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ON AN OPEN PROBLEM OF VALUE SHARING AND DIFFERENTIAL EQUATIONS

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Abstract Let f be a non-constant meromorphic function. We define its linear differential polynomial $L_k[f]$ by

$$L_k[f] = b_{-1} + \sum_{j=0}^{k} b_j f^{(j)}, \text{where } b_j (j = 0, 1, 2, \ldots, k) \text{ are constants with } b_k \neq 0.$$

In this paper, we solve an open problem posed by Li [*J. Math. Anal. Appl.* 310 (2005) 412-423] in connection with the problem of sharing a set by entire functions f and their linear differential polynomials $L_k[f]$. Furthermore, we study the Fermat-type functional equations of the form $f^n + g^n = 1$ to find the meromorphic solutions (f, g) which enable us to answer the question of Li completely. This settles the long-standing open problem of Li.

Keywords: meromorphic function; differential polynomial; solutions; shared set; Fermat-type functional equation

2020 Mathematics subject classification: Primary 30D35
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1. Introduction

A meromorphic (respectively, entire) function always means a non-constant function meromorphic (resp. analytic) in the complex plane $\mathbb{C}$. It is also assumed that the reader is familiar with the basic concepts of Nevanlinna's value distribution theory (see [7, 25, 26]) and in particular the most usual of its symbol:

$$m(r, f),\ m\left(r, \frac{1}{f-a}\right),\ N(r, f), N\left(r, \frac{1}{f-a}\right),\ T(r, f).$$

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Let f_1 and f_2 be meromorphic, let $a \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and let $E_j(a)$ be the set of points $z \in \mathbb{C}$ for which $f_j(z) = a$, $j = 1, 2$. We say that f_1 and f_2 share the value a ignoring multiplicities (IM), if $E_1(a) = E_2(a)$ without being counted multiplicities. Moreover, if the multiplicities of $f_1(z) = a$ and $f_2(z) = a$ are pairwise the same for each $z \in \mathbb{C}$, then f_1 and f_2 are said to share the value a counting multiplicities (CM).

The classical results due to Nevanlinna [21] have prompted research activity on shared value problems even today. The famous five-point uniqueness theorem due to Nevanlinna [21] says that if two meromorphic functions f and g share five distinct values $a_1, a_2, a_3, a_4, a_5 \in \widehat{\mathbb{C}}$ IM, then $f \equiv g$. The beauty of this result lies on the fact that it has no counterpart in function theory in $\mathbb{R}$. Another famous result is the four-point uniqueness theorem which states that if two meromorphic functions f and g share four values $a_1, a_2, a_3, a_4 \in \widehat{\mathbb{C}}$ CM, then $f = \mathcal{T} \circ g$, where $\mathcal{T}$ is a Möbius map. These results initiated the study of the relation between two meromorphic maps and the theory of value distribution of meromorphic functions in $\mathbb{C}$ which has numerous applications in the study of complex differential equations.

In 1977, Rubel and Yang [22] first demonstrated that the Nevanlinna five-point uniqueness theorem could be significantly improved. Rubel and Yang [22] have proved that if entire functions f and their derivatives f' share two values a, b CM, then $f \equiv f'$. Consequently, the number of shared values in Nevanlinna four-value theorem is reduced from four to two. Moreover, it is easy to see that the number 'two' is the best possible in the result of Rubel and Yang. In fact, we see that $f' - 1 = \mathrm{e}^z(f - 1)$, and hence f and f' share the value 1 CM, where

$$f(z) = \mathrm{e}^{\mathrm{e}^z} \int_0^z \mathrm{e}^{-\mathrm{e}^t}(1 - \mathrm{e}^t)\mathrm{d}t,$$

but $f \not\equiv f'$.

In 1979, Mues and Steinmetz [20] improved the result of Rubel and Yang establishing that if entire functions f and f' share two distinct values IM, then $f \equiv f'$. It is noteworthy that the result of Mues and Steinmetz is not valid for meromorphic functions. For example, a simple computation shows that the function

$$f(z) = \left(\frac{1}{2} - \frac{\sqrt{5}i}{2} \tan\left(\frac{\sqrt{5}i}{4} z\right)\right)^2$$

and $f'(z)$ share the values 0 and 1 IM but $f \not\equiv f'$.

A special topic widely studied in value distribution theory is the case when $f(z)$ share values with their higher-order derivatives or differential polynomials (see [2, 5, 6, 10–12, 14, 16, 17, 19, 24]). In 2008, Heittokangas *et al.* [9] studied the linear differential equation of the form

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0,$$

with coefficients in weighted Bergman or Hardy spaces. Throughout the paper, for a meromorphic function f, we define its linear differential polynomial $L_k[f]$ by

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$$L_k[f] := b_{-1} + \sum_{j=0}^{k} b_j f^{(j)}, \text{where } b_j \in \mathbb{C}\ (j = 0, 1, 2, \ldots, k), \text{with } b_k \neq 0. \tag{1.1}$$

The result of Rubel and Yang [22] can be extended up to a linear differential polynomial $L_k[f]$ for entire functions, but the same is not possible for meromorphic functions. For example, suppose that $f(z) = e^z/(e^z - 1)$ and $L_1[f] = f' - f$. It is easy to see that f and $L_1[f]$ share -1 and 0 CM, but $f \not\equiv L_1[f]$. On the other hand, such extension of the result of Mues and Steinmetz is not always possible for entire as well as meromorphic functions.

(i) (Entire function:) Let $f(z) = \cos z$ and $L_k[f] = b_k f^{(k)} - f''/2 + f/2$, where

$$\begin{cases} i, & \text{when } k = 4m - 1 \\ -i, & \text{when } k = 4m - 3 \end{cases}$$

for some $m \in \mathbb{N}$. It is easy to see that f and $L_k[f]$ share the values -1 and 1 , IM, but $f \not\equiv L_k[f]$.

(ii) (Meromorphic function:) Let

$$f(z) = 2\frac{e^{2\sqrt{2}iz} + 4e^{\sqrt{2}iz} + 1}{\left(e^{\sqrt{2}iz} - 1\right)^2}$$

and $L_2[f] = f''$ share the values 0 and 1 IM, but note that $f \not\equiv L_2[f]$.

We now demonstrate the following well-known set sharing, which is a general setting of value sharing. Let $\mathcal{S}$ be a set of complex numbers and $E_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}}\{z : f(z) = a\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicities, then the set $\bigcup_{a\in\mathcal{S}}\{z : f(z) = a\}$ is denoted by $\bar{E}_f(\mathcal{S})$. If $E_f(\mathcal{S}) = E_g(\mathcal{S})$, where f and g are said to share the set $\mathcal{S}$ CM. On the other hand, if $\bar{E}_f(\mathcal{S}) = \bar{E}_g(\mathcal{S})$, then we say that f and g share the set $\mathcal{S}$ IM. It is easy to see that, if $\mathcal{S}$ contains only one element, then it coincides with the usual definition of sharing of values.

Studying meromorphic functions in view of sharing of a set with its derivatives and finding a possible relationship between them is an interesting topic in the value distribution theory of Nevanlinna. For example, in case of sharing a set of two elements by entire functions f and their derivatives f', in 1999, Li and Yang [15] proved that if $E_f(\mathcal{S}) = E_{f'}(\mathcal{S})$, where $\mathcal{S} = \{a, b\}$, then (i) $f \equiv f'$ or (ii) $f + f' = a + b$ or (iii) $f(z) = c_1 e^{cz} + c_2 e^{-cz}$, where $a + b = 0$, and c, c_1, c_2 are non-zero constants satisfying $c^2 \neq 1$ and $c_1 c_2 = \left(1 - c^{-2}\right) a^2/4$. Later, in 2013, considering the general derivatives of entire functions, Li [18] proved that if the zeros of $f(z)$ are of multiplicities at least k, and $E_f(\mathcal{S}) = E_{f^{(k)}}(\mathcal{S})$, then (i) $f(z) = Ae^{Bz}$, where $\mathcal{S} = \{a, b\}$ and $A \neq 0$ and B are constants satisfying $B^k = \pm 1$, or (ii) $f(z) + f^{(k)}(z) = a + b$.

In view of the above results of Li and Yang [15], and Li [18], it turns out that the following question is inevitable.

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Questions 1.2. *What can we say if entire functions f share the set $\mathcal{S} = \{a, b\}$ with its linear differential polynomials $L_k[f]$?*

In 2005, Li [13] gave a possible answer to Question 1.2 and proved the following result.

Theorem A. [13] *Suppose that f is a non-constant entire function and*

$$g = L_k[f] = b_{-1} + \sum_{j=0}^{k} b_j f^{(j)}, \text{ where } b_j (j = 0, 1, 2, \ldots, k) \text{ are constants, with } b_k \neq 0.$$

If $E_f(\mathcal{S}) = E_g(\mathcal{S})$, where $\mathcal{S} = \{a, b\}$, then one of the following holds:

(i) $f \equiv g$;
(ii) $f + g = a + b$;
(iii) *The functions f and g take the forms*

$$f(z) = (h_1(z) + h_2(z) + a + b)/2,$$
$$g(z) = (-h_1(z)e^{\gamma} + h_2(z)e^{\gamma} + a + b)/2,$$

where γ is an entire function satisfying $T(r, e^{\gamma}) = S(r, f)$ and h_1, h_2 are two entire functions satisfying $h_1 h_2 = ((a-b)/2)^2 \left(1 - e^{-2\gamma}\right)$ and the following two differential equations:

$$\sum_{j=0}^{k} b_j h_1^{(j)} = -e^{\gamma} h_1, \quad \sum_{j=0}^{k} b_j h_2^{(j)} = e^{\gamma} h_2. \tag{1.2}$$

Moreover, $2b_{-1} + (a + b)(b_0 - 1) = 0$.

By considering the existence of solutions for the two equations in equation (1.2), in [13], Li obtained the following result, providing explicit forms for h_1 and h_2 in certain special cases.

