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Equidistribution in the complex plane and self-similar measures

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We establish the pointwise equidistribution of self-similar measures in the complex plane. Let $\beta \in \mathbb{Z}[\mathrm{i}]$, whose complex conjugate $\overline{\beta}$ is not a divisor of β, and $T \subset \mathbb{Z}[\mathrm{i}]$ a finite subset. Let μ be a non-atomic self-similar measure with respect to the IFS $\{f_t(z) = \frac{z+t}{\beta} : t \in T\}$. For $\alpha \in \mathbb{Z}[\mathrm{i}]$, if α and β are relatively prime, then we show that the sequence $(\alpha^n z)_{n\geq 1}$ is equidistributed modulo one for μ-almost everywhere $z \in \mathbb{C}$. We also discuss normality of radix expansions in Gaussian integer base, and obtain pointwise normality. Our results generalize partially the classical results in the real line to the complex plane.

Keywords: equidistributed modulo one; radix expansions; self-similar measures; normal number; p-adic interpolation

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1. Introduction

1.1. Equidistribution of self-similar measures

For a real number $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. Let $\#A$ denote the cardinality of set A.

DEFINITION 1.1. *A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be* equidistributed modulo one *or* uniformly distributed modulo one *if for any interval $[u, v) \subset [0, 1)$ we have*

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \{x_n\} \in [u, v)\}}{N} = v - u.$$

The equidistribution is closely related to normality. For an integer $b \geq 2$, a real number x is called *b-normal* or *normal in base b* if in the b-ary expansion of $\{x\}$, each possible block with length ℓ occurs with the same asymptotic frequency $b^{-\ell}$. Due to Wall [39], a real number x is b-normal if and only if the sequence $(b^n x)_{n\geq 1}$ is equidistributed modulo one.

In 1909, Borel [11] showed that Lebesgue almost every $x \in \mathbb{R}$ is *absolutely normal*, i.e., normal in all integer bases $b \geq 2$. However, for some specific numbers (such as π, e), it's quiet difficult to say something about normality. In the late 1950s, Cassels [15] and Schmidt [36] proved independently that in the sense of Cantor-Lebesgue measure, almost every x is b-normal whenever $\log b/\log 3 \notin \mathbb{Q}$. Since then, a considerable number of studies have concentrated on investigating normality of typical elements with respect to appropriate measures, particularly fractal measures. For convenience, for an integer $b \geq 2$, we call a Borel probability measure μ on $\mathbb{R}$ *pointwise b-normal* if μ-almost every x is b-normal.

Let $\Phi = \{f_1, f_2, \ldots, f_n\}$ be a finite collection of contractive similarity maps on $\mathbb{R}^d$, which is called a *self-similar iterated function system* (IFS). By Hutchinson [26], there exists a unique nonempty compact subset $K \subset \mathbb{R}^d$ satisfying

$$K = \bigcup_{j=1}^{n} f_j(K).$$

Moreover, given a probability vector $(p_1, p_2, \ldots, p_n)$, there exists a unique Borel probability measure μ satisfying the equation

$$\mu = \sum_{j=1}^{n} p_j \cdot f_j\mu,$$

where $f\mu$ denotes the image measure of μ under the map f. The set K and the measure μ are called *self-similar set and measure*, respectively. The self-similar measure μ is supported on the self-similar set K. Feldman and Smorodinsky [19] showed the following generalization of Cassels-Schmidt's results.

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THEOREM A ([15, 19, 36]). *Let $q, b \geq 2$ be two integers with $\log q / \log b \notin \mathbb{Q}$. Let μ be a non-atomic self-similar measure with respect to the self-similar IFS*

$$\left\{ f_j(x) = \frac{x + j - 1}{q} \right\}_{j=1}^{q}.$$

Then the measure μ is pointwise b-normal.

For an integer $q \geq 2$, define the transformation T_q on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by $T_q(x) = qx \pmod 1$. The measure μ in **Theorem A** was extended by Host [24] to T_q-invariant and ergodic probability measures with positive entropy under the assumption $\gcd(q, b) = 1$. Lindenstrauss [30] derived the same conclusion under the weaker assumption that $q \nmid b^n$ for all $n \in \mathbb{N}$. In 2015, Host's theorem was generalized by Hochman and Shmerkin [23] to the natural assumption that $\log q / \log b \notin \mathbb{Q}$, see [22] for a short proof.

THEOREM B ([22–24, 30]). *Let $q, b \geq 2$ be two integers with $\log q / \log b \notin \mathbb{Q}$. Let μ be a T_q-invariant and ergodic probability measure with positive entropy. Then the measure μ is pointwise b-normal.*

A classical theorem to establish pointwise normality was obtained by Davenport, Erdős, and LeVeque [16] (see theorem 4.3), provided that the average decay rate of the Fourier transform is fast enough. It has many applications for Riesz products or measures with an infinite convolution structure [9, 10, 12–14, 20, 33]. Recently, Algom, Rodriguez Hertz, and Wang [5, theorem 1.3] improved the criterion for self-similar measures in $\mathbb{R}$, where they only require that the Fourier transform vanishes at infinity (see also [3, theorem 1.2]). A recent breakthrough work of Hochman and Shmerkin [23] gave a general criterion for pointwise normality of uniformly scaling measures, which goes beyond the classical approach of Davenport, Erdős, and LeVeque [16]. We refer the reader to [7] for uniformly scaling measures. In 2022, based on Hochman-Shmerkin's results, Algom, Baker and Shmerkin [4] conducted a disintegration method to obtain the following general result. For a non-integer $\beta > 1$, the transformation $T_\beta : x \to \beta x \pmod 1$ admits a unique absolutely continuous invariant measure, which is called the Parry measure. We say that a real number x is β-normal if the orbit $(T_\beta^n x)_{n \geq 1}$ equidistributes for the Parry measure (see [4] for details).

THEOREM C ([4, 23]). *Let $\beta > 1$ be a Pisot number, and let μ be a non-atomic self-similar measure on $\mathbb{R}$ with respect to the self-similar IFS $\Phi = \{f_j(x) = s_j x + t_j\}_{j=1}^n$ and a strictly positive probability vector. If $\log |s_j| / \log \beta \notin \mathbb{Q}$ for some $1 \leq j \leq n$, then the measure μ is pointwise β-normal.*

Very recently, Dayan, Ganguly, and Weiss [17] gave sufficient conditions in terms of the translations of the IFS to ensure pointwise normality. In this paper, we focus on equidistribution of self-similar measures in the plane and try to extend **Theorem A** for Gaussian integers in the complex plane.

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1.2. Main results

For $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$, the notation $x \in E \pmod 1$ means that

$$x \in \bigcup_{k\in\mathbb{Z}^d} (E+k).$$

DEFINITION 1.2. *A sequence $(x_n)_{n\geq 1}$ of vectors in $\mathbb{R}^d$ is said to be* equidistributed modulo one *if, for any Cartesian product $I := \prod_{j=1}^d [u_j, v_j)$ of intervals $[u_j, v_j) \subset [0,1)$, we have*

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N\colon\ x_n \in I \pmod 1\}}{N} = \prod_{j=1}^d (v_j - u_j).$$

Let $\mathbb{Z}[\mathrm{i}] = \{n + m\mathrm{i} : n, m \in \mathbb{Z}\}$ be the Gaussian integer ring. The units of a ring are the nonzero elements with a multiplicative inverse. The units in $\mathbb{Z}[\mathrm{i}]$ are ± 1 and $\pm\mathrm{i}$. For $a, b \in \mathbb{Z}[\mathrm{i}]$ we say that a is a *divisor* of b, denoted by $a \mid b$, if there exists an element $c \in \mathbb{Z}[\mathrm{i}]$ such that $b = ca$. Two elements a, b in $\mathbb{Z}[\mathrm{i}]$ are called *relatively prime* if each common divisor of a and b is a unit, and we also use the notation $\gcd(a, b) = 1$ as done in $\mathbb{Z}$.

An element $p \in \mathbb{Z}[\mathrm{i}] \setminus \{0, \pm 1, \pm\mathrm{i}\}$ is said to be *prime* if we write $p = ab$ with $a, b \in \mathbb{Z}[\mathrm{i}]$ then either a or b is a unit. Two prime elements p_1 and p_2 are said to be *essentially different* if p_1/p_2 is not a unit. Each element $z \in \mathbb{Z}[\mathrm{i}] \setminus \{0, \pm 1, \pm\mathrm{i}\}$ can be written as

$$z = u p_1^{n_1} p_2^{n_2} \dots p_k^{n_k},$$

where u is a unit, $k \in \mathbb{N}$, $n_1, n_2, \dots, n_k \in \mathbb{N}$, and $p_1, p_2, \dots, p_k$ are essentially different prime elements.

Now we state the main result of this paper. Here and in the following, we identify $\mathbb{C}$ with $\mathbb{R}^2$, and write $\overline{z}$ for the complex conjugate of $z \in \mathbb{C}$.

THEOREM 1.1 *Let $\alpha, \beta \in \mathbb{Z}[\mathrm{i}]$ with $|\alpha|, |\beta| > 1$. Let $T \subset \mathbb{Z}[\mathrm{i}]$ be a finite subset with $\#T \geq 2$. Let μ be a non-atomic self-similar measure with respect to the IFS*

$$\left\{ f_t(z) = \frac{z+t}{\beta} : t \in T \right\}.$$

Suppose that $\overline{\beta}$ is not a divisor of β and $\gcd(\alpha, \beta) = 1$. Then the sequence $(\alpha^n z)_{n\geq 1}$ is equidistributed modulo one for μ-a.e. $z \in \mathbb{C}$.

REMARK 1.1. (i) Pointwise normality will be discussed in § 5. We naturally extend the notion of normality to radix expansions of complex numbers in a Gaussian integer base, and obtain an equivalent definition of normality analogous to Wall's theorem [39], see proposition 5.1 for details.

(ii) In theorem 1.1, we restrict the translation set T in $\mathbb{Z}[\mathrm{i}]$. In fact, we will prove a little more general theorem (see theorem 4.1), where we only require that there are two distinct translations $t_1, t_2 \in T$ such that $t_1 - t_2 \in \mathbb{Z}[\mathrm{i}]$. However, we cannot remove completely the restriction on T. This is because we need to employ p-adic interpolation in $\mathbb{Z}[\mathrm{i}]$.

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(iii) The assumptions that $\overline{\beta}$ is not a divisor of β and $\gcd(\alpha, \beta) = 1$ are essential in Lemma 3.1, which is the key estimation to establish equidistribution.

