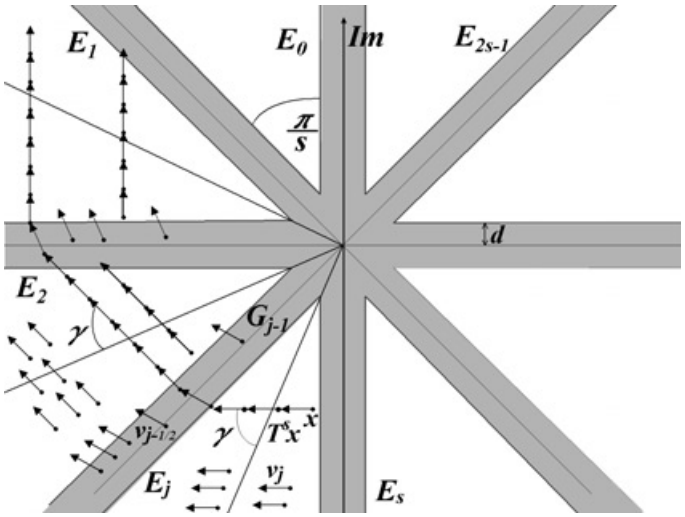


FIGURE 2. An illustration of the proof when α is a rational multiple of π. The angle $\gamma = \frac{1}{2}\pi + \frac{1}{2}\alpha - \beta$. If $\gamma < \frac{1}{2}\pi$, then orbits starting sufficiently far away will diverge to infinity.

LEMMA 4. (Directions of the translation vectors)

(a) *The action of the map $V = T^{2s}$ restricted to the cone E_j is a translation by*

$$v_j = ve^{i(\beta-\alpha/2)}e^{i\pi j/s}e^{i\pi/(2s)}, \quad \textit{for some } v > 0. \tag{17}$$

(b) *Let G_{j+1} be the d-neighborhood of the line $C_j \cap C_{j+1}$. The action of the map $V = T^{2s}$ restricted to G_{j+1} outside a bounded set is a translation by either v_j, v_{j+1} or by*

$$v_{j+1/2} = v_j e^{i\pi/(2s)}.$$

Proof of Lemma 4. We first prove part (a). In order to show that in each cone E_j, the map $V = T^{2s}$ is a translation by v, we observe that the first $(2s-1)$ iterates of E_j are contained entirely in either P_0 or P_1. From the definition of T, note that $|Tx - e^{i\alpha}x| < 2\max(|S_0|, |S_1|)$. Hence, for $l = 0, \ldots, 2s-1$, $|T^l x - e^{i\alpha l}x| < 4s\max(|S_0|, |S_1|)$. It follows that if $x \in E_j$,

$$T^p x \in C_{j+pk} \quad \text{where } p = 0, \ldots, 2s-1, \tag{18}$$

and the addition in the index of C_{j+pk} is taken modulo $2s$. This means that the first $(2s-1)$ iterates of E_j are contained entirely in either P_0 or P_1, and thus $T^{2n}|_{E_j}$ is a translation by a vector, which we will now compute.

Define

$$\eta[t] = \begin{cases} 0 & \text{if } j < s \mod 2s \\ 1 & \text{if } j \geq s \mod 2s. \end{cases}$$

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We assume that $x \in E_j$. We now compute the translation vector v_j explicitly

$$T^{2s}x = T_{\eta[j+(2s-1)k]} \dots T_{\eta[j+k]} T_{\eta[j]} x = x + (1 - e^{i\alpha}) \sum_{p=0}^{2s-1} S_{\eta[j+pk]} e^{i\alpha(2s-p-1)}$$

$$= x + (1 - e^{i\pi k/s}) \sum_{p=0}^{2s-1} S_{\eta[j+pk]} e^{-i\pi(kp+k)/s}$$

$$= x + (1 - e^{i\pi k/s}) \sum_{t=0}^{2s-1} S_{\eta[t]} e^{-i\pi(t-j+k)/s}$$

(Since $\{j + pk \bmod 2s \mid p = 0, \dots, 2s-1\} = \{0, \dots, 2s-1\}$
substitute $t = j + pk$)