Theorem B. [13] *Suppose that f is a non-constant entire function and $g = L_k[f]$ is the differential polynomial defined by equation (1.1), where $k = 1$ or k is an even number. Let a_1 and a_2 be two distinct numbers in $\mathbb{C}$. If $E_f(\mathcal{S}) = E_g(\mathcal{S})$, where $\mathcal{S} = \{a, b\}$, then one of the following cases holds:*

(i) $f = g$;
(ii) $f + g = a + b$;
(iii) $f(z) = \frac{a+b}{2} + \frac{1}{2}Ae^{\lambda z} + \frac{1}{2}Be^{-\lambda z}$ *and* $g(z) = \frac{a+b}{2} - \frac{1}{2}\rho Ae^{\lambda z} + \frac{1}{2}\rho Be^{-\lambda z}$ *with* $(a + b)(b_0 - 1) = 2b_{-1}$, *where A, B, λ and ρ are non-zero constants satisfying*

$$\sum_{j=0}^{k} b_j \lambda^j = -\rho, \quad \sum_{j=0}^{k} b_j (-\lambda)^j = \rho, \quad AB = \left(\frac{a-b}{2}\right)^2 \frac{\rho^2 - 1}{\rho^2}.$$

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Furthermore, if f is an entire function of finite order, then the conclusion is also true for odd number k.

For further study, Li [13] posed the following question which was unanswered for several years.

Questions 1.4. [13] *Suppose $\mathcal{S}$ is a set in $\mathbb{C}$ with more than two elements, f is an entire function and $L_k[f]$ is a linear differential polynomial in f with constant coefficients. What can we say ith constant coefficients. What can we say when f shares the set $\mathcal{S}$ CM with $L_k[f]$?*

Investigated in depth, we realized that the key problem with answering Question 1.3 is to find the explicit form of the functions f and $g = L_k[f]$ in case when they share the set $\mathcal{S}$ containing more than two elements. In this paper, instead of considering a set of more than three elements, we shall concentrate to find a possible answer to Question 1.3 when $\mathcal{S}$ contains exactly three elements, i.e. when $\mathcal{S} = \{a, b, c\}$. Since the functions f and g satisfy an algebraic equation, applying suitable transformation, we can transform the equation into its canonical form to find their explicit form using the theories used to solve Fermat-type functional equations. Henceforth, we study Fermat-type functional equations $f^n + g^n = 1$, and the main aim of this paper is to find certain meromorphic solutions (f, g), with some specific values of $n \in \mathbb{N}$.

The organization of the paper is follows: taking help of some transformations in $\mathbb{C}^2$, we give a complete answer to Question 1.3 with a full characterization of the function f when the set $\mathcal{S}$ contains three elements. In § 2, we state the main results of this paper with corollaries of them. In § 3, we present the key lemmas and their proofs that are necessary to prove the main results. Section 4 is devoted to proving the main results of the paper.

2. Main results

The circle functional equation $f^2+g^2 = 1$ and the Fermat cubic $f^3+g^3 = 1$ are the basic fundamental equations. Generalizations of these power equations are called Fermat-type functional equations, which are associated with diagonal varieties, and have been the subject of interest in global complex analysis. In 2013, Saleeby [23] studied the quadratic trinomial functional equation $f^2+2\alpha fg+g^2 = 1$, $\alpha^2 \neq 1$, $\alpha \in \mathbb{C}$, and showed that under the transformation

$$g = \frac{\mathcal{U}+\mathcal{V}}{\sqrt{2}} \quad \text{and} \quad f = \frac{\mathcal{U}-\mathcal{V}}{\sqrt{2}},$$

the trinomial equation is transformed to $(1+\alpha)\mathcal{U}^2 + (1-\alpha)\mathcal{V}^2 = 1$, which is associated with an ellipse in $\mathbb{C}^2$. The entire solution of the trinomial equations is

$$f = \frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}} + \frac{\sin(h)}{\sqrt{1-\alpha}}\right) \quad \text{and} \quad g = \frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}} - \frac{\sin(h)}{\sqrt{1-\alpha}}\right).$$

In this paper, our aim is to answer Question 1.3 and also to find the solution (f, g) of the equation $f^2 + fg + g^2 - (a+b+c)(f+g) + (ab+bc+ca) = 0$ with the help of some

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transformation in $\mathbb{C}^2$. Therefore, from the point of view of Question 1.3 and solution of Fermat-type functional equation $F^2+G^2=1$, considering the set $\mathcal{S}=\{a,b,c\}$, we prove the following two results.

Theorem 2.1. *Let f be an entire function and $\mathcal{S}=\{a,b,c\}$, where a,b and c are three distinct complex numbers. If f and $g=L_k[f]$ (differential polynomial defined as in equation (1.1) satisfy $E_f(\mathcal{S})=E_{L_k[f]}(\mathcal{S})$, then one of the following cases hold.*

(i) *$f\equiv g$. Furthermore, the function f assumes the form $f(z)=a_1e^{\lambda_1 z}+\cdots+a_ke^{\lambda_k z}$, where λ_j $(j=1,2,\ldots,k)$ are roots of the equation $b_kz^k+\cdots+b_1z+b_0=1$.*

(ii) *$f^2+fg+g^2-(a+b+c)(f+g)+(ab+bc+ca)=0$. Furthermore, we have one of the following:*
(a)

$$f(z)=\pm\sqrt{\eta/54}\left(\sqrt{3}\sin(h(z))-3\cos(h(z))\right)+(a+b+c)/4,$$
$$g(z)=\pm\sqrt{\eta/54}\left(\sqrt{3}\sin(h(z))+3\cos(h(z))\right)+(a+b+c)/4.$$

(b)

$$f(z)=\pm\sqrt{\eta/54}\left(3\sin(h(z))-\sqrt{3}\cos(h(z))\right)+(a+b+c)/4,$$
$$g(z)=\pm\sqrt{\eta/54}\left(3\sin(h(z))+\sqrt{3}\cos(h(z))\right)+(a+b+c)/4,$$

where $\eta=\left((a-b)^2+(b-c)^2+(c-a)^2\right)$.

(iii)

$$f(z)=\quad(h_1(z)+h_2(z)+h_3(z)+a+b+c)/3,$$
$$g(z)=-\left([h_1(z)+\omega^2h_2(z)+\omega h_3(z)]e^{2\gamma/3}-(a+b+c)\right)/3,$$

where $\omega^2(2a-b-c)=\omega(2b-c-a)=(2c-a-b)$, with $\omega^3=1$, γ is an entire function satisfying $T(r,e^{\gamma})=S(r,f)$, and h_j $(j=1,2,3)$ are three entire functions satisfying $h_1h_2h_3=((2a-b-c)/3)^3\left(1-e^{-2\gamma}\right)$ and the following

$$\begin{cases}\displaystyle\sum_{j=0}^{k}b_jh_1^{(j)}(z)=-h_1(z)e^{\frac{2}{3}\gamma},\\ \displaystyle\sum_{j=0}^{k}b_jh_2^{(j)}(z)=-\omega^2h_2(z)e^{\frac{2}{3}\gamma},\\ \displaystyle\sum_{j=0}^{k}b_jh_3^{(j)}(z)=-\omega h_3(z)e^{\frac{2}{3}\gamma}.\end{cases}\tag{2.2}$$

Moreover, we have $3b_{-1}+(b_0-1)(a+b+c)=0$.

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Remark 2.3. Theorem 2.1 is not valid when "CM" sharing of the set $\mathcal{S}$ is replaced by "IM" sharing.

Example 2.4. Let $f(z) = \cos z$, and we choose $g(z) = L_k[f] = f^{(k)} + f^{(k-1)} - f'$, where $k(\geqslant 3)$ is an integer. We see that $\bar{E}_f(\mathcal{S}) = \bar{E}_g(\mathcal{S})$, where $\mathcal{S} = \{-1, 0, 1\}$, but none of the conclusions (i), (ii) and (iii) of Theorem 2.1 holds.

In particular, when $\mathcal{S} = \{a, a\omega,\ a\omega^2\}$, $a \in \mathbb{C} \setminus \{0\}$ and ω is a non-real cube root of unity, we have the following corollary of Theorem 2.1.

Corollary 2.5. *Suppose that f is an entire function and $g = L_k[f]$. If $E_f(\mathcal{S}) = E_{L_k[f]}(\mathcal{S})$ where $\mathcal{S} = \{a, a\omega,\ a\omega^2\}$, $a \in \mathbb{C} \setminus \{0\}$ and ω is a non-real cube root of unity, then one of the following cases hold:*

(i) *Conclusion (i) of Theorem 2.1 holds.*

(ii) $f^2 + fg + g^2 = 0$. *Furthermore, we have one of the following:*

(a) $g \equiv \omega f$. *Furthermore,* $f(z) = c_1 e^{\mu_1 z} + \cdots + c_k e^{\mu_k z}$, *where* μ_j $(j = 1, 2, \ldots, k)$ *are all simple roots of the equation* $b_k z^k + \cdots + b_1 z + b_0 = \omega$. *Also,*

$$f(z) = \sum_{j=1}^{p} \left(\sum_{r=1}^{m_j} a_{jr} z^{r-1} \right) e^{\alpha_j z} + \sum_{s=1}^{t} c_s e^{\beta_s z},$$

where $a_{jr} \in \mathbb{C}$, $\alpha_1, \ldots, \alpha_p$ *are roots of multiplicities* $m_1, \ldots, m_p$, *respectively, and* $\beta_1, \ldots, \beta_t$ $(t = k - (m_1 + \cdots + m_p))$ *are all simple roots of the equation* $b_k z^k + \cdots + b_1 z + b_0 = \omega$..

(b) $g \equiv \omega^2 f$. *Furthermore,* $f(z) = c_1 e^{\delta_1 z} + \cdots + c_k e^{\delta_k z}$, *where* δ_j $(j = 1, 2, \ldots, k)$ *are roots of the equation* $b_k z^k + \cdots + b_1 z + b_0 = \omega^2$. *Also,*

$$f(z) = \sum_{j=1}^{q} \left(\sum_{r=1}^{n_j} d_{jr} z^{r-1} \right) e^{\gamma_j z} + \sum_{s=1}^{y} c_s e^{\delta_s z},$$

where $d_{jr} \in \mathbb{C}$, $\gamma_1, \ldots, \gamma_q$ *are roots of multiplicities* $n_1, \ldots, n_q$, *respectively, and* $\delta_1, \ldots, \delta_y$ $(y = k - (n_1 + \cdots + n_q))$ *are all simple roots of the equation* $b_k z^k + \cdots + b_1 z + b_0 = \omega^2$..