(iv) One can check that $\overline{\beta}$ is not a divisor of β if and only if

$$\beta \in \mathbb{Z}[\mathrm{i}] \setminus (\mathbb{Z} \cup \mathrm{i}\mathbb{Z} \cup (1+\mathrm{i})\mathbb{Z} \cup (1-\mathrm{i})\mathbb{Z}).$$

If $\beta \in \mathbb{Z} \cup \mathrm{i}\mathbb{Z} \cup (1+\mathrm{i})\mathbb{Z} \cup (1-\mathrm{i})\mathbb{Z}$, then we have $\beta^4 \in \mathbb{Z}$. The measure μ can also be viewed as a self-similar measure with respect to the IFS

$$\left\{ f_{\tilde{t}}(z) = \frac{z + \tilde{t}}{\beta^4} : \tilde{t} \in \widetilde{T} \right\},$$

where $\widetilde{T} = \beta^3 T + \beta^2 T + \beta T + T$. This means that the matrix corresponding to β^4 in the IFS is a diagonal integer matrix. See [1, 2, 25, 31, 32] for related results.

(v) Compared to **Theorem A**, the assumption that $\gcd(\alpha, \beta) = 1$ is much more stronger. In the language of matrix in $\mathbb{R}^2$, $\alpha, \overline{\alpha}$ and $\beta, \overline{\beta}$ stand for eigenvalues of corresponding matrices. We have no ideas of what the condition $\gcd(\alpha, \beta) = 1$ means in geometry. We conjecture that theorem 1.1 holds under the assumption that α and β are multiplicatively independent, i.e., $\alpha^n \neq \beta^m$ for any $n, m \in \mathbb{N}$.

(vi) Due to homogeneity of the IFS, the self-similar measure is non-atomic if and only if all probability weights are strictly less than 1.

(vii) In theorem 1.1, we only consider $\alpha \in \mathbb{Z}[\mathrm{i}]$. Generally, for $\alpha \in \mathbb{C}$ with $|\alpha| > 1$, consider the transformation T_α on $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}[\mathrm{i}]$ defined by $T_\alpha(z) = \alpha z \pmod{\mathbb{Z}[\mathrm{i}]}$. If $\alpha \notin \mathbb{Z}[\mathrm{i}]$, then the Lebesgue measure is not T_α-invariant. Like the T_β-transformation in $\mathbb{R}$, the equidistribution of orbit $(T_\alpha^n z)_{n \geq 1}$ should be considered for the T_α-invariant absolutely continuous measure. We do not delve into this aspect in this paper.

Our approach is derived from Host [24] and Meiri [31]. Because the partition generated by the transformation T_β on $\mathbb{C}/\mathbb{Z}[\mathrm{i}]$ is too complicated, we fail to generalize **Theorem B** to the complex plane. Compared with other higher-dimensional results [1, 2, 25, 32], we only consider a class of specific self-similar measures. However, the action induced by a Gaussian integer is no longer a diagonal endomorphism, and it is more intimately connected with radix expansions. As shown in [16], Fourier decay for measures is often sufficient, but not necessary, to establish equidistribution. We recommend the reader to recent works [6, 8, 35] for polynomial Fourier decay for self-conformal measures. Roughly speaking, non-linearity is necessary to obtain polynomial Fourier decay. So, theorem 1.1 cannot be deduced from these general results. It is worth mentioning that the self-similar measures considered in theorem 1.1 are homogeneous. The random walk approach in [29, 37] may be more appropriate to deal with non-homogeneous self-similar measures.

Our strategy can be summarized as follows:

Step 1. Exploiting the infinite convolution structure. According to Weyl's criterion [40, 41] and Davenport-Erdös-LeVeque's approach [16], it suffices to establish the convergence of the series in (4.1), which essentially requires that the Fourier

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transform of self-similar measures has average decay. By utilizing the self-similarity, the Fourier transform can be expressed as an infinite product (see (4.2) and (4.3)).

Step 2. Estimating the distance to nearest integer after projecting. To find out small factors in (4.3), we need to estimate the distance to the nearest integer after projecting the complex number $(\alpha^n - \alpha^m)(t_1 - t_2)\beta^{-j}$ onto a given line, which is determined only by the first j digits when we write the Gaussian integer $(\alpha^n - \alpha^m)(t_1 - t_2)$ in base β. Thus, the problem is transformed into studying radix expansions of $(\alpha^n - \alpha^m)(t_1 - t_2)$.

Step 3. Constructing p-adic interpolation. When m is fixed, we develop the p-adic interpolation in $\mathbb{Z}[\mathrm{i}]$ to analysis radix expansions of $(t_1 - t_2)\alpha^n$. Since α and β is relatively prime, we show that radix expansions of $(t_1 - t_2)\alpha^n$ in base β exhibit sufficient diversity when n varies. Then we give an upper bound of the number of relative big values in the desired summation, which yields average Fourier decay.

The rest of paper is organized as follows. In the next section, we recall some known results of radix expansion and give some basic properties. In § 3, we develop the p-adic interpolation in $\mathbb{Z}[\mathrm{i}]$ and obtain some delicate estimates of radix expansions. Section 4 is devoted to the proof of theorem 1.1. In the last section, we discuss normality of radix expansions and apply theorem 1.1.

2. Radix expansion

In this section, we introduce some basic facts about radix expansions.

For $\beta \in \mathbb{Z}[\mathrm{i}]$ with $|\beta| > 1$, if there exists a finite set $D_\beta \subset \mathbb{Z}[\mathrm{i}]$ with $0 \in D_\beta$ such that every Gaussian integer $\alpha \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}$ has a unique representation of the form

$$\alpha = \sum_{k=0}^{\ell} a_k \beta^k,$$

where $\ell \geq 0$, $a_k \in D_\beta$ for all $0 \leq k \leq \ell$, and $a_\ell \neq 0$, then we call (β, D_β) a *number system in* $\mathbb{Z}[\mathrm{i}]$. A necessary condition for (β, D_β) to be a number system in $\mathbb{Z}[\mathrm{i}]$ is that D_β is a set of coset representatives of the quotient $\mathbb{Z}[\mathrm{i}]/(\beta\mathbb{Z}[\mathrm{i}])$; in particular, we have $\#D_\beta = |\beta|^2$.

As an additive group, we can identify $\mathbb{Z}[\mathrm{i}]$ with $\mathbb{Z}^2$. For $\beta = n + \mathrm{i}m \in \mathbb{Z}[\mathrm{i}]$ with $n, m \in \mathbb{Z}$, the multiplication by β on $\mathbb{Z}[\mathrm{i}]$ can be identified with the matrix

$$T_\beta := \begin{pmatrix} n & -m \\ m & n \end{pmatrix}$$

acting on $\mathbb{Z}^2$. Note that all singular values of T_β are $|\beta|$. Applying [38, corollary 5], we can obtain the following proposition.

PROPOSITION 2.1. *Given $\beta \in \mathbb{Z}[\mathrm{i}]$ with $|\beta| > 2$, there exists a finite set $D_\beta \subset \mathbb{Z}[\mathrm{i}]$ with $0 \in D_\beta$ such that (β, D_β) is a number system in $\mathbb{Z}[\mathrm{i}]$.*

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REMARK 2.1. For the following cases, (β, D_β) is a number system in $\mathbb{Z}[i]$:

- $\beta = -1 \pm \mathrm{i}$ and $D_\beta = \{0, 1\}$ [27];
- $\beta = -2, \pm 2\mathrm{i}$ and $D_\beta = \{0, 1, \mathrm{i}, 1 + \mathrm{i}\}$ [38, corollary 2].

But, there is no number systems in $\mathbb{Z}[\mathrm{i}]$ for $\beta = 2, 1 \pm \mathrm{i}$ [38, corollary 1].

In the following, fix $\beta \in \mathbb{Z}[\mathrm{i}]$ with $|\beta| > 2$ and a finite set $D_\beta \subset \mathbb{Z}[\mathrm{i}]$ satisfying that (β, D_β) is a number system in $\mathbb{Z}[\mathrm{i}]$. Define $|D_\beta| := \max\{|z| : z \in D_\beta\}$, and let m_β be the smallest integer not less than $|D_\beta|/(|\beta| - 1)$. Recall that $\{x\}$ denotes the fractional part of a real number $x \in \mathbb{R}$. For $z = x + \mathrm{i}y \in \mathbb{C}$ with $x, y \in \mathbb{R}$, define $\{z\} := \{x\} + \mathrm{i}\{y\}$. Write $B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for the open disc in $\mathbb{C}$ for $z_0 \in \mathbb{Z}$ and $r > 0$.

Next, we give two basic estimations concerning radix expansions with a fixed length.

LEMMA 2.1. *Given $M \in \mathbb{N}$, for $z_0 = x + \mathrm{i}y$ with $x, y \in [0, 1)$ and $\rho > 1$, define*

$$\mathcal{N}(z_0, \rho) := \left\{ (a_j)_{j=1}^M \in D_\beta^M : \left| \Big\{ \sum_{j=1}^M a_j \beta^{-j} \Big\} - z_0 \right| < \rho |\beta|^{-M} \right\}.$$

Then we have

$$\#\mathcal{N}(z_0, \rho) \le 36 \rho^2 m_\beta^2.$$

Proof. The set $\mathcal{N}(z_0, \rho)$ can be rewritten as

$$\mathcal{N}(z_0, \rho) = \left\{ (a_j)_{j=1}^M \in D_\beta^M : \left| \beta^M \Big\{ \sum_{j=1}^M a_j \beta^{-j} \Big\} - \beta^M z_0 \right| < \rho \right\}.$$

According to the integer part of $\sum_{j=1}^M a_j \beta^{-j}$, we only need to count Gaussian integers in a circle of radius ρ. This is why we obtain an upper bound independent of M.

Note that for all $(a_j)_{j=1}^M \in D_\beta^M$,

$$\Big| \sum_{j=1}^M a_j \beta^{-j} \Big| \le \sum_{j=1}^M |a_j| |\beta|^{-j} < \frac{|D_\beta|}{|\beta| - 1} \le m_\beta.$$

Let $L_\beta := \big\{ -m_\beta, -m_\beta + 1, \ldots, 0, 1, \ldots, m_\beta - 1 \big\}$. Then we have

$$\sum_{j=1}^M a_j \beta^{-j} \in \bigcup_{(n_1, n_2) \in L_\beta^2} [n_1, n_1 + 1) \times [n_2, n_2 + 1).$$

Note that we identify $\mathbb{C}$ with $\mathbb{R}^2$. For each pair $(n_1, n_2) \in L_\beta^2$, let

$$\mathcal{N}_{n_1, n_2} := \left\{ (a_j)_{j=1}^M \in D_\beta^M : \sum_{j=1}^M a_j \beta^{-j} \in [n_1, n_1 + 1) \times [n_2, n_2 + 1) \right\} \cap \mathcal{N}(z_0, \rho).$$

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It follows that

$$\#\mathcal{N}(z_0,\rho) \le \sum_{(n_1,n_2)\in L_\beta^2} \#\mathcal{N}_{n_1,n_2}. \tag{2.1}$$

Fix a pair $(n_1,n_2)\in L_\beta^2$ with $\mathcal{N}_{n_1,n_2}\neq\emptyset$. For $(a_j)_{j=1}^M\in\mathcal{N}_{n_1,n_2}$, we have

$$\Big\{\sum_{j=1}^M a_j\beta^{-j}\Big\} = \sum_{j=1}^M a_j\beta^{-j} - (n_1+\mathrm{i}n_2),$$

and hence,

$$\Big|\sum_{j=1}^M a_j\beta^{-j} - (n_1+\mathrm{i}n_2+z_0)\Big| < \rho\beta^{-M}.$$

It follows that

$$\sum_{j=1}^M a_j\beta^{M-j} \in B\big(z_{n_1,n_2},\rho\big),$$

where $z_{n_1,n_2}=\beta^M(n_1+\mathrm{i}n_2+z_0)$. Note that $\sum_{j=1}^M a_j\beta^{M-j}$ is a Gaussian integer, and (β,D_β) is a number system. Therefore, we conclude that

$$\#\mathcal{N}_{n_1,n_2} \le (2\rho+1)^2 < 9\rho^2.$$

It follows from (2.1) that

$$\#\mathcal{N}(z_0,\rho) < 9\rho^2\cdot(\#L_\beta)^2 = 36\rho^2m_\beta^2,$$

as desired. □

For two vectors $x,y\in\mathbb{R}^2$, we use $\langle x,y\rangle$ to denote the inner product of x and y. Let $\|x\|$ denote the distance between x and the set $\mathbb{Z}$, i.e., $\|x\| := \min_{z\in\mathbb{Z}}|x-z|$. The following lemma derives an upper bound for the number of the projections onto a given line which are close to integers.

