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# Generalised (2+1)-dimensional Super MKdV Hierarchy for Integrable Systems in Soliton Theory

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**Abstract.** Much attention has been given to constructing Lie and Lie superalgebra for integrable systems in soliton theory, which often have significant scientific applications. However, this has mostly been confined to (1+1)-dimensional integrable systems, and there has been very little work on (2+1)-dimensional integrable systems. In this article, we construct a class of generalised Lie superalgebra that differs from more common Lie superalgebra to generate a (2+1)-dimensional super modified Korteweg-de Vries (mKdV) hierarchy, via a generalised Tu scheme based on the Lax pair method where the Hamiltonian structure derives from a generalised supertrace identity. We also obtain some solutions of the (2+1)-dimensional mKdV equation using the $G'/G^2$ method.

**AMS subject classifications**: 35Q51, 37K05, 37K10

**Key words**: Soliton theory, generalised Lie superalgebra, (2+1)-dimensional super mKdV hierarchy, supertrace identity, $G'/G^2$ method.

## 1. Introduction

In soliton theory, new physically justified integrable and super integrable systems are often sought, and different methods for generating integrable and super integrable systems have previously been proposed — e.g. Refs. [1–4]. An efficient approach due to Tu [5] generates integrable systems as well as Hamiltonian structures based on the Lax pair method, and following Ma [6] we call this approach the Tu scheme. Some researchers have applied the Tu method to obtain (1+1)-dimensional integrable and super integrable systems [7–12]. With the development of solitary wave theory, attention has turned to more applicable (2+1)-dimensional integrable systems. Tu himself [13] obtained the Kadomtsev-Petviashvili (KP) and Davey-Stewartson(DS) equations, and recently Zhang *et al.* [14, 15] discussed several other (2+1)-dimensional equations. Various methods have been used to solve the (1+1)-dimensional equations [16–21], and solutions of (2+1)-dimensional and even (3+1)-dimensional nonlinear partial differential equations have also been found [22, 23].

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In this article, we use the generalised Tu scheme to generate a (2+1)-dimensional super mKdV hierarchy and obtain some solutions. The main steps of the Tu scheme are as follows.

(1) Define an appropriate matrix $B = \{(b_{ij})_{n\times n}\} = \{f_1, f_2, \cdots, f_p\}$, to express the loop Lie superalgebra as $\widetilde{B} = \{B\lambda^n\} = \{f_1(n), f_2(n), \cdots, f_p(n)\} = \{f_1\lambda^n f_2\lambda^n, f_3\lambda^n, \cdots f_p\lambda^n\}$.
(2) Construct the isospectral problem $\varphi_x = UM\varphi$, $\varphi_t = VM\varphi$ involving two matrices $U$ and $V$, where $U = u_1f_1(n) + u_2f_2(n) + u_3f_3(n) + \cdots + u_pf_p(n)$ and $u_1, u_2, u_3, \cdots, u_p$ are functions of $x, y$ and $t$, $\psi$ is a vector function, and $M \in B$.
(3) Solve the static zero curvature equation $V_x = [U, V] = UMV - VMU$ where $V = \sum V_m\lambda^{-m}$, to get recursive equations.
(4) Establish $-V^n_{+x} + [U, V^{(n)}_+] \in C_1f_1 + C_2f_2 + C_3f_3 + \cdots + C_pf_p$; or if not, construct a correction term $\Delta_n \in \widetilde{B}$ such that $V^{(n)} = V^{(n)}_+ + \Delta_n$ and $-V^{(n)}_x + [U, V^{(n)}] \in C_1f_1 + C_2f_2 + C_3f_3 + \cdots + C_pf_p$.
(5) Solve the equation $U_t - V^n_x + [U, V^{(n)}] = 0$ to obtain $U_{t_n} = Jg^{(n)}$, where $g^{(n)}$ satisfies $\langle V, \partial U/\partial u_i\rangle = \sum g^{(n)}\lambda^{-n}$ and $J$ is a Hamiltonian operator; and then calculate $g^{(n)} = Lg^{(n-1)}$ to determine the recurrence operator $L$.
(6) Use the (2+1)-dimensional supertrace identity to find the Hamiltonian structure.

Further, in proceeding to the (2+1)-dimensional super mKdV hierarchy and its Hamiltonian structure of interest here, some further operators and ideas are needed as follows.

(1) We constitute an associative algebra $\mathscr{A}[\xi]$, which includes all pseudo-differential operators $\sum_{i=-\infty}^{N} a_i\xi^i$, $a_i \in \mathscr{A}$, where $\xi$ represent an operator defined by

$$\xi f = f\xi + \partial f, \quad f \in \mathscr{A}; \tag{1.1}$$

and on the basis of Eq. (1.1) infer that

$$\xi^n f = \sum_{n\geq 0}\binom{n}{i}(\partial^i f)\xi^{n-i},$$
$$f\xi^n = \sum_{n\geq 0}(-1)\binom{n}{i}\xi^{n-i}(\partial^i f), \quad n \in Z,$$

and introduce a residue operator $R : \mathscr{A}[\xi] \longrightarrow \mathscr{A}, R(\sum a_i\xi^i) = a_{-1}$.

