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A uniqueness theorem for the coagulation-fragmentation equation

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1. *Introduction*

This paper presents a uniqueness result for solutions to the general nonlinear coagulation-fragmentation equation

$$\left.\begin{aligned} \frac{\partial c}{\partial t}(x,t) &= \frac{1}{2}\int_0^x J(x-y,y,c(t))\,dy - \int_0^\infty J(x,y,c(t))\,dy \\ c(x,0) &= c_0(x) \geqslant 0, \end{aligned}\right\} \tag{1·1}$$

where

$$J(x,y,c(t)) = K(x,y)\,c(x,t)\,c(y,t) - F(x,y)\,c(x+y,t). \tag{1·2}$$

Equation (1·1) has many applications in the applied sciences (cf. [**1**, **3**, **8**, **13**, **15**]) and a brief physical interpretation can be found in Melzak [**12**] or the survey article by Drake [**7**]. $c(x,t)$, for $x \geqslant 0$, $t \geqslant 0$, denotes the number of particles of size x at time t and the non-negative kernels K and F describe, respectively, the rates at which particles of size x coalesce with those of size y and particles of size $(x+y)$ break-up into those of sizes x and y.

Global existence and uniqueness of solutions to (1·1) has been proved by Aizenman and Bak [**1**] for constant kernels and by Melzak [**12**] for particular classes of bounded kernels K and F. Various existence and uniqueness results for unbounded K (with $F \equiv 0$) have been derived in [**4**, **5**, **8**, **9**, **10**, **11**, **13**] and Ziff and McGrady [**15**] have demonstrated existence for certain explicit bounded and unbounded kernels F (with $K \equiv 0$). Stewart [**14**] removed many of the restrictive conditions imposed in the above results (e.g. [**9**] requires $\int_0^\infty x^k c(x,0)\,dx < \infty$ for all $k \geqslant 2$) and proved global existence where K and F may be simultaneously unbounded functions satisfying certain growth conditions. The uniqueness theorem given in this paper similarly examines K and F together and aims to remove the stringent hypotheses introduced in the aforementioned references.

It is known that non-uniqueness can arise for an analogous discrete system of equations (cf. [**2**]) and this, together with the following example, motives the study of uniqueness presented below. If $K \equiv 0$, $F \equiv 2$ and $c_0(x) = (\lambda+x)^{-3}$ for some $\lambda > 0$, then it can be verified that

$$c(x,t) = \exp\{\lambda t\}\,c_0(x) \tag{1·3}$$

is a solution of (1·1). However, Ziff and McGrady [**15**] show that

$$c(x,t) = \exp\{-tx\}\left(c_0(x) + \int_x^\infty c_0(y)\,[2t+t^2(y-x)]\,dy\right) \tag{1·4}$$

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is also a solution, thus demonstrating non-uniqueness for (1·1). Furthermore, it can be shown that (1·4) is a density conserving solution (i.e. $\int_0^\infty xc(x,t)\,dx = \int_0^\infty xc_0(x)\,dx$ for all $t \geqslant 0$) while (1·3) is clearly not density conserving. It is hoped the author will return to the problem of density conservation in a future paper.

In order to investigate uniqueness in Section 2 below we now introduce some definitions and notation. The phrase 'almost everywhere' will be abbreviated to the customary 'a.e.'. It will be assumed throughout that the kernels K and F are continuous non-negative functions with $K(x,y) = K(y,x)$ and $F(x,y) = F(y,x)$ for all $x \geqslant 0$ and $y \geqslant 0$. Let X be the following Banach space with norm $\|\cdot\|$:

$$X = \{y \in L^1(0,\infty) : \|y\| < \infty\} \quad \text{where} \quad \|y\| = \int_0^\infty (1+x)\,|y(x)|\,dx$$

and set $X^+ = \{y \in X : y \geqslant 0 \text{ a.e.}\}$.