LEMMA 2.2. *For $h=(h_1,h_2)\in\mathbb{Z}^2\setminus\{(0,0)\}$, $\rho>0$ and $M\in\mathbb{N}$, let*

$$D_{h,\rho,M} := \left\{(a_j)_{j=1}^M\in D_\beta^M:\ \Big\|\Big\langle h,\sum_{j=1}^M a_j\beta^{-j}\Big\rangle\Big\| < \rho|\beta|^{-M}\right\}.$$

Then we have

$$\#D_{h,\rho,M} \le 72m_\beta^2(2|h_1|+2|h_2|+1)(2\rho+\sqrt{2})^2|\beta|^M.$$

Proof. Without loss of generality, we assume that $|h_1|\le|h_2|$. Since $h\neq(0,0)$, we have $h_2\neq0$. For a word $\mathfrak{a}=(a_j)_{j=1}^M\in D_{h,\rho,M}$, we define $x_\mathfrak{a},y_\mathfrak{a}\in[0,1)$ by

$$x_\mathfrak{a}+\mathrm{i}y_\mathfrak{a} := \Big\{\sum_{j=1}^M a_j\beta^{-j}\Big\}.$$

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Then we have

$$\Big\|\Big\langle h, \sum_{j=1}^{M} a_j \beta^{-j}\Big\rangle\Big\| = \|h_1 x_{\mathfrak{a}} + h_2 y_{\mathfrak{a}}\| < \rho|\beta|^{-M}.$$

Note that $|h_1 x_{\mathfrak{a}} + h_2 y_{\mathfrak{a}}| < |h_1| + |h_2|$. Thus, there is an integer $n_{\mathfrak{a}} \in \{0, \pm 1, \ldots, \pm(|h_1| + |h_2|)\}$ such that

$$|h_1 x_{\mathfrak{a}} + h_2 y_{\mathfrak{a}} - n_{\mathfrak{a}}| < \rho|\beta|^{-M}.$$

For any integer $|n| \le |h_1| + |h_2|$, let $D_{h,\rho,M}^{(n)} := \big\{\mathfrak{a} \in D_{h,\rho,M} \colon |h_1 x_{\mathfrak{a}} + h_2 y_{\mathfrak{a}} - n| < \rho|\beta|^{-M}\big\}$. Thus, we have

$$D_{h,\rho,M} \subset \bigcup_{|n| \le |h_1| + |h_2|} D_{h,\rho,M}^{(n)}. \tag{2.2}$$

In the following, fix an integer $n \in \{0, \pm 1, \ldots, \pm(|h_1| + |h_2|)\}$, and define $\varphi_n(x) := (-h_1 x + n)/h_2$. Let $[x]$ denote the integer part of a real number $x \in \mathbb{R}$. For $\ell = 0, 1, \ldots, [|\beta|^M]$, define

$$D_{h,\rho,M}^{(n,\ell)} := \Big\{\mathfrak{a} \in D_{h,\rho,M}^{(n)} \colon \ell|\beta|^{-M} \le x_{\mathfrak{a}} < (\ell+1)|\beta|^{-M}\Big\}.$$

If $D_{h,\rho,M}^{(n,\ell)} = \emptyset$, then there is nothing to prove. If $D_{h,\rho,M}^{(n,\ell)} \ne \emptyset$, then we fix a word $\mathfrak{a} = (a_j)_{j=1}^M \in D_{h,\rho,M}^{(n,\ell)}$. For any word $\mathfrak{b} = (b_j)_{j=1}^M \in D_{h,\rho,M}^{(n,\ell)}$, we have

$$\big|x_{\mathfrak{b}} + \mathrm{i}\varphi_n(x_{\mathfrak{b}}) - \big(x_{\mathfrak{a}} + \mathrm{i}\varphi_n(x_{\mathfrak{a}})\big)\big| = \big|x_{\mathfrak{b}} - x_{\mathfrak{a}}\big|\sqrt{1 + (h_1/h_2)^2} < \sqrt{2}\beta^{-M}.$$

Note that

$$\Big|\Big\{\sum_{j=1}^{M} a_j \beta^{-j}\Big\} - \big(x_{\mathfrak{a}} + \mathrm{i}\varphi_n(x_{\mathfrak{a}})\big)\Big| = \frac{1}{|h_2|}\big|h_2 y_{\mathfrak{a}} + h_1 x_{\mathfrak{a}} - n\big| < \rho|\beta|^{-M},$$

and also

$$\Big|\Big\{\sum_{j=1}^{M} b_j \beta^{-j}\Big\} - \big(x_{\mathfrak{b}} + \mathrm{i}\varphi_n(x_{\mathfrak{b}})\big)\Big| = \frac{1}{|h_2|}\big|h_2 y_{\mathfrak{b}} + h_1 x_{\mathfrak{b}} - n\big| < \rho|\beta|^{-M}.$$

It follows that

$$\Big|\Big\{\sum_{j=1}^{M} b_j \beta^{-j}\Big\} - \Big\{\sum_{j=1}^{M} a_j \beta^{-j}\Big\}\Big| < (2\rho + \sqrt{2})|\beta|^{-M}.$$

By Lemma 2.1, we conclude that

$$\#D_{h,\rho,M}^{(n,j)} \le 36(2\rho + \sqrt{2})^2 m_\beta^2.$$

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It follows from (2.2) that

$$\begin{aligned}\#D_{h,\rho,M} &\le \sum_{|n|\le|h_1|+|h_2|} \#D^{(n)}_{h,\rho,M} = \sum_{|n|\le|h_1|+|h_2|} \sum_{\ell=0}^{[|\beta|^M]} \#D^{(n,\ell)}_{h,\rho,M} \\ &\le (2|h_1|+2|h_2|+1)\cdot 2|\beta|^M \cdot 36(2\rho+\sqrt{2})^2 m_\beta^2,\end{aligned}$$

as desired. □

3. *p*-adic analysis

To prove theorem 1.1, we need the following key estimation of radix expansions, and we shall develop the *p-adic interpolation* to prove it in this section.

LEMMA 3.1. *Let $\alpha,\beta\in\mathbb{Z}[\mathrm{i}]$ with $|\alpha|,|\beta|>1$, and let $\tau\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}$. For $z_0\in\mathbb{Z}[\mathrm{i}]$ and $k\in\mathbb{N}$, define*

$$E(z_0,k):=\Big\{1\le n\le|\beta|^{2k}:\tau\alpha^n\equiv z_0 \pmod{\beta^k}\Big\}.$$

Suppose that $\overline{\beta}$ is not a divisor of β and $\gcd(\alpha,\beta)=1$. Then there exist $C=C(\tau,\alpha,\beta)>0$ and $\varepsilon=\varepsilon(\beta)>0$ such that

$$\#E(z_0,k)\le C|\beta|^{(1-\varepsilon)k},$$

for any $z_0\in\mathbb{Z}[\mathrm{i}]$ and any $k\in\mathbb{N}$.

REMARK 3.1. For $k\in\mathbb{N}$, define $\mathrm{Ord}_{\beta^k}(\alpha)$ be the smallest positive integer n such that $\alpha^n\equiv 1 \pmod{\beta^k}$. Taking $z_0=1$ in Lemma 3.1, we obtain that

$$\mathrm{Ord}_{\beta^k}(\alpha)\ge C^{-1}|\beta|^{(1+\varepsilon)k}\quad\forall k\in\mathbb{N}.$$

The analogous result about order in $\mathbb{R}$ has appeared in previous works [15, 24, 31, 36].

For a prime number $p\in\mathbb{Z}$, the p-adic valuation can be defined on $\mathbb{Z}$, and then is extended to $\mathbb{Q}$. Then we define the p-adic absolute value on $\mathbb{Q}$, which is not complete. By the completion process, we obtain the p-adic field $\mathbb{Q}_p$, and then we can do the p-adic analysis in $\mathbb{Q}_p$. We refer the reader to [21] for details. Note that $\mathbb{Z}[\mathrm{i}]$ is a unique factorization domain, so that its properties are much like those of the usual integers. We can develop the analogous p-adic analysis.

Fix a prime element $p\in\mathbb{Z}[\mathrm{i}]$. The p-adic valuation on $\mathbb{Z}[\mathrm{i}]$ is defined as follows: for $z\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}$, let $v_p(z)$ be the maximal non-negative integer satisfying

$$p^v \mid z.$$

It can be extended to the field $\mathbb{Q}(\mathrm{i})$. For $z=a/b\in\mathbb{Q}(\mathrm{i})\setminus\{0\}$ with $a,b\in\mathbb{Z}[\mathrm{i}]$, let

$$v_p(z):=v_p(a)-v_p(b).$$

It is often convenient to set $v_p(0)=+\infty$. One can easily check that for all $z_1,z_2\in\mathbb{Q}(\mathrm{i})$, we have

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- $v_p(z_1 z_2) = v_p(z_1) + v_p(z_2)$;
- $v_p(z_1 + z_2) \geq \min\{v_p(z_1), v_p(z_2)\}$.

We define the p-adic absolute value on $\mathbb{Q}(\mathrm{i})$ by

$$|z|_p := e^{-v_p(z)},\ z \in \mathbb{Q}(\mathrm{i}),$$

with the convention $|0|_p = 0$. The completion of $\mathbb{Q}(\mathrm{i})$ with respect to the p-adic absolute value $|\cdot|_p$ is denoted by $\mathbb{K}_p$, which is also a field. The extended absolute value on $\mathbb{K}_p$ is also denoted by $|\cdot|_p$. The p-adic valuation can also be extended to $\mathbb{K}_p$ by $v_p(z) := -\log|z|_p$ for $z \in \mathbb{K}_p \setminus \{0\}$. Note that we also have $v_p(z) \in \mathbb{Z}$ for $z \in \mathbb{K}_p \setminus \{0\}$.