(2) Adjusting the derivation $\partial : \partial = \partial_y$, we obtain the algebra $\mathscr{A}[\xi]$ involving the operator $\xi$ such that

$$\xi f = f\xi + \partial_y f, \quad f \in \mathscr{A}, \tag{1.2}$$

and expand the operators $\partial_x$ and $\partial_y$ from $\mathscr{A}$ to $\mathscr{A}[\xi]$:

$$\partial_x\left(\sum a_i\xi^i\right) = \sum(\partial_x a_i)\,\xi^i,$$
$$\partial_y\left(\sum a_i\xi^i\right) = \sum\left(\partial_y a_i\right)\xi^i.$$

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(3) Setting $\xi : \mathscr{A}[\xi] \to \mathscr{A}[\xi]$, a mapping from $\mathscr{A}[\xi]$ to itself, for $\xi : a \in \mathscr{A}[\xi] \to \xi a \in \mathscr{A}[\xi]$ we obtain $\xi f = \xi^* f + \partial_y f$ involving the conjugate mapping $\xi^*$ where

$$\xi^* a = a\xi\,,\ a \in \mathscr{A}[\xi]\,. \tag{1.3}$$

(4) By extension $\langle V, \partial U/\partial u_i\rangle = \Sigma_i g_i^{(n)} \lambda^{-n}$, and consequently via recurrence relations we have $g^{(n)} \equiv (g_1^n, g_2^n, \cdots, g_p^n)^T$ where

$$\langle a, b\rangle = \operatorname{tr} R(ab)\,.$$

(5) Assume that $U$ and $V$ satisfy the (2+1)-dimensional supertrace identity

$$\frac{\delta}{\delta u_i}\langle V, U_\lambda\rangle = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^\gamma\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle. \tag{1.4}$$

(6) Under Eq. (1.4) and setting $G(V) = f(V,V)$, we have that $G(V)$ is independent of $x$, $(G(V))_x = 0$ and $G(V)$ is a power function in $\lambda$ given by $G(V) = c\lambda^{-2\gamma}$ where $c$ is a constant, such that

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|G(V)|\,. \tag{1.5}$$

In Section 2, the (2+1)-dimensional super integrable system is given, and on applying the generalised supertrace identity the corresponding Hamiltonian structure is obtained. In Section 3, the $G'/G^2$ method [24] and solutions of (2+1)-dimensional super mKdV hierarchy obtained are discussed, and some conclusions are drawn in Section 4.

## 2. The (2+1)-dimensional Super MKdV Integrable Hierarchy

The Lie superalgebra adopted is

$$B = \operatorname{span}\{f_1, f_2, f_3, f_4, f_5\}\,,$$

where

$$f_1 = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},\quad f_2 = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad f_3 = \begin{pmatrix} 1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$
$$f_4 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 1 & 0 & 0\end{pmatrix},\quad f_5 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 1 & 0\end{pmatrix},\quad M = \begin{pmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and the commutation and anti-commutation relations may be defined by

$$[a, b] = aMb - (-1)^{p(a)p(b)}bMa\,, \tag{2.1}$$

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such that

$$\begin{array}{llll}
[f_1,f_2]=-2f_2, & [f_1,f_3]=2(f_2+f_3), & [f_2,f_3]=-2f_1, & [f_1,f_4]=-f_4,\\
[f_1,f_5]=f_5-f_4, & [f_2,f_4]=0, & [f_2,f_5]=-f_4, & [f_3,f_4]=f_4-2f_5,\\
[f_3,f_5]=f_4-f_5, & [f_4,\ f_4]_+=-2f_2, & [f_4,f_5]_+=f_1-f_2, & [f_5,\ f_5]_+=f_1+f_3.
\end{array}$$

The loop algebra of the Lie superalgebra B can be expressed as

$$\widetilde{B}=span\left\{f_1(n),f_2(n),f_3(n),f_4(n),f_5(n)\right\},$$

where $f_i(n)=f_i\lambda^n$, $[f_i(n),f_j(m)]=[f_i,f_j]\lambda^{n+m}$. In terms of $f_1(n)$, $f_2(n)$, $f_3(n)$, $f_4(n)$, $f_5(n)$, any element in $\widetilde{B}$ can be written as

$$\begin{pmatrix} b_1+b_3 & 2b_3 & b_5\\ b_2 & -b_1+b_3 & -b_4\\ b_4+b_5 & b_5 & 0 \end{pmatrix}.$$

Let us consider the following isospectral problem:

$$\varphi_x=UM\varphi, U=\begin{pmatrix} \lambda+u_1+\xi+u_3 & 2u_3 & u_5\\ u_2 & -\lambda-u_1-\xi+u_3 & -u_4\\ u_4+u_5 & u_5 & 0 \end{pmatrix}, \tag{2.2}$$

$$\varphi_t=VM\varphi, V=Af_1(0)+Bf_2(0)+Cf_3(0)+Df_4(0)+Ef_5(0). \tag{2.3}$$

According to the generalised Tu scheme and Eqs. (2.1)-(2.3), we solve the stationary zero curvature equation

$$V_x=[U,V] \tag{2.4}$$

and can then set

$$A_y=0,\quad D_y=0,\quad E_y=0,$$

to arrive at the recursion relation

$$\begin{cases}
A_x=-2u_2C+2u_3B+u_4E+u_5D+u_5E,\\
B_x=-2\lambda B-2u_1B-2B\xi-B_y+2\lambda C+2u_1C+2C\xi+C_y+2u_2A\\
\quad -2u_3A-2u_4D-u_4E-u_5D,\\
C_x=2\lambda C+2u_1C+2C\xi+C_y-2u_3A+u_5E,\\
D_x=-\lambda D-u_1D-D\xi-\lambda E-u_1E-E\xi-u_2E+u_3D+u_3E+u_4A\\
\quad -u_4C+u_5A+u_5B-u_5C,\\
E_x=\lambda E+u_1E+E\xi-2u_3D-u_3E+2u_4C-u_5A+u_5C.
\end{cases} \tag{2.5}$$

Taking

$$A=\sum_{m\geq 0}A_m\lambda^{-m},\ B=\sum_{m\geq 0}B_m\lambda^{-m},\ C=\sum_{m\geq 0}C_m\lambda^{-m},\ D=\sum_{m\geq 0}D_m\lambda^{-m},\ E=\sum_{m\geq 0}E_m\lambda^{-m},$$