Definition (cf. Stewart[**14**]). Let $0 < T \leqslant \infty$. A solution c of (1·1) is a function $c : [0,T) \to X^+$ such that for a.e. $x \in [0,\infty)$

(i) $c(x,t) \geqslant 0$ for all $t \in [0,T)$,

(ii) $c(x,\cdot)$ is continuous on $[0,T)$,

(iii) for all $t \in [0,T)$,

$$\int_0^t \int_0^\infty K(x,y)\,c(y,s)\,dy\,ds < \infty \quad \text{and} \quad \int_0^t \int_0^\infty F(x,y)\,c(x+y,s)\,dy\,ds < \infty,$$

(iv) for all $t \in [0,T)$,

$$c(x,t) = c(x,0) + \int_0^t \left[\frac{1}{2}\int_0^x J(x-y,y,c(s))\,dy - \int_0^\infty J(x,y,c(s))\,dy\right] ds.$$

By (iv), $c(x,\cdot)$ is absolutely continuous on $[0,T)$ for a.e. $x \in [0,\infty)$ and therefore $c(x,t)$ satisfies (1·1) for a.e. $t \in [0,T)$. Notice that the initial data c_0 need only be in X^+. Further comments on the choice of the space X^+ can be found in [**14**].

2. *Uniqueness*

The proof of the following theorem relies on detailed manipulations of integrals and is adapted from the method used by Carr[**6**] for uniqueness of solutions to an analogous discrete system of equations.

THEOREM. *If* (H1): $K(x,y) \leqslant \phi(x)\,\phi(y)$ *for all* x *and* y *where* $\phi(x) \leqslant k(1+x)^{\frac{1}{2}}$ *for some constant* k, *and*; (H2): *for all* $x \geqslant 0$ *there is a constant* m *such that*

$$\int_0^x (1+y)^{\frac{1}{2}} F(x-y,y)\,dy \leqslant m(1+x)^{\frac{1}{2}}$$

then solutions to (1·1) *are unique.*

Proof. For $\lambda \in \mathbb{R}$ define $\operatorname{sgn}(\lambda)$ to equal $1, 0, -1$ whenever $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively. Let c and d be two solutions to (1·1) on $[0,T)$, where $T > 0$, with $c(0) = d(0)$, and set $Y = c - d$. For $n = 1, 2, 3 \ldots$ define

$$u^n(t) = \int_0^n (1+x)^{\frac{1}{2}}\,|Y(x,t)|\,dx.$$

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Multiplying $|Y|$ by $(1+x)^{\frac{1}{2}}$ and applying Fubini's theorem to (iv) above we obtain for each n and $0 < t < T$

$$u^n(t) = \int_0^t \int_0^n (1+x)^{\frac{1}{2}} \operatorname{sgn}(Y(x,s)) \left[\frac{1}{2} \int_0^x (J(x-y,y,c(s)) - J(x-y,y,d(s)))\, dy \right.$$
$$\left. - \int_0^\infty (J(x,y,c(s)) - J(x,y,d(s)))\, dy \right] dx\, ds. \quad (2{\cdot}1)$$

Using the substitution $x' = x-y, y' = y$ in the first integral on the right-hand side of (2·1) we find that

$$u^n(t) = \int_0^t \int_0^n \int_0^{n-x} [\tfrac{1}{2}(1+x+y)^{\frac{1}{2}} \operatorname{sgn}(Y(x+y,s)) - (1+x)^{\frac{1}{2}} \operatorname{sgn}(Y(x,s))]$$
$$\times [J(x,y,c(s)) - J(x,y,d(s))]\, dy\, dx\, ds$$
$$- \int_0^t \int_0^n \int_{n-x}^\infty (1+x)^{\frac{1}{2}} \operatorname{sgn}(Y(x,s))\, [J(x,y,c(s)) - J(x,y,d(s))]\, dy\, dx\, ds. \quad (2{\cdot}2)$$

We note that by interchanging the order of integration (and interchanging the roles of x and y) the symmetry of J yields the identity

$$\int_0^n \int_0^{n-x} (1+x)^{\frac{1}{2}} \operatorname{sgn}(Y(x,s))\, J(x,y,c(s))\, dy\, dx$$
$$= \int_0^n \int_0^{n-x} (1+y)^{\frac{1}{2}} \operatorname{sgn}(Y(y,s))\, J(x,y,c(s))\, dy\, dx. \quad (2{\cdot}3)$$