Note that $|\cdot|_p$ is a non-archimedean absolute value on $\mathbb{K}_p$. Thus, a sequence $(z_n)_{n\geq 1}$ in $\mathbb{K}_p$ is a Cauchy sequence if and only if

$$\lim_{n\to\infty} |a_{n+1} - a_n|_p = 0.$$

It follows that an infinite series $\sum_{n=1}^{\infty} z_n$ with $z_n \in \mathbb{K}_p$ is convergent if and only if

$$\lim_{n\to\infty} z_n = 0.$$

By [21, proposition 2.3.4], we have

$$|z_1 + z_2|_p = \max\{|z_1|_p, |z_2|_p\} \quad \text{if } z_1, z_2 \in \mathbb{K}_p \text{with } |z_1|_p \neq |z_2|_p. \tag{3.1}$$

Next, we use power series to define the p-adic logarithm and exponential functions in $\mathbb{K}_p$. Consider the formal power series for the logarithm function:

$$\mathbf{log}(1+Z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{Z^n}{n}. \tag{3.2}$$

LEMMA 3.2. *For any $|z-1|_p < 1$, the p-adic logarithm function can be defined as*

$$\mathbf{log}_p(z) := \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.$$

Moreover, if $|z-1|_p < e^{-1/(|p|-1)}$, then we have

$$|\mathbf{log}_p(z)|_p = |z-1|_p.$$

Proof. As done in [21, proposition 5.4.1], the convergence radius of the series in (3.2) is

$$r = \frac{1}{\limsup\limits_{n\to\infty} |a_n|_p^{1/n}},$$

where $a_n = (-1)^{n+1}/n$. Note that $|a_n|_p^{1/n} = e^{v_p(n)/n}$ and $\frac{v_p(n)}{n} \leq \frac{\log n}{n \log|p|} \to 0$. We conclude that $r = 1$. Thus, for any $|z|_p < 1$, the series in (3.2) converges in $\mathbb{K}_p$.

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Therefore, for $|z-1|_p < 1$, we can define the p-adic logarithm function as

$$\mathbf{log}_p(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.$$

Note that for $n \geq 2$ with $v_p(n) \geq 1$,

$$\frac{v_p(n)}{n-1} \leq \frac{v_p(n)}{|p|^{v_p(n)}-1} = \frac{1}{|p|-1} \cdot \frac{v_p(n)}{|p|^{v_p(n)-1}+\cdots+|p|+1} \leq \frac{1}{|p|-1}. \tag{3.3}$$

If $|z-1|_p < e^{-1/(|p|-1)}$, then we have

$$v_p(z-1) > \frac{1}{|p|-1} \geq \frac{v_p(n)}{n-1} \quad \forall n \geq 2,$$

which implies that

$$\left|\frac{(z-1)^{n-1}}{n}\right|_p < 1 \quad \forall n \geq 2.$$

It follows that

$$\left|\frac{(z-1)^n}{n}\right|_p < |z-1|_p \quad \forall n \geq 2.$$

Thus, by (3.1) we conclude that

$$|\mathbf{log}_p(z)|_p = \lim_{N\to\infty} \left| \sum_{n=1}^{N} (-1)^{n+1} \frac{(z-1)^n}{n} \right|_p = |z-1|_p.$$

The proof is completed. □

As done in [21, proposition 5.7.3], we have

$$\mathbf{log}_p(z_1 z_2) = \mathbf{log}_p(z_1) + \mathbf{log}_p(z_2) \quad \forall |z_1-1|_p < 1,\ |z_2-1|_p < 1. \tag{3.4}$$

To define the p-adic exponential function, consider the following formal power series:

$$\mathbf{exp}(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{n!}. \tag{3.5}$$

We need the following lemma to calculate the convergence radius.

LEMMA 3.3. *Let $p \in \mathbb{Z}[\mathrm{i}]$ be a prime element. Then for any $n \in \mathbb{N}$, we have*

$$v_p(n!) \leq \frac{n-1}{|p|-1}.$$

Proof. Let $\widetilde{v}_{p'}(\cdot)$ denote the p'-adic valuation on $\mathbb{Z}$ where $p' \in \mathbb{N}$ is a prime number. By [21, problem 180], we have that for $n \in \mathbb{N}$

$$\widetilde{v}_{p'}(n!) \leq \frac{n-1}{p'-1}.$$

By the description for prime elements in $\mathbb{Z}[\mathrm{i}]$ (see [34, proposition 3.68]), we have

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(i) p is an associate of a prime number in $\mathbb{Z}$ of the form $4m+3$; or
(ii) p is an associate of $1+\mathrm{i}$ or its conjugate $1-\mathrm{i}$; or
(iii) $p = a+\mathrm{i}b$, where a^2+b^2 is a prime number in $\mathbb{Z}$ of the form $4m+1$.

For case (i), we have $|p|$ is a prime number in $\mathbb{Z}$, and $p = \pm|p|$ or $\pm\mathrm{i}|p|$. Note that $p^{v_p(n!)} \mid n!$ in $\mathbb{Z}[\mathrm{i}]$. It follows that $|p|^{v_p(n!)} \mid n!$ in $\mathbb{Z}$, and hence

$$v_p(n!) \le \widetilde{v}_{|p|}(n!) \le \frac{n-1}{|p|-1}.$$

For case (iii), we have $\overline{p}$ is also a prime element in $\mathbb{Z}[\mathrm{i}]$ and $|p|^2$ is a prime number in $\mathbb{Z}$. Note that $p^{v_p(n!)} \mid n!$ in $\mathbb{Z}[\mathrm{i}]$. It follows that $\overline{p}^{v_p(n!)} \mid n!$ in $\mathbb{Z}[\mathrm{i}]$. Since $p^{v_p(n!)}$ and $\overline{p}^{v_p(n!)}$ are relatively prime, we conclude that $|p|^{2v_p(n!)} = p^{v_p(n!)}\overline{p}^{v_p(n!)} \mid n!$ in $\mathbb{Z}$. Thus, we have

$$v_p(n!) \le \widetilde{v}_{|p|^2}(n!) \le \frac{n-1}{|p|^2-1} < \frac{n-1}{|p|-1}.$$

For case (ii), we have $p\overline{p} = 2$. Note that $p^{v_p(n!)} \mid n!$ implies that $\overline{p}^{v_p(n!)} \mid n!$. Hence $2^{v_p(n!)} = (p\overline{p})^{v_p(n!)} \mid (n!)^2$. By the definition of $\widetilde{v}_2(\cdot)$, this implies that

$$v_p(n!) \le \widetilde{v}_2\big((n!)^2\big) = 2\widetilde{v}_2(n!) \le 2\frac{n-1}{2-1} < \frac{n-1}{|p|-1},$$

which completes the proof. □

LEMMA 3.4. *For any $|z|_p < e^{-1/(|p|-1)}$, the p-adic exponential function can be defined as*

$$\mathbf{exp}_p(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Moreover, we have

$$|\mathbf{exp}_p(z) - 1|_p = |z|_p,$$

and hence, $|\mathbf{exp}_p(z)|_p = \max\{|\mathbf{exp}_p(z) - 1|_p, |1|_p\} = 1$.

Proof. As done in [21, proposition 5.4.1], the convergence radius of the series in (3.5) is

$$r = \frac{1}{\limsup_{n\to\infty} |a_n|_p^{1/n}},$$

where $a_n = 1/n!$. By Lemma 3.3, we have $r \ge e^{-1/(|p|-1)}$. Thus, for $|z|_p < e^{-1/(|p|-1)}$, we can define the p-adic exponential function as

$$\mathbf{exp}_p(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Moreover, by again Lemma 3.3, for $n \ge 2$ we have

$$|z^{n-1}|_p = |z|_p^{n-1} < e^{-\frac{n-1}{|p|-1}} \le e^{-v_p(n!)} = |n!|_p.$$

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It follows that

$$\left|\frac{z^n}{n!}\right|_p < |z|_p \quad \forall n \geq 2.$$

Thus, by (3.1) we conclude that

$$|\mathbf{exp}_p(z) - 1|_p = \lim_{N\to\infty}\left|\sum_{n=1}^{N}\frac{z^n}{n!}\right|_p = |z|_p.$$

The proof is completed. □

As done in [21, proposition 5.7.7], we have

$$\mathbf{exp}_p(z_1 + z_2) = \mathbf{exp}_p(z_1)\mathbf{exp}_p(z_2) \quad \forall |z_1|_p < e^{-1/(|p|-1)}, |z_2|_p < e^{-1/(|p|-1)}.$$

Moreover, if $|z-1|_p < e^{-1/(|p|-1)}$, then we have $|\mathbf{log}_p(z)|_p = |z-1|_p < e^{-1/(|p|-1)}$ so that $\mathbf{log}_p(z)$ is in the domain of $\mathbf{exp}_p$. As done in [21, proposition 5.7.8], we have

$$\mathbf{exp}_p\big(\mathbf{log}_p(z)\big) = z \quad \forall |z-1|_p < e^{-1/(|p|-1)}. \tag{3.6}$$

The valuation ring

$$\mathcal{O}_p := \{z \in \mathbb{K}_p : |z|_p \leq 1\}.$$

Clearly, we have $\mathbb{Z}[\mathrm{i}] \subset \mathcal{O}_p$. We can define the power function on $\mathcal{O}_p$ by using the p-adic logarithm and exponential functions. Choose $\alpha \in \mathbb{K}_p$ with $|\alpha-1|_p < e^{-1/(|p|-1)}$. Note that for all $z \in \mathcal{O}_p$, $|z\mathbf{log}(\alpha)|_p \leq |\mathbf{log}(\alpha)|_p = |\alpha-1|_p < e^{-1/(|p|-1)}$. We can define the power function on $\mathcal{O}_p$ as

$$G_\alpha(z) := \mathbf{exp}_p\big(z\mathbf{log}_p(\alpha)\big),\ z \in \mathcal{O}_p.$$

For all $k \in \mathbb{Z}$, by (3.4) and (3.6), we have

$$G_\alpha(k) = \mathbf{exp}_p(k\mathbf{log}_p(\alpha)) = \mathbf{exp}_p(\mathbf{log}_p(\alpha^k)) = \alpha^k.$$

For any $z_1, z_2 \in \mathcal{O}_p$, by Lemma 3.4 we obtain

$$\begin{aligned} |G_\alpha(z_1) - G_\alpha(z_2)|_p &= \big|\mathbf{exp}_p\big(z_2\mathbf{log}_p(\alpha)\big)\big|_p \cdot \big|\mathbf{exp}_p\big((z_1-z_2)\mathbf{log}_p(\alpha)\big) - 1\big|_p \\ &= \big|\mathbf{exp}_p\big((z_1-z_2)\mathbf{log}_p(\alpha)\big) - 1\big|_p = |\mathbf{log}_p(\alpha)|_p|z_1-z_2|_p. \end{aligned} \tag{3.7}$$

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Since $\overline{\beta}$ is not a divisor of β, there exists a prime element $p_0 \in \mathbb{Z}[\mathrm{i}]$ such that $v_{p_0}(\beta) > v_{\overline{p}_0}(\beta)$. This means that p_0 and $\overline{p}_0$ are essentially different prime elements in $\mathbb{Z}[\mathrm{i}]$. Write

$$\beta = up_0^{k_0}\overline{p_0}^{k_0'}p_1^{k_1}p_2^{k_2}\dots p_m^{k_m},$$

where u is a unit, $p_0, \overline{p_0}, p_1, \dots, p_m$ are essentially different prime elements in $\mathbb{Z}[\mathrm{i}]$, $k_0, \dots, k_m \in \mathbb{N}$, $k_0' \in \mathbb{N} \cup \{0\}$ and $k_0 > k_0'$.