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Eq. (2.5) can be written as

$$\begin{cases} A_{mx} = -2u_2C_m + 2u_3B_m + u_4E_m + u_5D_m + u_5E_m\,, \\ B_{mx} = -2B_{m+1} - 2u_1B_m - 2B_m\xi - B_{my} + 2C_{m+1} + 2u_1C_m + 2C_m\xi \\ \qquad +C_{my} + 2u_2A_m - 2u_3A_m - 2u_4D_m - u_4E_m - u_5D_m\,, \\ C_{mx} = 2C_{m+1} + 2u_1C_m + 2C_m\xi + C_{my} - 2u_3A_m + u_5E_m\,, \\ D_{mx} = -D_{m+1} - u_1D_m - D_m\xi - E_{m+1} - u_1E_m - E_m\xi - u_2E_m + u_3D_m \\ \qquad +u_3E_m + u_4A_m - u_4C_m + u_5A_m + u_5B_m - u_5C_m\,, \\ E_{mx} = E_{m+1} + u_1E_m + E_m\xi - 2u_3D_m - u_3E_m + 2u_4C_m - u_5A_m + u_5C_m\,. \end{cases}$$

On adopting the initial values

$$A_0 = \xi^{-1}\,, \qquad B_0 = C_0 = D_0 = E_0 = 0\,,$$

we consequently obtain

$$\begin{cases} A_1 = \partial_x^{-1}\left(2u_4u_5\xi^{-1} + u_5^2\xi^{-1}\right)\,, \\ B_1 = u_2\xi^{-1}\,, \\ C_1 = u_3\xi^{-1}\,, \\ D_1 = u_4\xi^{-1}, \\ E_1 = u_5\xi^{-1}\,, \\ A_2 = \partial_x^{-1}\left\{\left[-u_2(u_{3x} - u_{3y}) - u_3(u_{2x} + u_{2y}) + u_2u_5^2 + u_3u_{3x} - u_3u_5^2 - 2u_3u_4^2 - 2u_3u_4u_5\right.\right. \\ \qquad \left.\left. +u_4u_{5x} - u_5u_{4x} - 2u_1u_4u_5 - u_1u_5^2 + (2u_4u_5 + u_5^2)\partial_x^{-1}(2u_4u_5 + u_5^2) + u_5u_{5xx}\right]\xi^{-1}\right. \\ \qquad \left. -2u_4u_5 - u_5^2\right\}\,, \\ B_2 = \left[-\frac{1}{2}u_{2x} - \frac{1}{2}u_{2y} - u_1u_2 + \frac{1}{2}u_{3x} - \frac{1}{2}u_5^2 + u_2\partial_x^{-1}(2u_4u_5 + u_5^2) - u_4^2 - u_4u_5\right]\xi^{-1} - u_2\,, \\ C_2 = \left[\frac{1}{2}u_{3x} - \frac{1}{2}u_{3y} - u_1u_3 + u_3\partial_x^{-1}(2u_4u_5 + u_5^2) - \frac{1}{2}u_5^2\right]\xi^{-1} - u_3\,, \\ D_2 = \left[-u_{4x} - u_1u_4 + u_4\partial_x^{-1}(2u_4u_5 + u_5^2) - u_{5x}\right]\xi^{-1} - u_4\,, \\ E_2 = \left[u_{5x} - u_1u_5 + u_5\partial_x^{-1}(2u_4u_5 + u_5^2)\right]\xi^{-1} - u_5\,. \end{cases} \tag{2.6}$$

Under the above assumptions, on setting

$$\begin{aligned} V_+^{(n)} &= \sum_{m\geq 0}^{n}\left[A_mf_1(n-m) + B_mf_2(n-m) + C_mf_3(n-m) + D_mf_4(n-m) + E_mf_5(n-m)\right]\,, \\ V_-^{(n)} &= \lambda^nV - V_+^{(n)}\,, \end{aligned} \tag{2.7}$$

Eq. (2.4) can be decomposed into the form

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}],$$

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whence

$$-V_{+x}^{(n)}+[U,V_{+}^{(n)}]=(2B_{n+1}-2C_{n+1})f_2(0)-2C_{n+1}f_3(0)+(D_{n+1}+E_{n+1})f_4(0)-E_{n+1}f_5(0)\,.$$

Denoting

$$V^{(n)}=V_{+}^{(n)}+\Delta_n\,,\quad \Delta_n=-A_nf_1(0)\,,$$

we obtain

$$\begin{aligned}-V_x^n+[U,V^{(n)}]=&A_{nx}f_1(0)+(2B_{n+1}-2C_{n+1}-2u_2A_n+2u_3A_n)f_2(0)\\&+(-2C_{n+1}+2u_3A_n)f_3(0)+(D_{n+1}+E_{n+1}-u_4A_n-u_5A_n)f_4(0)\\&+(-E_{n+1}+u_5A_n)f_5(0)\,.\end{aligned}$$

Thus the curvature equation

$$U_t-V_x^n+[U,V^{(n)}]=0,$$

admits the relations

$$\begin{cases}u_{1t}=-A_{nx}\,,\\ u_{2t}=B_{nx}+2u_1B_n+B_n\xi+\xi B_n-2u_1C_n-C_n\xi-\xi C_n+2u_4D_n+u_4E_n+u_5D_n\,,\\ u_{3t}=C_{nx}-2u_1C_n-C_n\xi-\xi C_n-u_5E_n\,,\\ u_{4t}=D_{nx}+u_1D_n+D_n\xi+u_1E_n+E_n\xi+u_2E_n-u_3D_n-u_3E_n+u_4C_n-u_5B_n+u_5C_n\,,\\ u_{5t}=E_{nx}-u_1E_n-E_n\xi+2u_3D_n+u_3E_n-2u_4C_n-u_5C_n\,.\end{cases}\tag{2.8}$$

