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Semi-Discrete and Fully Discrete Hybrid Stress Finite Element Methods for Elastodynamic Problems

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Abstract. This paper proposes and analyzes semi-discrete and fully discrete hybrid stress finite element methods for elastodynamic problems. A hybrid stress quadrilateral finite element approximation is used in the space directions. A second-order center difference is adopted in the time direction for the fully discrete scheme. Error estimates of the two schemes, as well as a stability result for the fully discrete scheme, are derived. Numerical experiments are done to verify the theoretical results.

AMS subject classifications: 65N12; 65N15;65N30
Key words: elastodynamic problem, hybrid stress finite element, semi-discrete, fully discrete.

1. Introduction

Consider the following vibration model of plane elasticity:

$$\begin{cases} \mathbf{u}_{tt} - \mathbf{div}\sigma = \mathbf{f}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T], \\ \sigma = \mathbb{C}\epsilon(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\ \mathbf{I}, & (\mathbf{x},t) \in \Omega \times [0,T], \\ \mathbf{u} = 0, & (\mathbf{x},t) \in \Gamma \times [0,T], \\ \mathbf{u}(\mathbf{x},0) = \varphi_0(\mathbf{x}), \mathbf{u}_t(\mathbf{x},0) = \varphi_1(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^2$ is a bounded open set with boundary Γ, T a positive constant, $\mathbf{u} = (u_1, u_2)^T$ the displacement field, $\sigma = (\sigma_{ij})_{2\times 2}$ the symmetric stress tensor. $\epsilon(\mathbf{u}) = \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)/2$ the strain tensor. λ, μ are the Lamé parameters given by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

for plane strain problems and $\lambda = \frac{E\nu}{(1+\nu)(1-\nu)}$ for plane stress problems,with $0 < \nu < 0.5$ the Poisson ratio and E the Young's modulus. $\mathbf{f} \in (L^2(\Omega \times [0,T]))^2$ denotes the body force, initial data $\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x}) \in (H_0^1(\Omega))^2$.

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Numerical simulation of time-dependent problems plays an important role in vibration analysis of elastic structures ([7], [21]). Basically, the corresponding discretization approach consists of getting a semi-discrete scheme by means of the finite element method in the space direction and then discretizing time derivatives to obtain a fully discrete scheme [15]. Douglas and Gupta [8] discussed the superconvergence for a mixed finite element method of wave propagation in a plane domain, where a quasi-projection analysis was given to obtain error estimates in Sobolev spaces of nonpositive index. In [1, 2], B$\acute{e}$cache, Joly and Tsogka changed the second-order system (1.1) to a first order system, and constructed a new family of quadrangular or cubic mixed finite elements for the new system. In [9], Lai, Huang and Chen established a new C^0-continuous time stepping displacement-type finite element method to solve the vibration problem of plane elasticity. Boulaajine, Farhloul and Paquet [3, 4] used a new dual mixed finite element method to solve the linear elastodynamic system with explicit/implicit Newmark schemes for the time discretizaton. The explicit Newmark scheme was shown to be stable under an appropriate CFL condition and the implicit Newmark scheme was unconditionally stable. In [6], Cheng and Xie considered a space-time nonconforming finite element method for the vibration problem.

The assumed-stress hybrid approach, pioneered by Pian [11], is a kind of mixed finite element method based on the Hellinger-Reissner variational principle which includes unknowns of displacements and stresses. In [12] Pian and Sumihara derived the famous hybrid stress element (abbr. PS) through a rational choice of stress terms. Pian and Tong [13] discussed the similarity and basic difference between the incompatible displacement model and the hybrid stress model. In the direction of determining the optimal stress parameters, there have been many other research efforts, cf. [14, 16–18, 22]. Xie and Zhou [17, 18] derived robust 4-node hybrid stress quadrilateral elements by optimizing stress modes with a so-called energy-compatibility condition, i.e. the assumed stress terms are L^2-orthogonal to the enhanced strains caused by Wilson bubble displacements. Yu, Xie and Carstensen [19] derived uniform convergence and a posteriori error estimation for the hybrid stress methods.

In this paper, we shall apply the hybrid stress finite element method to the elastodynamic system (1.1) to obtain a semi-discrete scheme, then use a second-order center difference scheme in the time direction to get a fully discrete scheme. We will derive error estimates for the two schemes and a stability result for the fully discrete explicit scheme.

The paper is organized as follows. Section 2 gives notations and weak formulations. Section 3 is devoted to the error analysis of the semi-discrete hybrid stress method. In Section 4, we show a stability result and the error estimation of the fully discrete hybrid stress scheme. Some numerical results are provided in the last section.

2. Notations and weak formulations

Throughout this paper, we denote by $H^r(\Omega)$ the standard Sobolev spaces with norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$, and $H^0 = L^2$ is the space of square integrable functions. For

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a time dependent function $v(\mathbf{x},t)$, we simply write $\|v\|_r = \|v(\cdot,t)\|_r$. We then define

$$\begin{aligned} &L^p([0,T];H^r(\Omega)) \\ =&\Big\{v:[0,T]\to H^r(\Omega); \|v\|_{L^p([0,T];H^r(\Omega))}<\infty\Big\}, \quad 1\le p<\infty, \end{aligned}$$

where

$$\|v\|_{L^p([0,T];H^r(\Omega))} = \left(\int_0^T \|v(\cdot,t)\|_r^p dt\right)^{1/p},$$

when $p=\infty$, $\|v\|_{L^\infty([0,T];H^r(\Omega))} = \text{esssup}_{0\le t\le T}\|v(\cdot,t)\|_r$.

The corresponding semi-norm is

$$|v|_{L^p([0,T];H^r(\Omega))} = \left(\int_0^T |v(\cdot,t)|_r^p dt\right)^{1/p}.$$

We note that all the notations presented above can also be extended to vector-valued/tensor-valued functions naturally.

For convenience, we use the notation $a \lesssim b$ to represent that there exists a generic positive constant C, independent of the space mesh parameter h and time mesh parameter $\triangle t$, such that $a \le Cb$.

We recall two Gronwall's inequalities [15]:

Continuous Gronwall's inequality. Let $\phi(\cdot)\ge 0$ be such that

$$\phi_t(t) \le \rho\phi(t) + \eta(t) \quad \text{for} \quad 0\le t\le T,$$

where $\rho\ge 0$ is some constant and $\eta(\cdot)\ge 0$, $\eta\in L^1([0,T])$. Then it holds

$$\phi(t) \le e^{\rho T}\left(\phi(0) + \int_0^T \eta(s)ds\right), \quad \forall t\in[0,T]. \tag{2.1}$$

Discrete Gronwall's inequality. Let two nonnegative sequences $(k_n)_{n\ge 0}, (p_n)_{n\ge 0}$ be given and $g_0\ge 0$ also be given. Suppose that the sequence $(\phi_n)_{n\ge 0}$ satisfies

$$\begin{cases} \phi_0 \le g_0, \\ \phi_n \le g_0 + \sum\limits_{s=0}^{n-1} p_s + \sum\limits_{s=0}^{n-1} k_s\phi_s, \quad \forall n\ge 1. \end{cases} \tag{2.2}$$

Then it holds

$$\phi_n \le \left(g_0 + \sum_{s=0}^{n-1} p_s\right)\exp\left(\sum_{s=0}^{n-1} k_s\right), \quad \forall n\ge 1. \tag{2.3}$$

We define two spaces as follows

$$V := (H_0^1(\Omega))^2 = \Big\{u\in(H^1(\Omega))^2 : u\mid_\Gamma = 0\Big\},$$

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$$\Sigma := \Big\{\tau \in L^2(\Omega;\mathbb{R}^{2\times 2}_{sym}) : \int_\Omega tr\tau d\mathbf{x} = 0\Big\},$$

where $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$ denotes the space of square-integrable symmetric tensors with the norm $\|\cdot\|_0$ defined by $\|\tau\|_0^2 := \int_\Omega \tau : \tau d\mathbf{x}$, and $tr\tau := \tau_{11} + \tau_{22}$ represents the trace of tensor τ. Notice that on the space V, the semi-norm $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_1$.

The weak problem for the elastodynamic system (1.1) reads as: Find $(\sigma(t), \mathbf{u}(t)) \in L^2([0,T];\Sigma) \times L^2([0,T];V)$ such that

$$\begin{cases} a(\sigma(t),\tau) + b(\tau,\mathbf{u}(t)) = 0, & \forall \tau \in \Sigma, \\ b(\sigma(t),\mathbf{v}) + (\mathbf{f}(t),\mathbf{v}) = (\mathbf{u}_{tt}(t),\mathbf{v}), & \forall \mathbf{v} \in V, \\ \mathbf{u}(0) = \varphi_0, \quad \mathbf{u}_t(0) = \varphi_1, \end{cases} \tag{2.4}$$

where

$$a(\sigma,\tau) := \int_\Omega \sigma : \mathbb{C}^{-1}\tau d\mathbf{x} = \frac{1}{2\mu}\int_\Omega \sigma : \tau - \frac{\lambda}{2(\mu+\lambda)} tr\sigma tr\tau d\mathbf{x},$$
$$b(\tau,\mathbf{v}) := -\int_\Omega \tau : \epsilon(\mathbf{v}) d\mathbf{x},$$

and $(\cdot,\cdot)$ is the L^2-inner product on Ω.

