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Discrete least squares approximation and prewavelets from radial function spaces

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Abstract

In this article we study the convergence behaviour of least squares approximations of various types by radial basis functions, i.e. least squares approximations from spaces spanned by radially symmetric functions $\phi(\|\cdot - x_j\|)$. Here the x_j are given 'centres' in $\mathbb{R}^n$ which we assume to lie on a grid. The inner products with respect to which the least squares problem is considered are discrete and Sobolev, i.e. may involve derivative information. Favourable estimates for the least squares errors are found that are shown to decrease as powers of the gridspacing. The work on discrete least squares approximations also gives rise to the construction of prewavelets from radial function spaces with respect to a discrete Sobolev inner product, which are discussed in the paper as well.

1. *Introduction*

Radial basis function (or, shortly, radial function) methods approximate functions $f: \mathbb{R}^n \to \mathbb{R}$ by approximants from linear spaces spanned by functions

$$\phi(\|\cdot - x_j\|),$$

where the x_j are given centres in $\mathbb{R}^n$. Further, $\phi: \mathbb{R}_+ \to \mathbb{R}$ is a given continuous function for which $\phi(r) = \sqrt{(r^2+c^2)}$ (c is a real parameter) is the most popular example. It is called the multiquadric function and it is considered to be highly useful for applications (Hardy [**19**]). Another frequently used example is $\phi(r) = \exp(-c^2r^2)$. The norm $\|\cdot\|$ used is almost always the Euclidean norm $\|x\| := [|x_1|^2+|x_2|^2+\ldots+|x_n|^2]^{\frac{1}{2}}$. That shall be our choice here. We shall also use the notation $\|\cdot\|$ for the Euclidean norm on finite-dimensional spaces of dimension other than n, which will remain the notation for the dimension of the ambient space of our approximation problems.

In applications, the x_j are usually 'scattered' in $\mathbb{R}^n$ but for the analysis one often assumes that they lie on a (scaled) multi-integer grid (Powell [**22**], Buhmann [**10**], [**11**]). Radial basis functions have been applied most successfully for interpolation at the centres (same references and Hardy [**19**]), but the fact that the most widely used radial basis functions are globally supported and unbounded still creates serious problems when one wants to solve the interpolation problem for many centres. To

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explain this further, note that solving the interpolation problem means that one has to solve a linear system with the interpolation matrix

$$\{\phi(\|x_k - x_j\|)\}_{jk},$$

so that unbounded ϕ make the linear system awkward to solve. Good progress has been made on this issue recently by preconditioning the matrix (see Baxter[**2**], also for references to earlier work by Dyn, Levin and Rippa), but the problem has not been completely overcome so far. Another problem is the efficient evaluation of radial function interpolants with many centres for which special considerations are needed too (Powell[**23**], [**24**]).

With these remarks in mind, it is reasonable to experiment with a least squares approach to the approximation problem from radial basis function spaces. The use of least squares approximations with radial basis functions in the theory of neural networks gives further justification to this idea (Broomhead[**9**]). We choose discrete inner products with respect to which the problem is solved in order to make the approach more realistic, although we are still working on a grid. Specifically, we are going to consider radial functions whose centres are elements of a scaled multi-integer grid $h\mathbb{Z}^n$ and inner products where the functions are sampled on a (finer) grid $h'h\mathbb{Z}^n$. We shall take always $h' = L^{-1}$, L a positive integer, so that $h'\mathbb{Z}^n \supset \mathbb{Z}^n$. We also admit Sobolev inner products, i.e. both function values and derivatives may be taken into account. The derivatives to be considered are defined by a vector $q(D_t)$, where q is a (finite length) vector of monomials in n variables. The first component is assumed to be 1, so that the function values themselves are always used. Here and throughout,

$$D_t^\alpha := \left(\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}}, \frac{\partial^{\alpha_2}}{\partial t_2^{\alpha_2}}, \ldots, \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}}\right) \quad \text{for } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n,$$

$\mathbb{Z}_+^n$ being the lattice of vectors whose coordinates are non-negative integers, $D_t^{\mathbf{1}}$, $\mathbf{1}$ denoting $(1, 1, \ldots, 1)$, will be abbreviated to D_t, and, of course, $D_t^0 = 1$. We are dealing first with the convergence question, i.e. we ask whether the best least squares approximations from spaces spanned by orthonormal functions

$$L_{h'}\left(\frac{x}{h} - k\right) = \sum_{m \in \mathbb{Z}^n} c_m^{h'} \phi\left(\frac{x}{h} - m - k\right), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{Z}^n, \quad h > 0, \tag{1·1}$$

with respect to inner products, for positive h and h',

$$(f, g)_{q(h\cdot)} := h'^n \sum_{j \in \mathbb{Z}^n} \{q(hD_t) f(t)\}_{t - jhh'} \overline{\{q(hD_t)\, g(t)\}}_{t - jhh'} \tag{1·2}$$

converge to the approximated function in various (semi-)norms, and how fast, as $h \to 0$. In (1·1) we have neglected radial symmetry – and we shall do so from now on because it is not important to the analysis. The dilation by h^{-1} and the integer translates associate the orthonormal function $L_{h'}$ with centres on the scaled grid $h\mathbb{Z}^n$. The sampling of the functions on $hh'\mathbb{Z}^n$ in (1·2) reflects our view that it is usual in practice to take many fewer centres than point evaluations of the function when a least squares approach is used. The functions f and g must satisfy suitable decay and differentiability conditions so that (1·2) is well-defined. If the degree of the highest order monomial in q is Q, then it is sufficient e.g. to assume f and g to be in $\mathscr{C}^Q(\mathbb{R}^n)$

