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Geometric classification of simple graph algebras

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Abstract. Inspired by Franks' classification of irreducible shifts of finite type, we provide a short list of allowed moves on graphs that preserve the stable isomorphism class of the associated C^*-algebras. We show that if two graphs have stably isomorphic and simple unital algebras then we can use these moves to transform one into the other.

1. *Introduction*

In [**PS75**] Parry and Sullivan answered Bowen's question, 'what is the equivalence relation on non-negative integral matrices induced by flow equivalence?'. They did this by showing that flow-equivalence is generated by two moves on matrices, *strong shift equivalence* and what is now known as the *Parry–Sullivan move*. Using this result, Franks classified irreducible subshifts of finite type up to flow-equivalence [**Fra84**].

Cuntz and Krieger noticed a connection between C^*-algebras and subshifts of finite type, and in [**CK80**] they associated to any 0–1 valued square matrix A with no zero rows or columns a C^*-algebra $\mathcal{O}_A$. These algebras are now known as Cuntz–Krieger algebras. They reflect many of the properties of the matrix, for instance if A is an irreducible non-permutation matrix then $\mathcal{O}_A$ is simple, and they are an invariant for flow-equivalence in the following sense: if B and C are flow-equivalent irreducible matrices then $\mathcal{O}_B$ is stably isomorphic to $\mathcal{O}_C$. The latter observation raised the question whether flow-equivalence of irreducible matrices is equivalent to stable isomorphism of the associated C^*-algebras. Building on an idea by Cuntz, Rørdam answered this question in the negative in [**Rør95**]. Rørdam showed that the relation on (irreducible non-permutation 0–1 valued square) matrices A and B, given by $A \sim B$ if and only if $\mathcal{O}_A$ is stably isomorphic to $\mathcal{O}_B$, is generated not only by strong shift equivalence and the Parry–Sullivan move, but also by the relation

$$A \sim A_{_} = \left(\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ \hline 1 & 0 & & & & \\ 0 & 0 & & & A & \\ \vdots & \vdots & & & & \end{array}\right).$$

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The operation that takes A to $A_$ is called a *Cuntz splice*. It is known, due to the work of Franks, that two irreducible matrices A, B are flow-equivalent if and only if:

(1) $\operatorname{coker}(I - A) \cong \operatorname{coker}(I - B)$; and

(2) $\det(I - A) = \det(I - B)$.

In fact, given $\operatorname{coker}(I - A)$ we can easily construct the Smith normal form, S say, of $I - A$, and since $|\det(I - A)| = \det(S)$ we can reduce (2) to $\operatorname{sgn}\det(I - A) = \operatorname{sgn}\det(I - B)$. The importance of the relation $A \sim A_$ becomes clear once we notice that $\operatorname{coker}(I - A) \cong \operatorname{coker}(I - A_)$ but $\det(I - A) = -\det(I - A_)$. Hence, Rørdam's result states that the determinant of $I - A$ is not an obstruction for stable isomorphism of simple Cuntz–Krieger algebras, so they are classified by their K-theory, as $K_0(\mathcal{O}_A) \cong \operatorname{coker}(I - A)$.

In the reducible matrix/non-simple C^*-algebra case, Huang has classified reducible subshifts of finite type using the so-called K-web [**Hua94**]. Restorff used results of Boyle and Huang [**BH03, Boy02**] to prove that Cuntz–Krieger algebras with finitely many ideals are classified by ideal related K-theory, [**Res06**].

Graph algebras are a generalization of Cuntz–Krieger algebras. For some of the first steps towards this generalization see [**EW80, KPRR97**]. The simple graph algebras are either purely infinite or approximately finite dimensional, and so are classifiable by K-theory. Certain non-simple graph algebras have also been classified using K-theoretic methods, for instance the case of precisely one ideal is handled in [**ET10**], and some linear ideal latices are considered in [**ERR10**]. These results are proved using heavy classification machinery, and the fact that the class of graph algebras behaves nicely with respect to classification.

In this paper we take an approach that is less K-theoretic and much closer to how the subshifts of finite type were classified, as we study moves on graphs that preserve the stable isomorphism class of the graph algebras involved. Our focus will be on graphs with finitely many vertices, i.e. on unital graph algebras, since this makes it seem plausible that a finite number of moves can transform related graphs into one another. One might hope to find a (short) list of moves such that two graphs have stably isomorphic C^*-algebras if and only if we can transform one into the other using these moves, similar to the situation for flow-equivalence. Moves on special types of graphs have been studied in [**DS01, EFW81**] where the graphs must have either one or zero edges between any two vertices, and in [**ERS11**] where the graphs must have either infinitely many or zero edges between any two vertices. Leavitt path algebras, the algebraic cousins of the analytic graph C^*-algebras, have also been studied using flow-equivalence techniques and moves. In [**ALPS11**] the isomorphism question for purely infinite simple algebras is studied using Franks' invariant, and in [**AÁLP08**] an algebraic version of the gauge-invariant uniqueness theorem is used to provide moves on graphs that preserve isomorphism of Leavitt path algebras. (The author is grateful to Enrique Pardo for calling his attention to this work.)

Inspired by the moves used in [**ERS11**] to control infinite emitters, we will study arbitrary graphs and find a short list of moves that suffice if the graph algebras involved are simple and unital (for a precise statement, see Theorem 4.3). We stress that the moves (with one exception) preserve the stable isomorphism class of any graph algebra, but that we only prove they generate the equivalence relation given by stable isomorphism of the algebras in the class of unital simple graph algebras. As already noted, simple unital graph

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algebras are classified by K-theory, so the chronology here is backwards when compared to subshifts of finite type. Hopefully classification of graph algebras by moves will one day catch up and perhaps even overtake classification by K-theoretic means.

The basic moves we use to manipulate graphs are:

(S) remove a source if it is a regular vertex;
(I) in-split the graph (as described in Theorem 3.5);
(O) out-split the graph (as described in Theorem 3.3);
(R) reduction (as described in Proposition 3.2).

In- and out-splittings for graph algebras originate from [**BP04**]. The moves (S), (I), (O), and (R) alone are not enough: we also need a version of the Cuntz splice (see the Definition following Theorem 3.5) to fix the determinant of the adjacency matrices. Somewhat surprisingly, the Cuntz splice is only needed when we study graphs without any infinite emitters.

2. *Notation*

We use this short section to fix notation and give standard definitions. First and foremost: in this paper we follow the convention for graph algebras used in (for instance) [**BP04**], but not in (for instance) [**Rae05**].

We now define a few graph concepts, and give the definition of a graph algebra.

Definition. A graph G is a 4-tuple $G = (G^0, G^1, r, s)$ consisting of a set of vertices, G^0, a set of edges, G^1, and two maps $r, s\colon G^1 \to G^0$ specifying the range and source of any edge.

Definition. A path in a graph is a finite sequence of edges $e_1e_2\cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \ldots, n-1$. We extend the range and source maps to paths by putting $s(e_1e_2\cdots e_n) = s(e_1)$ and $r(e_1e_2\cdots e_n) = r(e_n)$.

A loop is a path $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ such that $s(\alpha) = r(\alpha)$, and a simple loop is a loop such that $r(\alpha_i) \neq r(\alpha)$ for any $i \neq n$.

We say that a vertex v supports a loop or is the base of a loop, if there is some loop α such that $s(\alpha) = v = r(\alpha)$.

Notation. Let G be a graph, and let u, v be vertices. We write $u \geq v$ if there is a path from u to v or $u = v$.

If H is a set of vertices we write $H \geq v$ if there is a vertex $w \in H$ such that $w \geq v$, and we write $v \geq H$ if there is some vertex $w \in H$ such that $v \geq w$.