(2·3) similarly holds for solution d. For $x, y \geqslant 0$ and $t \in [0,T)$ define f by

$$f(x,y,t) = (1+x+y)^{\frac{1}{2}} \operatorname{sgn}(Y(x+y,t)) - (1+x)^{\frac{1}{2}} \operatorname{sgn}(Y(x,t)) - (1+y)^{\frac{1}{2}} \operatorname{sgn}(Y(y,t)).$$

Using (2·3) we can show that (2·2) can be rewritten as

$$u^n(t) = \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} f(x,y,s)\, K(x,y)\, c(x,s)\, Y(y,s)\, dy\, dx\, ds$$
$$+ \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} f(x,y,s)\, K(x,y)\, d(y,s)\, Y(x,s)\, dy\, dx\, ds$$
$$- \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} f(x,y,s)\, F(x,y)\, Y(x+y,s)\, dy\, dx\, ds$$
$$- \int_0^t \int_0^n \int_{n-x}^\infty (1+x)^{\frac{1}{2}} \operatorname{sgn}(Y(x,s))\, [J(x,y,c(s)) - J(x,y,d(s))]\, dy\, dx\, ds$$
$$= \int_0^t \sum_{i=1}^4 S_i^n(s)\, ds, \quad (2{\cdot}4)$$

where S_i^n, for $i = 1, \ldots, 4$, are the corresponding integrands in the preceding line.

We now consider each S_i^n individually. Noting that for all $\lambda, \gamma \in \mathbb{R}$ we have $\operatorname{sgn}(\lambda) \operatorname{sgn}(\gamma) = \operatorname{sgn}(\lambda\gamma)$ and $|\lambda| = \lambda \operatorname{sgn}(\lambda)$, we find that

$$f(x,y,s)\, Y(y,s) \leqslant [(1+x+y)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}} - (1+y)^{\frac{1}{2}}]\, |Y(y,s)| \leqslant 2(1+x)^{\frac{1}{2}} |Y(y,s)|. \quad (2{\cdot}5)$$

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Thus by Hölder's inequality, hypothesis (H1) and (2·5),

$$\int_0^t S_1^n(s)\,ds \leqslant L_1 \int_0^t u^n(s)\,ds, \tag{2·6}$$

where $$L_1 = k^2 \sup_{s\in[0,t]} \|c(s)\|.$$

Similarly, there is a constant L_2 such that

$$\int_0^t S_2^n(s)\,ds \leqslant L_2 \int_0^t u^n(s)\,ds. \tag{2·7}$$

To consider S_3^n we first observe that

$$-f(x,y,s)\,Y(x+y,s) \leqslant [(1+x)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}-(1+x+y)^{\frac{1}{2}}]\,|Y(x+y,s)|. \tag{2·8}$$

By (2·8), Fubini's theorem, hypothesis (H2) and the symmetry of F

$$\begin{aligned}
\int_0^t S_3^n(s)\,ds &\leqslant \frac{1}{2}\int_0^t\int_0^n\int_0^{n-x} [(1+x)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}-(1+x+y)^{\frac{1}{2}}]F(x,y)\,|\,Y(x+y,s)\,|\,dy\,dx\,ds \\
&= \frac{1}{2}\int_0^t\int_0^n\int_x^n [(1+x)^{\frac{1}{2}}+(1+y-x)^{\frac{1}{2}}-(1+y)^{\frac{1}{2}}]F(x,y-x)\,|\,Y(y,s)|\,dy\,dx\,ds \\
&= \frac{1}{2}\int_0^t\int_0^n\int_0^x [(1+x-y)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}-(1+x)^{\frac{1}{2}}]F(x-y,y)\,|\,Y(x,s)|\,dy\,dx\,ds \\
&\leqslant \frac{1}{2}\int_0^t\int_0^n\int_0^x (1+y)^{\frac{1}{2}}F(x-y,y)\,|Y(x,s)|\,dy\,dx\,ds \\
&\leqslant L_3\int_0^t u^n(s)\,ds
\end{aligned} \tag{2·9}$$

where $L_3 = \frac{1}{2}m$. Let χ denote the characteristic function, i.e. for any set E we have $\chi_E(x) = 1$ if $x\in E$ and zero otherwise. Define

$$g_n(x,s) = \chi_{[0,n]}(x)\,(1+x)\,c(x,s)\int_{n-x}^{\infty}(1+y)^{\frac{1}{2}}c(y,s)\,dy$$