For a.e. $t \in [0,T]$, the following continuity conditions are immediate:

$$|a(\sigma,\tau)| \lesssim \|\sigma\|_0\|\tau\|_0, \quad \sigma,\tau \in L^2([0,T];\Sigma), \tag{2.5}$$

$$|b(\tau,\mathbf{v})| \lesssim \|\tau\|_0|\mathbf{v}|_1, \quad \tau \in L^2([0,T];\Sigma), \mathbf{v} \in L^2([0,T];V). \tag{2.6}$$

Define an energy norm $\|\cdot\|_a$ on Σ as

$$\|\tau\|_a^2 := a(\tau,\tau) = \frac{1}{2\mu}\|\tau - \frac{1}{2}tr\tau I\|_0^2 + \frac{1}{4(\mu+\lambda)}\|tr\tau\|_0^2 \quad \text{for } \tau \in \Sigma.$$

Then it holds

$$\frac{1}{\sqrt{2(\mu+\lambda)}}\|\tau\|_0 \le \|\tau\|_a \le \frac{1}{\sqrt{2\mu}}\|\tau\|_0. \tag{2.7}$$

3. Semi-discrete hybrid stress finite element method

3.1. Geometric properties of quadrilaterals

In what follows we assume that Ω is a convex polygonal domain. Let $\mathscr{T}_h$ be a conventional quadrilateral mesh of Ω. Let $K \in \mathscr{T}_h$ be an arbitrary quadrilateral with diameter h_K. Denote $h := \max\limits_{K\in\mathscr{T}_h}\{h_K\}$. Let $Z_i(x_i,y_i), 1 \le i \le 4$ be the four vertices of K

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and T_i denotes the sub-triangle of K with vertices Z_{i-1}, Z_i and Z_{i+1}(the index on Z_i is modulo 4). Define

$$\rho_K = \min_{1\le i\le 4} \text{diameter of circle inscribed in } T_i.$$

Throughout the paper, we assume that the partition $\mathscr{T}_h$ satisfies the following "shape-regularity" hypothesis: there exists a constant $\varrho > 2$ independent of h such that for all $K \in \mathscr{T}_h$,

$$h_K \le \varrho \rho_K. \tag{3.1}$$

We define the isoparametric bilinear mapping $F_K : \widehat{K} = [-1,1]^2 \longrightarrow K$ (see Figure 1) as

$$\begin{pmatrix} x \\ y \end{pmatrix} = F_K(\xi,\eta) = \begin{pmatrix} \sum\limits_{i=1}^{4} x_i N_i(\xi,\eta) \\ \sum\limits_{i=1}^{4} y_i N_i(\xi,\eta)) \end{pmatrix}, \tag{3.2}$$

where ξ, η are the local isoparametric coordinates, and

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta),$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

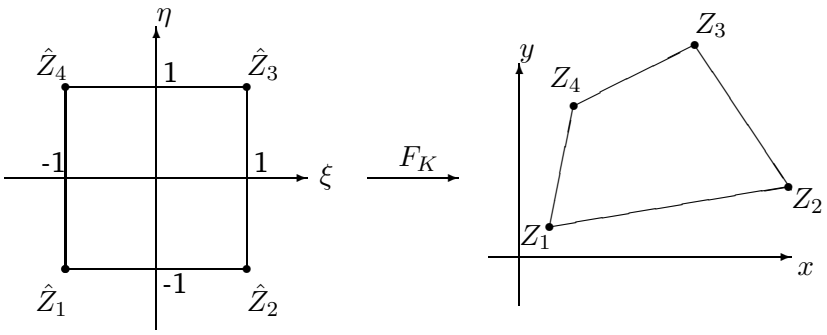


Figure 1: The mapping F_K

We can rewrite (3.2) as

$$x = a_0 + a_1\xi + a_2\eta + a_{12}\xi\eta, \quad y = b_0 + b_1\xi + b_2\eta + b_{12}\xi\eta,$$

where

$$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_{12} & b_{12} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}.$$

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3.2. Preliminary results

Let $\Sigma_h \subset \Sigma$ and $V_h \subset V$ be finite dimensional spaces respectively for stress and displacement approximations. For the PS element [12], the isoparametric bilinear interpolation is used for the displacement approximation, i.e. the displacement space V_h is chosen as

$$V_h = \Big\{ \mathbf{v} \in V : \hat{\mathbf{v}} = \mathbf{v} \mid_K \circ F_K \in span\{1, \xi, \eta, \xi\eta\}^2 \ \text{ for all } \ K \in \mathscr{T}_h \Big\}.$$

For simplicity we abbreviate the symmetric stress tensor $\tau = (\tau_{ij})_{2\times 2}$ as $\tau = (\tau_{11}, \tau_{22}, \tau_{12})^T$. The 5-parameters stress mode on $\hat{K}$ for PS element takes the form

$$\tau = P\beta \text{ for } \beta := (\beta_1, \cdots, \beta_5)^T \in \mathbb{R}^5, \tag{3.3}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & \eta & \frac{a_2^2}{b_2^2}\xi \\ 0 & 1 & 0 & \frac{b_1^2}{a_1^2}\eta & \xi \\ 0 & 0 & 1 & \frac{b_1}{a_1}\eta & \frac{a_2}{b_2}\xi \end{pmatrix}.$$

Then the corresponding stress space for PS element is

$$\Sigma_h := \Big\{ \tau \in \Sigma : \hat{\tau} = \tau \mid_K \circ F_K \text{ is of form (3.3) for all } K \in \mathscr{T}_h \Big\}.$$

Introduce the spaces

$$W_h := \Big\{ q \in L^2(\Omega) : \hat{q} = q|_K \circ F_K \in span\{1, \xi, \eta\} \ \text{ for all } \ K \in \mathscr{T}_h \Big\},$$

$$\bar{W}_h := \Big\{ \bar{q} \in W_h : \bar{q}|_K = constant \ \text{ for all } \ K \in \mathscr{T}_h \Big\}.$$

Lemma 3.1. ([19]) *Assume that for any $\bar{q} \in \bar{W}_h$ there exists some $\mathbf{v} \in V_h$ with $\|\bar{q}\|_0^2 \lesssim \int_\Omega \bar{q} div \mathbf{v} d\mathbf{x}, |\mathbf{v}|_1^2 \lesssim \|\bar{q}\|_0^2$. Then for any $\tau \in Z_h := \{\tau \in \Sigma_h : \int_\Omega \tau : \epsilon(\mathbf{v}_h) d\mathbf{x} = 0, \ for \ all \ \mathbf{v}_h \in V_h\}$ it holds*

$$\|\tau\|_0^2 \lesssim a(\tau, \tau). \tag{3.4}$$

Remark 3.1. This Lemma states that any quadrilateral mesh which is stable for the Stokes element $Q_1 - P_0$ is sufficient for (3.4). As we know, the only unstable case for $Q_1 - P_0$ is checkerboard mode [10, 20]. Thereupon, any quadrilateral mesh which breaks the checkerboard mode is sufficient for the uniform stability (3.4).

Lemma 3.2. ([19]) *For any $\mathbf{v}_h \in V_h$, it holds*

$$|\mathbf{v}_h|_1 \lesssim \sup_{0 \neq \tau_h \in \Sigma_h} \frac{\int_\Omega \tau_h : \epsilon(\mathbf{v}_h) d\mathbf{x}}{\|\tau_h\|_0}.$$

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We note that it holds the inverse inequality

$$\|\epsilon(\mathbf{v}_h)\|_0 \leq \delta h^{-1}\|\mathbf{v}_h\|_0, \quad \forall \mathbf{v}_h \in V_h, \tag{3.5}$$

where δ is a positive constant independent of h.

Define an operator $\Pi_h : \Sigma \to \Sigma_h$ such that for $\tau \in \Sigma$ it holds

$$b(\tau, \mathbf{v}_h) = b(\Pi_h \tau, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h.$$

We easily have

$$\|\tau - \Pi_h \tau\|_0 \lesssim h|\tau|_1, \quad \forall \tau \in \Sigma. \tag{3.6}$$

3.3. Semi-discrete scheme

Let $\varphi_{0,h}$ and $\varphi_{1,h}$ be respectively approximations of initial data φ_0 and φ_1. The corresponding semi-discrete scheme for the problem (2.4) reads as: Find $(\sigma_h(t), \mathbf{u}_h(t)) \in L^2([0,T];\Sigma_h) \times L^2([0,T];V_h)$, such that

$$\begin{cases} a(\sigma_h(t), \tau_h) + b(\tau_h, \mathbf{u}_h(t)) = 0, & \forall \tau_h \in \Sigma_h, \\ b(\sigma_h(t), \mathbf{v}_h) + (\mathbf{f}(t), \mathbf{v}_h) = (\mathbf{u}_{h,tt}(t), \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h, \\ \mathbf{u}_h(0) = \varphi_{0,h}, \mathbf{u}_{h,t}(0) = \varphi_{1,h}. \end{cases} \tag{3.7}$$

Lemma 3.3. *The semi-discrete scheme* (3.7) *admits a unique solution* $(\sigma_h(\cdot), \mathbf{u}_h(\cdot))$.

Proof. Let $\mathbf{u}_h(t) \mid_K = N\alpha(t)$, $\sigma_h(t) \mid_K = P\beta(t)$ with $\alpha(t) := (\alpha_1(t), \cdots, \alpha_8(t))^T$, $\beta(t) := (\beta_1(t), \cdots, \beta_5(t))^T$ and

$$N := \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix},$$

where P is given in (3.3) and N_i $(i = 1, \cdots, 4)$ are the same as in (3.2).