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Consider the ring $R := \mathbb{Z}[\mathrm{i}]/(\beta^3\mathbb{Z}[\mathrm{i}])$ and the unit group $\mathrm{U}(R) = \big(\mathbb{Z}[\mathrm{i}]/\beta^3\mathbb{Z}[\mathrm{i}]\big)^\times$ consisting of all the elements $x \in R$ such that there exists an element $y \in R$ with $xy = 1_R$, where $1_R = 1 + \beta^3\mathbb{Z}[\mathrm{i}]$ is the identity element of the ring R. Since $\gcd(\alpha, \beta) = 1$, there exist $\xi, \eta \in \mathbb{Z}[\mathrm{i}]$ such that $\alpha\xi + \beta^3\eta = 1$, thus $\alpha + \beta^3\mathbb{Z}[\mathrm{i}] \in \mathrm{U}(R)$. Let ℓ be the order of $\alpha + \beta^3\mathbb{Z}[\mathrm{i}]$ in the finite group $\mathrm{U}(R)$. Then we have $\alpha^\ell \equiv 1 \pmod{\beta^3}$, that is, $\beta^3 \mid \alpha^\ell - 1$ (see [34, proposition 2.34] for details). It follows that for $r = 0, 1, \ldots, m$,

$$v_{p_r}(\alpha^\ell - 1) \geq v_{p_r}(\beta^3) = 3v_{p_r}(\beta) \geq 3k_r \geq 3,$$

and hence we have

$$|\alpha^\ell - 1|_{p_r} \leq e^{-3} < e^{-1/(|p_r|-1)}.$$

Thus, the power function $G_{\alpha^\ell}(z)$ can be defined on all the sets $\mathcal{O}_{p_r}$ for $r = 0, 1, \ldots, m$.

Let

$$N := \max\big\{ -\log\big|\tau\alpha^j \mathbf{log}_{p_r}(\alpha^\ell)\big|_{p_r} : \ j = 1, 2, \ldots, \ell; \ r = 0, 1, \ldots, m\big\}.$$

For $j = 1, 2, \ldots, \ell$, define

$$E(z_0, k, j) := \Big\{0 \leq n < |\beta|^{2k}/\ell : \tau\alpha^j(\alpha^\ell)^n \equiv z_0 \pmod{\beta^k}\Big\}.$$

For two distinct elements $n_1, n_2 \in E(z_0, k, j)$, we have $\beta^k \mid \tau\alpha^j(\alpha^{n_1\ell} - \alpha^{n_2\ell})$, i.e., $\beta^k \mid \tau\alpha^j\big(G_{\alpha^\ell}(n_1) - G_{\alpha^\ell}(n_2)\big)$. Thus for any $r = 0, 1, \ldots, m$, by (3.7) we have

$$e^{-N}|n_1 - n_2|_{p_r} \leq \Big|\tau\alpha^j\big(G_{\alpha^\ell}(n_1) - G_{\alpha^\ell}(n_2)\big)\Big|_{p_r} \leq e^{-kk_r}.$$

This implies that for any $r = 0, 1, \ldots, m$,

$$p_r^{kk_r - N} \mid n_1 - n_2.$$

Since $n_1 - n_2 \in \mathbb{Z}$, we also have

$$\overline{p_0}^{kk_0 - N} \mid n_1 - n_2.$$

It follows that

$$\beta^k \cdot \overline{p_0}^{(k_0 - k_0')k}(p_0\overline{p_0}p_1 \ldots p_m)^{-N} \mid n_1 - n_2,$$

which implies that

$$|n_1 - n_2| \geq |\beta^k \cdot \overline{p_0}^{(k_0 - k_0')k}(p_0\overline{p_0}p_1 \ldots p_m)^{-N}|.$$

Therefore, we conclude that

$$\begin{aligned}\#E(z_0, k, j) \leq& \frac{|\beta|^{2k}/\ell}{|\beta^k \cdot \overline{p_0}^{(k_0 - k_0')k}(p_0\overline{p_0}p_1 \ldots p_m)^{-N}|} + 1\\ =& |\beta|^k |p_0\overline{p_0}p_1 \ldots p_m|^N/(\ell|\overline{p_0}|^{(k_0 - k_0')k}) + 1.\end{aligned}$$

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It follows that

$$\#E(z_0,k) \le \sum_{j=1}^{\ell} \#E(z_0,k,j) \le |p_0\overline{p_0}p_1\dots p_m|^N \cdot \Big(\frac{|\beta|}{|\overline{p_0}|^{k_0-k_0'}}\Big)^k + \ell. \tag{3.8}$$

Note that $|\overline{p_0}| > 1$ and $k_0 - k_0' \ge 1$. Take sufficiently small $\varepsilon > 0$ such that

$$|\beta|^{\varepsilon} < |\overline{p_0}|^{k_0-k_0'},$$

and $C = |p_0\overline{p_0}p_1\dots p_m|^N + \ell$. The desired result follows from (3.8). □

4. Equidistribution in the complex plane

In this section, we will prove the following theorem, which is a little more general than theorem 1.1.

THEOREM 4.1 *Let $\alpha, \beta \in \mathbb{Z}[\mathrm{i}]$ with $|\alpha|, |\beta| > 1$. Let $T \subset \mathbb{C}$ be a finite subset with $\#T \ge 2$. Let μ be a self-similar measure with respect to the IFS*

$$\Big\{ f_t(z) = \frac{z+t}{\beta} : t \in T \Big\},$$

and a positive probability vector. Suppose that $\overline{\beta}$ is not a divisor of β, $\gcd(\alpha,\beta) = 1$, and there exist $t_1 \ne t_2 \in T$ such that $t_1 - t_2 \in \mathbb{Z}[\mathrm{i}]$. Then the sequence $(\alpha^n z)_{n\ge1}$ is equidistributed modulo one for μ-a.e. $z \in \mathbb{C}$.

Note that theorem 1.1 follows directly from theorem 4.1. The proof of theorem 4.1 relies on Weyl's criterion and the classical Davenport-Erdös-LeVeque approach. For convenience, we write $e(x) = e^{2\pi \mathrm{i} x}$ for $x \in \mathbb{R}$.

THEOREM 4.2 (Weyl's criterion [40, 41], see also [18, theorem 1.19]) *A sequence $(x_n)_{n\ge1}$ of vectors in $\mathbb{R}^2$ is equidistributed modulo one if and only if, for each $h \in \mathbb{Z}^2 \setminus \{(0,0)\}$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e\big(\langle h, x_n\rangle\big) = 0.$$

THEOREM 4.3 (Davenport, Erdös and LeVeque [16]) *Let $(X, \mathscr{A}, \mu)$ be a measure space and let $(f_n)_{n\ge1}$ be a sequence of uniformly bounded measurable functions defined on $(X, \mathscr{A})$. If*

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_X \Big| \frac{1}{N} \sum_{n=1}^{N} f_n \Big|^2 \mathrm{d}\mu < +\infty,$$

then for μ-a.e. $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(x) = 0.$$

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Now we begin to prove theorem 4.1. In the following, suppose that all assumptions in theorem 4.1 holds, and let $(p_t)_{t\in T}$ be the positive probability vector associated with the self-similar measure μ.

By theorems 4.2 and 4.3, it suffices to show that, for each $h \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

$$\Sigma_h := \sum_{N=1}^{\infty} \frac{1}{N} \int \Big| \frac{1}{N} \sum_{n=1}^{N} e\big(\langle h, \alpha^n z\rangle\big) \Big|^2 \mathrm{d}\mu(z) < +\infty.$$

We remind the reader that $\mathbb{C}$ is identified with $\mathbb{R}^2$. Fix $h = (h_1, h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. A direct calculation gives

$$\begin{aligned}\Sigma_h &= \sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{1\le m,n\le N} \int e\big(\langle h, (\alpha^n - \alpha^m) z\rangle\big) \mathrm{d}\mu(z) \\ &\le \sum_{N=1}^{\infty} \frac{1}{N^3} \Big(N + 2 \sum_{1\le m<n\le N} \Big| \int e\big(\langle h, (\alpha^n - \alpha^m) z\rangle\big) \mathrm{d}\mu(z) \Big| \Big).\end{aligned}$$

Therefore, we only need to prove that

$$\Sigma_h^* := \sum_{N=2}^{\infty} \frac{1}{N^3} \sum_{1\le m<n\le N} \Big| \int e\big(\langle h, (\alpha^n - \alpha^m) z\rangle\big) \mathrm{d}\mu(z) \Big| < +\infty. \tag{4.1}$$

Note that the measure μ is self-similar, i.e.,

$$\mu = \sum_{t\in T} p_t \cdot f_t \mu.$$

We have

$$\begin{aligned}\lambda_{n,m} &:= \int e\Big(\big\langle h, (\alpha^n - \alpha^m) z\big\rangle\Big) \mathrm{d}\mu(z) \\ &= \sum_{t\in T} p_t \int e\Big(\big\langle h, (\alpha^n - \alpha^m) f_t(z)\big\rangle\Big) \mathrm{d}\mu(z) \\ &= \Big(\sum_{t\in T} p_t e\Big(\big\langle h, (\alpha^n - \alpha^m) t\beta^{-1}\big\rangle\Big)\Big) \int e\Big(\big\langle h, (\alpha^n - \alpha^m)\beta^{-1} z\big\rangle\Big) \mathrm{d}\mu(z) \\ &= \Big(\prod_{j=1}^{\ell} \sum_{t\in T} p_t e\Big(\big\langle h, (\alpha^n - \alpha^m) t\beta^{-j}\big\rangle\Big)\Big) \int e\Big(\big\langle h, (\alpha^n - \alpha^m)\beta^{-\ell} z\big\rangle\Big) \mathrm{d}\mu(z) \\ &= \prod_{j=1}^{\infty} \sum_{t\in T} p_t e\Big(\big\langle h, (\alpha^n - \alpha^m) t\beta^{-j}\big\rangle\Big). \end{aligned} \tag{4.2}$$

To estimate (4.2), we need the following lemma.

