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A NEW STRATEGY FOR ROBBINS' PROBLEM OF OPTIMAL STOPPING

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Abstract

In this paper we study the expected rank problem under full information. Our approach uses the planar Poisson approach from Gnedin (2007) to derive the expected rank of a stopping rule that is one of the simplest nontrivial examples combining rank dependent rules with threshold rules. This rule attains an expected rank lower than the best upper bounds obtained in the literature so far, in particular, we obtain an expected rank of 2.326 14.

Keywords: Optimal stopping; Robbins' problem

2010 Mathematics Subject Classification: Primary 60G40

1. Introduction and motivation

Consider independent and identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$. The rank of X_k is defined as $R_k = \sum_{m=1}^{n} \mathbf{1}_{\{X_m \le X_k\}}$. Robbins' problem is to find the optimal stopping rule that minimizes the expected rank of the chosen realization. Although the optimal rule is unknown, some results on the properties of the optimal rule are available: Bruss and Ferguson [5, Section 4.2] proved full history dependence of the optimal rule for the n-period problem. That is to say, the optimal decision to stop at stage $m < n$ depends on all past realizations $x_1, \ldots, x_{m-1}$ as well as the current realization x_m. In addition, by using variational calculus Assaf and Samuel-Cahn [1] obtained a lower bound of 1.85 for the limit of the n-period problem. Using truncated loss functions, Bruss and Ferguson [4] derived a lower bound of 1.908 by computational methods (with the help of Hardwick and Schork). A further question raised in [6], which is associated with Robbins' problem, is whether the limit superior of n times the expected value of X_{τ_n} arising from the optimal n-period rule τ_n is finite. This question was settled by Gnedin and Iksanov [10], who showed that it is indeed finite. For an overview on Robbins' problem and related stopping problems we refer the reader to [2], [9], and [12].

Full information rules are stopping rules adapted to the filtration generated by all prior and current observations. Interesting subsets of full information rules are *rank rules*, where the decision to stop depends on time k and the relative rank $I_k = \sum_{m=1}^{k} \mathbf{1}_{\{X_m \le X_k\}}$, and *threshold rules*, where one stops with the first realization x_m such that $x_m \le f_m$, where f_m is some positive real number that depends only on m. When minimizing the expected rank under full information for the $n \to \infty$ case, Assaf and Samuel Cahn [1, Example 4.2] showed that a value of 2.3318 can be obtained with a threshold rule of a simple form. This value was replicated

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in [9] for the continuous-time Poisson embedding. A value of $\frac{7}{3}$ has been obtained in [4] with another approximately optimal threshold rule. The authors estimated that the optimal threshold rule gives a value of 2.326 59. Additionally, in [1] it was shown that in the $n \to \infty$ limit the expected loss for the optimal threshold rule has to be in the interval (2.295, 2.3267).

More recent literature, for example, [3], [6], [7]–[9], has shifted attention to Poisson embeddings of discrete-time optimal stopping problems. These articles also demonstrate how the continuous-time versions can be used to obtain upper bounds for n-period discrete-time problems. By using a continuous-time Poisson embedding of Robbins' problem, in [9] it was shown that full history dependence also persists in the $n \to \infty$ limit, so that even in the limit a threshold rule cannot be optimal. However, it is still not clear by how much the optimal rule is better than the optimal threshold rule.

This article combines a threshold rule with a rank dependent rule, which enables us to obtain an analytic solution with an expected rank smaller than 2.326 59. More specifically, up to a time α we use a rule as in [8] for the best choice problem, where one stops with the first observation that is below some threshold function $f_b(t)$ and has relative rank 1, but with a different threshold parameter b. From α on, we apply a threshold rule, where—given that the stopping criterion was not fulfilled for $t \leq \alpha$—the decision maker stops at the first observation below a function $f_c(t)$, as in [9, Section 3], again with a different threshold parameter c. The framework of [9] allows us to work 'directly in the limit'. It would be much more difficult to use a similar family of rules like our rule in the n-period problems, and then compute the expected rank of this family of rules as $n \to \infty$.