Definition. Let G be a graph. A vertex $v \in G^0$ is called a source if does not receive any edges, i.e. $r^{-1}(v) = \emptyset$. We call a vertex $u \in G^0$ a sink if it does not emit any edges, that is, $s^{-1}(u) = \emptyset$. If a vertex $w \in G^0$ emits infinitely many edges, meaning $|s^{-1}(w)| = \infty$, we say that w is an infinite emitter.

A vertex is called singular if it is either an infinite emitter or a sink. A regular vertex is a vertex that is not singular.

Definition. Let $G = (G^0, G^1, r, s)$ be a graph. The graph C^*-algebra of G, denoted by $C^*(G)$, is the universal C^*-algebra generated by a set of mutually orthogonal

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projections $\{p_v \mid v \in G^0\}$ and a set $\{s_e \mid e \in G^1\}$ of partial isometries satisfying the following conditions:

- $s_e^* s_f = 0$ if $e, f \in G^1$ and $e \neq f$;
- $s_e^* s_e = p_{r_G(e)}$ for all $e \in G^1$;
- $s_e s_e^* \leq p_{s_G(e)}$ for all $e \in G^1$; and
- $p_v = \sum_{e \in s_G^{-1}(v)} s_e s_e^*$ for all $v \in G^0$ with $0 < |s_G^{-1}(v)| < \infty$.

We would like to discuss matrices as well as graphs so we set up some notation and give a brief description of the Smith normal form.

Notation. Throughout this paper we only consider matrices with entries in $\mathbb{N} \cup \{0, \infty\}$.

Notation. Given a finite set X and an $X \times X$ matrix A, let G_A be the graph with $G_A^0 = X$ and $|s^{-1}(x) \cap r^{-1}(y)| = A(x, y)$ for all $x, y \in X$. Given a graph G, we let A_G be the $G^0 \times G^0$ matrix with $A(u, v) = |s^{-1}(u) \cap r^{-1}(v)|$.

Every integer matrix A without zero entries can be diagonalized, using row and column operations, in such a way that the diagonal entries $d_1, d_2, \ldots, d_n$ (listed from top to bottom) satisfy the condition that d_i divides d_{i+1} and that d_1 is the greatest common divisor of all the entries in A. The diagonal matrix is called the Smith normal form of A, and the d_i are called elementary divisors of A or invariant factors of A. The term 'elementary divisor' is used in [**Fra84**], so we also use it here. There is an algorithm for computing the Smith form S of a matrix A and since A and S have the same cokernel this makes for convenient computations of cokernels of matrices and therefore of K-groups of graph algebras. This also illustrates why the elementary divisors played an important role in [**Fra84**] and why they will feature prominently here; see Theorem 8.5. For details about the Smith form see [**Rot02**, §9.4].

3. *The moves*

In this section we discuss the basic moves that we will use to manipulate graphs. Our first two moves are special cases of the slightly complicated move described in [**CG06**, Theorem 3.1]. One does not need the full power of this theorem to prove that these moves preserve Morita equivalence, as this is easily proved by directly writing down explicit isomorphisms, for instance by following the proof given by Crisp and Gow and noticing that there are many simplifications in the two special cases.

PROPOSITION 3.1. (Move (S)) *Let $G = (G^0, G^1, r_G, s_G)$ be a graph, and let $u \in G^0$ be a regular vertex that is a source. Define a graph $E = (E^0, E^1, r_E, s_E)$ by $E^0 = G^0 \backslash \{u\}$, $E^1 = G^1 \backslash s^{-1}(v)$, $r_E = r_G|_{E^0}$, and $s_E = s_G|_{E^0}$. Then $C^*(E)$ is (isomorphic to) a full corner of $C^*(G)$.*

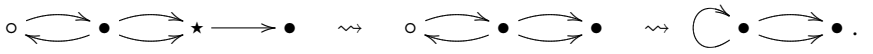
PROPOSITION 3.2. (Move (R)) *Let $G = (G^0, G^1, r_G, s_G)$ be a graph, and let $u \in G^0$ be a regular vertex such that $s_G^{-1}(u)$ and $s_G(r_G^{-1}(u))$ are one-point sets. Let v be the only vertex that emits to u and let f be the only edge u emits. Define a graph $E = (E^0, E^1, r_E, s_E)$ by $E^0 = G^0 \backslash \{u\}$,*

$$E^1 = (G^1 \backslash (r_G^{-1}(u)) \cup \{f\}) \cup \{[ef] \mid e \in r_G^{-1}(u)\},$$

and range and source maps that extend those of G and satisfy $r_E([ef]) = r_G(f)$ and $s_E([ef]) = s_G(e) = v$. If $r_G(f) \neq u$ then $C^*(E)$ is (isomorphic to) a full corner of $C^*(G)$.

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The following figure illustrates an application of move (R). First we use the move to remove the vertex $\star$, and then we use it to remove the vertex $\circ$. Here we use $\rightsquigarrow$ to denote an application of move (R).



Aside from these moves, we will also use the (proper) in- and out-splittings at one vertex from [**BP04**], which we record here for the convenience of the reader. The following are special cases of [**BP04**, Theorem 3.5 and Corollary 5.4].

THEOREM 3.3. (Move (O)) *Let $G = (G^0, G^1, r, s)$ be a graph, and let $v \in G^0$. Suppose that v is not a sink. Partition $s^{-1}(v)$ into a finite number, n say, of sets $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$. Define a graph G_{os} by:*

$$G^0_{\mathrm{os}} = (G^0 \backslash \{v\}) \cup \{v^1, v^2, \ldots, v^n\},$$
$$G^1_{\mathrm{os}} = (G^1 \backslash r^{-1}(v)) \cup \{e^1, e^2, \ldots, e^n \mid e \in G^1, r(e) = v\}.$$

For $e \notin r^{-1}(v)$ we let $r_{\mathrm{os}}(e) = r(e)$, for $e \in r^{-1}(v)$ we let $r_{\mathrm{os}}(e^i) = v^i$, $i = 1, 2, \ldots, n$. For $e \notin s^{-1}(v)$ we let $s_{os}(v) = s(e)$, for $e \in s^{-1}(v) \backslash r^{-1}(v)$ we let $s_{os}(e) = v^i$ if $e \in \mathcal{E}_i$, and for $e \in s^{-1}(v) \cap r^{-1}(v)$ we let $s_{os}(e^j) = v^i$ if $e \in \mathcal{E}$, $i, j = 1, 2, \ldots, n$.

If at most one of the $\mathcal{E}_i$ is infinite then $C^*(G) \cong C^*(G_{\mathrm{os}})$.

Remark 3.4. If we apply move (O) to G thus yielding a graph E, we say that E is an out-split of G. We also say that G is an out-amalgamation of E, and refer to using move (O) 'backwards' as out-amalgamating E into G.

THEOREM 3.5. (Move (I)) *Let $G = (G^0, G^1, r, s)$ be a graph, and let $v \in G^0$. Suppose that v is not a source. Partition $r^{-1}(v)$ into a finite number, n say, of sets $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$. Define a graph G_{is} by:*

$$G^0_{\mathrm{is}} = (G^0 \backslash \{v\}) \cup \{v^1, v^2, \ldots, v^n\},$$
$$G^1_{\mathrm{is}} = (G^1 \backslash s^{-1}(v)) \cup \{e^1, e^2, \ldots, e^n \mid e \in G^1, s(e) = v\}.$$

For $e \notin r^{-1}(v)$ we let $r_{is}(e) = r(e)$, for $e \in r^{-1}(v) \backslash s^{-1}(v)$ we let $r_{is}(e) = v^i$ if $e \in \mathcal{E}_i$, and if $e \in r^{-1}(v) \cap s^{-1}(v)$ then $r_{is}(e^j) = v^i$ for $e \in \mathcal{E}_i$, $i, j = 1, 2, \ldots, n$. For $e \notin s^{-1}(v)$ we let $s_{\mathrm{is}}(v) = s(e)$, for $e \in s^{-1}(v)$ we let $s_{\mathrm{is}}(e^i) = v^i$ for $i = 1, 2, \ldots, n$.