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LEMMA 4.1. ([42, lemma 2.1]) *Let $(q_1, \dots, q_\ell)$ be a probability vector and $\theta_1, \dots, \theta_\ell$ a sequence of real numbers. We have*

$$\left|\sum_{j=1}^{\ell} q_j e(\theta_j)\right| \le 1 - 8q_0 \max_{1\le j,k\le \ell} \|\theta_j - \theta_k\|^2,$$

where $q_0 = \min\{q_1, q_2, \dots, q_\ell\}$.

Let $p_* := \min\{p_t : t \in T\}$. Note that $t_1, t_2 \in T$. Applying Lemma 4.1 to (4.2), we obtain

$$|\lambda_{n,m}| \le \prod_{j=1}^{\infty} \left(1 - 8p_* \left\|\left\langle h, (\alpha^n - \alpha^m)(t_1 - t_2)\beta^{-j}\right\rangle\right\|^2\right). \tag{4.3}$$

Since $\overline{\beta}$ is not a divisor of β, we have $\beta \in \mathbb{Z}[\mathrm{i}] \setminus \big(\mathbb{Z} \cup \mathrm{i}\mathbb{Z} \cup (1+\mathrm{i})\mathbb{Z} \cup (1-\mathrm{i})\mathbb{Z}\big)$. This implies that $|\beta| > 2$. By proposition 2.1, we fix a finite subset $D_\beta \subset \mathbb{Z}[\mathrm{i}]$ such that (β, D_β) is a number system in $\mathbb{Z}[\mathrm{i}]$. Note that $0 \in D_\beta$ and $\#D_\beta = |\beta|^2$. Recall that $|D_\beta| = \max\{|z| : z \in D_\beta\}$, and m_β is the smallest integer not less than $|D_\beta|/(|\beta| - 1)$. Let

$$\rho = |D_\beta|(|h_1| + |h_2|)/(|\beta| - 1) + 1. \tag{4.4}$$

Choose an integer $M \in \mathbb{N}$ such that

$$72m_\beta^2(2|h_1| + 2|h_2| + 1)(2\rho + \sqrt{2})^2 \le |\beta|^{M\varepsilon/4}, \tag{4.5}$$

where $\varepsilon > 0$ is as in Lemma 3.1. Then define

$$N(n,m) := \#\Big\{j \in \mathbb{N} : \left\|\left\langle h, (\alpha^n - \alpha^m)(t_1 - t_2)\beta^{-Mj}\right\rangle\right\| \ge |\beta|^{-M}\Big\}. \tag{4.6}$$

It follows from (4.3) that

$$|\lambda_{n,m}| \le \left(1 - 8p_*|\beta|^{-2M}\right)^{N(n,m)} \le e^{-8p_*|\beta|^{-2M}N(n,m)},$$

where we use the inequality that $1 - x \le e^{-x}$ for $0 \le x \le 1$. Therefore, we conclude that

$$\Sigma_h^* \le \sum_{N=2}^{\infty} \frac{1}{N^3} \sum_{1\le m<n\le N} e^{-8p_*|\beta|^{-2M}N(n,m)}.$$

The remaining estimation is split into two parts:

$$\Sigma_{1,N} := \sum_{\substack{1\le m<n\le N \\ N(n,m) > c\log\log N}} e^{-8p_*|\beta|^{-2M}N(n,m)},$$

$$\Sigma_{2,N} := \sum_{\substack{1\le m<n\le N \\ N(n,m) \le c\log\log N}} e^{-8p_*|\beta|^{-2M}N(n,m)},$$

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where $c := |\beta|^{2M}/(4p_*)$. For the first part, we have

$$\sum_{N=2}^{\infty} \frac{1}{N^3}\Sigma_{1,N} \le \sum_{N=2}^{\infty} \frac{1}{N^3} N^2 e^{-2\log\log N} = \sum_{N=2}^{\infty} \frac{1}{N\log^2 N} < +\infty. \tag{4.7}$$

For the second part, we have $\Sigma_{2,N} \le \mathcal{N}(N)$, where

$$\mathcal{N}(N) := \#\big\{(n,m)\colon 1 \le m < n \le N,\ N(n,m) \le c\log\log N\big\}.$$

LEMMA 4.2. *Let $\varepsilon > 0$ be defined as in Lemma 3.1. There exists a constant $C > 0$ such that for sufficiently large N,*

$$\mathcal{N}(N) \le CN^{2-\frac{\varepsilon}{4}}.$$

Proof. For $m \in \{1, 2, \ldots, N-1\}$, let

$$\mathcal{N}_m(N) := \big\{m+1 \le n \le N\colon N(n,m) \le c\log\log N\big\}.$$

In the following, we fix $m \in \{1, 2, \ldots, N-1\}$.

Arbitrarily take $n \in \mathcal{N}_m(N)$. Note that (β, D_β) is a number system in $\mathbb{Z}[\mathrm{i}]$, and $t_1 - t_2 \in \mathbb{Z}[\mathrm{i}]$. We write

$$(t_1 - t_2)(\alpha^n - \alpha^m) = \sum_{j=0}^{\infty} a_j^{(n)} \beta^j \tag{4.8}$$

with each $a_j^{(n)} \in D_\beta$ and there are only finitely many $a_j^{(n)} \ne 0$. Note that the sequence $(a_j^{(n)})_{j\ge 1}$ is uniquely determined by n. It follows that for $j \in \mathbb{N}$,

$$(t_1 - t_2)(\alpha^n - \alpha^m)\beta^{-Mj} = \sum_{k=0}^{\infty} a_{Mj+k}^{(n)} \beta^k + \sum_{k=1}^{Mj} a_{Mj-k}^{(n)} \beta^{-k}.$$

Let $D_{h,\rho,M}$ be defined as in Lemma 2.2, where ρ is given in (4.4). If $(a_{Mj-k}^{(n)})_{k=1}^{M} \in D_\beta^M \setminus D_{h,\rho,M}$, then we have

$$\begin{aligned}
&\Big\|\Big\langle h, (t_1 - t_2)(\alpha^n - \alpha^m)\beta^{-Mj}\Big\rangle\Big\| \\
=&\Big\|\Big\langle h, \sum_{k=1}^{M} a_{Mj-k}^{(n)} \beta^{-k} + \sum_{k=M+1}^{Mj} a_{Mj-k}^{(n)} \beta^{-k}\Big\rangle\Big\| \\
\ge&\Big\|\Big\langle h, \sum_{k=1}^{M} a_{Mj-k}^{(n)} \beta^{-k}\Big\rangle\Big\| - (|h_1| + |h_2|) \sum_{k=M+1}^{Mj} |a_{Mj-k}^{(n)}||\beta|^{-k} \\
\ge&\Big\|\Big\langle h, \sum_{k=1}^{M} a_{Mj-k}^{(n)} \beta^{-k}\Big\rangle\Big\| - (|h_1| + |h_2|) \frac{|D_\beta|}{|\beta| - 1} |\beta|^{-M} \\
\ge&|\beta|^{-M}.
\end{aligned}$$

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By (4.6), we have

$$\#\Big\{j\in\mathbb{N}:(a^{(n)}_{Mj-k})_{k=1}^{M}\in D_\beta^M\setminus D_{h,\rho,M}\Big\}\le N(n,m)\le c\log\log N,$$

where the last inequality follows from $n\in\mathcal{N}_m(N)$. Now we choose $g\in\mathbb{N}$ such that

$$|\beta|^{2(g-1)M}<N\le|\beta|^{2gM}. \tag{4.9}$$

Clearly, the sequence $(a_j^{(n)})_{j=0}^{gM-1}$ satisfies

$$\#\Big\{1\le j\le g:(a^{(n)}_{Mj-k})_{k=1}^{M}\in D_\beta^M\setminus D_{h,\rho,M}\Big\}\le c\log\log N, \tag{4.10}$$

and by (4.8) we have

$$(t_1-t_2)(\alpha^n-\alpha^m)\equiv\sum_{j=0}^{gM-1}a_j^{(n)}\beta^j\pmod{\beta^{gM}}. \tag{4.11}$$

Let A be the set of sequences $(a_j)_{j=0}^{gM-1}\in D_\beta^{gM}$ satisfying

$$\#\Big\{1\le j\le g:(a_{Mj-k})_{k=1}^{M}\in D_\beta^M\setminus D_{h,\rho,M}\Big\}\le c\log\log N.$$

For $\mathfrak{a}=(a_j)_{j=0}^{gM-1}\in D_\beta^{gM}$, define

$$B(\mathfrak{a}):=\Big\{m+1\le n\le|\beta|^{2gM}:(t_1-t_2)(\alpha^n-\alpha^m)\equiv\sum_{j=0}^{gM-1}a_j\beta^j\pmod{\beta^{gM}}\Big\}.$$

If $n\in\mathcal{N}_m(N)$, then by (4.10) the sequence $\mathfrak{a}^{(n)}=(a_j^{(n)})_{j=0}^{gM-1}\in A$, and moreover, by (4.11) we have $n\in B(\mathfrak{a}^{(n)})$. Thus, we conclude that

$$\#\mathcal{N}_m(N)\le\sum_{\mathfrak{a}\in A}\#B(\mathfrak{a}).$$

By Lemma 3.1 and (4.9), we have

$$\#B(\mathfrak{a})\le c_1|\beta|^{gM(1-\varepsilon)}\le c_1|\beta|^M N^{\frac{1-\varepsilon}{2}} \tag{4.12}$$

for some $c_1>0$. Note by Lemma 2.2 that

$$\#D_{h,\rho,M}\le c_2|\beta|^M,$$

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where $c_2 = 72m_\beta^2(2|h_1|+2|h_2|+1)(2\rho+\sqrt{2})^2 > 0$. Note by (4.5) that $c_2 \le |\beta|^{M\varepsilon/4}$. Thus, we have

$$\begin{aligned}
\#A &\le \binom{g}{[c\log\log N]}(|\beta|^{2M})^{[c\log\log N]}(\#D_{h,\rho,M})^{g-[c\log\log N]}\\
&\le g^{[c\log\log N]}|\beta|^{M(g+[c\log\log N])}c_2^g\\
&\le g^{c\log\log N}|\beta|^{M(1+c\log\log N)}|\beta|^{gM\varepsilon/4}N^{1/2}\\
&\le g^{c\log\log N}|\beta|^{M(2+c\log\log N)}N^{\frac{1}{2}+\frac{\varepsilon}{8}}\\
&\le c_3N^{\frac{1}{2}+\frac{\varepsilon}{4}},
\end{aligned}$$

for some $c_3 > 0$ when N is sufficiently large. Together with (4.12), we obtain that

$$\#\mathcal{N}_m(N) \le \sum_{\mathfrak{a}\in A}\#B(\mathfrak{a}) \le CN^{1-\frac{\varepsilon}{4}},$$

for some $C > 0$. It follows that

$$\mathcal{N}(N) \le \sum_{m=1}^{N}\#\mathcal{N}_m(N) \le CN^{2-\frac{\varepsilon}{4}},$$

as desired. □

By Lemma 4.2, for sufficiently large N, we have

$$\frac{\mathcal{N}(N)}{N^3} \le \frac{C}{N^{1+\varepsilon/4}}.$$

It follows that

$$\sum_{N=2}^{\infty}\frac{1}{N^3}\Sigma_{2,N} \le \sum_{N=2}^{\infty}\frac{1}{N^3}\mathcal{N}(N) < +\infty.$$

Together with (4.7), we conclude that

$$\Sigma_h^* < +\infty.$$

Since h is arbitrary, we complete the proof of theorem 4.1.