Invoking Eqs. (1.2)-(1.3) and (2.8) and a generalised Tu scheme, we now form the (2+1)-dimensional hierarchy of evolution equations

$$\begin{aligned}U_{t_n}=\begin{pmatrix}u_{1t_n}\\u_{2t_n}\\u_{3t_n}\\u_{4t_n}\\u_{5t_n}\end{pmatrix}&=\begin{pmatrix}-\partial_x&0&0&0&0\\0&\partial_x&J_{23}&u_4&-2u_4-u_5\\0&J_{32}&0&-u_5&0\\0&u_4&-u_5&J_{44}&J_{45}\\0&-2u_4-u_5&0&J_{54}&-2u_3\end{pmatrix}\begin{pmatrix}R(A_n)\\R(C_n)\\R(B_n-C_n)\\R(E_n)\\R(-D_n)\end{pmatrix}\\&=J\begin{pmatrix}R(A_n)\\R(C_n)\\R(B_n-C_n)\\R(E_n)\\R(-D_n)\end{pmatrix}=Jg^n\,,\end{aligned}\tag{2.9}$$

involving $g^n=(R(A_n),R(C_n),R(B_n-C_n),R(E_n),R(-D_n))^T$ and the Hamiltonian operator $J$ such that

$$\begin{cases}J_{23}=\partial_x+2u_1+\xi+\xi^*\,,\\ J_{32}=\partial_x-2u_1-\xi-\xi^*\,,\\ J_{44}=u_1+u_2-u_3+\xi^*\,,\\ J_{45}=-\partial_x-u_1-\xi^*+u_3\,,\\ J_{54}=\partial_x-u_1-\xi^*+u_3\,.\end{cases}$$

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The recurrence relation

$$g^{n+1} = Lg^n,$$

involves the recurrence operator

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ u_3 & L_{21} & 0 & -\frac{1}{2}u_5 & 0 \\ u_2 - u_3 & -\frac{1}{2}\partial_x & L_{33} & -\frac{1}{2}u_4 & u_4 + \frac{1}{2}u_5 \\ u_5 & -2u_4 - u_5 & 0 & L_{44} & -2u_3 \\ -u_4 & -u_4 - u_5 & -u_5 & \partial_x + u_2 & L_{55} \end{pmatrix},$$

where

$$\begin{cases} L_{11} = \partial_x^{-1}(2u_4u_5 + u_5^2), \\ L_{12} = \partial_x^{-1}\left(-u_2(\partial_x - \partial_y) + 2u_1u_2 + 2u_2\xi^* - 2u_3u_1 - 2u_3\xi^* - u_3\partial_y - 2u_4^2 - 2u_4u_5\right), \\ L_{13} = \partial_x^{-1}(-u_3\partial_x - 2u_3u_1 - 2u_3\xi^* - u_3\partial_y + u_5^2), \\ L_{14} = \partial_x^{-1}(u_4\partial_x - u_1u_4 - u_4\xi^* - u_1u_5 - u_5\xi^*), \\ L_{15} = \partial_x^{-1}(u_5\partial_x + u_1u_5 + u_5\xi^*), \\ L_{21} = \dfrac{1}{2}(\partial_x - \partial_y) - u_1 - \xi^*, \\ L_{33} = -\dfrac{1}{2}(\partial_x + \partial_y) - u_1 - \xi^*, \\ L_{44} = \partial_x - u_1 - \xi^* + u_3, \\ L_{55} = -\partial_x - u_1 - \xi^* - u_3. \end{cases}$$

Consequently, Eq. (2.9) can be written as

$$U_{t_n} = \begin{pmatrix} u_{1t_n} \\ u_{2t_n} \\ u_{3t_n} \\ u_{4t_n} \\ u_{5t_n} \end{pmatrix} = JL^n g^0 = JL^n \begin{pmatrix} R(A_0) \\ R(C_0) \\ R(B_0 - C_0) \\ R(E_0) \\ R(-D_0) \end{pmatrix}, \tag{2.10}$$

and when $n = 1$ we obtain

$$U_{t_1} = \begin{pmatrix} u_{1t_1} \\ u_{2t_1} \\ u_{3t_1} \\ u_{4t_1} \\ u_{5t_1} \end{pmatrix} = Jg^1 = J \begin{pmatrix} R(A_1) \\ R(C_1) \\ R(B_1 - C_1) \\ R(E_1) \\ R(-D_1) \end{pmatrix}.$$

Invoking Eqs. (2.6) and (2.9), the solution to the above equation is

$$\begin{cases} u_{1t_1} = -2u_4u_5 - u_5^2, \\ u_{2t_1} = u_{2x} + 2u_1u_2 - 2u_1u_3 + u_{2y} - u_{3y} + 2u_4u_5 + 2u_4^2, \\ u_{3t_1} = u_{3x} - 2u_1u_3 - u_{3y} - u_5^2, \\ u_{4t_1} = u_1u_5 + u_{4x} + u_1u_4, \\ u_{5t_1} = u_{5x} - u_1u_5. \end{cases} \tag{2.11}$$

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On adopting the values $u_1 = u_4 = u_5 = 0$, from Eq. (2.11) we have

$$\begin{cases} u_{1t_1} = 0\,, \\ u_{2t_1} = u_{2x} + u_{2y} - u_{3y}\,, \\ u_{3t_1} = u_{3x} - u_{3y}\,, \\ u_{4t_1} = 0\,, \\ u_{5t_1} = 0\,. \end{cases}$$

When $n = 2$,

$$U_{t_2} = \begin{pmatrix} u_{1t_2} \\ u_{2t_2} \\ u_{3t_2} \\ u_{4t_2} \\ u_{5t_2} \end{pmatrix} = Jg^2 = J \begin{pmatrix} R(A_2) \\ R(C_2) \\ R(B_2 - C_2) \\ R(E_2) \\ R(-D_2) \end{pmatrix},$$

so on assuming $u_1 = u_4 = u_5 = 0$ and $J$, $P^2$ we similarly get

$$\begin{cases} u_{1t_2} = u_2(u_{3x} - u_{3y}) + u_3(u_{2x} - u_{2y}) - u_3u_{3x}\,, \\ u_{2t_2} = \dfrac{1}{2}u_{3xx} - \dfrac{1}{2}u_{2xx} - u_{2xy} - \dfrac{1}{2}u_{2yy} + \dfrac{1}{2}u_{3yy}\,, \\ u_{3t_2} = \dfrac{1}{2}u_{3xx} - u_{3xy} + \dfrac{1}{2}u_{3yy}\,, \\ u_{4t_2} = 0\,, \\ u_{5t_2} = 0\,. \end{cases}$$