If v is a regular vertex then $C^*(G)$ is stably isomorphic to $C^*(G_{\mathrm{is}})$.

We will also need the Cuntz splice.

Definition. (Move (C)) Let $G = (G^0, G^1, r_G, s_G)$ be a graph and let $v \in G^0$ be a regular vertex that supports at least two simple loops (recall that a loop α is simple if $r(\alpha_i) \neq r(\alpha)$ for all $i < |\alpha|$). Define a graph $E = (E^0, E^1, r_E, s_E)$ by $E^0 = G^0 \cup \{u_1, u_2\}$, $E^1 = G^1 \cup \{e_1, e_2, f_1, f_2, h_1, h_2\}$; r_E and s_E extend r_G and s_G respectively, and satisfy

$$s_E(e_1) = v, \quad s_E(e_2) = u_1, \quad s_E(f_i) = u_1, \quad s_E(h_i) = u_2,$$

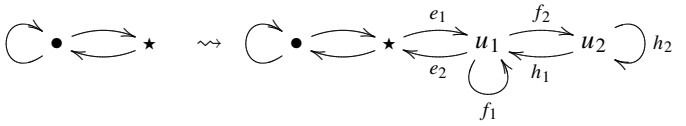
and

$$r_E(e_1) = u_1, \quad r_E(e_2) = v, \quad r_E(f_i) = u_i, \quad r_E(h_i) = u_i.$$

We say that E arises by applying move (C) to G at v.

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The following is an illustration of move (C) at vertex $\star$.



Remark 3.6. The Cuntz splice differs from the other moves in that we cannot show that it preserves stable isomorphism of the graph algebras. We can, however, show that it does not change the K-theory of the algebra. This is easily seen once we recall that the K-groups are computed as the kernel and cokernel of $I - A_G$; see [**DT02**]. If E arises from G by an application of move (C), $I - A_E$ can be transformed into

$$\left(\begin{array}{cc|ccccc} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & & & & \\ 0 & 0 & & I - A_G & & \\ \vdots & \vdots & & & & \end{array}\right)$$

using row and column operations, so neither the kernel nor the cokernel changes. Similarly, one can see that neither the ideal structure nor the order of K_0 is changed; the latter uses the fact that the Cuntz splice is done at a vertex that supports two simple loops.

4. *Some equivalence relations*

We will define four equivalence relations on graphs. First our two most important ones.

Definition. We define $\sim_M$ to be the smallest equivalence relation on graphs with finitely many vertices such that $G \sim_M E$ if G differs, up to isomorphism of graphs, from E by an application of one of the moves (S), (I), (O), or (R).

If $G \sim_M E$, we say that G is move-equivalent to E.

Definition. Given two graphs G and E, we say that G and E are C^*-equivalent, written $G \sim_{C^*} E$, if $C^*(G) \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators.

We have two more equivalence relations on graphs.

Definition. Given two graphs G and E, we say that G and E are K-equivalent, written $G \sim_K E$, if $FK(C^*(G)) \cong FK(C^*(E))$, where $FK(-)$ denotes the filtered K-theory. See [**ERR10**] for a definition of FK.

Remark 4.1. Suppose that $C^*(G)$ is simple (this is the case we are mainly interested in). Then

$$FK(C^*(G)) = (K_0(C^*(G)),\, K_0^+(C^*(G)),\, K_1(C^*(G))).$$

So $G \sim_K E$ means that $K_0(C^*(G))$ and $K_0(C^*(E))$ are isomorphic as ordered groups, and that $K_1(C^*(G))$ and $K_1(C^*(E))$ are isomorphic as groups.

Definition. We define $\sim_{M'}$ to be the smallest equivalence relation on graphs such that:

(1) $G \sim_{M'} E$ if $G \sim_M E$; and

(2) $G \sim_{M'} E$ if E arises by using move (C) on G.

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Remark 4.2. We have

$$G \sim_M E \implies G \sim_{C^*} E \implies G \sim_K E$$

and, by Remark 3.6,

$$G \sim_{M'} E \implies G \sim_K E.$$

If $C^*(G)$ and $C^*(E)$ are purely infinite simple then by the Kirchberg–Phillips theorem (for instance, [**Phi00**, Theorem 4.2.4]),

$$G \sim_K E \implies G \sim_{C^*} E.$$

We also define the corresponding equivalence relations on matrices.

Definition. Given two square matrices A and B, we say that A and B are C^*-equivalent, written $A \sim_{C^*} B$, if $G_A \sim_{C^*} G_B$. Likewise $A \sim_M B$ if $G_A \sim_M G_B$, $A \sim_K B$ if $G_A \sim_K G_B$, and $A \sim_{M'} B$ if $G_A \sim_{M'} G_B$.

Using the notation introduced in this section we can state our main result.

THEOREM 4.3. *Let G, E be graphs with simple unital algebras. If G has at least one singularity then*

$$G \sim_K E \iff G \sim_M E \iff G \sim_{C^*} E.$$

If G has no singularities then

$$G \sim_K E \iff G \sim_{M'} E \iff G \sim_{C^*} E.$$

We will prove this theorem in §9.

Remark 4.4. The case where the algebras are purely infinite and the graphs have no infinite emitters is essentially Franks' result [**Fra84**, Theorem 3.3] combined with Rørdam's result [**Rør95**, Theorem 6.5].

Remark 4.5. Suppose that G and E are graphs with simple unital algebras, that G has at least one singularity, and that $G \sim_K E$. By Theorem 4.3 we then have $G \sim_M E$. Since the maps showing that the moves (S), (R), (I) and (O) preserve stable isomorphism can be described explicitly, one could chase through the moves used to show $G \sim_M E$ and thereby obtain a chain of concrete isomorphism of full corners showing that $C^*(G)$ is stably isomorphic to $C^*(E)$. However, this is probably impractical in most cases.

5. *Derived moves*

We now describe two very usable moves that are in $\sim_M$ but not on our list, one involving infinite emitters and one involving regular vertices. We chose not to include them in the initial list of allowed moves to get a list as short and simple as possible. Though we will use these moves to study graphs with simple unital algebras, the use of the moves requires no such restriction. First we deal with regular vertices.

5.1. *Collapse.* This move brings us a lot closer to the full power of [**CG06**, Theorem 3.1]. We prove it in two steps. First we prove that we can handle the case where the vertex we collapse receives edges from more than one vertex.

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LEMMA 5.1. *Let $G = (G^0, G^1, r_G, s_G)$ be a graph with finitely many vertices, and let $v \in G^0$ be a regular vertex which is not a source and which emits precisely one edge, f_0 say. Define a graph $E = (E^0, E^1, r_E, s_E)$ by $E^0 = G^0 \backslash \{v\}$,*

$$E^1 = (G^1 \backslash (r^{-1}(v) \cup s^{-1}(v))) \cup (\{[ef_0] \mid e \in r^{-1}(v)\}),$$

where the range and source maps extend those of G and satisfy $r_E([ef_0]) = r_G(f_0)$ and $s_E([ef]) = s_G(e)$. If f_0 is not a loop of length one then $G \sim_M E$.