(iii)

$$f(z) = \frac{1}{3}\left(h_1(z) + h_2(z) + h_3(z),\right)$$
$$g(z) = -\frac{1}{3}\left(h_1(z) + \omega^2 h_2(z) + \omega h_3(z)\right) e^{2\gamma/3},$$

where γ is an entire function satisfying $T(r, e^{\gamma}) = S(r, f)$, h_j $(j = 1, 2, 3)$ are three entire functions satisfying $h_1 h_2 h_3 = a^3\left(1 - e^{-2\gamma}\right)$ and equation (2.1) in Theorem 2.1 holds.

Considering the existence of solutions of the set of equations in equation (2.1) of Theorem 2.1, we prove Theorem 2.5 finding the explicit forms of the functions h_1, h_2 and

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h_3 in particular cases. Consequently, Theorem 2.5 constitutes a significant improvement over Theorem B.

Theorem 2.6. *Suppose that f is an entire function and $g = L_k[f]$ is the differential polynomial defined in Theorem 2.1, where $k = 1$ or an even integer. If $E_f(\mathcal{S}) = E_{L_k[f]}(\mathcal{S})$, where $\mathcal{S} = \{a, b, c\}$, then one of the following cases holds:*

(i) *Conclusion (i) of Theorem 2.1 holds.*
(ii) *Conclusion (ii) of Theorem 2.1 holds.*
(iii) *The functions f and g will be of the forms*

$$f(z) = \frac{1}{3}\left(A_1 e^{\lambda z} + A_2 e^{\omega^2 \lambda z} + A_3 e^{\omega \lambda z} + (a+b+c),\right)$$
$$g(z) = -\frac{1}{3}\left(\rho A_1 e^{\lambda z} + \rho\omega^2 A_2 e^{\omega^2 \lambda z} + \rho\omega A_3 e^{\omega \lambda z} - (a+b+c)\right).$$

where A_1, A_2 and A_3 are constants satisfying

$$A_1 A_2 A_3 = ((2a-b-c)/3)^3\left((\rho^3-1)\rho^3\right) \text{ and}$$

$$\sum_{j=0}^{k} b_j \lambda^j = -\rho, \quad \sum_{j=0}^{k} b_j \omega^{2j} \lambda^j = -\omega^2 \rho, \quad \sum_{j=0}^{k} b_j \omega^j \lambda^j = -\omega\rho,$$

where ρ is a non-zero constant, with $3b_{-1} + (b_0 - 1)(a+b+c) = 0$.

Furthermore, if f is an entire function of finite order, then the conclusion also holds for odd positive integer k.

3. Key lemmas and their proofs

In this section, we present some useful lemmas which will play key role in proving the main results of the paper. The first lemma is obvious by the lemma of logarithmic derivative, i.e. $m(r, f'/f) = S(r, f)$; see, e.g. [7]. Lemma 3.3 is a Clunie type theorem which is often used to discuss related to differential polynomials. Clunie has proved his lemma under the stronger hypothesis that the coefficients are small functions relative to f, and we know his proof also works under the weaker assumption of functions, which has small proximity relative to f. We prove Lemma 3.5 independently to prove Theorem 2.5.

Lemma 3.1. *Let f be a transcendental meromorphic function, $P_k(f)$ denote a polynomial in f of degree k and b_i $(i = 1, 2, \ldots, n)$ be finite distinct complex numbers. Let*

$$H(f) = \frac{P_k(f) f'}{(f-b_1)(f-b_2)\ldots(f-b_n)}.$$

If $k < n$, then $m(r, H(f)) = S(r, f)$.

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Lemma 3.2. [8] *Let f be a transcendental meromorphic function and $b_j \equiv b_j(z)$ $(j = 0, 1, 2, \ldots, n)$ be small functions of f. If*

$$b_n f^n + \ldots + b_1 f + b_0 = 0,$$

then $b_j \equiv 0$ for all $j = 0, 1, 2, \ldots, n$.

Lemma 3.3. [1] *Let f be a meromorphic function and $\mathcal{Q}[f]$, $\mathcal{Q}^*[f]$ be two differential polynomials in f with $\mathcal{Q}[f] \neq 0$. Suppose that a_j $(j = 1, 2, \ldots, m_1)$ are coefficients of $\mathcal{Q}[f]$ and a_j^* $(j = 1, 2, \ldots, m_2)$ are coefficients of $\mathcal{Q}^*[f]$. Let $n \in \mathbb{N}$ and $f^n \mathcal{Q}^*[f] = \mathcal{Q}[f]$. If $\gamma_{\mathcal{Q}} \leqslant n$, here $\gamma_{\mathcal{Q}}$ is a degree of $\mathcal{Q}$, then*

$$m\left(r, \mathcal{Q}^*[f]\right) \leqslant \sum_{j=1}^{m_1} m(r, a_j) + \sum_{k=1}^{m_2} m\left(r, a_k^*\right) + S(r, f).$$

Lemma 3.4. [3] *In the following equation*

$$f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_1 f' + b_0 f = 0,$$

if b_0 is a transcendental entire function and $b_j (j = 1, 2, \ldots, k-1)$ are polynomials, and if furthermore $T(r, b_j) = S(r, f)$, for $j = 0, 1, \ldots, k-1$, then f must be infinite order entire function.

In view of Lemma 3.3 and Lemma 3.4, we prove the following lemma.

Lemma 3.5. *Suppose that h_1, h_2, h_3 and γ are entire functions satisfying $h_1 h_2 h_3 = 1 - e^{-2\gamma}$ and $T\left(r, e^{2\gamma/3}\right) = S(r, h_j)$ for $j = 1, 2, 3$. If h_1, h_2 and h_3 are, respectively, solutions of*

$$\begin{cases} \sum_{j=0}^{k} b_j h_1^{(j)}(z) = -h_1(z) e^{\frac{2}{3}\gamma}, \\ \sum_{j=0}^{k} b_j h_2^{(j)}(z) = -\omega^2 h_2(z) e^{\frac{2}{3}\gamma}, \\ \sum_{j=0}^{k} b_j h_3^{(j)}(z) = -\omega h_3(z) e^{\frac{2}{3}\gamma}, \end{cases}$$

where $\omega^3 = 1$, then h_1, h_2 and h_3 must be infinite order entire functions, and $k \geqslant 3$ must be an odd integer.

Proof. In view of Lemma 3.4, it is easy to see that $h_j (j = 1, 2, 3)$ are of infinite order. If $k = 1$, then we have $b_0 h_1 + b_1 h_1' = -h_1 \mathrm{e}^{2\gamma/3}$, $b_0 h_2 + b_1 h_2' = -\omega^2 h_2 e^{2\gamma/3}$ and

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$b_0h_3 + b_1h_3' = -\omega h_3 e^{2\gamma/3}$. Eliminating $e^{2\gamma/3}$ from these three equations, we obtain

$$\frac{(h_1h_2h_3)'}{h_1h_2h_3} = -\frac{3b_0}{b_1},$$

which shows that $h_1h_2h_3 = De^{-3b_0z/b_1}$, where D is a non-zero constant. We thus see that $h_1h_2h_3 = 1 - e^{-2\gamma}$ has no zeros, which is impossible. Hence, $k \geqslant 2$, and in the following, we will prove that k must be an odd integer.

Since $h_1h_2h_3 = 1 - e^{-2\gamma}$ and $T\left(r, e^{2\gamma/3}\right) = S(r, h_j)$ for $j = 1, 2, 3$, and therefore, we must have $S(r, h_1) = S(r, h_2) = S(r, h_3) = S(r)$, say. Let

$$\alpha_1 = h_1'/h_1, \quad \beta_1 = h_2'/h_2, \quad \gamma_1 = h_3'/h_3.$$

Since $h_j (j = 1, 2, 3)$ are infinite order entire functions, so α_1, β_1 and γ_1 must be non-constants. In view of $h_1h_2h_3 = 1 - e^{-2\gamma}$ and $T\left(r, e^{2\gamma/3}\right) = S(r)$, so we have $N\left(r, 1/h_j\right) = S(r)$ for $j = 1, 2, 3$, and we see that

$$T(r, \alpha_1) = S(r), \quad T(r, \beta_1) = S(r), \quad T(r, \gamma_1) = S(r).$$

Using the principle of mathematical induction, a simple computation shows that

$$h_1^{(j)} = \alpha_j h_1, \quad h_2^{(j)} = \beta_j h_2, \quad h_3^{(j)} = \gamma_j h_3,$$

where α_j, β_j and γ_j are meromorphic functions satisfying the following recurrence relations, respectively

$$\alpha_{j+1} = \alpha_j' + \alpha_1\alpha_j, \quad \beta_{j+1} = \beta_j' + \beta_1\beta_j, \quad \gamma_{j+1} = \gamma_j' + \gamma_1\gamma_j,$$

for $j = 0, 1, \ldots$, where $\alpha_0 = \beta_0 = \gamma_0 = 1$. In view of our assumption, it is easy to see that

$$\sum_{j=0}^{k} b_j\alpha_j = -e^{2\gamma/3}, \tag{3.6}$$

$$\sum_{j=0}^{k} b_j\beta_j = -\omega^2 e^{2\gamma/3,} \tag{3.7}$$

$$\sum_{j=0}^{k} b_j\gamma_j = -\omega e^{2\gamma/3}. \tag{3.8}$$

A simple computation shows that

$$\alpha_j = \alpha_1^j + \frac{(j-1)j}{2}\alpha_1^{j-2}\alpha_1' + \mathcal{P}_{j-2}(\alpha_1), \quad j \geqslant 2, \tag{3.9}$$