Taking $\tau_h|_K = P\tilde{\beta}$ with an arbitrary $\tilde{\beta} \in \mathbb{R}^5$ in the first equation of (3.7), we obtain

$$\int_K P^T \mathbb{C}^{-1} P d\mathbf{x}\beta(t) - \int_K P^T \epsilon(N) d\mathbf{x}\alpha(t) = 0,$$

where $\epsilon(N)$ is the strain on N. Denote $G = \int_K P^T \epsilon(N) d\mathbf{x}$, $H = \int_K P^T \mathbb{C}^{-1} P d\mathbf{x}$, then the above relation yields

$$\beta(t) = H^{-1} G \alpha(t). \tag{3.8}$$

Taking $\mathbf{v}_h|_K = N\tilde{\alpha}$ with an arbitrary $\tilde{\alpha} \in \mathbb{R}^8$ in the second equation of (3.7), we obtain

$$\begin{aligned} &\sum_K -\beta(t)^T \int_K P^T \epsilon(N) d\mathbf{x} \\ &= \sum_K \int_K -\mathbf{f}(t)^T N d\mathbf{x} + \sum_K \alpha_{tt}(t)^T \int_K N^T N d\mathbf{x}. \end{aligned}$$

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Denote

$$Q^T = \sum_K \int_K N^T N d\mathbf{x}, \quad \tilde{F}^T(t) = \sum_K \int_K -\mathbf{f}(t)^T N d\mathbf{x},$$

and substitute (3.8) into the above relation, we then have

$$Q\hat{\alpha}_{tt}(t) + G^T H^{-1} G\hat{\alpha}(t) + \tilde{F}(t) = 0 \tag{3.9}$$

with $\hat{\alpha}(t)|_K = \alpha(t)$. Because Q, $G^T H^{-1} G$ are symmetric positive defined constant matrices, we easily know that (3.9) has a unique solution provided that $\mathbf{f}(t)$ is continuous in t. This completes the proof. □

3.4. Error analysis

Let us introduce the following discrete elliptic projection problem: Find $(\widehat{\sigma_h}(t), \widehat{\mathbf{u}_h}(t)) \in L^2([0,T];\Sigma_h) \times L^2([0,T];V_h)$, such that

$$\begin{cases} a(\widehat{\sigma_h}(t), \tau_h) + b(\tau_h, \widehat{\mathbf{u}_h}(t)) = 0, & \forall \tau_h \in \Sigma_h, \\ b(\widehat{\sigma_h}(t), \mathbf{v}_h) + (\mathbf{f}(t), \mathbf{v}_h) = (\mathbf{u}_{tt}(t), \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h. \end{cases} \tag{3.10}$$

By standard arguments in the theory of mixed finite element methods [5] we can estimate as follows the error between $(\sigma(t), \mathbf{u}(t))$, the solution of weak problem (2.4), and $(\widehat{\sigma_h}(t), \widehat{\mathbf{u}_h}(t))$, the solution of the discrete elliptic projection problem (3.10).

Lemma 3.4. *For a.e. $t \in [0,T]$ it holds the following estimates:*

$$\|\sigma(t) - \widehat{\sigma_h}(t)\|_0 + \|\mathbf{u}(t) - \widehat{\mathbf{u}_h}(t)\|_1 \lesssim h(\|\mathbf{u}(t)\|_2 + \|\sigma(t)\|_1), \tag{3.11}$$

$$\|\sigma_{tt}(t) - \widehat{\sigma_{h,tt}}(t)\|_0 \lesssim h(\|\sigma_{tt}(t)\|_1 + \|\mathbf{u}_{tt}(t)\|_2), \tag{3.12}$$

$$\|\mathbf{u}_{tt}(t) - \widehat{\mathbf{u}_{h,tt}}(t)\|_1 \lesssim h(\|\sigma_{tt}(t)\|_1 + \|\mathbf{u}_{tt}(t)\|_2). \tag{3.13}$$

Remark 3.2. We note that in the estimates (3.11)-(3.13), the constants C in $\lesssim$ are not only independent of h and Δt, but also independent of the Lamé parameter λ.

For the convenience of analysis we choose initial data $\varphi_{0,h}$, $\varphi_{1,h}$ in the semi-discretized problem (3.7) as follows:

$$\varphi_{0,h} = \widehat{\mathbf{u}_h}(0), \varphi_{1,h} = \widehat{\mathbf{u}}_{h,t}(0). \tag{3.14}$$

We note that in actual computation one may take $\varphi_{0,h} = I_h\varphi_0$ and $\varphi_{1,h} = I_h\varphi_1$, where I_h denotes some interpolation from V onto V_h.

Theorem 3.1. *Let $(\sigma(t), \mathbf{u}(t))$ and $(\sigma_h(t), \mathbf{u}_h(t))$ be the solutions to the weak problem* (2.4) *and the semi-discrete scheme* (3.7)*, respectively. Suppose $\mathbf{u} \in L^\infty([0,T];(H^2(\Omega))^2 \cap$*

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$(H_0^1(\Omega))^2)$, $\mathbf{u}_{tt} \in L^\infty([0,T];(H^2(\Omega))^2)$ *and* $\sigma, \sigma_{tt} \in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$. *Then for a.e.* $t \in [0,T]$ *the following error estimate holds:*

$$\begin{aligned} &\|\sigma(t)-\sigma_h(t)\|_0 + \|\mathbf{u}(t)-\mathbf{u}_h(t)\|_1 \\ &\lesssim h\Big(\|\sigma(t)\|_1 + \|\mathbf{u}(t)\|_2 + \|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}\Big). \end{aligned} \tag{3.15}$$

Further more, it holds

$$\begin{aligned} &\|\sigma-\sigma_h\|_{L^\infty([0,T];(L^2(\Omega))^{2\times 2})} + \|\mathbf{u}-\mathbf{u}_h\|_{L^\infty([0,T];(H^1(\Omega))^2)} \\ &\lesssim h\Big(\|\sigma\|_{L^\infty([0,T];(H^1(\Omega))^{2\times 2})} + \|\mathbf{u}\|_{L^\infty([0,T];(H^2(\Omega))^2)} \\ &\quad + \|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2}\Big). \end{aligned} \tag{3.16}$$

Proof. Denoting $\mathbf{r}_h(t) := \mathbf{u}_h(t) - \widehat{\mathbf{u}_h}(t)$, $\psi_h(t) := \sigma_h(t) - \widehat{\sigma_h}(t)$, from (3.7) and (3.10) we write the error system in the form

$$\begin{cases} a(\psi_h(t),\tau_h) + b(\tau_h,\mathbf{r}_h(t)) = 0, & \forall \tau_h \in \Sigma_h, \\ b(\psi_h(t),\mathbf{v}_h) = (\mathbf{u}_{h,tt}(t) - \mathbf{u}_{tt}(t),\mathbf{v}_h), & \forall \mathbf{v}_h \in V_h. \end{cases} \tag{3.17}$$

By (3.14) it holds $\mathbf{r}_h(0) = 0$ and $\mathbf{r}_{h,t}(0) = 0$, then we obtain $\psi_h(0) = 0$. Differentiating the first equation of (3.17) with respect to t and taking $\tau_h = \psi_h(t)$, we obtain

$$a(\psi_{h,t}(t),\psi_h(t)) + b(\psi_h(t),\mathbf{r}_{h,t}(t)) = 0. \tag{3.18}$$

Take $\mathbf{v}_h = \mathbf{r}_{h,t}(t)$ in the second equation of (3.17), then it follows

$$b(\psi_h(t),\mathbf{r}_{h,t}(t)) = (\mathbf{u}_{h,tt}(t) - \mathbf{u}_{tt}(t),\mathbf{r}_{h,t}(t)). \tag{3.19}$$

Combining (3.18) with (3.19) we have

$$a(\psi_{h,t}(t),\psi_h(t)) + (\mathbf{u}_{h,tt}(t) - \mathbf{u}_{tt}(t),\mathbf{r}_{h,t}(t)) = 0, \tag{3.20}$$

then it holds

$$\begin{aligned} \frac{d}{dt}\|\psi_h(t)\|_a^2 = \frac{d}{dt}a(\psi_{h,t}(t),\psi_h(t)) &= (\mathbf{u}_{tt}(t) - \widehat{\mathbf{u}_{h,tt}}(t),\mathbf{r}_{h,t}(t)) - \frac{d}{dt}\|\mathbf{r}_{h,t}(t)\|_0^2 \\ &\le \|\mathbf{u}_{tt}(t) - \widehat{\mathbf{u}_{h,tt}}(t)\|_0^2 + \|\mathbf{r}_{h,t}(t)\|_0^2 - \frac{d}{dt}\|\mathbf{r}_{h,t}(t)\|_0^2, \end{aligned} \tag{3.21}$$

which indicates

$$\frac{d}{dt}\left(\|\psi_h(t)\|_a^2 + \|\mathbf{r}_{h,t}(t)\|_0^2\right) \le \|\mathbf{u}_{tt}(t) - \widehat{\mathbf{u}_{h,tt}}(t)\|_0^2 + \|\mathbf{r}_{h,t}(t)\|_0^2. \tag{3.22}$$

Using the continuous Gronwall's inequality (2.1) and the estimate (3.13) we have

$$\begin{aligned} \|\psi_h(t)\|_a^2 + \|\mathbf{r}_{h,t}(t)\|_0^2 &\le e^T \int_0^T \|\mathbf{u}_{tt}(s) - \widehat{\mathbf{u}_{h,tt}}(s)\|_0^2 ds \\ &\lesssim h^2\Big(\int_0^T \|\mathbf{u}_{tt}(s)\|_2^2 ds + \int_0^T \|\sigma_{tt}(s)\|_1^2 ds\Big), \end{aligned} \tag{3.23}$$

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which yields

$$\|\sigma_h(t)-\widehat{\sigma_h}(t)\|_a \lesssim h\Big(\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})}\Big). \tag{3.24}$$

From (2.7) we obtain

$$\begin{aligned}\|\sigma_h(t)-\widehat{\sigma_h}(t)\|_0 &\le \sqrt{2(\mu+\lambda)}\|\sigma_h(t)-\widehat{\sigma_h}(t)\|_a\\ &\lesssim h(\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})}).\end{aligned} \tag{3.25}$$