Proof. If v receives from only one vertex then this is move (R). So let us assume that v receives from the vertices $\{u_1, u_2, \ldots, u_n\}$, where $2 \leq n < \infty$. Define $\mathcal{E}_i = r^{-1}(v) \cap s^{-1}(u_i)$ for $i = 1, 2, \ldots, n$. Since v is regular we can use move (I) at v according to the partition $\{\mathcal{E}_i\}$. This will yield a graph G_{in} with v replaced by n vertices $v_1, v_2, \ldots, v_n$ each receiving from one of the vertices v received from and each emitting one edge to $r(f_0)$. We can now use move (R) to collapse the v_i, and this will yield the graph E. Hence $G \sim_M E$. □

We now deal with the general case.

THEOREM 5.2. (Collapse) *Let $G = (G^0, G^1, r_G, s_G)$ be a graph with finitely many vertices, and let $v \in G^0$ be a regular vertex which does not support a loop of length one. Define a graph $E = (E^0, E^1, r_E, s_E)$ by $E^0 = G^0 \backslash \{v\}$,*

$$E^1 = (G^1 \backslash (r^{-1}(v) \cup s^{-1}(v))) \cup (\{[ef] \mid e \in r^{-1}(v),\ f \in s^{-1}(v)\}),$$

where the range and source maps extend those of G, and satisfy $r_E([ef]) = r_G(f)$ and $s_E([ef]) = s_G(e)$. We have $G \sim_M E$.

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the edges with source v. If $n = 1$ then we appeal to Lemma 5.1, otherwise we define $\mathcal{E}_i = \{e_i\}$, for $i = 1, 2, \ldots, n$. We can use move (O) at v according to the partition $\{\mathcal{E}_i\}$ of $s^{-1}(v)$. This gives a graph, G_{os} say, with $G^0_{\text{os}} = G^0 \backslash \{v\} \cup \{v_1, v_2, \ldots, v_n\}$,

$$G^1_{\text{os}} = (G^1 \backslash (r^{-1}(v) \cup s^{-1}(v))) \cup \{f_i \mid f \in r^{-1}(v), i = 1, \ldots, n\} \cup \{\bar{e}_i \mid e = e_i\},$$

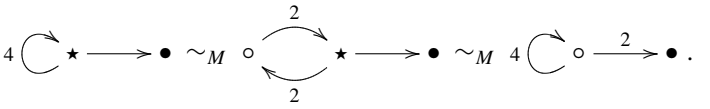
with range and source map that agree with those of G when the edge is in G^1 but with

$$r_{\text{os}}(f_i) = v_i, \quad r_{\text{os}}(\bar{e}_i) = r_G(e_i), \quad s_{\text{os}}(f_i) = s_G(f) \quad \text{and} \quad s_{\text{os}}(\bar{e}_i) = v_i.$$

Note that there are no edges between the v_i, that none of them support a loop of length one and that each of them emits exactly one edge. We now use Lemma 5.1 to collapse each of the v_i, and this will yield the graph E. Thus,

$$G \sim_M G_{\text{os}} \sim_M E.$$ □

Example 5.3. As an application of Theorem 5.2 we see that



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A number by an edge indicates that there are that many edges, so for instance in the leftmost graph there are four loops at $\star$. This simplifies drawings, especially when dealing with infinite emitters.

The two leftmost graphs are seen to be move-equivalent by collapsing the vertex $\circ$ in the middle graph. The two rightmost are equivalent by collapsing $\star$ in the middle graph. That the two outer graphs are C^*-equivalent was proved in [**ET10**, Example 5.2]. We return to this specific example in the final section.

5.2. Move (T). We now discuss a move that pertains to infinite emitters. The idea is to make an infinite emitter emit infinitely to as many vertices as possible. We prove that the move is in $\sim_M$, which implies that it preserves stable equivalence of graph algebras. It has been proved in [**ERS11**] that it in fact preserves isomorphism of graph algebras.

THEOREM 5.4. (Move (T)) *Let $G = (G^0, G^1, r_G, s_G)$ be a graph, let $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ be a path in G, and suppose that $A_G(s(\alpha_1), r(\alpha_1)) = \infty$. Let $E = (E^0, E^1, r_E, s_E)$ be the graph with vertex set G^0, edge set*

$$E^1 = G^1 \cup \{\alpha^m \mid m \in \mathbb{N}\},$$

and range and source maps that extend those of G, and have $r_E(\alpha^m) = r_G(\alpha)$ and $s_E(\alpha^m) = s_G(\alpha)$. Then $G \sim_M E$.

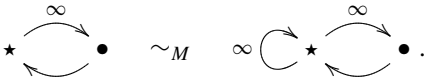
Proof. We proceed by induction on the length of α. If $|\alpha| = 1$ then since $A_G(s(\alpha_1), r(\alpha_1)) = \infty$, G and E are isomorphic.

Suppose that $|\alpha| = 2$ and let $v = s(\alpha_2)$. If α_2 is a loop of length one then $E \cong G$, so there is nothing to prove. We will therefore assume that $r(\alpha_2) \neq v$. If α_2 is the only edge v emits, we can collapse v (as in Theorem 5.2) in both E and G, and get isomorphic graphs. So we will assume that v emits at least two edges. Partition $s_G^{-1}(v)$ as $\mathcal{E}_1 = \{\alpha_2\}$, $\mathcal{E}_2 = s_G^{-1}(v)\backslash\{\alpha_2\}$. We can out-split G at v according to this partition and get a graph G_{os} where v is replaced by two vertices v_1 and v_2. The first will emit only a copy of α_2, whereas the second will emit copies of every other edge v emitted. They will both receive copies of everything v received. Arguing as above, we can add infinitely many edges from $s(\alpha)$ to $r(\alpha)$ without changing move-equivalence class. Doing so does not affect the edges going into or out of v_1 and v_2. Hence we can out-amalgamate them back to v (i.e. use move (O) backwards). This yields a graph isomorphic to E, and since we have only used our moves, we have $G \sim_M E$.

Suppose now that $k > 2$, that we have proved the theorem for paths of length less than k, and that α is a path of length k. Using the induction hypothesis on the path $\alpha_1\alpha_2 \cdots \alpha_{k-1}$, we can, without changing move-equivalence class, add infinitely many edges from $s(\alpha_1)$ to $s(\alpha_{k-1})$. Hence there will be a path, β, of length two from $s(\alpha)$ to $r(\alpha)$ with $A_G(s(\beta_1), r(\beta_1)) = \infty$. Using the induction hypothesis on β, we can add infinitely many edges from $s(\beta) = s(\alpha)$ to $r(\beta) = r(\alpha)$. Using the induction one final time, we remove the edges we added from $s(\alpha)$ to $s(\alpha_{k-1})$. Thus we have constructed a graph isomorphic to E using only allowed moves. □

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For nice applications of move (T) see [**ERS11**]. As a simple example to see how it works consider:



The equivalence follows from Theorem 5.4, since there are infinitely many edges from $\star$ to $\bullet$, and an edge from $\bullet$ to $\star$. Using Theorem 5.2 to collapse $\bullet$, we see that both graphs are move-equivalent to



Hence, the algebras are all stably isomorphic to $\mathcal{O}_\infty$.

6. *Reductions (on graphs)*

The purpose of this section is to show the following proposition.

PROPOSITION 6.1. *Let G be a graph with finitely many vertices such that $C^*(G)$ is purely infinite simple. There is some graph E with finitely many vertices, no sinks, no sources, and satisfying the following:*

(i) *every vertex in E supports a loop of length one;*
(ii) *any two regular vertices in E are connected by a path of regular vertices;*
(iii) *if $u \in E$ is an infinite emitter, then u emits infinitely many edges to every vertex in E; and*
(iv) *$E \sim_M G$.*

First we show that we do not have to worry about sinks and sources.