When $n = 3$,

$$U_{t_3} = \begin{pmatrix} u_{1t_3} \\ u_{2t_3} \\ u_{3t_3} \\ u_{4t_3} \\ u_{5t_3} \end{pmatrix} = Jg^3 = J \begin{pmatrix} R(A_3) \\ R(C_3) \\ R(B_3 - C_3) \\ R(E_3) \\ R(-D_3) \end{pmatrix}, \tag{2.12}$$

and taking $u_1 = u_4 = u_5 = 0$ we likewise obtain

$$\begin{cases} R(A_3) = \partial_x^{-1}\left[-2u_2\left(\dfrac{1}{4}u_{3xx} - \dfrac{1}{2}u_{3xy} + \dfrac{1}{4}u_{3yy}\right) + 2u_3\left(\dfrac{1}{4}u_{2xx} + \dfrac{1}{2}u_{2xy} + \dfrac{1}{4}u_{2xy} - \dfrac{1}{2}u_{3xy}\right)\right], \\ R(B_3) = \left(\dfrac{1}{4}u_{2xx} + \dfrac{1}{2}u_{2xy} + \dfrac{1}{4}u_{2xy} - \dfrac{1}{2}u_{3xy}\right) + u_2\partial_x^{-1}\left[-u_2(u_{3x} - u_{3y}) - u_3(u_{2x} + u_{2y}) \right. \\ \qquad \left. + u_3u_{3x}\right], \\ R(C_3) = \left(\dfrac{1}{4}u_{3xx} - \dfrac{1}{2}u_{3xy} + \dfrac{1}{4}u_{3yy}\right) + u_3\partial_x^{-1}\left[-u_2(u_{3x} - u_{3y}) - u_3(u_{2x} + u_{2y}) + u_3u_{3x}\right], \\ R(D_3) = 0\,, \\ R(E_3) = 0\,. \end{cases} \tag{2.13}$$

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From Eqs. (2.12) and (2.13) we have

$$\begin{aligned} u_{3t_3} =& J_{32}R(C_3) = \frac{1}{4}u_{3xxx} - \frac{3}{4}u_{3xxy} + \frac{3}{4}u_{3xyy} - \frac{1}{4}u_{3yyy} \\ &+ u_{3x}\partial_x^{-1}\left[-u_2(u_{3x}-u_{3y}) - u_3(u_{2x}+u_{2y}) + u_3u_{3x}\right] \\ &+ u_3\partial_x\partial_x^{-1}\left[-u_2(u_{3x}-u_{3y}) - u_3(u_{2x}+u_{2y}) + u_3u_{3x}\right] \\ &- u_{3y}\partial_x^{-1}\left[-u_2(u_{3x}-u_{3y}) - u_3(u_{2x}+u_{2y}) + u_3u_{3x}\right] \\ &- u_3\partial_y\partial_x^{-1}\left[-u_2(u_{3x}-u_{3y}) - u_3(u_{2x}+u_{2y}) + u_3u_{3x}\right], \end{aligned}$$

and on setting $u_2 = u_3 = r$ we get the (2+1)-dimensional mKdV equation

$$4r_t = r_{xxx} - 3r_{xxy} + 3r_{xyy} - r_{yyy} - 6r^2r_x + 6r^2r_y\,,$$

so we call the system as (2+1)-dimensional super mKdV hierarchy.

In order to find the super Hamiltonian structure of the (2+1)-dimensional super mKdV hierarchy, we introduce a linear functional

$$a = a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 + a_5f_5\,.$$

A direct calculation $ad_a b = [a, b]$ gives

$$ad_a = \begin{pmatrix} 0 & 2a_3 & -2a_2 & a_5 & a_4+a_5 \\ 2a_2-2a_3 & -2a_1 & 2a_1 & -2a_4-a_5 & -a_4 \\ -2a_3 & 0 & 2a_1 & 0 & a_5 \\ a_4+a_5 & a_5 & -a_4-a_5 & -a_1+a_3 & -a_1-a_2+a_3 \\ -a_5 & 0 & 2a_4+a_5 & -2a_3 & a_1-a_3 \end{pmatrix},$$

so we define the supertrace

$$\mathrm{str}\,(C) = R(c_{11}+c_{22}-c_{33}), \quad C = aMbM\,, \quad a, b \in \widetilde{B}\,,$$

where $C = (c_{ij})_{3\times3}$. If $D = (d_{ij})_{5\times5}$, then

$$\mathrm{str}\,(D) = R\left(d_{11}+d_{22}+d_{33}-d_{44}-d_{55}\right).$$

From the linearity, we get a relation in $\widetilde{B}$ — viz.

$$\mathrm{str}\,(ad_a ad_b) = 3\,\mathrm{str}\,(aMbM),$$

it is easy to see that

$$\begin{aligned} &\mathrm{str}\,(ad_v ad_{u_\lambda}) = R(6A), && \mathrm{str}\,(ad_v ad_{\partial U/\partial u_1}) = R(6A), \\ &\mathrm{str}\,(ad_v ad_{\partial U/\partial u_2}) = R(6C), && \mathrm{str}\,(ad_v ad_{\partial U/\partial u_3}) = R(6(B-C)), \\ &\mathrm{str}\,(ad_v ad_{\partial U/\partial u_4}) = R(6E), && \mathrm{str}\,(ad_v ad_{\partial U/\partial u_5}) = R(-6D), \end{aligned} \tag{2.14}$$

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and the supertrace identity

$$\frac{\delta}{\delta u_i}\langle V, U_\lambda\rangle = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle . \tag{2.15}$$