A similar rule for the discrete-time problem has already been proposed in [1, Remark 6.2]. There, an approximately optimal threshold rule is combined with the requirement to stop with a relative rank of one within a time span γn, where $\gamma \in [0, 1)$. An analytical investigation of this rule—in the planar Poisson framework of [8]—was performed in [13], where a value of 2.330 44, with $\alpha = 0.42$ and $b = c = 1.95$ was obtained. The main difference to our approach is that the same threshold function is applied during the whole time span, which simplifies the computations compared to ours.

2. A simple rank-threshold rule

We follow [8] and [9] and consider a scatter of atoms P arising from a continuous-time planar Poisson process on the strip $[0, 1] \times \bar{\mathbb{R}}_+$, where $\bar{\mathbb{R}}_+$ denotes the interval $[0, \infty]$. The intensity measure is the Lebesgue measure $\mathrm{d}t\,\mathrm{d}x$, which implies that the number of particles N in some subset with Lebesgue measure ν follows a Poisson distribution with density $\mathbb{P}(N = n) = \mathrm{e}^{-\nu}(\nu^n/n!)$. An atom (T, X) consists of the arrival time T and the value X. Ordering the atoms with respect to X_j in ascending order yields the increasing sequence of points $(X_{1,1}, X_{1,2}, X_{1,3}, \dots) =: \boldsymbol{X}_1$ of a unit Poisson process. We denote an atom of the ordered sequence by $(T_r, X_{1,r})$. By the properties of the planar Poisson process, the arrival times T_r are uniform i.i.d. on $[0, 1]$, and T_r, $r = 1, 2, \dots$, and $\boldsymbol{X}_1$ are independent. In addition, a stochastic process $(\tilde{X}_t,\ t \in [0, 1])$ with values in $[0, \infty]$ can be constructed by $\tilde{X}_t := X_r$, where r is the minimal r such that $t = T_r$, if such an r exists, and $\tilde{X}_t := \infty$ otherwise. For more technical details on this planar Poisson process we refer the reader to [8] and [9].

Let (t, x) and $\mathcal{P}$ stand for realizations of (T, X) and P. For a generic $(t, x) \in \mathcal{P}$ the absolute rank $R(t, x)$ is defined by the number of $\mathcal{P}$-points strictly south of (t, x) plus 1, while the relative rank $I(t, x)$ is the number of $\mathcal{P}$-points strictly south-west of (t, x) plus 1.

Let (τ, X) define a *stopping point*, where the event $\{\tau \leq t\}$ is measurable with respect to the sigma field generated by $(\tilde{X}_s)_{0 \leq s \leq t}$. Then, the problem considered in the following is to

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minimize $\mathbb{E}[R(\tau, X)] = \mathbb{E}[I(\tau, X) + (1 - \tau)X]$, where $\tau < 1$ almost surely. Consider two arbitrary positive, strictly increasing, and continuous functions f_1, f_2 on $[0, 1)$ with $\int_0^1 f_2(t) = \infty$. Let the random variable Y be the height of the lowest P-point above f_1 on $[0, \alpha]$, where $0 \le \alpha < 1$. That is,

$$Y := \min\{X\colon (T, X) \in P,\ T \in [0, \alpha],\ X > f_1(T)\}.$$

In the following, y denotes a realization of Y. Next we consider a stopping point (τ, X) with

$$\tau := \inf\{T\colon (T, X) \in P,\ X \le (f_1(T) \wedge Y)\mathbf{1}_{\{T \le \alpha\}} + f_2(T)\mathbf{1}_{\{T > \alpha\}}\}, \tag{1}$$

where the threshold function is random due to Y. Note that $\tau < 1$ almost surely by the condition $\int_0^1 f_2 = \infty$. By the stopping rule defined in (1), the relative rank of X is 1 in the event $\{\tau \le \alpha\}$. Since f_1 is increasing, $y \le f_1(t)$ implies that $(t', y) \in \mathcal{P}$ for some $0 \le t' < t$. Hence, τ is a stopping time for the filtration generated by $(\tilde{X}_s)_{0 \le s \le t}$.

Figure 1(a) provides a graphical description of the stopping rule τ. For $t \le \alpha$, a decision maker stops if a particle (t, x) with $x \le y$ and $x \le f_1(t)$ is observed, while for $t > \alpha$, the threshold f_2 is applied. In Figure 1(a) we show one realization of Y, where $y < f_1(\alpha)$. The decision whether to stop with the first (t, x) such that $x \le f_1(t)$ and $t \le \alpha$ depends on y only if $t \in (f_1^{-1}(y), \alpha]$. Figure 1(b) provides an example where $y > f_1(t)$ for all $t \le \alpha$. In this case the decision maker stops with the first (t, x) such that $0 \le t \le \alpha$ and $x \le f_1(t)$, if any such (t, x) exists.