LEMMA 6.2. *Let G be a graph with finitely many vertices. If $C^*(G)$ is purely infinite simple then $G \sim_M E$, where E is some graph with no sinks or sources.*

For the following proof we need to know about the ideal structure of graph algebras. For this, and the notions of hereditary and saturated sets, we refer the reader to [**BHRS02**].

Proof of Lemma 6.2. Since $C^*(G)$ is purely infinite simple, every vertex in G connects to a loop [**DT05**, Remark 2.16]. Hence G has no sinks. Let $u \in G^0$ be a source and put $H = \{v \in G^0 \mid u \geq v\} \backslash \{u\}$. Because u is a source H is hereditary, and since $C^*(G)$ is simple the saturation of H must be all of G. As no singular vertices are added when taking saturations, we must have that u is regular. We can now use move (S) to remove the sources from G. The resulting graph will have fewer vertices than G although it may have more sources. Again we see that none of these new sources can be infinite emitters. As there are only finitely many vertices in G, if we just keep removing sources we will eventually reach a graph with no sources. This graph, E say, has no sinks and, since we have only used move (S), we must have $G \sim_M E$. □

We now consider how the vertices interconnect.

LEMMA 6.3. *If G is a graph with finitely many vertices, no sinks, no sources and $C^*(G)$ is simple, then any vertex in G can reach any other vertex.*

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Proof. Since G has no sources and only finitely many vertices, every vertex can be reached by some loop. To see this, fix some vertex $u_0 \in G^0$. Either u_0 is on a loop or it is not. If it is we are done, so suppose it is not. As u_0 is not a source there is some vertex $u_1 \in G^0 \backslash \{u_0\}$ with an edge to u_0. Either u_1 is on a loop or it is not. If it is not then since u_1 is not a source there is some vertex $u_2 \in G^0 \backslash \{u_0, u_1\}$ that points to u_1. Continuing, we must eventually reach a vertex that is on a loop since there are only finitely many vertices. By [**DT05**, Corollary 2.15], G is cofinal because $C^*(G)$ is simple; in particular, every vertex in G can reach every loop in G. Let $u, v \in G^0$. There is some loop μ in G such that $\mu \geq u$ and so

$$v \geq \mu \geq u.$$

Hence $v \geq u$. □

LEMMA 6.4. *If G is a graph with finitely many vertices, no sinks, no sources and $C^*(G)$ is simple, then $G \sim_M E$ for some graph E with no more vertices than G, no sinks, no sources and where every regular vertex supports a loop of length one.*

Proof. Let $v \in G^0$ be a regular vertex that does not support a loop of length one. Then we can collapse v as in Theorem 5.2. Doing this will yield a graph F with one vertex less than G and with $F \sim_M G$. Repeating this process we will eventually get rid of all the regular vertices that do not support a loop of length one. □

Proof of Proposition 6.1. By Lemmas 6.4 and 6.2 we may assume that every regular vertex in G supports a loop of length one, and that G has neither sinks nor sources. Let $u \in G^0$ be an infinite emitter and let $v \in G^0$ be such that u emits infinitely many edges to v. Since $v \geq u$, by Lemma 6.3, we can use Theorem 5.4 to add infinitely many edges from u to itself. So without changing move-equivalence class, we can assume that every vertex in G supports a loop of length one.

Suppose that we are given three distinct vertices u, v, w such that u is regular, v is singular, there is an edge from u to v and one from v to w. Fix a loop of length one based at v, f say, and an edge from v to w, e say. Partition $s^{-1}(v)$ as $\mathcal{E}_1 = \{f, e\}$ and $\mathcal{E}_2 = s^{-1}(v) \backslash \{e, f\}$. Using move (O) on this partition, we replace v by the regular vertex v_1 that supports a loop of length one such that there is an edge from u to v_1 and one from v_1 to w, and the vertex v_2 which is just like v was before except it does not emit e. The vertex v_1 can only reach the vertices u can. Hence, repeating this process a finite number of times will eventually lead to a graph where all the regular vertices are connected by paths of regular vertices.

We have now shown that we can find a graph F with finitely many vertices, no sinks or sources and satisfying (i), (ii) and (iv). Let $u \in F^0$ be an infinite emitter and let v be a vertex to which u emits infinitely many edges. Let $w \in F^0$ be any vertex. By Lemma 6.3, $u \geq w$, so, using move (T) (Theorem 5.4), we can add infinitely many edges from u to w. Doing the same for every infinite emitter and every $w \in F^0$, we obtain a graph E with finitely many vertices, no sinks or sources, and satisfying (i), (ii), (iii) and (iv). □

7. *Moves on matrices*

We now turn to the matrices. At the heart of Franks' classification of irreducible subshifts [**Fra84**] is simple matrix manipulation. Since we wish to follow Franks, we

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now consider how the moves we have listed can be used to manipulate adjacency matrices. Reordering the rows (and then the columns accordingly) in a matrix does not change the move-equivalence class, so we may assume that the rows which contain ∞ are at the bottom. The non-zero rows with no ∞ entries (i.e. those that correspond to a regular vertex) will be called 'regular' rows; non-'regular' rows will be called 'singular' rows. Similarly for the columns.

While the matrix manipulations described in this section do not depend on the associated C^*-algebras being simple, it does depend on the matrices having a certain form. So unlike our moves on graphs, the utility of these matrix manipulations in the non-simple case is probably limited. Nevertheless, we optimistically state the results in as much generality as possible.

Let A be a square matrix and let m be the number of 'regular' rows in A. Define an $n \times n$ matrix with m regular rows by

$$J_{nm} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \infty & \infty & \cdots & \infty & \infty & \cdots & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \cdots & \infty & \infty & \cdots & \infty \end{pmatrix}.$$

Proposition 6.1 shows that if $C^*(G_A)$ is purely infinite simple we can find an $n \times n$ matrix B of the form

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} & b_{1(m+1)} & \cdots & b_{1n} \\ b_{12} & b_{22} & \cdots & b_{2m} & b_{2(m+1)} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} & b_{m(m+1)} & \cdots & b_{mn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

such that

$$A \sim_M J_{nm} + B,$$

and the top-left $m \times m$ corner of $J_{nm} + B$, as well as all of it, is irreducible.

We now show that we can manipulate matrices in a way similar to [**Fra84**, Corollary 2.2]. Due to the asymmetry between rows and columns we will need to do two proofs. This was not an issue for Franks, as A is flow-equivalent to B if and only if A^T is flow-equivalent to B^T. For us, there is a big difference between rows and columns, as seen for instance in the difference between out- and in-splittings, or the fact that we can remove (regular) sources, but not sinks.

We first handle rows.

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LEMMA 7.1. *Let A be a square matrix with no zeros on the diagonal. Let i, j be two distinct indices. If $a_{ij} \neq 0$ and row j contains no ∞, then*

$$A \sim_M \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & a_{i2} + a_{j2} & & a_{ij} + a_{jj} - 1 & & a_{in} + a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}.$$

The matrix on the right is A with the jth row added to the ith row but where we subtract 1 at the (i, j)th entry. Note that this matrix has no zeros on the diagonal, and if the top-left corner was irreducible in A then it is still irreducible. The same is true for all of A and all of the right-hand side.

Proof. In terms of graphs, what we are doing is choosing two vertices in the graph G_A, u and v say, with an edge, f say, from u to v. A new graph, E, is then formed by removing f but adding for each edge $e \in s^{-1}(v)$ an edge $\bar{e}$ with $s(\bar{e}) = u$ and $r(\bar{e}) = r(e)$. We claim that $E \sim_M G_A$ if v is not an infinite emitter.