$$= x + (1 - e^{i\pi k/s}) \left[S_0 \sum_{t=0}^{s-1} e^{-i\pi(t-j+k)/s} + S_1 \sum_{t=s}^{2s-1} e^{-i\pi(t-j+k)/s} \right]$$

$$= x + (1 - e^{i\pi k/s})(S_0 - S_1) \sum_{t=0}^{s-1} e^{-i\pi(t-j+k)/s}$$

$$= x + (1 - e^{i\pi k/s})(S_0 - S_1) \frac{2}{1 - e^{-i\pi/s}} e^{i\pi(j-k)/s} \quad \text{(a geometric summation)}$$

$$= x + e^{i\pi(k/2s)}(e^{-i\pi(k/2s)} - e^{i\pi(k/2s)})(S_0 - S_1) \frac{2e^{i\pi/2s}}{e^{i\pi/2s} - e^{-i\pi/2s}} e^{i\pi(j-k)/s}$$

$$= x + 2\frac{-2i \sin(k/2s)}{2i \sin(1/2s)}(S_0 - S_1) e^{i\pi(2j-k+1)/2s}$$

$$= x + 2\frac{\sin(k/2s)}{\sin(\pi/2s)}(|S_1 - S_0|) e^{i\beta} e^{i\pi(2j-k+1)/2s}$$

$$= x + 2\frac{\sin(\alpha/2)}{\sin(\pi/2s)}(|S_1 - S_0|) e^{i(\beta - \frac{1}{2}\alpha)} e^{i\pi j/2s} e^{i\pi(1/2s)}.$$

This concludes the proof of part (a).

In order to compute possible translation vectors in the d-neighborhood G_j, we write down the possible patterns of visits to P_0 and P_1. Let

$$w_0, w_1, \dots, w_{2s-1} \tag{19}$$

denote the coding pattern of visits to P_0 and P_1 by E_j (i.e. $T^l E_j \in P_{w_l}$). Then the coding of E_{j-1} differs from the coding E_j by two entries at slots p and q, where $p < q$, and $j + pk = s \bmod 2s$ and $j + qk = 0 \bmod 2s$ or $j + qk = s \bmod 2s$ and $j + pk = 0 \bmod 2s$; i.e. the coding of E_{j-1} is

$$w_0, w_1, \dots, (1 - w_p), \dots, (1 - w_q), \dots, w_{2s-1}. \tag{20}$$

The strip E_{j-1} may follow one of the codings (19) or (20). Otherwise it must follow one of the following codings:

$$w_0, w_1, \dots, w_p, \dots, (1 - w_q), \dots, w_{2s-1}, \tag{21}$$

or

$$w_0, w_1, \dots, (1 - w_p), \dots, w_q, \dots, w_{2s-1}. \tag{22}$$

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A similar calculation of the translation vector to the one just given yields that if x follows either of the two codings (21) or (22), then

$$T^{2s}x = x + v_j e^{i\pi/(2s)}.$$

The summation in square brackets in the fourth line of the calculation,

$$S_0 \sum_{t=0}^{s-1} e^{-i\pi(t-j+k)/s} + S_1 \sum_{t=s}^{2s-1} e^{-i\pi(t-j+k)/s},$$

needs to be replaced with

$$S_0 \sum_{t=0}^{s-1} e^{-i\pi(t-j+k)/s} + S_1 \sum_{t=s}^{2s-1} e^{-i\pi(t-j+k)/s} + (S_1 - S_0) e^{-i\pi(-j+k)/s} + (S_0 - S_1) e^{-i\pi(s-j+k)/s}.$$

We leave the details of this calculation to the reader. □

We now show that the conclusion of Theorem 1 in the rational cases follows from Lemma 4.

First, check that if

$$\frac{3\pi}{2s} < \beta - \frac{\alpha}{2} < \pi - \frac{3\pi}{2s} \bmod 2\pi,$$

then by Lemma 4,

$$|Vx| - |x| > 0$$

for all x sufficiently far from the origin. (This holds true for x in the cones E_j or in the strips G_j.) This implies that orbits starting sufficiently far away diverge to infinity and Theorem 1 holds in this case.