Substituting Eq. (2.14) into Eq. (2.15) gives

$$\frac{\delta}{\delta u}\int R(A)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} R(A)\\ R(C)\\ R(B-C)\\ R(E)\\ R(-D)\end{pmatrix},$$

and comparing the coefficient of $\lambda^{-n-1}$ yields

$$\frac{\delta}{\delta u}\int R(A_{n+1})\,dx = (\gamma-n)\begin{pmatrix} R(A_n)\\ R(C_n)\\ R(B_n-C_n)\\ R(E_n)\\ R(-D_n)\end{pmatrix}.$$

Since $\operatorname{str}(ad_v ad_v) = 3\operatorname{str}(VMVM) \neq 0$, we have

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\operatorname{str}(ad_v ad_v)| = 0,$$

and hence we obtain

$$\frac{\delta}{\delta u}\int \frac{-R(A_{n+1})}{n}\,dx = \begin{pmatrix} R(A_n)\\ R(C_n)\\ R(B_n-C_n)\\ R(E_n)\\ R(-D_n)\end{pmatrix}, \quad n\geq 1 .$$

The (2+1)-dimensional super mKdV hierarchy of Eq. (2.10) can be written as the super Hamiltonian form

$$U_{t_n} = \begin{pmatrix} u_{1t_n}\\ u_{2t_n}\\ u_{3t_n}\\ u_{4t_n}\\ u_{5t_n}\end{pmatrix} = J\frac{\delta\int -R(A_{n+1})/n\,dx}{\delta u} = J\frac{\delta H_n}{\delta u} = M\frac{\delta H_{n-1}}{\delta u},$$

where $H_n = \int -R(A_{n+1})/n\,dx$, $M = JL$.

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## 3. Some Solutions of the (2+1)-dimensional MKdV Equation

In order to obtain some solutions to the (2+1)-dimensional mKdV equation, let us first list the main steps of the $G'/G^2$ method we use.

1. Consider the nonlinear partial differential equation in three variables $x$, $y$, $t$:

$$P(r, r_x, r_y, r_t, r_{xx}, r_{xy}, r_{xt}, r_{yy}, r_{yt}, r_{tt}, \cdots) = 0\,, \tag{3.1}$$

   where $r = r(x, y, t)$.
2. Consider the wave transformation

$$r = r(\eta), \eta = kx + ly + wt\,, \tag{3.2}$$

   where $k, l$ and $w$ are constant (to be determined later) and put Eq. (3.2) into Eq. (3.1) to generate the ordinary nonlinear differential equation

$$P(r, kr^{'}, lr^{'}, wr^{'}, k^2r^{''}, klr^{''}, kwr^{''}, l^2r^{''}, lwr^{''}, w^2r^{''}, \cdots) = 0\,.$$

3. Expand $r(\eta)$ into a polynomial of $(g^{'}/g^2)$:

$$r(\eta) = \sum_{i=0}^{n} \alpha_i \left(\frac{g^{'}}{g^2}\right)^i (i = 0, 1, 2, \cdots, n)\,, \tag{3.3}$$

   where $g(\eta)$ satisfies

$$\left(\frac{g^{'}}{g^2}\right)^{'} = a + b\left(\frac{g^{'}}{g^2}\right)^2\,, \tag{3.4}$$

   with $\alpha_i, a, b$ constant and $n$ defined by an homogeneous balance method.
4. Substitute Eq. (3.4) into Eq. (3.3), combine the coefficients of $g^{'}/g^2$ and then equate each coefficient to zero, to get a set of algebraic equations in $\alpha_i (i = 0, 1, 2, \cdots, n), a, b$.

Let us now proceed to compute the solution $g^{'}/g^2$ to get solutions of the (2+1)-dimensional mKdV equation. From Eq. (3.4),

$$g^{''}g^2 - 2gg^{'2} = ag^4 + bg^{'2},$$

and setting $g = 1/y$ this equation can be written $y^{''} + by^{'2} + a = 0$. We obtain the following formulas: when $ab > 0$,

$$y(\eta) = \frac{1}{2b} \ln\left[\frac{b}{a}(A\sin(\sqrt{ab}\eta) - B\cos(\sqrt{ab}\eta))^2\right];$$

when $ab < 0$,

$$y(\eta) = -\frac{1}{2b}\left(2\eta\sqrt{|ab|} - \ln\left[\frac{b}{4a}(Ae^{2\eta\sqrt{|ab|}} - B)^2\right]\right);$$

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when $a = 0$, $b \neq 0$,

$$y(\eta) = \frac{1}{b}\ln(A\eta b + Bb).$$

Consequently, we have solutions for $g^{'}/g^2$ as follows, where $D$ and $E$ are constants:

(1) when $ab > 0$,

$$g(\eta) = \frac{2b}{\ln\left[\frac{b}{a}(E\sin(\eta\sqrt{ab}) - D\cos(\eta\sqrt{ab}))^2\right]},$$
$$\frac{g^{'}}{g^2} = \sqrt{\frac{a}{b}}\frac{E\cos(\eta\sqrt{ab}) + D\sin(\eta\sqrt{ab})}{E\sin(\eta\sqrt{ab}) - D\cos(\eta\sqrt{ab})};$$

(2) when $ab < 0$,

$$g(\eta) = -\frac{2b}{2\eta\sqrt{|ab|} - \ln\left[\frac{b}{4a}(Ee^{2\eta\sqrt{|ab|}} - D)\right]},$$
$$\frac{g^{'}}{g^2} = \frac{1}{2b}\left(2\sqrt{|ab|} - \frac{4\sqrt{|ab|}De^{2\eta\sqrt{|ab|}}}{Ee^{2\eta\sqrt{|ab|}} - D}\right),$$
$$\frac{g^{'}}{g^2} = \sqrt{\left|\frac{a}{b}\right|}\frac{E\sinh(\eta\sqrt{|ab|}) + D\cosh(\eta\sqrt{|ab|})}{E\cosh(\eta\sqrt{|ab|}) + D\sinh(\eta\sqrt{|ab|})};$$

(3) when $a = 0, b \neq 0$,

$$g(\eta) = \frac{b}{\ln(E\eta b + Db)}, \qquad \frac{g^{'}}{g^2} = -\frac{E}{E\eta b + Db}.$$

Using the $G^{'}/G^2$ method, let us now consider the (2+1)-dimensional mKdV equation

$$4r_t = r_{xxx} - 3r_{xxy} + 3r_{xyy} - r_{yyy} - 6r^2r_x + 6r^2r_y\,, \tag{3.5}$$

and the transformation

$$r(x, y, t) = r(\eta), \qquad \eta = kx + ly + wt \tag{3.6}$$

with constant $k$, $l$ and $w$, where $k, l$ are wave numbers in the $x, y$ directions and $w$ is the frequency in the solitary wave theory.