Since v is not singular, we can use move (I) on it. Partition $r^{-1}(v)$ as $\mathcal{E}_1 = \{f\}$ and $\mathcal{E}_2 = r^{-1}(v) \backslash \{f\}$. As there is a loop of length one based at v, $\mathcal{E}_2$ is not empty. In-splitting according to this partition replaces v with two new vertices, v_1 and v_2. Vertex v_1 only receives one edge, and that edge comes from u; vertex v_2 receives the edges that v received except f and also receives one edge from v_1 for each loop of length one based at v. Both vertices emit copies of the edges that v emitted, and do so in such a way that there is no loop of length one based at v_1: instead it emits one edge to v_2 for each loop of length one based at v. Collapsing v_1 as in Theorem 5.2 yields E. □

Above we proved move-equivalence of two matrices, hence their algebras are stably isomorphic. It follows from [**AÁLP08**, Corollary 2.5] that if the associated graphs are row-finite then the algebras are isomorphic. We now turn to columns.

LEMMA 7.2. *Let A be a matrix with no zeros on the diagonal. Let i, j be two distinct indices. If $a_{ji} \neq 0$, then*

$$A \sim_M \begin{pmatrix} a_{11} & \cdots & a_{1i} + a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} + a_{2j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & a_{ji} + a_{jj} - 1 & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} + a_{nj} & \cdots & a_{nn} \end{pmatrix}.$$

The matrix on the right is A with the jth column added to the ith column but where we subtract 1 at the (j, i)th entry. Note that the matrix on the right has no zeros on the diagonal, and that if the top-left corner was irreducible in A then it is still irreducible on the right-hand side. The same holds for all of A and all of the right-hand side.

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Proof. In terms of graphs, what we are doing is choosing two vertices in the graph G_A, u and v say, with an edge, f say, from v to u. A new graph, E, is then formed by removing f but adding for each edge $e \in r^{-1}(v)$ an edge $\bar{e}$ with $s(\bar{e}) = s(e)$ and $r(\bar{e}) = u$. We claim that $E \sim_M G_A$.

Partition $s^{-1}(v)$ as $\mathcal{E}_1 = \{f\}$ and $\mathcal{E}_2 = s^{-1}(v)\backslash\{f\}$. Since there is a loop of length one based at v, $\mathcal{E}_2$ is not empty, so we can use move (○). Doing so yields a graph just like G but where v is replaced by two vertices, v_1 and v_2. Vertex v_1 receives a copy of everything v did including an edge from v_2 for each loop of length one based at v; it emits only one edge, and that edge has range u. Vertex v_2 also receives a copy of everything v did, and it emits everything v did, except f. Since v_1 is regular and not the base of a loop of length one, we can collapse it (Theorem 5.2) and thus obtain E. □

Following Franks, we use the above to do matrix operations (see [**Fra84**, Theorem 2.4]).

PROPOSITION 7.3. *Suppose that we are given an irreducible matrix $J_{nm} + B$ where the top-left $m \times m$ corner is irreducible and the last $n - m$ rows of B are zero. If we form B' from B by adding any column to any other column, or by adding a non-zero row to any other non-zero row, then $J_{nm} + B \sim_M J_{nm} + B'$.*

Clearly adding a zero row to any other row changes nothing, and neither does adding any row to one of the zero rows since we add the matrix J_{nm} to B when we form A, so every entry of the last $n - m$ rows will be ∞. We have added the requirement, since there is very little gained by not doing so, and, more importantly, this is the way we intend to use it.

Proof of Proposition 7.3. First notice that we can add a 'singular' column to any other column by Lemma 7.2.

Let $A = J_{nm} + B$ and suppose that we are given two distinct indices i, j such that $j \leq m$. Let B' be the matrix obtained by adding the jth column of B to the ith. If a_{ji} is non-zero then $J_{nm} + B \sim_M J_{nm} + B'$ by Lemma 7.2. Hence applying Lemma 7.2 shows that for any two distinct indices l, k we can subtract row l in B from row k in B without changing move-equivalence class, provided that the resulting matrix is non-negative and $b_{kl} > 0$ after the subtraction.

Any two vertices in G_A are connected, so we can find a sequence of distinct indices $j = i_0, i_1, \ldots, i_k = i$ such that the entries $a_{i_l i_{l+1}}$ all are non-zero. Since $a_{j i_1}$ is non-zero we can add column i_1 to column j (in B). The new column j will be non-zero at entry $a_{j i_2}$, so we can add column i_2 to the new column j. Continuing in this way, we will eventually add column i to column j. The version of column j we now have is a sum of the columns $j = i_0, i_1, \ldots, i_k = i$ from B. We can now subtract column i_{k-1} from column j, then we subtract column i_{k-2} and so on. In the end, we will only have added column i to column j, as we wanted.

The proof for rows is very similar. The only change is that we need to use the fact that the regular vertices in G_A are connected, so we only work with non-zero rows of B. But that is the case, since the top-left $m \times m$ corner of A is irreducible. □

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8. *The standard form*

We now, following Franks, put our matrices in a standard form. There are two cases, one where the matrices have 'singular' rows, and one where they do not.

8.1. *Graphs with at least one singularity.* In this subsection we consider the case where we are given a matrix A of the form $A = J_{nm} + B$ with $m \neq n$ and the last $n - m$ rows of B filled with zeros. We copy the approach Franks takes in the last part of §2 and all of §3 in [**Fra84**]. Like Franks does, we first show that we can enlarge B without changing move-equivalence class. Moreover, we can, unlike Franks, get the determinant of the top-left regular corner of the enlarged matrix to be 0; this is why we will not need the Cuntz splice later. The key point is that we can adjoin almost any row to A without changing the $\sim_M$-class.

LEMMA 8.1. *Suppose that we are given an irreducible matrix $A = J_{nm} + B$ where the top-left $m \times m$ corner is irreducible, $m \neq n$ and the last $n - m$ rows of B are filled with zeros. Let $x_1, x_2, \ldots, x_n$ be non-negative integers that are not all zero and let j be given such that $m < j \leq n$. If*

$$A' = \begin{pmatrix} x_j & x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & \cdots & x_n \\ a_{1j} & a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{mj} & a_{m1} & \cdots & a_{m(j-1)} & a_{mj} & a_{m(j+1)} & \cdots & a_{mn} \\ \infty & \infty & \cdots & \infty & \infty & \infty & \cdots & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \cdots & \infty & \infty & \infty & \cdots & \infty \end{pmatrix},$$

then $A \sim_M A'$.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of G_A, the labelling chosen so that the ith row and column of A describe the edges respectively going out of and into u_i. Since $j > m$, u_j is an infinite emitter, and as $A = J_{mn} + B$, u_j emits infinitely many edges to every vertex in G_A. Partition $s^{-1}(u_j)$ into two sets, one containing x_1 edges to u_1, x_2 edges to u_2 and so on, the other containing everything else. Out-splitting according to this partition produces a graph with adjacency matrix A'. Hence $A' \sim_M A$. □

PROPOSITION 8.2. *Suppose that we are given an irreducible matrix $A = J_{nm} + B$ where the top-left $m \times m$ corner is irreducible, $m \neq n$ and the last $n - m$ rows of B are filled with zeros. For any $k \in \mathbb{N}$ such that $k \geq n + 2$, we can find a $k \times k$ matrix C satisfying the following:*

(i) *all the entries in the first $k - (n - m)$ rows of C are non-negative;*
(ii) *at least one of the entries in C is 1;*
(iii) *all the entries in the last $n - m$ rows of C are zero; and*
(iv) $J_{nm} + B \sim_M J_{k(k-(n-m))} + C$.