In view of Lemma 3.4, (3.25) and triangle inequality, we have

$$\begin{aligned}&\|\sigma(t)-\sigma_h(t)\|_0\\ \le&\|\sigma(t)-\widehat{\sigma_h}(t)\|_0+\|\sigma_h(t)-\widehat{\sigma_h}(t)\|_0\\ \lesssim&h(\|\sigma(t)\|_1+\|\mathbf{u}(t)\|_2)+h(\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})}+\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)})\\ \lesssim&h(\|\sigma(t)\|_1+\|\mathbf{u}(t)\|_2+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})}+\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}).\end{aligned} \tag{3.26}$$

Taking $\mathbf{v}_h=\mathbf{r}_h(t)$, $\tau_h=\psi_h(t)$ in (3.17) we have

$$\begin{cases} a(\psi_h(t),\psi_h(t))+b(\psi_h(t),\mathbf{r}_h(t))=0,\\ b(\psi_h(t),\mathbf{r}_h(t))=(\mathbf{u}_{h,tt}(t)-\mathbf{u}_{tt}(t),\mathbf{r}_h(t)).\end{cases} \tag{3.27}$$

Lemma 3.2 and (3.24) imply for a.e. $t\in[0,T]$,

$$\begin{aligned}\|\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t)\|_1 &\lesssim \sup_{0\neq\psi_h\in\Sigma_h}\frac{-b(\psi_h(t),\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t))}{\|\psi_h(t)\|_0}\\ &= \sup_{0\neq\psi_h\in\Sigma_h}\frac{a(\psi_h(t),\psi_h(t))}{\|\psi_h(t)\|_0}\lesssim\|\psi_h(t)\|_a\\ &\lesssim h\Big(\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})}\Big).\end{aligned} \tag{3.28}$$

This estimate, together with Lemma 3.4 and triangle inequality, leads to

$$\begin{aligned}&\|\mathbf{u}(t)-\mathbf{u}_h(t)\|_1\\ \lesssim&\|\mathbf{u}(t)-\widehat{\mathbf{u}_h}(t)\|_1+\|\mathbf{u}_h(t)-\widehat{\mathbf{u}_h}(t)\|_1\\ \lesssim&h(\|\mathbf{u}(t)\|_2+\|\sigma(t)\|_1)+h(\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})})\\ \lesssim&h(\|\mathbf{u}(t)\|_2+\|\sigma(t)\|_1+\|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})}).\end{aligned} \tag{3.29}$$

Finally, the estimates (3.26) and (3.29) yield the desired result (3.15), from which the estimate (3.16) follows naturally. □

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Remark 3.3. We note that the estimates (3.24) and (3.11) can yield the following estimate for the stress error in the energy norm for a.e. $t \in [0,T]$:

$$\|\sigma(t)-\sigma_h(t)\|_a \lesssim h\Big(\|\sigma(t)\|_1+|\mathbf{u}(t)|_2+\|\sigma_{tt}\|_{L^2([0,T];(H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_{tt}\|_{L^2([0,T];(H^2(\Omega))^2)}\Big). \tag{3.30}$$

where the constant C in $\lesssim$ is independent of the Lamé parameter λ.

4. Fully discrete hybrid stress finite element method

4.1. Fully discrete scheme

Denote by $\triangle t := \frac{T}{M} > 0$ the time step size , and take $t_i = i\triangle t$ $(i = 0,1,2,\cdots,M)$, $t_0 = 0$, $t_M = T$, $t_{i+\frac{1}{2}} := \frac{t_i+t_{i+1}}{2}$. For any function φ of t, define

$$\varphi^n := \varphi(t_n),\quad \varphi^{n+\frac{1}{2}} := \frac{\varphi^n+\varphi^{n+1}}{2},$$
$$\triangle_t\varphi^n := \frac{\varphi^{n+1}-\varphi^{n-1}}{2\triangle t},\quad \triangle_t\varphi^{n+\frac{1}{2}} := \frac{\varphi^{n+1}-\varphi^n}{\triangle t},$$
$$\triangle_t^2\varphi^n := \frac{\varphi^{n+1}-2\varphi^n+\varphi^{n-1}}{(\triangle t)^2}.$$

We can easily see that

$$\triangle_t^2\varphi^n = \frac{\triangle_t\varphi^{n+\frac{1}{2}}-\triangle_t\varphi^{n-\frac{1}{2}}}{\triangle t},\qquad \triangle_t\varphi^n = \frac{\triangle_t\varphi^{n+\frac{1}{2}}+\triangle_t\varphi^{n-\frac{1}{2}}}{2}. \tag{4.1}$$

The corresponding fully discrete mixed finite element scheme for the problem (2.4) reads as: For $n = 1,2,\cdots,M$, find $(\sigma_h^n,\mathbf{u}_h^n) \in \Sigma_h \times V_h$, such that

$$\begin{cases} a(\sigma_h^n,\tau_h)+b(\tau_h,\mathbf{u}_h^n) = 0, & \forall\tau_h\in\Sigma_h,\\ (\triangle_t^2\mathbf{u}_h^n,\mathbf{v}_h)-b(\sigma_h^n,\mathbf{v}_h) = (\mathbf{f}^n,\mathbf{v}_h), & \forall\mathbf{v}_h\in V_h, \end{cases} \tag{4.2}$$

where $\mathbf{u}_h^0 = \widehat{\mathbf{u}_h}(0)$ and $\mathbf{u}_h^{-1} = \widehat{\mathbf{u}_h}(-\triangle t)$. Similar to the semi-discrete case, the choice of initial data $\mathbf{u}_h^0$ and $\mathbf{u}_h^{-1}$ here is just for the convenience of analysis. In actual computation one may choose $\mathbf{u}_h^0 = I_h\varphi_0$ and $\mathbf{u}_h^{-1} = I_h\varphi_0 - \triangle t I_h\varphi_1$.

Lemma 4.1. *The fully discrete scheme* (4.2) *admits a unique solution* $(\sigma_h^n,\mathbf{u}_h^n)$ *for* $n = 1,2,\cdots,M$.

Proof. The proof is similar to that of Lemma 3.3. Let $\mathbf{u}_h^{n+1}|_K = N\alpha^{n+1}$,$\mathbf{u}_h^n|_K = N\alpha^n$,$\mathbf{u}_h^{n-1}|_K = N\alpha^{n-1}$, $\sigma_h^n|_K = P\beta^n$, where $\alpha^n = (\alpha_1^n,\cdots,\alpha_8^n)^T$, $\beta^n = (\beta_1^n,\cdots,\beta_5^n)^T$, N, P are the same as in Lemma 3.3.

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Taking $\tau_h|_K = P\tilde{\beta}$ with an arbitrary $\tilde{\beta} \in \mathbb{R}^5$ in the first equation of (4.2) we obtain

$$\beta^n = H^{-1}G\alpha^n, \tag{4.3}$$

where N, G are given in Lemma 3.3.

Taking $\mathbf{v}_h|_K = N\tilde{\alpha}$ with an arbitrary $\tilde{\alpha} \in \mathbb{R}^8$ in the second equation of (4.2) and using (4.3), we obtain

$$\begin{aligned}
&\sum_K (\alpha^{n+1})^T \int_K N^T N d\mathbf{x} \\
&= 2\sum_K (\alpha^n)^T \int_K N^T N d\mathbf{x} - \sum_K (\alpha^{n-1})^T \int_K N^T N d\mathbf{x} \\
&\quad - (\Delta t)^2 \sum_K (\beta^n)^T \int_K P^T \epsilon(N) d\mathbf{x} + (\Delta t)^2 \sum_K \int_K (\mathbf{f}^n)^T N d\mathbf{x}.
\end{aligned} \tag{4.4}$$

Denote $F^T := \sum\limits_K \int_K (\mathbf{f}^n)^T N d\mathbf{x}$, then we have

$$Q(\hat{\alpha}^{n+1}) = 2Q(\hat{\alpha}^n) - Q(\hat{\alpha}^{n-1}) - (\Delta t)^2 G^T H^{-1} G(\hat{\alpha}^n) + (\Delta t)^2 F, \tag{4.5}$$

with $\hat{\alpha}^n(t)|_K = \alpha^n(t)$, Q is the same as in Lemma 3.3.

Suppose u_h^{n-1}, u_h^n are given, then $\hat{\alpha}^{n-1}$, $\hat{\alpha}^n$ are given. Since the matrix Q is symmetric positive defined, the linear system (4.5) admits a unique solution $\hat{\alpha}^{n+1}$. □

4.2. Stability analysis

Lemma 4.2. *It holds*

$$\begin{aligned}
&\|\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}\|_0^2 + \|\sigma_h^{J+\frac{1}{2}}\|_a^2 - \frac{(\triangle t)^2}{4}\|\triangle_t \sigma_h^{J+\frac{1}{2}}\|_a^2 \\
&= 2\triangle t \sum_{n=1}^{J} (\mathbf{f}^n, \triangle_t \mathbf{u}_h^n) + \|\triangle_t \mathbf{u}_h^{\frac{1}{2}}\|_0^2 + \|\sigma_h^{\frac{1}{2}}\|_a^2 - \frac{(\triangle t)^2}{4}\|\triangle_t \sigma_h^{\frac{1}{2}}\|_a^2,
\end{aligned} \tag{4.6}$$

for $J = 1, \cdots, M-1$.