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$$\beta_j = \beta_1^j + \frac{(j-1)j}{2}\beta_1^{j-2}\beta_1' + \mathcal{P}_{j-2}(\beta_1), \quad j \geqslant 2, \tag{3.10}$$

$$\gamma_j = \gamma_1^j + \frac{(j-1)j}{2}\gamma_1^{j-2}\gamma_1' + \mathcal{P}_{j-2}(\gamma_1), \quad j \geqslant 2, \tag{3.11}$$

where $\mathcal{P}_{j-2}(\alpha_1)$, $\mathcal{P}_{j-2}(\beta_1)$ and $\mathcal{P}_{j-2}(\gamma_1)$ are differential polynomials, respectively, in α_1, β_1 and γ_1, and of same degrees $j-2$, with constant coefficients. It follows from equations (3.6) and (3.9) that $\mathrm{e}^{2\gamma/3}$ is a differential polynomial in α_1 of degree j with leading term $b_k\alpha_1^k$. Therefore, we have

$$\begin{aligned} T\left(r, e^{2\gamma/3}\right) &\leqslant kT(r,\alpha_1) + S(r,\alpha_1) \\ m(r,\alpha_1) &\leqslant m\left(r, \mathrm{e}^{2\gamma/3}\right) + S\left(r, \mathrm{e}^{\frac{2}{3}\gamma}\right). \end{aligned}$$

On the other hand, we see that the poles of α_1 come from the zeros of $\mathrm{e}^{-2\gamma}-1$, and hence, we have $N(r,\alpha_1) \leqslant 2T\left(r,\mathrm{e}^{\gamma}\right) + S\left(r,\mathrm{e}^{\gamma}\right)$. Therefore, we have

$$T(r,\alpha_1) \leqslant 3T\left(r,\mathrm{e}^{2\gamma/3}\right) + S\left(r,\mathrm{e}^{2\gamma/3}\right).$$

By a similar argument, we obtain

$$\begin{aligned} T(r,\beta_1) &\leqslant 3T\left(r,\mathrm{e}^{2\gamma/3}\right) + S\left(r,\mathrm{e}^{2\gamma/3}\right), \\ T(r,\gamma_1) &\leqslant 3T\left(r,\mathrm{e}^{2\gamma/3}\right) + S\left(r,\mathrm{e}^{2\gamma/3}\right). \end{aligned}$$

Thus, each of $S(r,\alpha_1)$, $S(r,\beta_1)$ and $S(r,\gamma_1)$ must be $S\left(r,\mathrm{e}^{2\gamma/3}\right)$ and vice versa.

By equations (3.6) and (3.9), we obtain

$$b_k\frac{\alpha_1}{\mathrm{e}^{2\gamma}-1}\alpha_1^{k-1} + \frac{1}{\mathrm{e}^{2\gamma}-1}\mathcal{P}(\alpha_1) = -\frac{\mathrm{e}^{2\gamma/3}}{\mathrm{e}^{2\gamma}-1},$$

where $\mathcal{P}(\alpha_1)$ is a differential polynomial in α_1 of degree $k-1$ with constant coefficients. We also see that

$$m\left(r, \frac{1}{\mathrm{e}^{2\gamma}-1}\right) = S\left(r,\mathrm{e}^{2\gamma/3}\right) = m\left(r, \frac{\mathrm{e}^{2\gamma/3}}{\mathrm{e}^{2\gamma}-1}\right).$$

By Lemma 3.3, we have $m\left(r,\alpha_1/(\mathrm{e}^{2\gamma}-1)\right) = S\left(r,\mathrm{e}^{2\gamma/3}\right)$. Similarly, from equations (3.7) and (3.10), it follows that

$$b_k\frac{\beta_1}{\mathrm{e}^{2\gamma}-1}\beta_1^{k-1} + \frac{1}{\mathrm{e}^{2\gamma}-1}\mathcal{P}(\beta_1) = -\frac{\mathrm{e}^{2\gamma/3}}{\mathrm{e}^{2\gamma}-1},$$

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and hence, we can show that $m\left(r, \beta_1/(\mathrm{e}^{2\gamma}-1)\right) = S\left(r, \mathrm{e}^{2\gamma/3}\right)$. Combining the above two, we have

$$m\left(r, \frac{\alpha_1 \pm \beta_1}{\mathrm{e}^{2\gamma}-1}\right) = S\left(r, \mathrm{e}^{2\gamma/3}\right). \tag{3.12}$$

Adding equations (3.6), (3.7) and (3.8), we obtain $\sum_{j=0}^{k} b_j(\alpha_j+\beta_j+\gamma_j) = 0$. From this, using equations (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned} &b_k\Bigg(\alpha_1^k + \frac{(k-1)k}{2}\alpha_1^{k-2}\alpha_1' + \mathcal{P}_{k-2}(\alpha_1) + \beta_1^k + \frac{(k-1)k}{2}\beta_1^{k-2}\beta_1' + \mathcal{P}_{k-2}(\beta_1) + \gamma_1^k \\ &+ \frac{(k-1)k}{2}\gamma_1^{k-2}\gamma_1' + \mathcal{P}_{k-2}(\gamma_1)\Bigg) + b_{k-1}\Bigg(\alpha_1^{k-1} + \frac{(k-2)(k-1)}{2}\alpha_1^{k-3}\alpha_1' + \mathcal{P}_{k-3}(\alpha_1) \\ &+ \beta_1^{k-1} + \frac{(k-2)(k-1)}{2}\beta_1^{k-3}\beta_1' + \mathcal{P}_{k-3}(\beta_1) + \gamma_1^{k-1} + \frac{(k-2)(k-1)}{2}\gamma_1^{k-3}\gamma_1' \\ &+ \mathcal{P}_{k-3}(\gamma_1)\Bigg) + \sum_{j=0}^{k-2} b_j(\alpha_j+\beta_j+\gamma_j) = 0. \end{aligned} \tag{3.13}$$

Moreover, we have

$$\alpha_1+\beta_1+\gamma_1 = \frac{(h_1h_2h_3)'}{h_1h_2h_3} = \frac{2\gamma'}{e^{2\gamma}-1},$$

which yields $\gamma_1 = 2\gamma'/(\mathrm{e}^{2\gamma}-1) - (\alpha_1+\beta_1)$. Substituting this γ_1 into equation (3.13) and using equation (3.12), we deduce that

$$\begin{aligned} &b_k\left(1+(-1)^k\right)\alpha_1^k + b_{k-1}\left(1-(-1)^k\right)\frac{(k-1)k}{2}\alpha_1^{k-2}\alpha_1' \\ &\quad + b_{k-1}\left(1-(-1)^k\right)\alpha_1^{k-1} \in \mathcal{D}_{k-2}(\alpha_1) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &b_k\left(1+(-1)^k\right)\beta_1^k + b_{k-1}\left(1-(-1)^k\right)\frac{(k-1)k}{2}\beta_1^{k-2}\beta_1' \\ &\quad + b_{k-1}\left(1-(-1)^k\right)\beta_1^{k-1} \in \mathcal{D}_{k-2}(\beta_1), \end{aligned} \tag{3.15}$$

where $\mathcal{D}_{k-2}(\alpha_1)$, $\mathcal{D}_{k-2}(\beta_1)$ are, respectively, differential polynomials in α_1, β_1 of same degree $k-2$, with coefficients being functions of small proximity relative to $e^{\frac{2}{3}\gamma}$. If k is even, then from equations (3.14) and (4.12), we notice $2b_k\alpha_1^k \in \mathcal{D}_{k-2}(\alpha_1)$ and $2b_k\beta_1^k \in \mathcal{D}_{k-2}(\beta_1)$.

By Lemma 3.3, we have $m(r,\alpha_1) = S\left(r, e^{2\gamma/3}\right)$ and $m(r,\beta_1) = S\left(r, e^{2\gamma/3}\right)$. With these and in view of equations (3.6) and (3.7), one can now deduce that $m\left(r, e^{2\gamma/3}\right) = S\left(r, \mathrm{e}^{2\gamma/3}\right)$, which leads to a contradiction. Hence, k must be an odd integer and $k \geqslant 3$. □

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4. Proof of the main results

Proof of Theorem 2.1. Let $g = L_k[f]$. Since $E_f(\mathcal{S}) = E_g(\mathcal{S})$, where $\mathcal{S} = \{a, b, c\}$, then there exists an entire function γ such that

$$\frac{(g-a)(g-b)(g-c)}{(f-a)(f-b)(f-c)} = \mathrm{e}^{2\gamma}. \tag{4.1}$$

By the Second Fundamental Theorem, we obtain

$$\begin{aligned} T(r,f) &\leqslant \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f-b}\right) + \bar{N}\left(r, \frac{1}{f-c}\right) + S(r,f) \\ &\leqslant \bar{N}\left(r, \frac{1}{g-a}\right) + \bar{N}\left(r, \frac{1}{g-b}\right) + \bar{N}\left(r, \frac{1}{g-c}\right) + S(r,f) \\ &\leqslant 3T(r,g) + S(r,f). \end{aligned}$$

With g being a linear differential polynomial, we have $T(r,g) \leqslant mT(r,f) + S(r,f)$, for some $m \in \mathbb{N}$. Therefore $S(r,f)$ can be replaced by $S(r,g)$ and vice versa. For the sake of simplicity, we assume $S(r) := S(r,f) = S(r,g)$. Hence equation (4.1) yields $T(r,\gamma) = S(r)$.