Substituting Eq. (3.6) into Eq. (3.5) yields

$$(k^3 - 3k^2l + 3kl^2 - l^3)r^{'''} + (6l - 6k)r^2r^{'} - 4wr^{'} = 0,$$

which on integrating with respect to $\eta$ gives

$$(k^3 - 3k^2l + 3kl^2 - l^3)r^{''} + (2l - 2k)r^3 - 4wr + C = 0\,, \tag{3.7}$$

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an balancing the nonlinear terms and the linear term with the highest order derivative of $r(\eta)$ implies $n+2=3n$, $n=1$. We therefore choose the formal solution

$$r(\eta)=\alpha_1\left(\frac{g^{'}}{g^2}\right)+\alpha_0 \tag{3.8}$$

from Eqs. (3.7) and (3.8) such that

$$r^{''}(\eta)=2\alpha_1 ab\left(\frac{g^{'}}{g^2}\right)+2\alpha_1 b^2\left(\frac{g^{'}}{g^2}\right)^3, \tag{3.9}$$

$$r^3(\eta)=\alpha_1^3\left(\frac{g^{'}}{g^2}\right)^3+3\alpha_0\alpha_1^2\left(\frac{g^{'}}{g^2}\right)^2+3\alpha_0^2\alpha_1\left(\frac{g^{'}}{g^2}\right)+\alpha_0^3, \tag{3.10}$$

so substituting Eqs. (3.8)-(3.10) into Eq. (3.7) we get

$$\begin{aligned}&(k^3-3k^2l+3kl^2-l^3)r^{''}+(2l-2k)r^3-wr+C\\ =&\left[(k^3-3k^2l+3kl^2-l^3)2\alpha_1 b+(2l-2k)\alpha_1^3\right](g^{'}/g^2)^3+\left[(2l-2k)3\alpha_0\alpha_1^2\right](g^{'}/g^2)^2\\ &+\left[(k^3-3k^2l+3kl^2-l^3)2\alpha_1 ab+(2l-2k)3\alpha_0^2\alpha_1-4w\alpha_1\right](g^{'}/g^2)\\ &+\left[(2l-2k)\alpha_0^3-4w\alpha_0+C\right].\end{aligned}$$

Then combining each coefficient of $g^{'}/g^2$ and equating the coefficients to zero,

$$\begin{cases}(k^3-3k^2l+3kl^2-l^3)2\alpha_1 b+(2l-2k)\alpha_1^3=0,\\ (2l-2k)3\alpha_0\alpha_1^2=0,\\ (k^3-3k^2l+3kl^2-l^3)2\alpha_1 ab+(2l-2k)3\alpha_0^2\alpha_1-4w\alpha_1=0,\\ (2l-2k)\alpha_0^3-4w\alpha_0+C=0.\end{cases}$$

and solving this set of equations we have

$$\alpha_1=\pm b(k-l),\quad \alpha_0=0,\quad C=0,\quad ab=\frac{2w}{(k-l)^3}. \tag{3.11}$$

In summary, we have obtained several different forms of solution as follows.
Case 1: when $ab<0$, we have an hyperbolic function general solution

$$r(\eta)=\pm(k-l)\sqrt{|ab|}\frac{E\sinh(\eta\sqrt{|ab|})+D\cosh(\eta\sqrt{|ab|})}{E\cosh(\eta\sqrt{|ab|})+D\sinh(\eta\sqrt{|ab|})},$$

and on setting

$$\sinh\lambda=\frac{D}{\sqrt{E^2-D^2}},\qquad \cosh\lambda=\frac{E}{\sqrt{E^2-D^2}}$$

the solitary wave solution of the (2+1)-dimensional mKdV equation is

$$r(\eta)=\pm(k-l)\sqrt{|ab|}\tanh\left(\eta\sqrt{|ab|}+\lambda\right). \tag{3.12}$$

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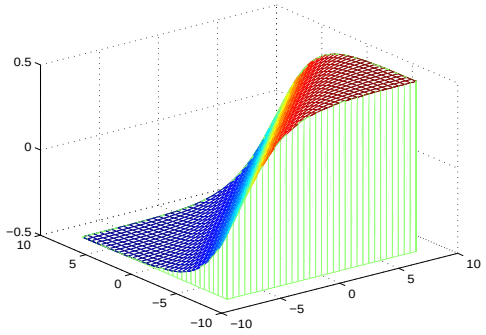



Figure 1: $t = 4$, $r(\eta)_{21} = \frac{1}{2}\tanh[\frac{1}{4}(x-y-1)]$.

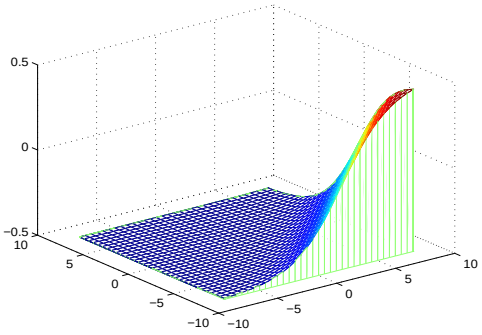



Figure 2: t=40, $r(\eta)_{22} = \frac{1}{2}\tanh[\frac{1}{4}(x-y-10)]$.