Proof. Subtracting the same equation at time $(n-1)\triangle t$ from the first equation of (4.2) at time $(n+1)\triangle t$, we obtain

$$a(\sigma_h^{n+1} - \sigma_h^{n-1}, \tau_h) + b(\tau_h, \mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}) = 0 \text{ for all } \tau_h \in \Sigma_h. \tag{4.7}$$

Taking $\tau_h = \frac{1}{2\triangle t}\sigma_h^n$ in (4.7) and $\mathbf{v}_h = \triangle_t \mathbf{u}_h^n$ in the second equation of (4.2) we get

$$a(\triangle_t \sigma_h^n, \sigma_h^n) + b(\sigma_h^n, \triangle_t \mathbf{u}_h^n) = 0, \tag{4.8}$$

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$$b(\sigma_h^n, \triangle_t \mathbf{u}_h^n) = (\triangle_t^2 \mathbf{u}_h^n, \triangle_t \mathbf{u}_h^n) - (\mathbf{f}^n, \triangle_t \mathbf{u}_h^n). \tag{4.9}$$

Subtracting (4.9) from (4.8) yields

$$a(\triangle_t \sigma_h^n, \sigma_h^n) + (\triangle_t^2 \mathbf{u}_h^n, \triangle_t \mathbf{u}_h^n) = (\mathbf{f}^n, \triangle_t \mathbf{u}_h^n). \tag{4.10}$$

It follows from (4.1)

$$\begin{aligned}(\triangle_t^2 \mathbf{u}_h^n, \triangle_t \mathbf{u}_h^n) &= \left(\frac{\triangle_t \mathbf{u}_h^{n+\frac{1}{2}} - \triangle_t \mathbf{u}_h^{n-\frac{1}{2}}}{\triangle t}, \frac{\triangle_t \mathbf{u}_h^{n+\frac{1}{2}} + \triangle_t \mathbf{u}_h^{n-\frac{1}{2}}}{2} \right) \\ &= \frac{1}{2\triangle t} \Big(\| \triangle_t \mathbf{u}_h^{n+\frac{1}{2}} \|_0^2 - \| \triangle_t \mathbf{u}_h^{n-\frac{1}{2}} \|_0^2 \Big).\end{aligned} \tag{4.11}$$

Since

$$\triangle_t \sigma_h^n = \frac{\sigma_h^{n+1} - \sigma_h^{n-1}}{2\triangle t} = \frac{\sigma_h^{n+1} + \sigma_h^n - \sigma_h^n - \sigma_h^{n-1}}{2\triangle t} = \frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\triangle t},$$

$$\sigma_h^n = \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} - \frac{(\triangle t)^2}{4} \Big(\frac{\sigma_h^{n+1} - 2\sigma_h^n + \sigma_h^{n-1}}{(\triangle t)^2} \Big),$$

we have

$$\begin{aligned}&a(\triangle_t \sigma_h^n, \sigma_h^n) \\ &= (\triangle_t \sigma_h^n, \mathbb{C}^{-1} \sigma_h^n) = \Big(\frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}, \mathbb{C}^{-1} \frac{\sigma_h^{n+\frac{1}{2}} + \sigma_h^{n-\frac{1}{2}}}{2} \Big) - \Big(\Delta_t \sigma_h^n, \mathbb{C}^{-1} \frac{(\Delta t)^2}{4} \Delta_t^2 \sigma_h^n \Big) \\ &= \frac{1}{2\Delta t} \Big(\| \sigma_h^{n+\frac{1}{2}} \|_a^2 - \| \sigma_h^{n-\frac{1}{2}} \|_a^2 \Big) - \frac{(\Delta t)^2}{4} \Big(\frac{\Delta_t \sigma_h^{n+\frac{1}{2}} + \Delta_t \sigma_h^{n-\frac{1}{2}}}{2}, \mathbb{C}^{-1} \frac{\Delta_t \sigma_h^{n+\frac{1}{2}} - \Delta_t \sigma_h^{n-\frac{1}{2}}}{\Delta t} \Big) \\ &= \frac{1}{2\triangle t} \Big(\| \sigma_h^{n+\frac{1}{2}} \|_a^2 - \| \sigma_h^{n-\frac{1}{2}} \|_a^2 \Big) - \frac{\triangle t}{8} \Big(\| \triangle_t \sigma_h^{n+\frac{1}{2}} \|_a^2 - \| \triangle_t \sigma_h^{n-\frac{1}{2}} \|_a^2 \Big).\end{aligned} \tag{4.12}$$

It follows from (4.10)-(4.12) that

$$\begin{aligned}&\frac{1}{2\triangle t} \Big(\| \sigma_h^{n+\frac{1}{2}} \|_a^2 - \| \sigma_h^{n-\frac{1}{2}} \|_a^2 \Big) - \frac{\triangle t}{8} \Big(\| \triangle_t \sigma_h^{n+\frac{1}{2}} \|_a^2 - \| \triangle_t \sigma_h^{n-\frac{1}{2}} \|_a^2 \Big) \\ &\quad + \frac{1}{2\triangle t} (\| \triangle_t \mathbf{u}_h^{n+\frac{1}{2}} \|_0^2 - \| \triangle_t \mathbf{u}_h^{n-\frac{1}{2}} \|_0^2) = (\mathbf{f}^n, \triangle_t \mathbf{u}_h^n),\end{aligned} \tag{4.13}$$

which yields (4.6). □

Theorem 4.1. *Assume the CFL condition*

$$\triangle t < \min \left\{ 1, \frac{h}{\delta \sqrt{\mu + \lambda}} \right\}$$

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holds with δ given in the inverse inequality (3.5)*, then the explicit scheme* (4.2) *is stable in the following sense:*

$$\begin{aligned}&\theta\|\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}\|_0^2 + \|\sigma_h^{J+\frac{1}{2}}\|_0^2\\ &\lesssim [\|\triangle_t \mathbf{u}_h^{\frac{1}{2}}\|_0^2 + \|\sigma_h^{\frac{1}{2}}\|_0^2 + T(\max_{t\in[0,T]}\|\mathbf{f}(t)\|_0)^2] \times \exp(T\theta^{-1})\end{aligned} \tag{4.14}$$

with $\theta := \frac{1}{2} - \frac{\delta^2(\triangle t)^2(\mu+\lambda)}{2h^2} > 0$*,* $J = 1, \cdots M-1$.

Proof. Subtracting the first equation of (4.2) at time $J\triangle t$ from the same equation at time $(J+1)\triangle t$ and dividing it by $\triangle t$, we get

$$a\Big(\frac{\sigma_h^{J+1} - \sigma_h^J}{\triangle t}, \tau_h\Big) + b\Big(\tau_h, \frac{\mathbf{u}_h^{J+1} - \mathbf{u}_h^J}{\triangle t}\Big) = 0. \tag{4.15}$$

Taking $\tau_h = \triangle_t \sigma_h^{J+\frac{1}{2}}$ in (4.15), we have

$$a(\triangle_t \sigma_h^{J+\frac{1}{2}}, \triangle_t \sigma_h^{J+\frac{1}{2}}) + b(\triangle_t \sigma_h^{J+\frac{1}{2}}, \triangle_t \mathbf{u}_h^{J+\frac{1}{2}}) = 0. \tag{4.16}$$

Combining (3.5), (2.7) with (4.16), we obtain

$$\begin{aligned}\|\triangle_t \sigma_h^{J+\frac{1}{2}}\|_a^2 &= a(\triangle_t \sigma_h^{J+\frac{1}{2}}, \triangle_t \sigma_h^{J+\frac{1}{2}})\\ &= -b(\triangle_t \sigma_h^{J+\frac{1}{2}}, \triangle_t \mathbf{u}_h^{J+\frac{1}{2}}) = (\triangle_t \sigma_h^{J+\frac{1}{2}}, \epsilon(\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}))\\ &\le \|\triangle_t \sigma_h^{J+\frac{1}{2}}\|_0 \|\epsilon(\triangle_t \mathbf{u}_h^{J+\frac{1}{2}})\|_0 \le \frac{\delta}{h}\|\triangle_t \sigma_h^{J+\frac{1}{2}}\|_0 \|\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}\|_0\\ &\le \frac{\delta\sqrt{2(\mu+\lambda)}}{h}\|\triangle_t \sigma_h^{J+\frac{1}{2}}\|_a \|\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}\|_0,\end{aligned} \tag{4.17}$$

thus it holds

$$\|\triangle_t \sigma_h^{J+\frac{1}{2}}\|_a \le \frac{\delta\sqrt{2(\mu+\lambda)}}{h}\|\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}\|_0. \tag{4.18}$$

From (4.6) and (4.18) we get

$$\begin{aligned}&\Big(1 - \frac{\delta^2(\triangle t)^2(\mu+\lambda)}{2h^2}\Big)\|\triangle_t \mathbf{u}_h^{J+\frac{1}{2}}\|_0^2 + \|\sigma_h^{J+\frac{1}{2}}\|_a^2\\ &\le 2\triangle t\sum_{n=1}^{J}(\mathbf{f}^n, \triangle_t \mathbf{u}_h^n) + \|\triangle_t \mathbf{u}_h^{\frac{1}{2}}\|_0^2 + \|\sigma_h^{\frac{1}{2}}\|_a^2 - \frac{(\triangle t)^2}{4}\|\triangle_t \sigma_h^{\frac{1}{2}}\|_a^2\\ &\le \triangle t\sum_{n=1}^{J}\|\mathbf{f}^n\|_0^2 + \triangle t\sum_{n=1}^{J}\|\triangle_t \mathbf{u}_h^n\|_0^2 + \|\triangle_t \mathbf{u}_h^{\frac{1}{2}}\|_0^2 + \|\sigma_h^{\frac{1}{2}}\|_a^2 - \frac{(\triangle t)^2}{4}\|\triangle_t \sigma_h^{\frac{1}{2}}\|_a^2.\end{aligned} \tag{4.19}$$