Then, we check that if

$$\pi + \frac{3\pi}{2s} < \beta - \frac{\alpha}{2} < 2\pi - \frac{3\pi}{2s} \bmod 2\pi,$$

then by Lemma 4,

$$|Vx| - |x| < 0$$

for x sufficiently far from the origin. This implies that all orbits must eventually enter a disc $B = B(0, M)$ for some $M > 0$, and then stay in the bounded set $B \cup S(M)$. Hence, Theorem 1 holds in this case.

Now, suppose that

$$0 < \beta - \frac{\alpha}{2} \leq \frac{3\pi}{2s}.$$

We check that orbits entering the strips G_j spend only a bounded amount of time there. (If $s > 2$, then the vector $x - Vx$ is never parallel to the strip. The case when $s = 2$ needs an additional simple argument and this is left to the reader.)

There is some number $\tau > 0$ such that outside of a fixed bounded set and in the complement of $\bigcup_{j=1}^{s} E_j$ (shaded set in Figure 2), every V-orbit ($V = T^{2s}$) spends, at most, time τ before leaving this set for the cone E_j. It follows that there is a number $a > 0$ such that an orbit passing through the strip G_j may not get closer to the origin than a units.

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Since an orbit passing through the strip G_j must then visit the cone E_j, this orbit after visiting E_j and G_j, is repelled from the origin by at least one unit. It finally follows that every orbit that starts sufficiently far away will diverge to infinity. The remaining cases are analogous.

This concludes the proof of Theorem 1 in the rational case. □

Remark. From Lemma 4, it follows that if the system is repelling, $\lim_{n\to\infty}(|T^n x|)/n$ does not exist (unlike the case when α is non-commensurate with π, where we obtained the value of this limit). However, it is possible to estimate $\liminf_{n\to\infty}(|T^n x|)/n$ and $\limsup_{n\to\infty}(|T^n x|)/n$.

5. *Concluding remarks*

5.1. *The main idea also applies to cases with distinct angles of rotation.* In this section, we briefly explain that the main idea from the proof of Theorem 1 can be applied to a much broader class of piecewise rotations for which $\alpha_0 \neq \alpha_1$.

Recall that Part 1 of the proof is based on Lemma 3 which holds because the map $g : [-\frac{1}{2}\pi, \frac{3}{2}\pi) \to [-\frac{1}{2}\pi, \frac{3}{2}\pi)$, $g(\theta) = \theta + \alpha \bmod 2\pi$ is uniquely ergodic on $[-\frac{1}{2}\pi, \frac{3}{2}\pi)$. In the proof of Theorem 1 the map g approximates the projection of the T-orbit onto the unit circle. Suppose now that $\alpha_0 \neq \alpha_1$. Then, in order to approximate the projection of T-orbits onto the unit circle, we will need to study the following piecewise translation:

$$g(\theta) = \begin{cases} \theta + \alpha_0 & \text{if } \theta \in [\frac{1}{2}\pi, \frac{3}{2}\pi) \\ \theta + \alpha_1 & \text{if } \theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi). \end{cases}$$

The piecewise translation g is not uniquely ergodic on $[-\frac{1}{2}\pi, \frac{3}{2}\pi)$ since for all choices of $\alpha_0 \neq \alpha_1$, g is non-invertible. However, if g is uniquely ergodic on its invariant set $\Omega = \bigcap_{n\geq 0} g^n([-\frac{1}{2}\pi, \frac{2}{3}\pi))$, and if g is of *finite type* in the sense of [**7**] (i.e. if for some $k \geq 0$, $\Omega = \bigcap_{n=0}^{n=k} g^n([-\frac{1}{2}\pi, \frac{3}{2}\pi))$, then our approach is applicable.

Here we shall not include the details. However, we briefly describe cases for which g is known to be of finite type and uniquely ergodic on its invariant set.

Case 1. $-\pi < \alpha_0 < 0 < \alpha_1 < \pi$, or $-\pi < \alpha_1 < 0 < \alpha_0 < \pi$, and (α_0, α_1) are rationally independent. In this case, it is not hard to show that Ω is an interval, g is of finite type and $g|_\Omega$ is an interval exchange with two intervals, hence $g|_\Omega$ is conjugate to an irrational rotation.