For example, let us consider the case when $E \neq 0$ but $D = 0$ with constants $k = 1$, $l = -1$, $w = -1/4$, and $ab = -1/16$, and illustrate the result graphically, see Figs. 1 and 2. In this case, according to Eqs. (3.11) and (3.12) we get $k > l$ if we take the negative sign of $w$, such that the wave number in the $x$-direction is greater than the wave number in the $y$-direction. Thus for a positive sign on the solution we find a solitary wave propagating forward, while for a negative sign the solitary wave propagates backward.

Case 2: when $ab > 0$, we have a trigonometric function general solution

$$r(\eta) = \pm(k-l)\sqrt{|ab|}\frac{E\cos(\eta\sqrt{ab}) + D\sin(\eta\sqrt{ab})}{E\sin(\eta\sqrt{ab}) - D\cos(\eta\sqrt{ab})},$$

and on setting

$$\sin\lambda = \frac{E}{\sqrt{E^2+D^2}}, \cos\lambda = \frac{D}{\sqrt{E^2+D^2}}$$

the solution of the (2+1)-dimensional mKdV equation is

$$r(\eta) = \pm(k-l)\sqrt{|ab|}\tan(\eta\sqrt{ab} + \lambda)\,. \tag{3.13}$$

For example, again considering $E \neq 0$ but $D = 0$ and now the constants $k = 1$, $l = 3$, $w = -1/4$ and $ab = 1/16$, and illustrating the result illustrate graphically, see Figs. 3 and 4.

In this case, according to Eqs. (3.11) and (3.13) we have $k < l$ for negative $w$, such that the wave number in the $x$-direction is less than the wave number in the $y$-direction. Thus taking a positive sign on the solution we find a solitary wave propagates backward, while for a negative sign the solitary wave propagates forward.

Comparing the graphics for the times $t = 4$ to $t = 40$, we note that the waveforms do not change with time, a characteristic property of solitary waves. We recall that the mKdV equation is commonly used to describe solitary waves in the atmosphere and ocean. For example, Yang & Yin [25,26] invoked the (1+1)-dimensional mKdV equation to describe the one-dimensional propagation of a Rossby solitary wave using scale and perturbation analysis. Since the (2+1)-dimensional mKdV equation is the corresponding appropriate form to

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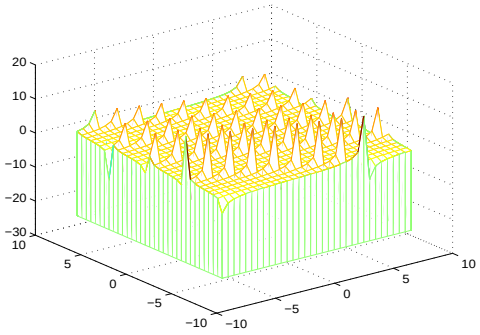


Figure 3: $t=4,\ r(\eta)_{11}=\frac{1}{2}\tan[\frac{1}{4}(x+3y-1)]$.

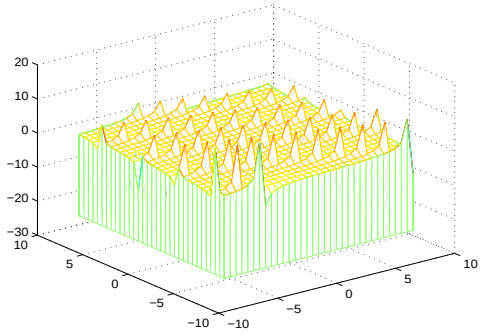


Figure 4: $t=40,\ r(\eta)_{12}=\frac{1}{2}\tan[\frac{1}{4}(x+3y-10)]$.

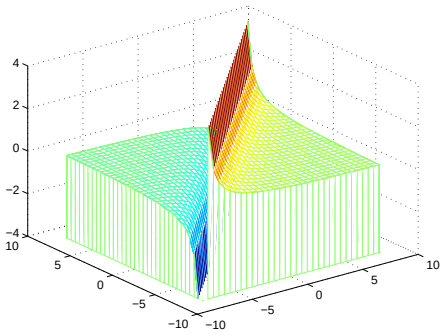


Figure 5: $r(\eta)_3=2/(x-y)$.

describe the two-dimensional propagation of solitary waves, the above two solutions of the (2+1)-dimensional mKdV equation are significant in the theory for the evolution of planar solitary waves.

Case 3: when $a=0$ but $b\neq 0$, we have the solution

$$r(\eta)=\pm(k-l)\frac{E}{E\eta+D} \tag{3.14}$$

for $ab=0,\ w=0,\ k=1$ and $l=-1$. This “algebraic solitary wave solution" represented in Eq. (3.14) defines a stationary solitary wave that has been identified with observations in the atmosphere and ocean, and previously investigate via differential and integral equations [27, 28]. Once again, this solution of the (2+1)-dimensional mKdV equation appropriately represents solitary waves in the plane (two dimensions).

## 4. Conclusions

We have discussed a new (2+1)-dimensional equation super hierarchy, together with its Hamiltonian structure. The key problem is to find an appropriate Lie superalgebra and Lax

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pair. While one can readily obtain the (1+1)-dimensional hierarchy using the Tu scheme, it is more difficult to do so for the (2+1)-dimensional super hierarchy. For example, when

$$U = \begin{pmatrix} \lambda + u_1 + u_3 & 2u_3 & u_5 \\ u_2 & -\lambda - u_1 + u_3 & -u_4 \\ u_4 + u_5 & u_5 & 0 \end{pmatrix}$$

one needs to replace $\lambda$ by $\lambda + \xi$ in the matrix $U$ in order to calculate the (2+1)-dimensional super hierarchy. Corresponding to $\xi f = f\xi + \partial_y f$, we need also need to consider contributions involving $\partial_y$, which requires several changes in the calculations. In this article, we have set $A_y = 0$ in obtaining the (2+1)-dimensional hierarchy, and in future work we intend to avoid this assumption in order to consider (2+1)-dimensional super hierarchy with wider application to isopectral problems and other Lie algebra.

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