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On the other hand,

$$\begin{aligned}\sum_{n=1}^{J}\|\triangle_t\mathbf{u}_h^n\|_0^2 &= \sum_{n=1}^{J}\|\frac{\triangle_t\mathbf{u}_h^{n+\frac{1}{2}}+\triangle_t\mathbf{u}_h^{n-\frac{1}{2}}}{2}\|_0^2\\ &\le \frac{1}{2}\Big(\sum_{n=1}^{J}\|\triangle_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\sum_{n=1}^{J}\|\triangle_t\mathbf{u}_h^{n-\frac{1}{2}}\|_0^2\Big)\\ &= \frac{1}{2}\Big(2\sum_{n=1}^{J-1}\|\triangle_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\|\triangle_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2+\|\triangle_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2\Big)\\ &\le \sum_{n=0}^{J-1}\|\triangle_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\frac{1}{2}\|\triangle_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2.\end{aligned} \tag{4.20}$$

Thus, combining (2.7), (4.19) with (4.20) we obtain

$$\begin{aligned}&\Big(\frac{1}{2}-\frac{\delta^2(\triangle t)^2(\mu+\lambda)}{2h^2}\Big)\|\triangle_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2+\|\sigma_h^{J+\frac{1}{2}}\|_0^2\\ \lesssim\ &\triangle t\sum_{n=1}^{J}\|\mathbf{f}^n\|_0^2+\triangle t\sum_{n=0}^{J-1}\|\triangle_t\mathbf{u}_h^{n+\frac{1}{2}}\|_0^2+\|\triangle_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\sigma_h^{\frac{1}{2}}\|_0^2-\frac{(\triangle t)^2}{4}\|\triangle_t\sigma_h^{\frac{1}{2}}\|_0^2.\end{aligned} \tag{4.21}$$

In light of the CFL condition, applying the discrete Gronwall's inequality (2.3) to (4.21) yields

$$\begin{aligned}&\theta\|\triangle_t\mathbf{u}_h^{J+\frac{1}{2}}\|_0^2+\|\sigma_h^{J+\frac{1}{2}}\|_0^2\\ \lesssim\ &\Big[\|\triangle_t\mathbf{u}_h^{\frac{1}{2}}\|_0^2+\|\sigma_h^{\frac{1}{2}}\|_0^2+T(\max_{t\in[0,T]}\|\mathbf{f}(t)\|_0)^2\Big]\times\exp(T\theta^{-1}).\end{aligned} \tag{4.22}$$

□

4.3. Error analysis

Lemma 4.3. *Under the CFL condition* $\triangle t<\min\{1,\zeta\frac{h}{\delta\sqrt{\mu+\lambda}}\}$ *with* $\zeta\in(0,1)$ *an arbitrarily given constant, it holds the following error estimate:*

$$\begin{aligned}&\max_{n=0}^{M}\|\mathbf{u}_h^n-\widehat{\mathbf{u}_h}(t_n)\|_1+\max_{n=0}^{M}\|\sigma_h^n-\widehat{\sigma_h}(t_n)\|_0\\ \lesssim\ &\triangle t\sum_{j=0}^{M}\|\mathbf{u}_{tt}(t_j)-\triangle_t^2\widehat{\mathbf{u}_h}(t_j)\|_0.\end{aligned} \tag{4.23}$$

Here $(\sigma_h^n,\mathbf{u}_h^n)$ *and* $(\widehat{\sigma_h},\widehat{\mathbf{u}_h})$ *are respectively the solutions to the fully discrete scheme* (4.2) *and the discrete elliptic projection problem* (3.10).

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Proof. Set $\psi_h^n = \sigma_h^n - \widehat{\sigma_h}(t_n)$, $\mathbf{r}_h^n = \mathbf{u}_h^n - \widehat{\mathbf{u}_h}(t_n)$, take $t = t_n$ in (3.10), and combine (4.2), we then have $\forall \tau_h \in \Sigma_h, \mathbf{v}_h \in V_h$

$$\begin{cases} a(\psi_h^n, \tau_h) + b(\tau_h, \mathbf{r}_h^n) = 0, \\ b(\psi_h^n, \mathbf{v}_h) = (\triangle_t^2 \mathbf{r}_h^n, \mathbf{v}_h) + (\triangle_t^2 \widehat{\mathbf{u}_h}(t_n) - \mathbf{u}_{tt}(t_n), \mathbf{v}_h). \end{cases} \tag{4.24}$$

Apply the difference operator $\triangle_t$ to the first equation of (4.24) and take $\tau_h = \psi_h^{n+\frac{1}{2}}$, we then obtain

$$a(\triangle_t \psi_h^{n+\frac{1}{2}}, \psi_h^{n+\frac{1}{2}}) + b(\psi_h^{n+\frac{1}{2}}, \triangle_t \mathbf{r}_h^{n+\frac{1}{2}}) = 0. \tag{4.25}$$

Take $\mathbf{v}_h = \frac{1}{2}(\triangle_t \mathbf{r}_h^{n+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{n-\frac{1}{2}})$ in the second equation of (4.24), and we get

$$\begin{aligned} & b\Big(\psi_h^n, \frac{1}{2}(\triangle_t \mathbf{r}_h^{n+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{n-\frac{1}{2}})\Big) \\ = & \Big(\triangle_t^2 \mathbf{r}_h^n, \frac{1}{2}(\triangle_t \mathbf{r}_h^{n+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{n-\frac{1}{2}})\Big) + \Big(\triangle_t^2 \widehat{\mathbf{u}_h}(t_n) - \mathbf{u}_{tt}(t_n), \frac{1}{2}(\triangle_t \mathbf{r}_h^{n+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{n-\frac{1}{2}})\Big). \end{aligned} \tag{4.26}$$

Subtracting (4.26) from (4.25), we arrive at

$$\begin{aligned} & a\Big(\Delta_t \psi_h^{n+\frac{1}{2}}, \psi_h^{n+\frac{1}{2}}\Big) + b\Big(\psi_h^{n+\frac{1}{2}}, \Delta_t \mathbf{r}_h^{n+\frac{1}{2}}\Big) - \frac{1}{2} b\Big(\psi_h^n, \Delta_t \mathbf{r}_h^{n+\frac{1}{2}}\Big) - \frac{1}{2} b\Big(\psi_h^n, \Delta_t \mathbf{r}_h^{n-\frac{1}{2}}\Big) \\ = & -\frac{1}{2}\Big(\Delta_t^2 \mathbf{r}_h^n, \Delta_t \mathbf{r}_h^{n+\frac{1}{2}} + \Delta_t \mathbf{r}_h^{n-\frac{1}{2}}\Big) + \frac{1}{2}\Big(\mathbf{u}_{tt}(t_n) - \Delta_t^2 \widehat{\mathbf{u}_h}(t_n), \Delta_t \mathbf{r}_h^{n+\frac{1}{2}} + \Delta_t \mathbf{r}_h^{n-\frac{1}{2}}\Big), \end{aligned} \tag{4.27}$$

which implies

$$\begin{aligned} & \frac{1}{2\triangle t}\Big(\|\psi_h^{n+1}\|_a^2 - \|\psi_h^n\|_a^2\Big) + \frac{1}{2} b\Big(\psi_h^{n+1}, \triangle_t \mathbf{r}_h^{n+\frac{1}{2}}\Big) - \frac{1}{2} b\Big(\psi_h^n, \triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\Big) \\ = & \frac{1}{2\triangle t}\Big(\|\triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\|_0^2 - \|\triangle_t \mathbf{r}_h^{n+\frac{1}{2}}\|_0^2\Big) + \frac{1}{2}\Big(\mathbf{u}_{tt}(t_n) - \triangle_t^2 \widehat{\mathbf{u}_h}(t_n), \triangle_t \mathbf{r}_h^{n+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\Big). \end{aligned} \tag{4.28}$$

Multiplying (4.28) by $2\triangle t$, replacing n by j and summing these equations from $j = 0$ to $n-1$, we have

$$\begin{aligned} & \|\psi_h^n\|_a^2 - \|\psi_h^0\|_a^2 + \|\triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\|_0^2 \\ & \quad - \|\triangle_t \mathbf{r}_h^{-\frac{1}{2}}\|_0^2 + \triangle t b\Big(\psi_h^n, \triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\Big) - \triangle t b\Big(\psi_h^0, \triangle_t \mathbf{r}_h^{-\frac{1}{2}}\Big) \\ = & \triangle t \sum_{j=0}^{n-1} \Big(\mathbf{u}_{tt}(t_j) - \triangle_t^2 \widehat{\mathbf{u}_h}(t_j), \triangle_t \mathbf{r}_h^{j+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{j-\frac{1}{2}}\Big). \end{aligned} \tag{4.29}$$

In view of $\psi_h^0 = 0$, $\mathbf{r}_h^0 = 0$ and $\triangle_t \mathbf{r}_h^{-\frac{1}{2}} = 0$, it follows

$$\begin{aligned} & \|\psi_h^n\|_a^2 + \|\triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\|_0^2 + \triangle t b\Big(\psi_h^n, \triangle_t \mathbf{r}_h^{n-\frac{1}{2}}\Big) \\ = & \triangle t \sum_{j=0}^{n-1} \Big(\mathbf{u}_{tt}(t_j) - \triangle_t^2 \widehat{\mathbf{u}_h}(t_j), \triangle_t \mathbf{r}_h^{j+\frac{1}{2}} + \triangle_t \mathbf{r}_h^{j-\frac{1}{2}}\Big), \end{aligned} \tag{4.30}$$

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which yields

$$\begin{aligned}&\frac{1}{2(\mu+\lambda)}\|\psi_h^n\|_0^2+\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0^2+\triangle tb(\psi_h^n,\triangle_t\mathbf{r}_h^{n-\frac{1}{2}})\\&\leq\triangle t\sum_{j=0}^{n-1}(\mathbf{u}_{tt}(t_j)-\triangle_t^2\widehat{\mathbf{u}_h}(t_j),\triangle_t\mathbf{r}_h^{j+\frac{1}{2}}+\triangle_t\mathbf{r}_h^{j-\frac{1}{2}}).\end{aligned}\tag{4.31}$$