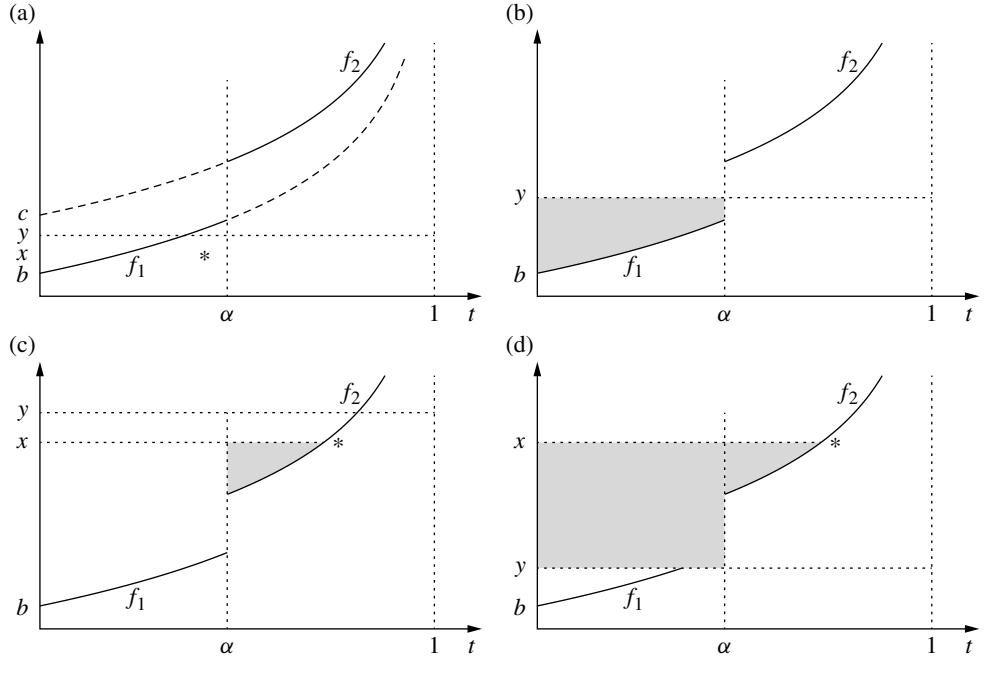


FIGURE 1: The stopping rule τ and loss components. (a) Stopping rule τ: $f_1 = b/(1-t)$, $f_2 = c/(1-t)$. (b) Area between y and the graph of $f_1(t)$. (c) Loss component, $y > f_2(t)$ is a realization of Y. (d) Loss component, $y < f_2(t)$ is a realization of Y. The scales of the y-axes are logarithmic.

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To obtain the risk $\mathbb{E}[R(\tau, X)]$, we consider the conditional risk $\mathbb{E}[R(\tau, X) \mid Y = y]$ and integrate out y. To do this, we define

$$F_1(t, y) := \int_0^t (f_1(s) \wedge y)\,\mathrm{d}s, \qquad F_2(t) := \int_\alpha^t f_2(s)\,\mathrm{d}s,$$
$$S_1(x) := \int_0^x (f_1^{-1}(z) \wedge \alpha)\,\mathrm{d}z = \alpha x - F_1(\alpha, x),$$

where $f_1^{-1}(z) = 0$ for $z < f_1(0)$. Furthermore,

$$S_2(x) := \int_0^x (f_2^{-1}(z) - \alpha)\,\mathrm{d}z,$$

where $f_2^{-1}(z) = \alpha$ for $z < f_2(\alpha)$. By the properties of the planar Poisson process, the probability that the area bounded by the horizontal line with height y and the graph of f_1 for the time interval $[0, \alpha]$ is empty is $\mathrm{e}^{-S_1(y)}$; see the shaded area in Figure 1(b). Hence, the density of Y is

$$\mathbb{P}(Y \in \mathrm{d}y) = \left[\frac{\mathrm{d}}{\mathrm{d}y}(1 - \mathrm{e}^{-S_1(y)})\right]\mathrm{d}y = \mathrm{e}^{-S_1(y)}(f_1^{-1}(y) \wedge \alpha)\,\mathrm{d}y.$$