Differentiating both sides of equation (4.1), we obtain

$$\begin{aligned} &\left(3g^2 - 2(a+b+c)g + (ab+bc+ca)\right)g' \\ &= \left(3f^2f' - 2(a+b+c)ff' + (ab+bc+ca)f' + 2\gamma'(f-a)(f-b)(f-c)\right)\mathrm{e}^{2\gamma}. \end{aligned} \tag{4.2}$$

If z_0 be a common zero of $(g-a)(g-b)(g-c)$ and $(f-a)(f-b)(f-c)$, then $g(\alpha) \in \mathcal{S}$ and $f(z_0) \in \mathcal{S}$. Next, our aim is to show that z_0 is a zero of $[g'(z)]^3 - [f'(z)\mathrm{e}^{2\gamma(z)}]^3$. In order to show this, we need to discuss several cases. Henceforth, we define

$$\rho_0 := \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \ \rho_1 := \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \ \rho_2 := \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix},$$
$$\rho_3 := \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \ \rho_4 := \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \ \rho_5 := \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}.$$

For the set $\mathcal{S} = \{a, b, c\}$, let us introduce the notation $f \Leftrightarrow_\rho g$ by which we understand that $f(a) = g(\alpha)$, $f(b) = g(\beta)$ and $f(c) = g(\gamma)$, where $\alpha, \beta, \gamma \in \mathcal{S}$ when ρ is defined by

$$\rho = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Since $\mathcal{S}$ consists of three elements and by the assumption of the result as $E_f(\mathcal{S}) = E_g(\mathcal{S})$, we have exactly $3! = 6$ possible cases, which are $f \Leftrightarrow_{\rho_j} g$ $(j = 0, 1, 2, \ldots, 5)$. We now discuss them one by one.

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Case 1. If $f \Leftrightarrow_{\rho_0} g$, then we see that $f(a) = g(a)$, $f(b) = g(b)$ and $f(c) = g(c)$. Hence, for these three situations, from equation (4.2), a simple computation shows that, in each case, we obtain $g'(z_0) = f'(z_0)e^{2\gamma(z_0)}$. It is clear that $[g'(z_0)]^3 = [f'(z_0)e^{2\gamma(z_0)}]^3$, which shows that z_0 is a zero of $[g'(z)]^3 - [f'(z)e^{2\gamma(z)}]^3$.

Case 2. If $f \Leftrightarrow_{\rho_1} g$, then we see that $f(a) = g(b)$, $f(b) = g(c)$ and $f(c) = g(a)$. Consequently, by employing equation (4.2), a simple calculation indicates that $(a-c)g'(z_0) = (c-b)f'(z_0)e^{2\gamma(z_0)}$, $(b-a)g'(z_0) = (a-c)f'(z_0)e^{2\gamma(z_0)}$. and $(c-b)g'(z_0) = (b-a)f'(z_0)e^{2\gamma(z_0)}$, respectively. Clearly, we have

$$(a-c)(b-a)(c-b)[g'(z_0)]^3 = (c-b)(a-c)(b-a)[f'(z_0)e^{2\gamma(z_0)}]^3,$$

and a fact that becomes apparent upon dividing out the factor $(a-c)(b-a)(c-b)$ shows that $[g'(z_0)]^3 = [f'(z_0)e^{2\gamma(z_0)}]^3$; hence, z_0 is a zero of $[g'(z)]^3 - [f'(z)e^{2\gamma(z)}]^3$.

Case 3. If $f \Leftrightarrow_{\rho_2} g$, then we see that $f(a) = g(c)$, $f(b) = g(a)$ and $f(c) = g(b)$. Henceforth, with the use of equation (4.2), a basic computation illustrates that $(a-b)g'(z_0) = (b-c)f'(z_0)e^{2\gamma(z_0)}$, $(b-c)g'(z_0) = (c-a)f'(z_0)e^{2\gamma(z_0)}$ and $(c-a)g'(z_0) = (a-b)f'(z_0)e^{2\gamma(z_0)}$, respectively. It is evident that

$$(a-b)(b-c)(c-a)[g'(z_0)]^3 = (b-c)(c-a)(a-b)[f'(z_0)e^{2\gamma(z_0)}]^3,$$

which shows upon eliminating the factor $(a-b)(b-c)(c-a)$ that z_0 is a zero of $[g'(z)]^3 - [f'(z)e^{2\gamma(z)}]^3$.

Case 4. If $f \Leftrightarrow_{\rho_3} g$, then we see that $f(a) = g(a)$, $f(b) = g(c)$ and $f(c) = g(b)$. Hence, from equation (4.2), a simple computation shows that $g'(z_0) = f'(z_0)e^{2\gamma(z_0)}$, $(b-a)g'(z_0) = (a-c)f'(z_0)e^{2\gamma(z_0)}$ and $(c-a)g'(z_0) = (a-b)f'(z_0)e^{2\gamma(z_0)}$, respectively. It is clear that

$$(b-a)(c-a)[g'(z_0)]^3 = (a-c)(a-b)[f'(z_0)e^{2\gamma(z_0)}]^3,$$

which shows upon canceling the factor $(b-a)(c-a)$ that z_0 is a zero of $[g'(z)]^3 - [f'(z)e^{2\gamma(z)}]^3$.

Case 5. If $f \Leftrightarrow_{\rho_4} g$, then we see that $f(a) = g(c)$, $f(b) = g(b)$ and $f(c) = g(a)$. Therefore, by referencing equation (4.2), a simple analysis demonstrates that $(a-b)g'(z_0) = (b-c)f'(z_0)e^{2\gamma(z_0)}$, $g'(z_0) = f'(z_0)e^{2\gamma(z_0)}$ and $(c-b)g'(z_0) = (b-a)f'(z_0)e^{2\gamma(z_0)}$, respectively. It is evident that

$$(a-b)(c-b)[g'(z_0)]^3 = (b-c)(b-a)[f'(z_0)e^{2\gamma(z_0)}]^3,$$

which indicates upon removing the factor $(a-b)(c-b)$ that z_0 is a zero of $[g'(z)]^3 - [f'(z)e^{2\gamma(z)}]^3$.

Case 6. If $f \Leftrightarrow_{\rho_5} g$, then we see that $f(a) = g(b)$, $f(b) = g(a)$ and $f(c) = g(c)$. Consequently, derived from equation (4.2), a simple calculation shows that $(a-c)g'(z_0) = (c-b)f'(z_0)e^{2\gamma(z_0)}$, $(b-c)g'(z_0) = (c-a)f'(z_0)e^{2\gamma(z_0)}$ and

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$g'(z_0) = f'(z_0)\mathrm{e}^{2\gamma(z_0)}$, respectively. Certainly, we have

$$(a-c)(b-c)[g'(z_0)]^3 = (c-b)(c-a)[f'(z_0)\mathrm{e}^{2\gamma(z_0)}]^3,$$

which shows upon canceling the factor $(a-c)(b-c)$ that z_0 is a zero of $[g'(z)]^3 - [f'(z)\mathrm{e}^{2\gamma(z)}]^3$.

The estimates in the above cases show that z_0 is also a zero of $(g')^3 - \left(f'\mathrm{e}^{2\gamma}\right)^3$. If z_0 is a zero of $(g-a)(g-b)(g-c)$ of multiplicity $p \geqslant 2$, then z_0 must be a zero of $(g')^3 - \left(f'\mathrm{e}^{2\gamma}\right)^3$ of multiplicity $3(p-1)$. Thus, z_0 is not a pole of the function ψ, where ψ is defined by

$$\psi := \frac{\left(f'\mathrm{e}^{2\gamma}\right)^3 - (g')^3}{(g-a)(g-b)(g-c)} = \frac{\left(f'\mathrm{e}^{2\gamma} - g'\right)\left(f'\mathrm{e}^{2\gamma} - \omega g'\right)\left(f'\mathrm{e}^{2\gamma} - \omega^2 g'\right)}{(g-a)(g-b)(g-c)}, \tag{4.3}$$

where ω is a non-real cube root of unity. Clearly, ψ is an entire function. Now, we can write

$$\frac{f'e^{2\gamma} - g'}{g-a} = \frac{(g-b)(g-c)f'}{(f-a)(f-b)(f-c)} - \frac{g'}{g-a}.$$

By the lemma of logarithmic derivative and Lemma 3.1, we obtain

$$m\left(r, \frac{f'\mathrm{e}^{2\gamma} - g'}{g-a}\right) = S(r).$$

Similarly, we obtain

$$m\left(r, \frac{f'\mathrm{e}^{2\gamma} - \omega g'}{g-b}\right) = S(r) = m\left(r, \frac{f'\mathrm{e}^{2\gamma} - \omega^2 g'}{g-c}\right).$$

Thus, we have $T(r,\psi) = S(r)$.

Next, we distinguish two cases as below.

Case A. Suppose $T\left(r, \mathrm{e}^{2\gamma}\right) \neq S(r)$. If $\psi = 0$, then we can write

$$\mathrm{e}^{2\gamma} = \omega\left(b_0 + b_1\frac{f''}{f'} + b_2\frac{f'''}{f'} + \ldots + b_k\frac{f^{(k+1)}}{f'}\right).$$

It follows from the lemma of the logarithmic derivative that $m\left(r, \mathrm{e}^{2\gamma}\right) = S(r)$, which contradicts the assumption.