Case 2. The rational rank of $(\alpha_0, \alpha_1, \pi)$ is two. From [**7**] it follows that, in this case, g is of finite type, hence its restriction to the invariant set is an interval exchange of rational rank two. From [**6**], it follows that g is uniquely ergodic on its invariant set. This case covers the following: $\alpha_0/\pi \in \mathbb{Q}$ and $\alpha_1/\pi \notin \mathbb{Q}$, $\alpha_1/\pi \in \mathbb{Q}$ and $\alpha_0/\pi \notin \mathbb{Q}$ or $\alpha_1/\alpha_0 \in \mathbb{Q}$ together with either $\alpha_0/\pi \notin \mathbb{Q}$ or $\alpha_1/\pi \notin \mathbb{Q}$.

5.2. *The invertible case.* The case when T is invertible is very intriguing and, as seen from the computer graphics, also very complicated (Figure 3). Using techniques from this paper, one can show that the orbits may not escape to infinity at a rate greater than $n^{1/2}$. However, computer-generated images suggest a stronger statement.

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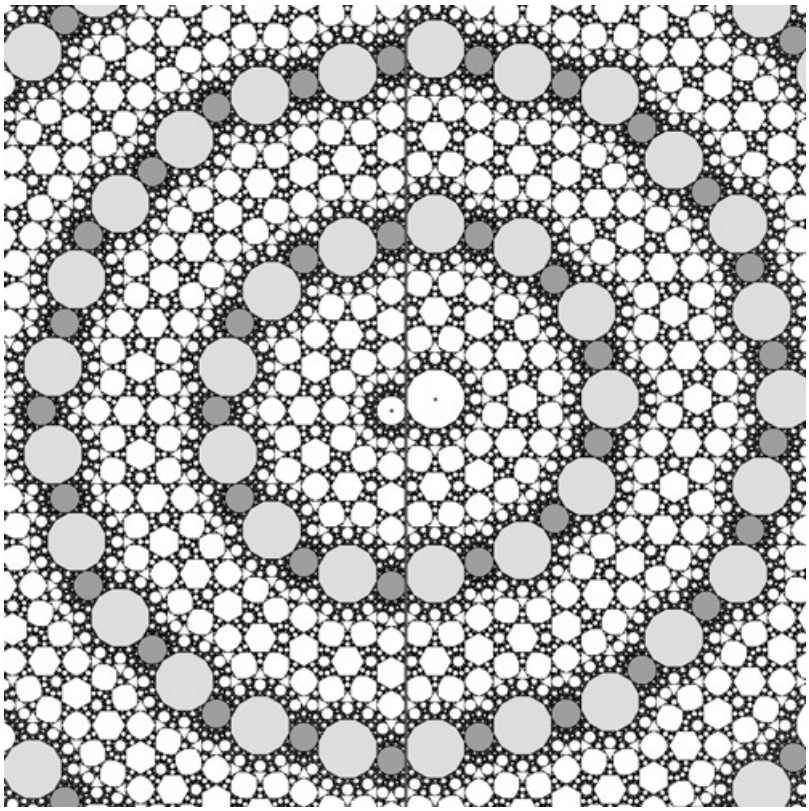


FIGURE 3. An invertible case. White and shaded regions denote sets that follow the same pattern of visits to the half-planes. The center of rotations were chosen so that S_1 is twice as far from the imaginary axis as S_0. The angle $\alpha = \frac{1}{6}\pi$ is a rational multiple of π. The shaded rings of polygons separate invariant regions in which all orbits are bounded.

If the angle of rotation α is a rational multiple of π, all orbits appear to be bounded between invariant rings of polygons. This situation is similar to the behavior of orbits in polygonal quasi-rational dual billiards [**25**], for which it is known that all orbits are bounded.

Finally, if the angle α is an irrational multiple of π, then we conjecture that there are points x such that $\liminf_{n\to\infty} |T^n x|$ is finite but $\limsup_{n\to\infty} |T^n x|$ is infinite (i.e. unbounded orbits that do not approach infinity).

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