By Cauchy-Schwarz inequality, the inverse inequality (3.5) and the CFL condition, we obtain

$$\begin{aligned}&|\triangle tb(\psi_h^n,\triangle_t\mathbf{r}_h^{n-\frac{1}{2}})|\\&\leq\triangle t\|\psi_h^n\|_0\|\epsilon(\triangle_t\mathbf{r}_h^{n-\frac{1}{2}})\|_0\leq\triangle t\delta h^{-1}\|\psi_h^n\|_0\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0\\&\leq\zeta\frac{1}{\sqrt{\mu+\lambda}}\|\psi_h^n\|_0\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0\leq\zeta(\frac{1}{2(\mu+\lambda)}\|\psi_h^n\|_0^2+\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0^2).\end{aligned}\tag{4.32}$$

As $0<\zeta<1$, it holds

$$\begin{aligned}&\frac{1}{2(\mu+\lambda)}\|\psi_h^n\|_0^2+\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0^2\\&\lesssim\triangle t\sum_{j=0}^{n-1}(\mathbf{u}_{tt}(t_j)-\triangle_t^2\widehat{\mathbf{u}_h}(t_j),\triangle_t\mathbf{r}_h^{j+\frac{1}{2}}+\triangle_t\mathbf{r}_h^{j-\frac{1}{2}})\\&\lesssim\triangle t\sum_{j=0}^{n-1}\|\mathbf{u}_{tt}(t_j)-\triangle_t^2\widehat{\mathbf{u}_h}(t_j)\|_0\|\triangle_t\mathbf{r}_h^{j+\frac{1}{2}}+\triangle_t\mathbf{r}_h^{j-\frac{1}{2}}\|_0\\&\lesssim\triangle t\max_{0\leq n\leq M}\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0\sum_{j=0}^{n-1}\|\mathbf{u}_{tt}(t_j)-\triangle_t^2\widehat{\mathbf{u}_h}(t_j)\|_0.\end{aligned}\tag{4.33}$$

Then it follows

$$\max_{0\leq n\leq M}\|\psi_h^n\|_0+\max_{0\leq n\leq M}\|\triangle_t\mathbf{r}_h^{n-\frac{1}{2}}\|_0\lesssim\triangle t\sum_{j=0}^{M}\|\mathbf{u}_{tt}(t_j)-\triangle_t^2\widehat{\mathbf{u}_h}(t_j)\|_0.\tag{4.34}$$

Now taking $\tau_h=\psi_h^n$ in the first equation of (4.24), we get

$$a(\psi_h^n,\psi_h^n)+b(\psi_h^n,\mathbf{r}_h^n)=0.\tag{4.35}$$

From Lemma 3.2, (2.7) and (4.35) we have

$$\|\mathbf{r}_h^n\|_{1,\Omega}\lesssim\sup_{0\neq\psi_h^n\in\Sigma_h}\frac{\int_\Omega\psi_h^n:\epsilon(\mathbf{r}_h^n)d\mathbf{x}}{\|\psi_h^n\|_0}=\sup_{0\neq\psi_h^n\in\Sigma_h}\frac{a(\psi_h^n,\psi_h^n)}{\|\psi_h^n\|_0}\lesssim\|\psi_h^n\|_0,\tag{4.36}$$

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which leads to

$$\max_{0\le n\le M}\|\mathbf{r}_h^n\|_1 \lesssim \max_{0\le n\le M}\|\psi_h^n\|_0. \tag{4.37}$$

Combining (4.34) with (4.37) completes the proof. □

Lemma 4.4. *Suppose* $\mathbf{u}_{tt}, \mathbf{u}_{ttt} \in L^\infty([0,T];(H^2(\Omega))^2)$, $\mathbf{u}_{tttt} \in L^\infty([0,T];(L^2(\Omega))^2)$ *and* $\sigma_{tt}, \sigma_{ttt} \in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$. *Then the following error estimate holds:*

$$\begin{aligned}&\max_{0\le n\le M}\|\mathbf{u}_h^n - \widehat{\mathbf{u}_h}(t_n)\|_1 + \max_{0\le n\le M}\|\sigma_h^n - \widehat{\sigma_h}(t_n)\|_0\\ &\lesssim h\left(\|\sigma_{tt}\|_{L^\infty([0,T];(H^1(\Omega))^{2\times 2})} + \|\sigma_{ttt}\|_{L^\infty([0,T];(H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_{tt}\|_{L^\infty([0,T];(H^2(\Omega))^2)}\right.\\ &[1ex] + \left.\|\mathbf{u}_{ttt}\|_{L^\infty([0,T];(H^2(\Omega))^2)}\right) + (\triangle t)^2\|\mathbf{u}_{tttt}\|_{L^\infty([0,T];(L^2(\Omega))^2)}.\end{aligned} \tag{4.38}$$

Proof. The proof follows from the same lines as in [3]. □

We are now in a position to show an error estimate for the solution of the fully discrete hybrid stress finite element scheme.

Theorem 4.2. *Suppose* $\mathbf{u}, \mathbf{u}_{tt}, \mathbf{u}_{ttt} \in L^\infty([0,T];(H^2(\Omega))^2)$, $\mathbf{u}_{tttt} \in L^\infty([0,T];(L^2(\Omega))^2)$ *and* $\sigma, \sigma_{tt}, \sigma_{ttt} \in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$. *Under the CFL condition* $\triangle t < \min\{1, \zeta\frac{h}{\delta\sqrt{\mu+\lambda}}\}$ *with* $0<\zeta<1$, *it holds the following error estimate:*

$$\begin{aligned}&\max_{0\le n\le M}\|\sigma(t_n) - \sigma_h^n\|_0 + \max_{0\le n\le M}\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_1\\ &\lesssim h\big(\|\mathbf{u}\|_{L^\infty([0,T];(H^2(\Omega))^2)} + \|\sigma\|_{L^\infty([0,T];(H^1(\Omega))^{2\times 2})}\\ &\quad + \|\mathbf{u}_{tt}\|_{L^\infty([0,T];(H^2(\Omega))^2)} + \|\sigma_{tt}\|_{L^\infty([0,T];(H^1(\Omega))^{2\times 2})}\\ &\quad + \|\mathbf{u}_{ttt}\|_{L^\infty([0,T];(H^2(\Omega))^2)} + \|\sigma_{ttt}\|_{L^\infty([0,T];(H^1(\Omega))^{2\times 2})}\big)\\ &\quad + (\triangle t)^2\|\mathbf{u}_{tttt}\|_{L^\infty([0,T];(L^2(\Omega))^2)}.\end{aligned} \tag{4.39}$$

Proof. The desired result follows from Lemma 3.4, Lemma 4.4 and triangle inequality. □

5. Numerical tests

Example 5.1. Consider the problem (1.1) with $\Omega = [0,1]\times[0,1]$, $T=1$, $\mu=1$, $\lambda=1$, $f=[f_1,f_2]^T$,

$$f_1(\mathbf{x},t) = e^{-t}[(x^4-2x^3-11x^2+12x-2)(4y^3-6y^2+2y) - (x^2-x)^2(24y-12)],$$

$$f_2(\mathbf{x},t) = -e^{-t}[(y^4-2y^3-11y^2+12y-2)(4x^3-6x^2+2x) - (y^2-y)^2(24x-12)].$$

The initial displacement and velocity are respectively given by

$$\varphi_0 = [(x^2-x)^2(4y^3-6y^2+2y), -(y^2-y)^2(4x^3-6x^2+2x)]^T,$$

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Table 1: Results of $E_{\mathbf{u}}$ and E_σ for HSFEM: Example 1.

	$(h,\triangle t)$	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1$
$E_{\mathbf{u}}$	(0.25, 0.0125)	0.5335	0.5334	0.5347	0.5491	0.5400
	(0.125, 0.0088)	0.2698	0.2701	0.2714	0.2747	0.2734
	(0.0625, 0.00625)	0.1350	0.1351	0.1351	0.1356	0.1357
	(0.03125, 0.0044)	0.0675	0.0675	0.0675	0.0676	0.0676
E_σ	(0.25, 0.0125)	0.5252	0.5227	0.5238	0.5570	0.5518
	(0.125, 0.0088)	0.2673	0.2669	0.2684	0.2720	0.2704
	(0.0625, 0.00625)	0.1335	0.1334	0.1334	0.1338	0.1342
	(0.03125, 0.0044)	0.0667	0.0667	0.0667	0.0667	0.0667

Table 2: Results of $E_{\mathbf{u}}$ for bilinear element and HSFEM: Example 1, $\Delta t = 10^{-3}$.

	h	$t=0.02$	$t=0.04$	$t=0.06$	$t=0.08$	$t=0.1$
Bilinear	0.25	0.5372	0.5369	0.5366	0.5364	0.5363
	0.125	0.2704	0.2703	0.2701	0.2700	0.2700
	0.0625	0.1351	0.1351	0.1350	0.1350	0.1350
	0.03125	0.0675	0.0675	0.0675	0.0675	0.0675
HSFEM	0.25	0.5371	0.5365	0.5357	0.5352	0.5349
	0.125	0.2704	0.2702	0.2700	0.2698	0.2698
	0.0625	0.1351	0.1351	0.1350	0.1350	0.1350
	0.03125	0.0675	0.0675	0.0675	0.0675	0.0675

$$\varphi_1 = [-(x^2-x)^2(4y^3-6y^2+2y), (y^2-y)^2(4x^3-6x^2+2x)]^T.$$

The exact displacement solution, $\mathbf{u} = (u_1, u_2)^T$, to this problem is as follows:

$$u_1 = e^{-t}(x^2-x)^2(4y^3-6y^2+2y), u_2 = -e^{-t}(y^2-y)^2(4x^3-6x^2+2x).$$

The domain Ω is divided equally into $m \times m$ meshes with $h = 1/m$. Table 1 shows the relative errors,

$$E_{\mathbf{u}} = \frac{|\mathbf{u}(t_n) - \mathbf{u}_h^n|_1}{|\mathbf{u}(t_n)|_1}, \quad \text{and } E_\sigma = \frac{\|\sigma(t_n) - \sigma_h^n\|_0}{\|\sigma(t_n)\|_0},$$

of the proposed fully discrete hybrid stress finite element method (abbr. HSFEM) at $t = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$. We note that the time step $\triangle t$ is chosen much smaller than the space step h so as to satisfy the CFL condition, and that, as the mesh is refined, the mesh parameters $(h,\triangle t)$ are decreased to $(\frac{h}{2}, \frac{\Delta t}{\sqrt{2}})$. Numerical results indicate that $E_{\mathbf{u}}$ and E_σ are both of $O(h + (\Delta t)^2)$-accuracy, as is conformable to the error estimate in Theorem 4.2.