By the definition of Y, $\mathbb{P}(Y \in \mathrm{d}y) > 0$ for $y > f_1(0)$ and 0 otherwise. Given $\{Y = y\}$, the conditional joint density of (τ, X) is

$$\mathbb{P}((\tau, X) \in (\mathrm{d}t, \mathrm{d}x) \mid Y = y) = \begin{cases} \mathrm{e}^{-F_1(t,y)}\,\mathrm{d}t\,\mathrm{d}x, & 0 \le t \le \alpha,\ 0 \le x \le f_1(t) \wedge y, \\ \mathrm{e}^{-F_1(\alpha,y)-F_2(t)}\,\mathrm{d}t\,\mathrm{d}x, & \alpha < t \le 1,\ 0 \le x \le f_2(t). \end{cases} \tag{2}$$

By means of the density (2), we obtain the conditional probability $\mathbb{P}(\tau > t \mid Y = y) = \mathrm{e}^{-F_1(t,y)}$ for $t \le \alpha$, and $\mathbb{P}(\tau > t \mid Y = y) = \mathrm{e}^{-F_1(\alpha,y)-F_2(t)}$ for $\alpha < t \le 1$. Splitting the time at α, the two components of the risk given $Y = y$ are computed as

$$\mathbb{E}[R(\tau, X)\mathbf{1}_{\{\tau \le \alpha\}} \mid Y = y] = \int_0^\alpha \mathrm{e}^{-F_1(t,y)}\left(\int_0^{f_1(t)\wedge y} (1 + x(1 - t))\,\mathrm{d}x\right)\mathrm{d}t,$$

$$\mathbb{E}[R(\tau, X)\mathbf{1}_{\{\tau > \alpha\}} \mid Y = y] = \int_\alpha^1 \mathrm{e}^{-F_1(\alpha,y)-F_2(t)}\left[\int_0^{f_2(t)\wedge y} (1 + x(1 - t))\,\mathrm{d}x + \int_{f_2(\alpha)}^{f_2(t)} S_2(x)\,\mathrm{d}x + \int_{f_2(t)\wedge y}^{f_2(t)} (2 + x(1 - t) + (x - y)\alpha)\,\mathrm{d}x\right]\mathrm{d}t.$$

For $t \le \alpha$, the integrand of the inner integral describes the conditional expected loss contribution when we stop at (t, x). In this case, the relative rank of (t, x) is 1 and the conditional expected loss of (t, x) is 1 plus the expected number of atoms in the south-east of (t, x). That is, $1 + x(1 - t)$. An example for such a (t, x) is provided by '$*$' in Figure 1(a). For $t \ge \alpha$, we have to account for atoms in the area above the curve $f_2(s)$, $\alpha < s \le t$, and below the level obtained by x. In the Figures 1(c) and 1(d) this corresponds to the shaded almost triangular area. Formally, the size of this area is $S_2(x)$, where $S_2(x) = 0$ for $x \le f_2(\alpha)$, and $S_2(x) > 0$ for $x > f_2(\alpha)$. In particular, in Figure 1(c) we show $S_2(x)$ for some $x > f_2(\alpha)$. In Figures 1(c)

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and 1(d) we describe two generic cases where a decision maker stops at (t, x) with $t > \alpha$. In Figure 1(c) we describe the $y > x$ case. The loss coming from the past is described by the shaded almost triangular area. The conditional expected loss is $1 + x(1-t) + S_2(x)$. The 1 accounts for the particle itself and $x(1-t)$ for the expected loss in the future, which is the size of the area in the south-east of the particle (t, x). Next, Figure 1(d) is the case where $y \le x$. In this case, the loss coming from the past is described by the shaded almost triangular area and the shaded rectangular area. The conditional expected loss is $2 + x(1-t) + S_2(x) + (x-y)\alpha$. The 2 accounts for the particle itself and the particle (t', y) such that y is the realization of Y. In addition, $x(1-t)$ accounts for the expected loss in the future. Last, but not least, $F_1(t, y)$ and $F_2(t)$ measure the size of areas below $f_1(s) \wedge y$, where $0 \le s \le t \le \alpha$, and $f_2(s)$, where $\alpha < s \le t$. Then $S_1(y)$ measures the difference of the size of the area of the rectangle described by the points $(0, 0)$, $(\alpha, 0)$, $(\alpha, f_1(\alpha) \wedge y)$, and $(0, f_1(\alpha) \wedge y)$ and the size of the area below the graph $\{(t, f_1(t) \wedge y) : 0 \le t \le \alpha\}$. The shaded area in Figure 1(b) provides an example for this area.