Proof. If $m = 0$, we can use Lemma 8.1 to replace A with an irreducible $(n + 1) \times (n + 1)$ matrix with one 'regular' row, and the regular top-left 1×1 corner irreducible, i.e. non-zero.

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Suppose now that $m > 0$ and that $k \geq n + 2$ is given. Since A is irreducible, there is at least one non-zero entry of B. Using that entry and row and column additions (Proposition 7.3), we can find a matrix $A' = J_{nm} + B'$ such that $A' \sim_M A$ and every entry in the first m rows of B is non-zero and every entry in the last $n - m$ rows is 0. We can now use Lemma 8.1 to add rows of the form $(2, 1, 1, 1, \ldots, 1, 2, 1, 1, \ldots, 1)$, where the second 2 is at the $(m + 1)$th position, to A' until we have a $k \times k$ matrix A''. Since $k \geq n + 2$ we add at least one such row (in the case where the original matrix had $m \neq 0$, we actually add at least two such rows). We have that $A'' \sim_M A$, that $A'' = J_{k(k-(m-n))} + C$ for some matrix C with all the entries in the first $k - (n - m)$ rows non-zero, all the entries in the last $n - m$ rows zero, and one row that looks like $(1, 1, \ldots, 1, 1, 2, 1, 1, \ldots, 1)$, with the 2 in a singular column. In particular, C contains a 1. □

Next we show that we can get a column of 1s.

PROPOSITION 8.3. *Suppose that we are given a matrix $A = J_{nm} + C$ where $m \neq n$. Suppose that all the entries in the first m rows of C are non-zero, all the entries in the last $n - m$ rows are zero, and C has 1 in at least one entry. There exists an $(n + 2) \times (n + 2)$ matrix D such that:*

(i) *the last $(n + 2) - (m + 2)$ rows of D are identically zero;*
(ii) *any entry in any other row is strictly positive;*
(iii) *$d_{i1} = 1$ for all $1 \leq i \leq m + 2$;*
(iv) *the determinant of the top-left $(m + 2) \times (m + 2)$ corner of D is zero; and*
(v) *$J_{nm} + C \sim_M J_{(n+1)(m+1)} + D$.*

Proof. We can use the algorithm described by Franks in [**Fra84**, Proposition 2.9] to obtain an $n \times n$ matrix C' with the last $n - m$ rows identically zero, any entry in any other row non-zero, a column consisting only of 1s in the first n places, and such that $J_{nm} + C \sim J_{nm} + C'$. If the column of 1s described the edges going into a regular vertex, we would just be a permutation away from having it be the first column. To ensure that this happens, we use Lemma 8.1. Let j_0 be the index of the column consisting only of 1s and let $j = \max\{j_0, m + 1\}$. Then $j > m$ so we can use Lemma 8.1 to add the row $(2, 1, 1, \ldots, 1, 2, 1, \ldots, 1)$ to $A' = J_{nm} + C'$, thereby getting a matrix $A'' = J_{(n+1)(m+1)} + C''$. If $j = j_0$ then we now have that the first column of C'' consists entirely of 1s. If $j = m + 1$ then some regular column consists entirely of 1s, since the added 2 in C'' will be in a singular column. So after a permutation the first column of C'' consists entirely of 1s. Note that C'' now satisfies (i), (ii), (iii) and (v), and that the top row of C'' is $(1, 1, \ldots, 1, 2, 1, \ldots, 1)$ with the 2 in a singular column. Using Lemma 8.1 we can add a row of the form $(3, 1, 1, \ldots, 1, 3, 1, \ldots, 1)$ to A'' resulting in $A''' = J_{(n+2)(m+2)} + D'$, where D' satisfies (i), (ii), and (v), and the regular part of the two top rows of D' are $(2, 1, \ldots, 1)$ and the regular part of the second column of D consists of 1s. Hence the top-left $(m + 2) \times (m + 2)$ corner of D' has two identical rows, and therefore has determinant zero. Preforming a permutation on D' that interchanges the second and first rows, we get a matrix D that satisfies (i), (ii), (iii), (iv), and (v). □

We can now put the matrices in a canonical form. We argue by induction on the rank of the matrix.

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LEMMA 8.4. *Suppose that we are given an $n \times n$ matrix C where all the entries in the first m rows are non-zero, all the entries in the last $n - m$ rows are zero, and the first column is $(d, d, \ldots, d, 0, 0, \ldots, 0)^T$, where d is the* gcd *of the non-zero entries of C. If C is rank one, then*

$$J_{mn} + C \sim J_{mn} + D,$$

where D is the $n \times n$ matrix where all the entries in the first m rows are d, and all the entries in the last $n - m$ rows are zero.

Proof. Since C is rank one and integer valued, each column is a multiple of $(d, d, \ldots, d, 0, 0, \ldots, 0)^T$. We can use column subtraction (Proposition 7.3) to form D. □

THEOREM 8.5. *Suppose that we are given a matrix $A = J_{nm} + C$ where $m \neq n$. Suppose that all the entries in the first m rows of C are non-zero, all the entries in the last $n - m$ rows are zero, and the first column is $(d, d, \ldots, d, 0, 0, \ldots, 0)^T$, where d is the* gcd *of the non-zero entries of C. Let $d_1, d_2, \ldots, d_k$ denote the non-zero elementary divisors of the first m rows of C, ordered so that d_i is a factor of d_{i+1}. Define*

$$B = \left(\begin{array}{ccccccccc}
0 & 0 & 0 & \cdots & 0 & d_k & d_k & \cdots & d_k \\
d_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & d_{k-1} & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & d_k & d_k & \cdots & d_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & d_k & d_k & \cdots & d_k \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{array}\right).$$

There are m rows above the second line and $n - m$ below it.

If the 'regular' top-left corner of $A = J_{nm} + C$ has determinant 0*, then $A \sim_M J_{nm} + B$.*

Proof. We mimic Franks' techniques and do the proof by induction on the rank of C. The case of rank one is Lemma 8.4.

Let us assume that the rank of C is at least 2. Since $\det(C) = 0$, C has at least three rows. Doing exactly what Franks does in the proof of [**Fra84**, Proposition 3.1], we get a matrix B' of the form

$$B' = \begin{pmatrix}
0 & * & \cdots & * \\
d_1 & 0 & \cdots & 0 \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix},$$

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such that $J_{nm} + B \sim_M J_{nm} + B'$. The matrix of $*$ entries has smaller rank than B', the determinant of its top-left 'regular' corner is 0, and the first row will be $(d', d', \ldots, d', 0, 0, \ldots, 0)$, where d' is the gcd of the remaining non-zero entries. So, by induction, we can put it in the desired form. □

8.2. *Regular graphs.* In this subsection we consider graphs (and corresponding matrices) with no infinite emitters. That is, matrices of the form $A = J_{nn} + B = I + B$ for some matrix B. Since there are no 'singular' rows (or columns) Proposition 7.3 gives us the following theorem.

THEOREM 8.6. *Suppose that B is a non-negative square integer matrix and $A = I + B$ is irreducible. If B' is obtained from B by adding any row to a different row or adding any column to a different column then $A \sim_M I + B'$.*

This is parallel to [**Fra84**, Theorem 2.4]. Since the proofs Franks gives really only depend on matrix operations, we get the following two results [**Fra84**, Corollary 2.6 and Theorem 3.3] by copying proofs.