5. Application

In this section, let (α, D) be a number system in $\mathbb{Z}[\mathrm{i}]$ with $|\alpha| > 1$. Let E_α be the self-similar set generated by the IFS

$$\left\{f_d(z) = \frac{z+d}{\alpha} : d \in D\right\}. \tag{5.1}$$

Then the set E_α can be written as

$$E_\alpha = \left\{\sum_{j=1}^{\infty}d_j\alpha^{-j} : d_j \in D \ \ \forall j \in \mathbb{N}\right\}. \tag{5.2}$$

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Let $\mathcal{L}$ denote the Lebesgue measure in $\mathbb{R}^2$. For a set $A \subset \mathbb{R}^2$, let A° denote the set of interior points of A, let ∂A denote the set of boundary points of A, and let $\overline{A} = A^\circ \cup \partial A$ be the closure of A. We remind the reader that $\mathbb{R}^2$ is identified with $\mathbb{C}$. By [28, theorems 1.1 and 1.2], we have

- $E_\alpha = \overline{E_\alpha^\circ}$ and $\mathcal{L}(\partial E_\alpha) = 0$;
- E_α tiles $\mathbb{C}$ with tiling set $\mathbb{Z}[\mathrm{i}]$. That is,

$$\mathbb{C} = \bigcup_{z \in \mathbb{Z}[\mathrm{i}]} (E_\alpha + z), \tag{5.3}$$

and $\mathcal{L}\big((E_\alpha + z_1) \cap (E_\alpha + z_2)\big) = 0$ for any $z_1 \neq z_2 \in \mathbb{Z}[\mathrm{i}]$.

It follows that $\mathcal{L}(E_\alpha) = \mathcal{L}(E_\alpha^\circ) = 1$ and $E_\alpha^\circ \cap (E_\alpha^\circ + z) = \emptyset$ for any $z \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}$. Note that $E_\alpha = \overline{E_\alpha^\circ}$. We conclude that

$$E_\alpha \cap (E_\alpha^\circ + z) = \emptyset \quad \text{for any } \ z \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}. \tag{5.4}$$

By (5.2) and (5.3), for each $z \in \mathbb{C}$, we can find a sequence $(d_j)_{j=1}^\infty \in D^{\mathbb{N}}$ such that

$$z - \sum_{j=1}^{\infty} d_j \alpha^{-j} \in \mathbb{Z}[\mathrm{i}]. \tag{5.5}$$

Since (α, D) is a number system in $\mathbb{Z}[\mathrm{i}]$, we can write uniquely

$$z - \sum_{j=1}^{\infty} d_j \alpha^{-j} = d_{-\ell}\alpha^{\ell} + \ldots + d_{-1}\alpha + d_0 \quad \text{with } d_0, d_{-1}, \ldots, d_{-\ell} \in D,$$

where $d_{-\ell} \neq 0$ or $\ell = 0$. We are interested in the "fractional part". The sequence $(d_j)_{j=1}^\infty$ in (5.5) is called an (α, D)-*radix expansion of* z. Note that each $z \in \mathbb{C}$ has an (α, D)-radix expansion, but not unique. A complex number $z \in \mathbb{C}$ is called (α, D)-*normal* if z has an (α, D)-radix expansion $(d_j)_{j=1}^\infty$ satisfying that for any finite block $d_1' d_2' \ldots d_k' \in D^k$,

$$\lim_{N\to\infty} \frac{1}{N} \#\big\{0 \le j \le N-1 : d_{j+1} = d_1', d_{j+2} = d_2', \ldots, d_{j+k} = d_k'\big\} = \frac{1}{|\alpha|^{2k}}.$$

We show that normality is equivalent to the equidistribution of sequence $(\alpha^n z)_{n\ge 1}$.

PROPOSITION 5.1. *A complex number $z \in \mathbb{C}$ is (α, D)-normal if and only if the sequence $(\alpha^n z)_{n\ge 1}$ is equidistributed modulo one.*

For any finite block $d_1 d_2 \ldots d_k \in D^k$, define $f_{d_1 d_2 \ldots d_k} := f_{d_1} \circ f_{d_2} \circ \cdots \circ f_{d_k}$. We first prove the sufficiency of proposition 5.1.

Proof of sufficiency of proposition 5.1. Assume that the sequence $(\alpha^n z)_{n\ge 1}$ is equidistributed modulo one. Fix a finite block $d_1' d_2' \ldots d_k' \in D^k$ and consider the bounded open set $f_{d_1' d_2' \ldots d_k'}(E_\alpha^\circ)$.

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Since $\mathcal{L}(E_\alpha) = \mathcal{L}(E_\alpha^\circ) = 1$, we have

$$\mathcal{L}\big(f_{d_1'd_2'\dots d_k'}(E_\alpha)\big) = \mathcal{L}\big(f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ)\big) = \frac{1}{|\alpha|^{2k}}.$$

It follows that the boundary of $f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ)$ has Lebesgue measure zero. Thus, for any $\varepsilon > 0$ there exist two sets E_ε and F_ε consisting of finitely many dyadic cubes such that

$$E_\varepsilon \subset f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \subset F_\varepsilon \quad \text{and} \quad \mathcal{L}(F_\varepsilon \setminus E_\varepsilon) < \varepsilon.$$

Applying the equidistribution of sequence $(\alpha^n z)_{n\geq 1}$ to sets E_ε and F_ε, and letting $\varepsilon \to 0$, we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \#\big\{1 \leq n \leq N : \alpha^n z \in f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \pmod 1\big\} = \frac{1}{|\alpha|^{2k}}. \tag{5.6}$$

Let $(d_j)_{j=1}^\infty \in D^{\mathbb{N}}$ be an (α, D)-radix expansion of z, i.e.,

$$z - \sum_{j=1}^\infty d_j \alpha^{-j} \in \mathbb{Z}[\mathrm{i}].$$

Suppose that $\alpha^n z \in f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \pmod 1$ for some $n \in \mathbb{N}$. Then we have

$$\sum_{j=1}^\infty d_{n+j}\alpha^{-j} \in f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \pmod 1.$$

It follows that

$$\sum_{j=1}^\infty d_{n+j}\alpha^{-j} \in \big(f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) + \mathbb{Z}[\mathrm{i}]\big) \cap E_\alpha.$$

Note that $f_{d_1'd_2'\dots d_k'}(E_\alpha) \subset E_\alpha$. It follows that $f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \subset E_\alpha^\circ$. By (5.4) we have $\big(f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) + \mathbb{Z}[\mathrm{i}]\big) \cap E_\alpha = f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ)$. Thus, we obtain

$$\sum_{j=1}^\infty d_{n+j}\alpha^{-j} \in f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ),$$

which implies that

$$f_{d_{n+1}d_{n+2}\dots d_{n+k}}(E_\alpha) \cap f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \neq \emptyset. \tag{5.7}$$

Suppose that $d_{n+1}d_{n+2}\dots d_{n+k} \neq d_1'd_2'\dots d_k'$. Let $\tau = \min\{1 \leq j \leq k : d_{n+j} \neq d_j'\}$. It follows from (5.7) that

$$f_{d_{n+\tau}}(E_\alpha) \cap f_{d_\tau'}(E_\alpha^\circ) \neq \emptyset.$$

That is, $E_\alpha \cap (E_\alpha^\circ + d_\tau' - d_{n+\tau}) \neq \emptyset$, which contradicts (5.4). Therefore, for any $n \in \mathbb{N}$ with $\alpha^n z \in f_{d_1'd_2'\dots d_k'}(E_\alpha^\circ) \pmod 1$, we conclude that

$$d_{n+1}d_{n+2}\dots d_{n+k} = d_1'd_2'\dots d_k'.$$

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It follows from (5.6) that

$$\liminf_{N\to\infty}\frac{1}{N}\#\{0\le n\le N-1: d_{n+1}=d'_1,\dots,d_{n+k}=d'_k\}\ge\frac{1}{|\alpha|^{2k}}. \tag{5.8}$$

Since (5.8) holds for any finite block $d'_1d'_2\dots d'_k\in D^k$, and $\#D=|\alpha|^2$, we conclude that

$$\lim_{N\to\infty}\frac{1}{N}\#\{0\le n\le N-1: d_{n+1}=d'_1,\dots,d_{n+k}=d'_k\}=\frac{1}{|\alpha|^{2k}},$$

for any finite block $d'_1d'_2\dots d'_k\in D^k$. It follows from definition that z is (α,D)-normal. □

Next, we use Weyl's criterion to show the necessity of proposition 5.1.

Proof of necessity of proposition 5.1. Let μ be the restriction of Lebesgue measure on E_α. Let λ be the Lebesgue measure on $\mathbb{T}^2=\mathbb{C}/\mathbb{Z}[\mathrm{i}]$ (or $\mathbb{R}^2/\mathbb{Z}^2$), and let π be the natural projection from $\mathbb{C}$ (or $\mathbb{R}^2$) to $\mathbb{T}^2$. Since E_α tiles $\mathbb{C}$ with tiling set $\mathbb{Z}[\mathrm{i}]$, we have $\pi\mu=\lambda$. Recall that $e(x)=e^{2\pi\mathrm{i}x}$ for $x\in\mathbb{R}$. It's well known that for any $h\in\mathbb{Z}^2\setminus\{(0,0)\}$,

$$\int_{\mathbb{T}^2}e(\langle h,x\rangle)\,\mathrm{d}\lambda(x)=0.$$

It follows that for any $h\in\mathbb{Z}^2\setminus\{(0,0)\}$,

$$\int_{\mathbb{C}}e(\langle h,x\rangle)\,\mathrm{d}\mu(x)=\int_{\mathbb{C}}e(\langle h,\pi x\rangle)\,\mathrm{d}\mu(x)=\int_{\mathbb{T}^2}e(\langle h,x\rangle)\,\mathrm{d}\pi\mu(x)=0. \tag{5.9}$$

Note that

$$E_\alpha=\bigcup_{d\in D}f_d(E_\alpha).$$

Then we have

$$\mathcal{L}(E_\alpha)\le\sum_{d\in D}\mathcal{L}\big(f_d(E_\alpha)\big)=(\#D)\cdot\frac{\mathcal{L}(E_\alpha)}{|\alpha|^2}=\mathcal{L}(E_\alpha).$$