and $$l_n(s) = \int_{\mathbf{R}} g_n(x,s)\,dx.$$

Clearly, for each $s\in[0,t]$ and a.e. $x\in[0,\infty)$,

$$|g_n(x,s)| \leqslant (1+x)\,c(x,s)\,\|c(s)\|$$

for all n and $g_n(x,s)\to 0$ as $n\to\infty$. Thus by the dominated convergence theorem $l_n(s)\to 0$ as $n\to\infty$. Moreover

$$|l_n(s)| \leqslant \sup_{s\in[0,t]}\|c(s)\|^2,$$

and therefore a further application of the dominated convergence theorem leads to

$$\int_0^t l_n(s)\,ds \to 0 \quad \text{as } n\to\infty. \tag{2·10}$$

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Using the symmetry of F, Fubini's theorem and hypothesis (H2) we have for each $s\in[0,t]$

$$\begin{aligned}\int_0^n\int_{n-x}^{\infty}(1+x)^{\frac{1}{2}}F(x,y)\,c(x+y,s)\,dy\,dx &= \int_0^n\int_n^{\infty}(1+x)^{\frac{1}{2}}F(x,y-x)\,c(y,s)\,dy\,dx\\ &= \int_n^{\infty}\int_0^n(1+y)^{\frac{1}{2}}F(x-y,y)\,c(x,s)\,dy\,dx\\ &\leqslant \int_n^{\infty}\int_0^x(1+y)^{\frac{1}{2}}F(x-y,y)\,c(x,s)\,dy\,dx\\ &\leqslant m\int_n^{\infty}(1+x)^{\frac{1}{2}}c(x,s)\,dx\\ &\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{2·11}$$

The right-hand side of (2·11) is always bounded by the constant $\sup_{s\in[0,t]}\|c(s)\|\,m$ and therefore

$$\int_0^t\int_0^n\int_{n-x}^{\infty}(1+x)^{\frac{1}{2}}F(x,y)\,c(x+y,s)\,dy\,dx\,ds\to 0 \quad \text{as } n\to\infty. \tag{2·12}$$

(2·10) and (2·12) are similarly true for the solution d and consequently

$$\int_0^t S_4^n(s)\,ds\to 0 \quad \text{as } n\to\infty. \tag{2·13}$$

We now set $L=L_1+L_2+L_3$. Applying (2·6), (2·7), (2·9) and (2·13) to equation (2·4) and taking limits as $n\to\infty$ gives

$$\begin{aligned}\int_0^{\infty}(1+x)^{\frac{1}{2}}|Y(x,t)|\,dx &= \lim_{n\to\infty}u^n(t) \leqslant \lim_{n\to\infty}L\int_0^t u^n(s)\,ds+\lim_{n\to\infty}\int_0^t S_4^n(s)\,ds\\ &= L\int_0^t\int_0^{\infty}(1+x)^{\frac{1}{2}}|Y(x,s)|\,dx\,ds. \end{aligned} \tag{2·14}$$

An application of Gronwall's inequality to (2·14) shows that

$$\int_0^{\infty}(1+x)^{\frac{1}{2}}|Y(x,t)|\,dx=0$$

for all $t\in[0,T)$ and therefore $c(x,t)=d(x,t)$ for a.e. $x\in[0,\infty)$, which completes the proof of the theorem.

Remarks. It should be pointed out that (H2) is not satisfied by constant kernels F. It would be interesting to find an improved type of estimate hypothesis to replace (H2) or to find conditions under which constant kernels F produce unique solutions. A method of proof different from the above may be required for such results.

To demonstrate the existence and uniqueness of solutions for certain unbounded K and bounded F it is easily verified that with $c_0\in X^+$ symmetric kernels of the form

$$K(x,y)\leqslant\phi(x)\,\phi(y), \quad \phi(x)\leqslant k(1+x)^{\alpha}, \quad \alpha<\tfrac{1}{2}$$

and

$$F(x,y)\leqslant m(1+x+y)^{-\beta}, \quad \beta>1.$$

where k and m are constants, satisfy (H1), (H2) and the hypotheses of the global existence theorem in Stewart [**14**].

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