Next, we work with $f_1(t) = b/(1-t)$ and $f_2(t) = c/(1-t)$. For these functions, we were able to obtain the conditional expected losses

$$\mathbb{E}[R(\tau, X)\mathbf{1}_{\{\tau \le \alpha\}} \mid Y = y] \quad \text{and} \quad \mathbb{E}[R(\tau, X)\mathbf{1}_{\{\tau > \alpha\}} \mid Y = y],$$

as well as the expected loss

$$\mathbb{E}[R(\tau, X)] = \int_b^\infty (\mathbb{E}[R(\tau, X)\mathbf{1}_{\{\tau \le \alpha\}} \mid Y = y] + \mathbb{E}[R(\tau, X)\mathbf{1}_{\{\tau \le \alpha\}} \mid Y = y])\mathrm{e}^{-S_1(y)}(f_1^{-1}(y) \wedge \alpha)\,\mathrm{d}y$$

in closed form by using the MATHEMATICA® 8.0 package. (To do this, the MATHEMATICA expressions were converted to MATLAB® R2012a code. For a version of this paper including the MATHEMATICA code, see [11].)

To minimize $\mathbb{E}[R(\tau, X)]$, the parameters α, b, and c have to be chosen optimally. By numerical tools we observed that the value of the loss function is approximately minimized with $(\alpha_*, b_*, c_*) = (0.343\,28, 1.825\,71, 2.000\,00)$. With these parameters we calculated an expected loss of 2.326 14.

Remark 1. To check the above results we performed various simulation studies. We observe that our rule with $(\alpha_*, b_*, c_*) = (0.343\,28, 1.825\,71, 2.000\,00)$ dominates the rule of [13], where $(\alpha_T, b_T, c_T) = (0.42, 1.95, 1.95)$, as well as the threshold rule presented in [9], where $(\alpha_G, b, c_G) = (0, b, 1.9469)$ and $b > 0$ is arbitrary, in the mean. For this threshold rule the expected rank is $1 + c_G/2 + 1/(c_G^2 - 1) = 2.3318$.

At the end of their paper, Assaf and Samuel-Cahn [1, Remark 6.2] mentioned that they tried a rule that would, in retrospect, be the n-period analogue of our rule, but with $b = c$ (as already mentioned, this rule has been investigated in [13], where a value of 2.330 44, with $\alpha_T = 0.42$ and $b_T = c_T = 1.95$, has been obtained in closed form). That is, they tried thresholds, where in the beginning fraction of time γ the additional condition of stopping only with relative rank 1 is imposed, and then a threshold rule is used for the remaining time. However, they left the thresholds at $\min\{2/(n-k+1), 1\}$ for $k = 1, \ldots, n$ (these thresholds had been known to be good approximations for thresholds of the optimal threshold rule for large enough n).

Assaf and Samuel-Cahn [1] reported that 'the improvement is however very small, and even with 10 000 simulations the standard error is too large to determine whether the improvement

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is real.' Also in our simulation runs (with 50 000 steps), we observed that the standard errors are large compared to the improvements obtained with our rule and the rule of [13], which also implies that a reliable comparison of rules can hardly be performed by means of a simulation analysis.

In addition, we inserted the parameters (α_T, b_T, c_T) and (α_G, b, c_G) into our closed-form expression obtained for $\mathbb{E}[R(\tau, X)]$. For the threshold rule, the difference between the number we obtain by inserting the corresponding parameters into the expression obtained for $\mathbb{E}[R(\tau, X)]$ and the number we obtain by means of $1 + c_G/2 + 1/(c_G^2 - 1)$ is smaller than 4×10^{-8}. For (α_T, b_T, c_T), we obtained 2.330 45 instead of 2.330 44 derived by [13]. That is, up to a numerical error, the expected losses obtained in [9] and [13] can also be replicated by means of the closed-form expression in this article.

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