THEOREM 8.7. *Let A be a non-negative integer matrix such that $C^*(G_A)$ is purely infinite simple. There exists an $N \in \mathbb{N}$ such that for all $n > N$, there is a strictly positive $n \times n$ integer matrix B with $A \sim_M I + B$, and* $\det(I - A) = \det(-B)$

Proof. We copy Franks' proof to get $A \sim_M I + B$. The determinant condition follows since A and $I + B$ are not only move-equivalent but also flow-equivalent. □

THEOREM 8.8. *Suppose that B is an $n \times n$, $n > 1$, strictly positive matrix with elementary divisors $d_1, d_2, \ldots, d_n$, each d_i a factor of d_{i+1}. Let $A = I + B$ and let $k = \operatorname{rank}(-B)$. If* $\det(-B) \leq 0$ *then $A \sim_M I + B'$ where*

$$B' = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & d_k & d_k & \cdots & d_k \\
d_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & d_{k-1} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & d_k & d_k & \cdots & d_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & d_k & d_k & \cdots & d_k
\end{pmatrix}.$$

9. *Geometric classification*

We can now prove our main theorem. We begin by giving short proofs of two lemmas that are certainly well known, though the author has been unable to find a good reference.

LEMMA 9.1. *Suppose that G and E are finite graphs. If $G \sim_K E$ and $C^*(G)$ (and hence $C^*(E)$) is simple, then $C^*(G)$ and $C^*(E)$ are either both AF or both purely infinite.*

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Proof. A simple graph algebra is either purely infinite or AF [**DT05**, Remark 2.16]. The positive cone of the K_0-group will tell us which case we are in: if it is all of K_0, then the algebras are purely infinite; if it is not, then they are AF. □

LEMMA 9.2. *Suppose that G and E are finite graphs. If $G \sim_K E$ then G and E have the same number of singularities.*

Proof. The difference between the rank of the free abelian part of the K_0-group and the rank of the K_1-group is the number of singularities. See [**DT02**, Theorem 3.1]. □

We now can now prove our main theorem.

Proof of Theorem 4.3. We clearly have $G \sim_{C^*} E \implies C \sim_K E$ and $G \sim_M E \implies G \sim_{C^*} E$, and in the purely infinite simple case we also have $G \sim_{M'} E \implies G \sim_{C^*} E$. Hence, the non-trivial part of the theorem is to show that only if the algebra is purely infinite simple can it have no singularities, and the implications $G \sim_K E \implies G \sim_M E$ when G has at least one singularity and $G \sim_K E \implies G \sim_{M'} E$ when G has no singularities. Suppose that we are given two graphs G and E with $G \sim_K E$.

By Lemma 9.1 either both $C^*(G)$ and $C^*(E)$ are purely infinite or they are both AF. We deal with the AF case first. Since $C^*(G)$ is AF, G has no loops [**DT05**, Remark 2.16], and since $C^*(G)$ is unital, G has finitely many vertices. Hence, G has a sink. Let $v \in G^0$ be a sink and let $H = \{v\}$. Since H is hereditary and $C^*(G)$ is simple, all vertices in G must be in the saturation of H, so G has precisely one sink and no infinite emitters. The same argument shows that E has exactly one sink and no infinite emitters. Using again the fact that G and E have finitely many vertices, we see by repeated applications of move (S) that

$$G \sim_M \bullet \sim_M E.$$

Suppose now that $C^*(G)$ (and so $C^*(E)$) is purely infinite. As before, both G and E have finitely many vertices. By Lemma 9.2, G and E have the same number of singularities, and since the graph algebras are purely infinite simple, the graphs have no sinks (see [**DT05**, Remark 2.16]). Hence G and E have the same number, k say, of infinite emitters.

Suppose that G has at least one infinite emitter, i.e. $k \geq 1$. By Propositions 8.2 and 8.3 we can find matrices C and D of the same size, n say, such that:

(i) all the entries in the first $n - k$ rows of C and D are non-zero;
(ii) all the entries in the last k rows of C and D are zero;
(iii) the first column of both C and D is $(1, 1, \ldots, 1, 0, 0, \ldots, 0)^T$;
(iv) the determinant of the top-left 'regular' corner is zero;
(v) $J_{n(n-k)} + C \sim_M A_G$; and
(iv) $J_{n(n-k)} + D \sim_M A_E$.

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From [**DT02**, Theorem 3.1] we get that

$$\operatorname{coker} C^T \cong \operatorname{coker} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{(n-k)1} & \cdots & c_{(n-k)n} \end{pmatrix}^T \cong K_0(C^*(G))$$
$$\cong K_0(C^*(E)) \cong \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{(n-k)1} & \cdots & d_{(n-k)n} \end{pmatrix}^T \cong \operatorname{coker} D^T.$$

Hence, C and D have the same elementary divisors. By Theorem 8.5 we then have $J_{n(n-k)} + C \sim_M J_{n(n-k)} + D$, so

$$A_G \sim_M J_{n(n-k)} + C \sim_M J_{n(n-k)} + D \sim_M A_E.$$

Thus, $G \sim_M E$.

We now consider the case where $k = 0$. Using the Cuntz splice we can find graphs $\widetilde{G}$ and $\widetilde{E}$ such that $\det(I - A_{\widetilde{G}})$ and $\det(I - A_{\widetilde{E}})$ both are non-positive, and $G \sim_{M'} \widetilde{G}$ and $E \sim_{M'} \widetilde{E}$. We can now use Theorem 8.7 to find matrices C, D of the same size such that $C \sim_M A_{\widetilde{G}}$, $D \sim_M A_{\widetilde{E}}$ and $\det(-C)$ and $\det(-D)$ both are non-positive. The K-theory argument from the previous case again works to show that C and D have the same elementary divisors. Hence, by Theorem 8.8,

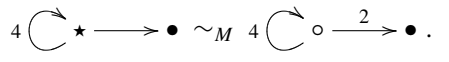
$$A_{\widetilde{G}} \sim_M I + C \sim_M I + D \sim_M A_{\widetilde{E}}.$$

Therefore,

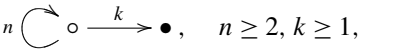
$$G \sim_{M'} \widetilde{G} \sim_M \widetilde{E} \sim_{M'} E. \qquad \square$$

Remark 9.3. It is interesting to note that in the presence of an infinite emitter we do not need the Cuntz splice to fix the sign of the determinant. This suggests, at least to the author, that even in the non-simple case we should be able to do something like a Cuntz splice when we have infinite emitters. We most likely need to assume that the infinite emitter and the vertex at which we wish to do a Cuntz splice interconnect in some way, as this will not always be the case when the graph algebra is not simple.

Remark 9.4. In Example 5.3 we saw that



In [**ET10**, Example 5.2] these graphs are studied as examples of graphs whose algebras have exactly one ideal, and where it would be hard to show stable isomorphism of the associated algebras without using K-theoretic classification. Since we were able to do this, one might hope that we can extend Theorem 4.3 to the one-ideal, or even general non-simple, case. However, in the interest of full disclosure we should note that the results in [**ET10**] can be used to classify all graphs of the form



$n \geq 2, k \geq 1,$

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and many others. That we could produce a move-equivalence between the two graphs considered in [**ET10**, Example 5.2] hinges on the fact that 2 divides 4. So even though



by [**ET10**, Theorem 5.1], it is unclear (to the author) if the graphs are move-equivalent.

We can say something in the non-simple case; for instance, we have the following proposition.

PROPOSITION 9.5. *If $C^*(G)$ is a unital AF algebra then there is a graph E with only singular vertices such that $G \sim_M E$.*

Proof. Since $C^*(G)$ is unital, G only has finitely many vertices. We can, in a finite number of steps, collapse (using Theorem 5.2) all the regular vertices of G. □

In [**ERS11**] it is shown that move (T) can be used to show that any two singular graphs with no breaking vertices which are K-equivalent actually are move-equivalent. This will not always be useful here, as we expect the graph E from Proposition 9.5 often will have breaking vertices. Seen together with Remark 9.4, this suggests that we might need more moves to handle the non-simple case.

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