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Hence, $\psi \not\equiv 0$, and from equation (4.3), we can write

$$\begin{cases} \dfrac{1}{\mathrm{e}^{2\gamma}f' - g'} = \dfrac{1}{\psi}\dfrac{(\mathrm{e}^{2\gamma}f' - \omega g')(\mathrm{e}^{2\gamma}f' - \omega^2 g')}{(g-a)(g-b)(g-c)}; \\ \dfrac{1}{\mathrm{e}^{2\gamma}f' - \omega g'} = \dfrac{1}{\psi}\dfrac{(\mathrm{e}^{2\gamma}f' - g')(\mathrm{e}^{2\gamma}f' - \omega^2 g')}{(g-a)(g-b)(g-c)}; \\ \dfrac{1}{\mathrm{e}^{2\gamma}f' - \omega^2 g'} = \dfrac{1}{\psi}\dfrac{(\mathrm{e}^{2\gamma}f' - g')(\mathrm{e}^{2\gamma}f' - \omega g')}{(g-a)(g-b)(g-c)}. \end{cases}$$

It is easy to see that

$$\begin{aligned} & m\left(\frac{1}{(\mathrm{e}^{2\gamma}f' - g')\,(g-a)(g-b)(g-c)}\right) \\ & \leqslant m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{\mathrm{e}^{2\gamma}f' - \omega g'}{(g-a)(g-b)(g-c)}\right) + m\left(r, \frac{e^{2\gamma}f' - \omega^2 g'}{(g-a)(g-b)(g-c)}\right) \\ & \leqslant m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{f'}{(f-a)(f-b)(f-c)} - \frac{\omega g'}{(g-a)(g-b)(g-c)}\right) \\ & + m\left(r, \frac{f'}{(f-a)(f-b)(f-c)} - \frac{\omega^2 g'}{(g-a)(g-b)(g-c)}\right) \\ & \leqslant m\left(r, \frac{1}{\psi}\right) + 2\,m\left(r, \frac{f'}{(f-a)(f-b)(f-c)}\right) + 2\,m\left(r, \frac{g'}{(g-a)(g-b)(g-c)}\right) \\ & \leqslant S(r). \end{aligned} \tag{4.4}$$

Similarly, we can obtain

$$m\left(\frac{1}{(\mathrm{e}^{2\gamma}f' - \omega g')\,(g-a)(g-b)(g-c)}\right) \leqslant S(r), \tag{4.5}$$

$$m\left(\frac{1}{(\mathrm{e}^{2\gamma}f' - \omega^2 g')\,(g-a)(g-b)(g-c)}\right) \leqslant S(r). \tag{4.6}$$

Therefore, in view of equations (4.3), (4.4), (4.5) and (4.6), we obtain

$$\begin{aligned} & m\left(r, \frac{1}{(g-a)^4(g-b)^4(g-c)^4}\right) \\ & = m\left(r, \frac{\psi}{(\mathrm{e}^{2\gamma}f' - g')\,(\mathrm{e}^{2\gamma}f' - \omega g')\,(\mathrm{e}^{2\gamma}f' - \omega^2 g')\,(g-a)^3(g-b)^3(g-c)^3}\right) \\ & \leqslant m\left(\frac{1}{(\mathrm{e}^{2\gamma}f' - g')\,(g-a)(g-b)(g-c)}\right) + m\left(\frac{1}{(\mathrm{e}^{2\gamma}f' - \omega g')\,(g-a)(g-b)(g-c)}\right) \\ & \quad + m\left(\frac{1}{(\mathrm{e}^{2\gamma}f' - \omega^2 g')\,(g-a)(g-b)(g-c)}\right) + m(r, \psi) \\ & \leqslant S(r), \end{aligned}$$

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which yields

$$m\left(r,\frac{1}{(g-a)(g-b)(g-c)}\right)=S(r). \tag{4.7}$$

It follows from equation (4.7) that

$$m\left(r,\frac{1}{g-a}\right)=S(r),\quad m\left(r,\frac{1}{g-b}\right)=S(r)\ \text{ and }\ m\left(r,\frac{1}{g-c}\right)=S(r).$$

Therefore, we obtain

$$3T(r,g)=N\left(r,\frac{1}{g-a}\right)+N\left(r,\frac{1}{g-b}\right)+N\left(r,\frac{1}{g-c}\right)+S(r). \tag{4.8}$$

A simple computation using equations (4.1) and (4.2) shows that

$$\frac{\phi(g)g'}{(g-a)(g-b)(g-c)}=\frac{\phi(f)f'}{(f-a)(f-b)(f-c)}+2\gamma', \tag{4.9}$$

where $\phi(h)=3h^2-2(a+b+c)h+(ab+bc+ca)$. It follows from equation (4.9) that

$$\begin{aligned}&\frac{\phi^3(g)\,(g')^3}{(g-a)^3(g-b)^3(g-c)^3}\\&=\frac{\phi^3(f)\,(f')^3}{(f-a)^3(f-b)^3(f-c)^3}+\frac{6\phi^2(f)\,(f')^2\,\gamma'}{(f-a)^2(f-b)^2(f-c)^2}+\frac{12\phi(f)f'\,(\gamma')^2}{(f-a)(f-b)(f-c)}\\&\quad+8\,(\gamma')^3.\end{aligned} \tag{4.10}$$

By equation (4.1) and eliminating γ, we obtain from equation (4.3) that

$$(g')^3=\frac{(g-a)^3(g-b)^3(g-c)^3}{(f-a)^3(f-b)^3(f-c)^3}\,(f')^3-\psi(g-a)(g-b)(g-c)$$

from which we obtain

$$\begin{aligned}&\frac{(g')^3}{(g-a)^3(g-b)^3(g-c)^3}\\&=\frac{(f')^3}{(f-a)^3(f-b)^3(f-c)^3}-\frac{\psi}{(g-a)^2(g-b)^2(g-c)^2}\end{aligned} \tag{4.11}$$

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By equations (4.10) and (4.11), we deduce that

$$\begin{aligned}&\frac{\psi\phi^3(g)}{(g-a)^2(g-b)^2(g-c)^2}\\&=\frac{[\phi(g)-\phi(f)][\phi(g)-\omega\phi(f)][\phi(g)-\omega^2\phi(f)]\,(f')^3}{(f-a)^3(f-b)^3(f-c)^3}-\frac{6\phi^2(f)\,(f')^2\,\gamma'}{(f-a)^2(f-b)^2(f-c)^2}\\&\quad-\frac{12\phi(f)f'\,(\gamma')^2}{(f-a)(f-b)(f-c)}-8\,(\gamma')^3.\end{aligned}\tag{4.12}$$

Since $E_f(\mathcal{S})=E_g(\mathcal{S})$, i.e. the functions $f$and g share the set $\mathcal{S}=\{a,b,c\}$ CM, and hence, all the zeros of $(f-a)(f-b)(f-c)$ must be zeros of $(g-a)(g-b)(g-c)$. Let z_1 be such a zero of $(f-a)(f-b)(f-c)$ and $(g-a)(g-b)(g-c)$. In view of equation (4.1), we see that equation (4.12) can be written as

$$\begin{aligned}\psi\phi^3(g)(g-a)(g-b)(g-c)&=[\phi(g)-\phi(f)][\phi(g)-\omega\phi(f)][\phi(g)-\omega^2\phi(f)]\,(f')^3\left(e^{2\gamma}\right)^3\\&\quad-6\phi^2(f)\,(f')^2\,\gamma'\left(e^{2\gamma}\right)^2(g-a)(g-b)(g-c)\\&\quad-12\phi(f)f'\,(\gamma')^2\,e^{2\gamma}(g-a)^2(g-b)^2(g-c)^2\\&\quad-8\,(\gamma')^3\,(g-a)^3(g-b)^3(g-c)^3.\end{aligned}\tag{4.13}$$

Considering equation (4.13) and noticing that the term associated with $(g-a)(g-b)(g-c)$ will vanish at z_1, it becomes evident that z_1 is a zero of $[\phi(g)-\phi(f)][\phi(g)-\omega\phi(f)][\phi(g)-\omega^2\phi(f)]\,(f')^3\left(\mathrm{e}^{2\gamma}\right)^3$. Consequently, this implies that z_1 must be a zero of $(\phi(g)-\phi(f))\,(\phi(g)-\omega\phi(f))\left(\phi(g)-\omega^2\phi(f)\right)(f')^3$. We observe that the poles of the right-hand side of equation (4.12) are all simple. Hence the zeros of $(g-a)(g-b)(g-c)$ are simple as long as they are not the zeros of ψ. Therefore, we deduce that

$$\begin{cases}N\left(r,\dfrac{1}{g-a}\right)=\bar{N}\left(r,\dfrac{1}{g-a}\right)+S(r);\\N\left(r,\dfrac{1}{g-b}\right)=\bar{N}\left(r,\dfrac{1}{g-b}\right)+S(r);\\N\left(r,\dfrac{1}{g-c}\right)=\bar{N}\left(r,\dfrac{1}{g-c}\right)+S(r).\end{cases}$$

Differentiating both sides of equation (4.3) and then eliminating $\mathrm{e}^{2\gamma}$, we obtain

$$\begin{aligned}&(3\psi(f''+2f'\gamma')-f'\psi')\,(g-a)(g-b)(g-c)\\&=\left(\psi\phi(g)f'-(6f'\gamma'+3f'')\,(g')^2+3f'g'g''\right)g'\end{aligned}\tag{4.14}$$

Using equation (4.14), it is easy to see that any simple zero of $(g-a)(g-b)(g-c)$ must be a zero of

$$\psi\phi(g)f'-(6f'\gamma'+3f'')\,(g')^2+3f'g'g''.$$

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Set

$$\psi_1 := \frac{\psi\phi(g)f' - (6f'\gamma' + 3f'')\,(g')^2 + 3f'g'g''}{(f-a)(f-b)(f-c)}. \tag{4.15}$$

Note that the functions f and g share the set $\{a, b, c\}$ CM and almost all zeros of $(g-a)(g-b)(g-c)$ are simple; hence, we have $N(r, \psi_1) = S(r)$. Again by the lemma of logarithmic derivative, we have $m(r, \psi_1) = S(r)$. Therefore, it is easy to see that $T(r, \psi_1) = N(r, \psi_1) + m(r, \psi_1) = S(r)$.

If $\psi_1 \not\equiv 0$, then from equation (4.15), we obtain

$$\begin{aligned} 3T(r,f) &\leqslant m\left(r, (f-a)(f-b)(f-c)\psi_1\right) + S(r) \\ &\leqslant m\left(r, f'\right) + 2m(r,g) + S(r) \\ &\leqslant T(r,f) + 2T(r,g) + S(r) \\ &\leqslant 3T(r,f) + S(r). \end{aligned}$$

It follows that $T(r, f) = T(r, g) + S(r)$. In view of equation (4.8) and the estimates just before equation (4.14), a simple computation shows that

$$3T(r,f) = \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f-b}\right) + \bar{N}\left(r, \frac{1}{f-c}\right) + S(r),$$

which implies that

$$m\left(r, \frac{1}{f-a}\right) = S(r), \quad m\left(r, \frac{1}{f-b}\right) = S(r) \text{ and } m\left(r, \frac{1}{f-c}\right) = S(r).$$

Therefore, from equation (4.1) and in view of the lemma of the logarithmic derivative, we must have $T\left(r, \mathrm{e}^{2\gamma}\right) = S(r, f)$, which contradicts Case A.