This implies that $\mathcal{L}\big(f_{d_1}(E_\alpha)\cap f_{d_2}(E_\alpha)\big)=0$ for any $d_1\ne d_2\in D$. For any Borel set $B\subset\mathbb{C}$, we have

$$\begin{aligned}\mu(B)=\mathcal{L}(B\cap E_\alpha)&=\sum_{d\in D}\mathcal{L}(B\cap f_d(E_\alpha))\\&=\frac{1}{|\alpha|^2}\sum_{d\in D}\mathcal{L}(f_d^{-1}(B)\cap E_\alpha)\\&=\frac{1}{|\alpha|^2}\sum_{d\in D}\mu(f_d^{-1}(B)).\end{aligned}$$

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That is,

$$\mu = \frac{1}{|\alpha|^2} \sum_{d\in D} f_d \mu.$$

It follows that

$$\begin{aligned}\int_{\mathbb{C}} e(\langle h, x\rangle)\, \mathrm{d}\mu(x) &= \left(\frac{1}{|\alpha|^2} \sum_{d\in D} e(\langle h, \alpha^{-1} d\rangle)\right) \int_{\mathbb{C}} e(\langle h, \alpha^{-1} x\rangle)\, \mathrm{d}\mu(x) \\ &= \prod_{j=1}^{k} \left(\frac{1}{|\alpha|^2} \sum_{d\in D} e(\langle h, \alpha^{-j} d\rangle)\right) \int_{\mathbb{C}} e(\langle h, \alpha^{-k} x\rangle)\, \mathrm{d}\mu(x).\end{aligned}$$

Note that

$$\lim_{k\to\infty} \int_{\mathbb{C}} e(\langle h, \alpha^{-k} x\rangle)\, \mathrm{d}\mu(x) = 1.$$

By (5.9), we conclude that for any $h \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

$$\lim_{k\to\infty} \prod_{j=1}^{k} \left(\frac{1}{|\alpha|^2} \sum_{d\in D} e(\langle h, \alpha^{-j} d\rangle)\right) = 0,$$

which can be rewritten as

$$\lim_{k\to\infty} \frac{1}{|\alpha|^{2k}} \sum_{d_1 d_2 \dots d_k \in D^k} e\left(\left\langle h, \sum_{j=1}^{k} d_j \alpha^{-j} \right\rangle\right) = 0. \tag{5.10}$$

Assume that z is (α, D)-normal with a normal (α, D)-radix expansion $(d'_j)_{j=1}^{\infty} \in D^{\mathbb{N}}$. Fix $h \in \mathbb{Z}^2 \setminus \{(0,0)\}$. For $k \in \mathbb{N}$, define

$$\varepsilon_k := \sup_{z\in E_\alpha} |e\big(\langle h, \alpha^{-k} z\rangle\big) - 1|.$$

Then we have

$$\begin{aligned}&\left|\frac{1}{N} \sum_{n=1}^{N} e\big(\langle h, \alpha^n z\rangle\big)\right| = \left|\frac{1}{N} \sum_{n=1}^{N} e\left(\left\langle h, \sum_{j=1}^{\infty} d'_{n+j} \alpha^{-j} \right\rangle\right)\right| \\ \le\ &\left|\frac{1}{N} \sum_{n=1}^{N} e\left(\left\langle h, \sum_{j=1}^{k} d'_{n+j} \alpha^{-j} \right\rangle\right)\right| + \frac{1}{N} \sum_{n=1}^{N} \left|e\left(\left\langle h, \sum_{j=k+1}^{\infty} d'_{n+j} \alpha^{-j} \right\rangle\right) - 1\right| \\ \le\ &\left|\frac{1}{N} \sum_{n=1}^{N} e\left(\left\langle h, \sum_{j=1}^{k} d'_{n+j} \alpha^{-j} \right\rangle\right)\right| + \varepsilon_k.\end{aligned}$$

Note that the expansion $(d'_j)_{j=1}^{\infty}$ is normal. We obtain

$$\limsup_{N\to\infty} \left|\frac{1}{N} \sum_{n=1}^{N} e\big(\langle h, \alpha^n z\rangle\big)\right| \le \left|\frac{1}{|\alpha|^{2k}} \sum_{d_1 d_2 \dots d_k \in D^k} e\left(\left\langle h, \sum_{j=1}^{k} d_j \alpha^{-j} \right\rangle\right)\right| + \varepsilon_k.$$

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Note that

$$\lim_{k\to\infty} \varepsilon_k = 0.$$

Together with (5.10), we conclude that

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e\big(\langle h, \alpha^n z\rangle\big) \right| = 0.$$

By theorem 4.2, the sequence $(\alpha^n z)_{n\geq 1}$ is equidistributed modulo one. □

Finally, we obtain the following theorem by theorem 1.1 and proposition 5.1.

THEOREM 5.1 *Let $\beta \in \mathbb{Z}[\mathrm{i}]$ with $|\beta| > 1$. Let $T \subset \mathbb{Z}[\mathrm{i}]$ be a finite subset with $\#T \geq 2$. Let μ be a non-atomic self-similar measure with respect to the IFS*

$$\left\{ f_t(z) = \frac{z+t}{\beta} : t \in T \right\}.$$

Suppose that $\overline{\beta}$ is not a divisor of β and $\gcd(\alpha, \beta) = 1$. Then μ-almost every $z \in \mathbb{C}$ is (α, D)-normal.

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References

1. A. Algom. A simultaneous version of Host's equidistribution theorem. *Trans. Amer. Math. Soc.* **373** (2020), 8439–8462.
2. A. Algom. Actions of diagonal endomorphisms on conformally invariant measures on the 2-torus. *Monatsh. Math.* **195** (2021), 545–564.
3. A. Algom. Recent progress on pointwise normality of self-similar measures. (arXiv:2504.18192), 2025.
4. A. Algom, S. Baker and P. Shmerkin. On normal numbers and self-similar measures. *Adv. Math.* **399** (2022), 108276.
5. A. Algom, F. Rodriguez Hertz and Z. Wang. Pointwise normality and Fourier decay for self-conformal measures. *Adv. Math.* **393** (2021), 108096.
6. A. Algom, F. Rodriguez Hertz and Z. Wang. Spectral gaps and Fourier decay for self-conformal measures in the plane. (arXiv:2407.11688), 2024.
7. B. Bárány, A. Käenmäki, A. Pyörälä and M. Wu. Scaling limits of self-conformal measures. (arXiv:2308.11399), 2023.
8. S. Baker, O. Khalil and T. Sahlsten. Fourier decay from l^2-flattening. (arXiv:2407.16699), 2024.
9. A. Bisbas. Normal numbers from infinite convolution measures. *Ergodic Theory Dynam. Systems*. **23** (2003), 389–393.
10. A. Bisbas. A Cassels-Schmidt theorem for non-homogeneous Markov chains. *Bull. Sci. Math*. **129** (2005), 25–37.
11. E. Borel. Les probabilités dénombrables et leurs applications arithmétiques. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*. **27** (1909), 247–271.

https://doi.org/10.1017/prm.2025.10067 Published online by Cambridge University Press

12. G. Brown, W. Moran and C. E. M. Pearce. Riesz products and normal numbers. *J. London Math. Soc.* **32** (1985), 12–18.
13. G. Brown, W. Moran and C. E. M. Pearce. Riesz products, Hausdorff dimension and normal numbers. *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 529–540.
14. G. Brown, W. Moran and A. D. Pollington. Normality to noninteger bases. *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), 1241–1244.
15. J. W. S. Cassels. On a problem of Steinhaus about normal numbers. *Colloq. Math.* **7** (1959), 95–101.
16. H. Davenport, P. Erdos and W. J. LeVeque. On Weyl's criterion for uniform distribution. *Michigan Math. J.* **10** (1963), 311–314.
17. Y. Dayan, A. Ganguly and B. Weiss. Random walks on tori and normal numbers in self-similar sets. *Amer. J. Math.* **146** (2024), 467–493.
18. M. Drmota and R. F. Tichy. *Sequences, Discrepancies and applications* (*Lecture Notes in Mathematics*). Vol.1651, (Springer-Verlag, 1997).
19. J. Feldman and M. Smorodinsky. Normal numbers from independent processes. *Ergodic Theory Dynam. Systems*. **12** (1992), 707–712.
20. X. Gao, J. Ma, K. Song and Y. Zhang. On the Fourier transform of coin-tossing type measures. *J. Math. Anal. Appl.* **484** (2020), 14 :123706.
21. F. Q. Gouvêa. *P -Adic numbers* (*Universitext*). 3rd edn, (Springer, 2020).
22. M. Hochman. A short proof of Host's equidistribution theorem. *Israel J. Math.* **251** (2022), 527–539.
23. M. Hochman and P. Shmerkin. Equidistribution from fractal measures. *Invent. Math.* **202** (2015), 427–479.
24. B. Host. Nombres normaux, entropie, translations. *Israel J. Math.* **91** (1995), 419–428.
25. B. Host. Some results of uniform distribution in the multidimensional torus. *Ergodic Theory Dynam. Systems*. **20** (2000), 439–452.
26. J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.
27. I Kátai and J. Szabó. Canonical number systems for complex integers. *Acta Sci. Math.* **37** (1975), 255–260.
28. J. C. Lagarias and Y. Wang. Self-affine tiles in $< b > R < /b >^n$. *Adv. Math.* **121** (1996), 21–49.
29. J. Li and T. Sahlsten. Trigonometric series and self-similar sets. *J. Eur. Math. Soc.* **24** (2022), 341–368.
30. E. Lindenstrauss. p-adic foliation and equidistribution. *Israel J. Math.* **122** (2001), 29–42.
31. D. Meiri. Entropy and uniform distribution of orbits in $< b > T < /b >^d$. *Israel J. Math.* **105** (1998), 155–183.
32. D. Meiri and Y. Peres. Bi-invariant sets and measures have integer Hausdorff dimension. *Ergodic Theory Dynam. Systems*. **19** (1999), 523–534.
33. W. Moran and A. D. Pollington. Metrical results on normality to distinct bases. *J. Number Theory*. **54** (1995), 180–189.
34. J. J. Rotman. *Advanced Modern Algebra* (Prentice Hall, Inc., 2002).
35. T. Sahlsten. Fourier transforms and iterated function systems, *Recent Developments in Fractals and Related Fields: Conference on Fractals and Related Fields IV* (*ile de Porquerolles, France, 2022*), (Springer Nature, 2025), p. 297.
36. W. M. Schmidt. On normal numbers. *Pacific J. Math.* **10** (1960), 661–672.
37. P. Varjú and H. Yu. Fourier decay of self-similar measures and self-similar sets of uniqueness. *Anal. PDE*. **15** (2022), 843–858.
38. A. Vince. Replicating tessellations. *SIAM J. Discrete Math.* **6** (1993), 501–521.
39. D. D. Wall. Normal numbers. *PhD Thesis* University of California, 1949.
40. H. Weyl. Über ein problem aus dem gebiet der diophantischen approximationen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*. (1914), 234–244.
41. H. Weyl. Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.* **77** (1916), 313–352.
42. Y. Xiong and J. Zhao. Equidistribution for measures defined by digit restrictions. *Nonlinearity*. **35** (2022), 4064–4084.

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