We also list in Tables 2-3 some numerical results with $\Delta t = 10^{-3}$ for the bilinear element and our method. We can see that our method gives more accurate stress approximations than the bilinear element.

Example 5.2. Consider the problem (1.1) in the plane strain case with $\Omega = [0,1]\times[0,1]$, $T = 1$, $E = 1$. The exact displacement solution, $\mathbf{u} = (u_1, u_2)^T$, to this problem is as

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Table 3: Results of E_σ for bilinear element and HSFEM: Example 1, $\Delta t = 10^{-3}$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
Bilinear	0.25	0.7513	0.7508	0.7503	0.7504	0.7510
	0.125	0.4017	0.4016	0.4015	0.4015	0.4017
	0.0625	0.2041	0.2040	0.2040	0.2040	0.2040
	0.03125	0.1024	0.1024	0.1024	0.1024	0.1024
HSFEM	0.25	0.5247	0.5231	0.5215	0.5212	0.5226
	0.125	0.2667	0.2661	0.2656	0.2657	0.2661
	0.0625	0.1333	0.1332	0.1332	0.1332	0.1333
	0.03125	0.0666	0.0666	0.0666	0.0666	0.0666

Table 4: Results of $E_{\mathbf{u}}$ and E_σ for HSFEM: Example 2, $\nu = 0.25$.

	$(h, \triangle t\,)$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1$
$E_{\mathbf{u}}$	(0.25, 0.0125)	0.2276	0.2345	0.2633	0.2868	0.2502
	(0.125, 0.0088)	0.1135	0.1144	0.1186	0.1224	0.1176
	(0.0625, 0.00625)	0.0567	0.0568	0.0574	0.0579	0.0572
	(0.03125, 0.0044)	0.0283	0.0284	0.0285	0.0285	0.0284
E_σ	(0.25, 0.0125)	0.2456	0.2515	0.2738	0.2903	0.2643
	(0.125, 0.0088)	0.1235	0.1243	0.1275	0.1298	0.1262
	(0.0625, 0.00625)	0.0618	0.0620	0.0624	0.0627	0.0621
	(0.03125, 0.0044)	0.0309	0.0310	0.0310	0.0310	0.0310

Table 5: Results of $E_{\mathbf{u}}$ for bilinear element and HSFEM: Example 2, $\Delta t = 10^{-3}$, $\nu = 0.25$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
bilinear	0.5	0.4755	0.4748	0.4736	0.4720	0.4700
	0.25	0.2308	0.2308	0.2312	0.2325	0.2353
	0.125	0.1140	0.1146	0.1172	0.1233	0.1341
	0.0625	0.0569	0.0592	0.0668	0.0807	0.0999
HSFEM	0.5	0.4754	0.4745	0.4729	0.4708	0.4682
	0.25	0.2308	0.2306	0.2304	0.2301	0.2297
	0.125	0.1139	0.1139	0.1139	0.1138	0.1138
	0.0625	0.0568	0.0567	0.0567	0.0567	0.0567

follows:

$$u_1 = (1-\nu^2)e^{-t}\sin(\pi x)\sin(\pi y), \quad u_2 = \nu(1+\nu)e^{-t}\sin(\pi x)\sin(\pi y).$$

Table 4 shows some results of relative errors, $E_{\mathbf{u}}$ and E_σ, for the fully discrete HSFEM at $t = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$ with $\nu = 0.25$. We note that the numerical results are conformable to the estimate of Theorem 4.2.

We list in Tables 5-10 some numerical results with $\Delta t = 10^{-3}$ and $\nu = 0.25, 0.499,$ 0.4999 for the bilinear element and our method. Tables 5-6 show that, at $\nu = 0.25$, both methods give displacement results of the same accuracy, and our method yields much more accurate stress approximations than the bilinear element. Tables 7-8 show

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Table 6: Results of E_σ for bilinear element and HSFEM: Example 2, $\Delta t = 10^{-3}$, $\nu = 0.25$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
bilinear	0.5	0.5152	0.5145	0.5134	0.5119	0.5101
	0.25	0.2887	0.2886	0.2886	0.2893	0.2912
	0.125	0.2033	0.2036	0.2049	0.2084	0.2146
	0.0625	0.1767	0.1774	0.1799	0.1851	0.1935
HSFEM	0.5	0.5051	0.5042	0.5026	0.5004	0.4977
	0.25	0.2498	0.2496	0.2493	0.2489	0.2484
	0.125	0.1241	0.1241	0.1240	0.1240	0.1239
	0.0625	0.0619	0.0619	0.0619	0.0619	0.0619

Table 7: Results of $E_{\mathbf{u}}$ for bilinear element and HSFEM: Example 2, $\Delta t = 10^{-3}$, $\nu = 0.499$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
bilinear	0.5	0.4585	0.4506	0.4746	0.4736	0.4497
	0.25	0.3632	0.5426	0.6109	0.6491	0.5957
	0.125	0.3996	0.8153	1.2325	1.4336	0.9553
	0.0625	0.4953	1.1476	1.8942	2.1468	1.2708
HSFEM	0.5	0.4490	0.5444	0.7837	0.9502	0.7876
	0.25	0.2298	0.2478	0.2787	0.3145	0.2970
	0.125	0.1142	0.1173	0.1243	0.1298	0.1211
	0.0625	0.0568	0.0575	0.0591	0.0595	0.0577

Table 8: Results of E_σ for bilinear element and HSFEM: Example 2, $\Delta t = 10^{-3}$, $\nu = 0.499$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
bilinear	0.5	0.6384	0.6343	0.6475	0.6469	0.6338
	0.25	0.5612	0.5996	0.5893	0.5941	0.6023
	0.125	0.5518	0.6025	0.6579	0.7646	0.5878
	0.0625	0.5515	0.5993	0.7024	0.8585	0.5927
HSFEM	0.5	0.5870	0.6280	0.7817	0.8693	0.7844
	0.25	0.3142	0.3234	0.3348	0.3435	0.3365
	0.125	0.1593	0.1605	0.1618	0.1631	0.1620
	0.0625	0.0799	0.0801	0.0802	0.0804	0.0803

Table 9: Results of $E_{\mathbf{u}}$ for bilinear element and HSFEM: Example 2, $\Delta t = 10^{-3}$, $\nu = 0.4999$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
bilinear	0.5	0.4755	0.4704	0.4621	0.4537	0.4488
	0.25	0.6426	0.2703	0.4354	0.5546	0.5244
	0.125	1.3348	0.4573	0.7804	1.0339	0.3181
	0.0625	2.86×10^4	5.35×10^{12}	1.02×10^{21}	1.94×10^{29}	3.70×10^{37}
HSFEM	0.5	0.8123	0.4814	0.4586	0.8681	0.4886
	0.25	0.2841	0.2499	0.2277	0.3017	0.2691
	0.125	0.1263	0.1148	0.1163	0.1241	0.1158
	0.0625	8.61×10^1	1.33×10^{10}	2.18×10^{18}	3.57×10^{26}	5.85×10^{34}

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Table 10: Results of $E_{\mathbf{u}}$ for bilinear element and HSFEM: Example 2, $\Delta t = 10^{-3}$, $\nu = 0.4999$.

	h	$t = 0.02$	$t = 0.04$	$t = 0.06$	$t = 0.08$	$t = 0.1$
bilinear	0.5	0.6478	0.6449	0.6404	0.6359	0.6333
	0.25	0.6022	0.5403	0.5577	0.5893	0.6196
	0.125	0.6994	0.5571	0.6160	0.5991	0.5426
	0.0625	3.05×10^4	5.72×10^{12}	1.09×10^{21}	2.37×10^{29}	3.94×10^{37}
HSFEM	0.5	0.8033	0.5967	0.5873	0.8434	0.6202
	0.25	0.2841	0.2499	0.2277	0.3017	0.2691
	0.125	0.1263	0.1148	0.1163	0.1241	0.1158
	0.0625	1.21×10^2	1.88×10^{10}	3.07×10^{18}	5.04×10^{26}	8.25×10^{34}

that at $\nu = 0.499$ the bilinear element gives divergent results due to Poisson locking, but our method still have the first-order convergence for both the displacement and stress approximations. Tables 9-10 show that, at $\nu = 0.4999$, our method still gives good results when the mesh parameter h is not so small, but it behaves unstablely as h becomes small enough. This phenomenon indicates that our fully discrete method still suffers from 'Poisson locking' for plane strain problems, at the nearly incompressible limit ($\lambda \to \infty$ or $\nu \to 0.5$).

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