If $\psi_1 \equiv 0$, then from equations (4.14) and (4.15), we obtain

$$\frac{\psi'}{\psi} = 3\left(2\gamma' + \frac{f''}{f',}\right)$$

which shows that

$$\left(\mathrm{e}^{2\gamma} f'\right)^3 = \eta\psi, \tag{4.16}$$

where η is a non-zero complex number. Using equations (4.3) and (4.16), a simple computation shows that

$$(g')^3 = -\psi((g-a)(g-b)(g-c) - \eta) = -\psi(g-\sigma_1)(g-\sigma_2)(g-\sigma_3), \tag{4.17}$$

where σ_1, σ_2 and σ_3 are three complex numbers.

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If $\sigma_1 \neq \sigma_2$, then it is easy to see that

$$m\left(r, \frac{1}{g'}\right) = m\left(r, \frac{-(g')^2}{\psi(g-\sigma_1)(g-\sigma_2)(g-\sigma_3)}\right) = S(r,g) = S(r). \tag{4.18}$$

Again, since g is a differential polynomial in f, it follows that

$$\begin{aligned} m\left(r, \frac{1}{(f-a)(f-b)(f-c)}\right) &\leqslant m\left(r, \frac{g'}{(f-a)(f-b)(f-c)}\right) + m\left(r, \frac{1}{g'}\right) + S(r) \\ &\leqslant S(r). \end{aligned}$$

By equation (4.1), we obtain

$$m\left(r, \mathrm{e}^{2\gamma}\right) = m\left(r, \frac{1}{(f-a)(f-b)(f-c)}\right) + S(r) = S(r),$$

which shows that $T\left(r, \mathrm{e}^{2\gamma}\right) = m\left(r, \mathrm{e}^{2\gamma}\right) = S(r)$, and this contradicts the assumption of Case A. Hence, we must have $\sigma_1 = \sigma_2$. Similarly, we can prove that $\sigma_2 = \sigma_3$ and $\sigma_3 = \sigma_1 = \zeta$(say), by following the above calculation. Therefore, it is easy to see that $\sigma_1 = \sigma_2 = \sigma_3$. Thus, we have

$$(g')^3 = -\psi(g-\zeta)^3. \tag{4.19}$$

Combining equations (4.1), (4.16) and (4.17), we deduce that

$$(g-a)(g-b)(g-c)(g-\zeta) = c_1\psi_2(f-a)(f-b)(f-c), \tag{4.20}$$

where c_1 is a non-zero constant satisfying $c_1^3 = -\eta$ and $\psi_2 = g'/f'$. From equation (4.16), we see that $N\left(r, 1/f'\right) = S(r)$. Thus, we have $N(r, \psi_2) = S(r)$. By the lemma of the logarithmic derivative, we obtain $m(r, \psi_2) = S(r)$, and hence, $T(r, \psi_2) = S(r)$. Since g cannot be a constant, we obtain $\psi_2 \not\equiv 0$. It follows from equation (4.20) that

$$4T(r,g) = 3T(r,f) + S(r). \tag{4.21}$$

We define

$$\psi_3 := \frac{g'}{g-\zeta}. \tag{4.22}$$

Hence from equation (4.19), we have $\psi_3^3 = -\psi$, which implies that $T\left(r, \psi_3\right) = S(r)$ and $\psi_3 \not\equiv 0$. By equations (4.1) and (4.2), we have

$$\phi(f)f'\mathrm{e}^{2\gamma} = \phi(g)(g-\zeta)\psi_3 - 2\gamma'(g-a)(g-b)(g-c). \tag{4.23}$$

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By equation (4.16), it is easy to see that $T\left(f'e^{2\gamma}\right) = S(r)$. Therefore, it follows from equation (4.23) that

$$2T(r,f) = 2T(r,g) + S(r) \quad \text{or} \quad 2T(r,f) = 3T(r,g) + S(r) \tag{4.24}$$

according to $\psi_3 = 2\gamma'/3$ or $\psi_3 = \gamma'$. Combining equations (4.21) and (4.24), we obtain $T(r,f) = S(r)$, which contradicts the assumption in Case A.

Case B. Let $T\left(r, e^{2\gamma}\right) = S(r)$. If $\mathrm{e}^{2\gamma} = 1$, then by equation (4.1), we see that

$$(f-a)(f-b)(f-c) \equiv (g-a)(g-b)(g-c),$$

which implies that either $f \equiv g$ or

$$f^2 + fg + g^2 - (a+b+c)(f+g) + (ab+bc+ca) = 0. \tag{4.25}$$

If $f \equiv g$, then it can be easily solved because it is an ordinary differential equation with constant coefficients. A simple computation shows that the general entire solution is $f(z) = \sum_{j=1}^{k} a_j \mathrm{e}^{\lambda_j z}$, where λ_j $(j = 1, \ldots, k)$ are distinct roots of the equation $b_k z^k + \cdots + b_1 z + b_0 = 1$.

In the next case, since equation (4.25) is of second degree, in order to obtain f and g explicitly, we require some transformations to reduce the equation in its canonical form. By comparing equation (4.25) with the standard second-degree equation $Af^2 + 2Hfg + Bg^2 + 2Gf + 2Fg + C = 0$, we have $A = 1 = B$, $H = 1/2$, $G = -(a+b+c)/2 = F$ and $C = ab + bc + ca$. We need to calculate Δ and D to find the nature of equation (4.25), where

$$\Delta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = ABC + 2FGH - AF^2 - BG^2 - CH^2, \quad D = AB - H^2.$$

It is easy to see that $\Delta = -\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)/8$ and $D = 3/4$. Since a, b and c are distinct constants, it is easy to see that $\Delta \neq 0$ and $D > 0$. Therefore, equation (4.25) represents an ellipse. To obtain the canonical form of equation (4.25), we first translate the origin $(0,0)$ to the point (h,k). Then the transformation formulae are $f(z) = \mathcal{F}(z) + h$ and $g(z) = \mathcal{G}(z) + k$, where (h,k) satisfies the equations $Ah + Bk + G = 0$ and $Hh + Bk + F = 0$. Solving these two equations, we obtain

$$(h,k) = \left(\frac{HF - BG}{AB - H^2}, \frac{HG - AF}{AB - H^2}\right) = \left(\frac{a+b+c}{4}, \frac{a+b+c}{4}\right).$$

Under this transformation, equation (4.25) transformed to $\mathcal{F}^2 + \mathcal{F}\mathcal{G} + \mathcal{G}^2 + \tau = 0$, where

$$\tau = \frac{\Delta}{D} = -\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{6}.$$

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To remove the product term $\mathcal{F}\mathcal{G}$, we take the transformation, namely, rotation through an angle θ, and the transformation formulae are

$$\begin{aligned}\mathcal{F} &= \mathcal{U}\cos\theta - \mathcal{V}\sin\theta,\\ \mathcal{G} &= \mathcal{U}\cos\theta + \mathcal{V}\sin\theta,\end{aligned}$$

where $\theta = (1/2)\tan^{-1}(2H/(A-B)) = \pi/4$. Thus, we have $\mathcal{F} = (\mathcal{U}-\mathcal{V})/\sqrt{2}$ and $\mathcal{G} = (\mathcal{U}+\mathcal{V})/\sqrt{2}$. Applying this transformation, the equation takes the form $\alpha\mathcal{U}^2+\beta\mathcal{V}^2+\tau = 0$, and by the invariance of this transformation, we must have $\alpha+\beta = A+B = 2$ and $\alpha\beta - 0^2 = AB - H^2 = 3/4$. A simple computation shows that $(\alpha,\beta) = (3/2,\ 1/2)$ or $(\alpha,\beta) = (1/2,\ 3/2)$. Therefore, the transformed equation is one of the following

$$\begin{aligned}\frac{3\mathcal{U}^2}{2} + \frac{\mathcal{V}^2}{2} &= \frac{1}{6}\left((a-b)^2+(b-c)^2+(c-a)^2\right),\\ \frac{\mathcal{U}^2}{2} + \frac{3\mathcal{V}^2}{2} &= \frac{1}{6}\left((a-b)^2+(b-c)^2+(c-a)^2\right).\end{aligned}$$

Denoting η by $\eta = \left((a-b)^2+(b-c)^2+(c-a)^2\right)$, we discuss the following two cases.
Subcase 1. If $3\mathcal{U}^2/2 + \mathcal{V}^2/2 = \left((a-b)^2+(b-c)^2+(c-a)^2\right)/6$, then we have

$$\left(\frac{\mathcal{U}}{\pm\frac{\sqrt{\eta}}{3}}\right)^2 + \left(\frac{\mathcal{V}}{\pm\sqrt{\frac{\eta}{3}}}\right)^2 = 1. \tag{4.27}$$

This is a Fermat-type equation of the form $F^2(z)+G^2(z) = 1$, and hence, by the result of Gross [4], the only non-constant entire solutions are $F(z) = \sin(h(z))$ and $G(z) = \cos(h(z))$, where $h(z)$ is an entire function. Therefore, by equation (4.27), we obtain $\mathcal{U}(z) = \pm(\sqrt{\eta}/3)\sin(h(z))$ and $\mathcal{V}(z) = \pm(\sqrt{\eta/3})\cos(h(z))$. Hence, f and g can be expressed as

$$\begin{aligned}f(z) &= \mathcal{F}(z)+h\\ &= (\mathcal{U}-\mathcal{V})/\sqrt{2} + (a+b+c)/4\\ &= \pm\sqrt{\eta/54}\left(\sqrt{3}\sin(h(z)) - 3\cos(h(z))\right) + (a+b+c)/4\end{aligned}$$

and

$$\begin{aligned}g(z) &= \mathcal{G}(z)+k\\ &= (\mathcal{U}+\mathcal{V})/\sqrt{2} + (a+b+c)/4\\ &= \pm\sqrt{\eta/54}\left(\sqrt{3}\sin(h(z)) + 3\cos(h(z))\right) + (a+b+c)/4.\end{aligned}$$

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Case 2. If $\mathcal{U}^2/2 + 3\mathcal{V}^2/2 = \left((a-b)^2 + (b-c)^2 + (c-a)^2\right)/6$, then we have

$$\left(\frac{\mathcal{U}}{\pm\sqrt{\frac{\eta}{3}}}\right)^2 + \left(\frac{\mathcal{V}}{\pm\frac{\sqrt{\eta}}{3}}\right)^2 = 1. \tag{4.28}$$

By the similar argument as in Case 1, we have

$$\mathcal{U}(z) = \pm\sqrt{\frac{\eta}{3}}\sin(h(z)) \text{ and } \mathcal{V}(z) = \pm\frac{\sqrt{\eta}}{3}\cos(h(z)).$$

Therefore, f and g can be expressed as

$$\begin{aligned} f(z) &= \mathcal{F}(z) + h \\ &= (\mathcal{U} - \mathcal{V})/\sqrt{2} + (a+b+c)/4 \\ &= \pm\sqrt{\eta/54}\left(3\sin(h(z)) - \sqrt{3}\cos(h(z))\right) + (a+b+c)/4 \end{aligned}$$

and

$$\begin{aligned} g(z) &= \mathcal{G}(z) + k \\ &= (\mathcal{U} + \mathcal{V})/\sqrt{2} + (a+b+c)/4 \\ &= \pm\sqrt{\eta/54}\left(3\sin(h(z)) + \sqrt{3}\cos(h(z))\right) + (a+b+c)/4. \end{aligned}$$

We now assume that $e^{2\gamma} \neq 1$.

Under the transformation $f(z) = \mathfrak{F}(z) + (a+b+c)/3$ and $g(z) = \mathfrak{G}(z) + (a+b+c)/3$, equation (4.1) is transformed to

$$(\mathfrak{F} - \mathcal{B})(\mathfrak{F} - \omega\mathcal{B})(\mathfrak{F} - \omega^2\mathcal{B})e^{2\gamma} = (\mathfrak{G} - \mathcal{B})(\mathfrak{G} - \omega\mathcal{B})(\mathfrak{G} - \omega^2\mathcal{B}), \tag{4.29}$$

where $\mathcal{B} = (2a-b-c)/3$, $\mathcal{B}\omega = (2b-c-a)/3$ and $\mathcal{B}\omega^2 = (2c-a-b)/3$, and so we have

$$\omega^2(2a-b-c) = \omega(2b-c-a) = 2c-a-b.$$

We set three functions $h_j(z)$ $(j = 1, 23)$ as follows:

$$h_1(z) = \mathfrak{F}(z) - e^{-\frac{2}{3}\gamma}\mathfrak{G}(z), \tag{4.30}$$

$$h_2(z) = \mathfrak{F}(z) - \omega e^{-\frac{2}{3}\gamma}\mathfrak{G}(z), \tag{4.31}$$

$$h_3(z) = \mathfrak{F}(z) - \omega^2 e^{-\frac{2}{3}\gamma}\mathfrak{G}(z). \tag{4.32}$$

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Then, we have

$$h_1h_2h_3 = \mathcal{B}^3(1-\mathrm{e}^{-2\gamma}) \neq 0. \tag{4.33}$$

In view of equation (4.33), we note that $h_j(z)$ $(j = 1, 2, 3)$ all are non-constant entire functions satisfying $N(r, 1/h_j(z)) = S(r)$ for $j = 1, 2, 3$. Let $\alpha_1 = h_1'/h_1$, $\beta_1 = h_2'/h_2$ and $\gamma_1 = h_3'/h_3$. Then, α_1, β_1 and γ_1 are non-constant meromorphic functions, and employing the lemma of the logarithmic derivative, we obtain that $T(r, \alpha_1) = S(r)$, $T(r, \beta_1) = S(r)$ and $T(r, \gamma_1) = S(r)$. By the principal of the mathematical induction, it can be seen that

$$h_1^{(j)} = \alpha_j h_1, \quad h_2^{(j)} = \beta_j h_2 \ \text{ and } \ h_3^{(j)} = \gamma_j h_3,$$

where α_j, β_j and γ_j are meromorphic functions satisfying $T(r, \alpha_j) = T(r, \beta_j) = T(r, \gamma_j) = S(r)$, for $j = 1, 2, \ldots$, and the following recurrence formulas, respectively,

$$\alpha_{j+1} = \alpha_j' + \alpha_j\alpha_1, \ \beta_{j+1} = \beta_j' + \beta_j\beta_1 \text{ and } \gamma_{j+1} = \gamma_j' + \gamma_j\gamma_1,$$

for $j = 0, 1, 2, \ldots$ and $\alpha_0 = \beta_0 = \gamma_0 = 1$.

From equations (4.30), (4.31) and (4.32), we obtain

$$\begin{aligned}
\mathfrak{F}(z) &= \frac{1}{3}\left(h_1(z) + h_2(z) + h_3(z)\right)\\
\mathfrak{G}(z) &= -\frac{1}{3}\left(h_1(z)\mathrm{e}^{2\gamma/3} + \omega^2 h_2(z)\mathrm{e}^{2\gamma/3} + \omega h_3(z)\mathrm{e}^{2\gamma/3}\right).
\end{aligned}$$

Thus, we have

$$\begin{cases}
f(z) &= \dfrac{1}{3}\left(h_1(z) + h_2(z) + h_3(z) + (a+b+c)\right)\\
g(z) &= -\dfrac{1}{3}\left(\left(h_1(z) + \omega^2 h_2(z) + \omega h_3(z)\right)\mathrm{e}^{2\gamma/3} - (a+b+c)\right).
\end{cases} \tag{4.34}$$

It is easy to see that

$$T(r, h_j) = \frac{1}{3}T(r, f) + S(r) \ \text{ for } \ j = 1, 2, 3.$$

By equation (4.34) and in view of the definition of g, we deduce that

$$\begin{aligned}
&\sum_{j=0}^{k}\frac{b_j}{3}\left(h_1^{(j)}(z) + h_2^{(j)}(z) + h_3^{(j)}(z)\right) + \frac{b_0}{3}(a+b+c)\\
&\qquad = -\frac{\left(h_1(z) + \omega^2 h_2(z) + \omega h_3(z)\right)\mathrm{e}^{2\gamma/3} - (a+b+c)}{3}.
\end{aligned} \tag{4.35}$$

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Comparing both sides of equation (4.35) and then employing Lemma 3.2, we obtain the following set of equations:

$$\begin{cases} \sum_{j=0}^{k} b_j h_1^{(j)}(z) = -h_1(z)\mathrm{e}^{2\gamma/3}, \\ \sum_{j=0}^{k} b_j h_2^{(j)}(z) = -\omega^2 h_2(z)\mathrm{e}^{2\gamma/3}, \\ \sum_{j=0}^{k} b_j h_3^{(j)}(z) = -\omega h_3(z)\mathrm{e}^{2\gamma/3}, \end{cases}$$

with $3b_{-1} + (b_0 - 1)(a + b + c) = 0$. □

Proof of Theorem 2.6. Following the proof of Theorem 2.1, we notice that h_1, h_2 and h_3 are entire functions. Let $h_j^* = 3h_j/(2a - b - c)$, $j = 1, 2, 3$. Then we have $h_1^* h_2^* h_3^* = 1 - \mathrm{e}^{-2\gamma}$ and $T\left(r, \mathrm{e}^{2\gamma/3}\right) = S(r, h_j^*)$ for $j = 1,\ 2,\ 3$. We note that the functions h_j^* $j = 1, 2, 3$ are still satisfied with the equations in equation (2.1).

If $k = 1$ or k is any even integer, then by Lemma 3.5, we see that γ must be a constant. Let $\mathrm{e}^{2\gamma} = 1$, then by the same argument as in Case B, we have $f \equiv g$ or $f^2 + fg + g^2 - (a + b + c)(f + g) + ab + bc + ca = 0$, and hence, conclusions (1) and (2), respectively, hold in Theorem 2.5.

Suppose that $\mathrm{e}^{2\gamma} \neq 1$, then by equations (2.1) and (4.33), there exist non-zero constants A_1, A_2, A_3 and λ such that $h_1(z) = A_1 e^{\lambda z}$, and hence, we obtain $h_2(z) = A_2\mathrm{e}^{\omega^2\lambda z}$ and $h_3(z) = A_3\mathrm{e}^{\omega\lambda z}$, where $A_1A_2A_3 = ((2a - b - c)/3)^3\left(1 - \mathrm{e}^{-2\gamma}\right)$. Assuming $\rho = e^{2\gamma/3}$, we have

$$A_1A_2A_3 = ((2a - b - c)/3)^3\left((\rho^3 - 1)/\rho^3\right),$$

and a simple computation shows that

$$\sum_{j=0}^{k} b_j\lambda^j = -\rho, \quad \sum_{j=0}^{k} b_j\omega^{2j}\lambda^j = -\omega^2\rho, \quad \sum_{j=0}^{k} b_j\omega\lambda^j = -\omega\rho.$$

In view of the functional forms of f and g, respectively, in equation (4.34), we obtain

$$\begin{aligned} f(z) &= \frac{1}{3}\left(A_1\mathrm{e}^{\lambda z} + A_2\mathrm{e}^{\omega^2\lambda z} + A_3\mathrm{e}^{\omega\lambda z} + (a + b + c)\right), \\ g(z) &= -\frac{1}{3}\left(\rho A_1\mathrm{e}^{\lambda z} + \rho\omega^2 A_2\mathrm{e}^{\omega^2\lambda z} + \rho\omega A_3\mathrm{e}^{\omega\lambda z} - (a + b + c)\right). \end{aligned}$$

If f is a finite order entire function, then g is also of finite order because f', f'', …, $f^{(k)}$ are all entire functions of finite order. Using equations (4.1), (4.30), (4.31) and (4.32), we observe that h_1, h_2 and h_3 are also functions of finite order. Consequently, by utilizing equation (2.1) and Lemma 3.4, we can deduce that γ must be a constant. As a result, it

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becomes evident that f and g still satisfy one of the cases outlined in Theorem 2.5. This completes the proof. □

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