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and that they and their partial derivatives of total order at most Q are $O(\|x\|^{-n/2-\epsilon})$ for large $\|x\|$ and a fixed positive ϵ. Suitable assumptions will be made in all instances in the paper where the inner product (1·2) is applied.

Of course the said best least squares solution has the form

$$s_h(x) = \sum_{k\in\mathbb{Z}^n} \left(f, L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)} L_{h'}\left(\frac{x}{h}-k\right), \quad x\in\mathbb{R}^n, \quad h>0. \tag{1·3}$$

The question we ask is whether the uniform and the $L^2(\mathbb{R}^n)$ error between f and s_h can be bounded by a power of h as $h\to 0$. We shall prove that this is the case, and we shall prove that convergence estimates hold for the partial derivatives up to total degree Q of the error function as well. Convergence in the semi-norm associated with the inner product (1·2) will be easy to derive too.

Suitable hypotheses have to be made on f and ϕ for this process to be well-defined. On f, the assumptions mentioned in the penultimate paragraph may be made. Our assumptions on ϕ shall be those of Buhmann and Micchelli [**15**] which also appear in Buhmann [**10**]. They are as follows. First, we require that ϕ is a continuous real-valued function on $\mathbb{R}^n$ and a tempered distribution so that its distributional Fourier transform $\hat{\phi}$ is well-defined. We shall assume that this distributional Fourier transform agrees with a continuous function on $\mathbb{R}^n\backslash\{0\}$, so that we shall in particular ignore δ-functions or derivatives thereof that are part of that Fourier transform. This means that we shall consider all ϕ that differ by a polynomial the same, but the degree of this polynomial can only be of a certain maximal value. We shall see in the next section why this is a justified approach and we shall see below what the maximal degree of the polynomial is, modulo to which different ϕ are considered the same. We now require that there are positive real numbers μ, $\tilde{\mu}$ and $\delta\geqslant 1$, such that, for $\rho := \lceil \mu+n\rceil$, i.e. the smallest integer which is at least $\mu+n$, we have

(i) $\hat{\phi}\in\mathscr{C}^\rho(\mathbb{R}^n\backslash\{0\})$,

(ii) $|D_x^\alpha\hat{\phi}(x)| = O(\|x\|^{-n-\tilde{\mu}})$, $\|x\|>1$, for all $\alpha\in\{\gamma\in\mathbb{Z}_+^n\colon |\gamma| := \gamma_1+\gamma_2+\ldots+\gamma_n\leqslant\rho\}$,

(iii) $D_x^\alpha\hat{\phi}(x) = \hat{C}D_x^\alpha\|x\|^{-\mu}+O(\|x\|^{-\mu-|\alpha|+\delta})$, $0<\|x\|<1$, for all $\alpha\in\{\gamma\in\mathbb{Z}_+^n\colon |\gamma|\leqslant\rho\}$, where $\hat{C}$ is a non-zero constant,

(iv) $\sum_{k\in\mathbb{Z}^n}|\hat{\phi}(x+2\pi k)|^2$ is positive for all $x\notin 2\pi\mathbb{Z}^n$ and

(v) $|\phi(x)| = O(\|x\|^{\mu-n}\log\|x\|)$, $\|x\|>2$.

The assumption (iv) is weaker than the according assumption in the aforementioned papers, in that there it is required that $\hat{\phi}>0$, so that one is able to interpolate from the radial basis function space. We shall see that the assumptions imply the orthonormal functions (1·1) exist and that they and all their partial derivatives of total order less than $\tilde{\mu}$ satisfy

$$|D_x^\alpha L_{h'}(x)| = O((1+\|x\|)^{-n-\mu-|\alpha|}). \tag{1·4}$$

On the basis of (1·4) we shall see that (1·3) is indeed well-defined not just for the choice of functions f we have indicated above, but for $f\in\mathscr{C}^Q(\mathbb{R}^n)$ which grow slower than $\|\cdot\|^\mu$ for large argument, too.

Assumption (v) restricts the space of polynomials modulo to which we consider different ϕ the same, to $\mathbb{P}_n^{\leqslant\mu-n}\subset\mathbb{P}_n^{\leqslant\mu}$, $\mathbb{P}_n^{<k}$ being the space of polynomials in n unknowns of degree less than k.

For examples of ϕ that satisfy these conditions, see Buhmann [**10**], where it is shown that, among many other radial functions, $\phi(x) = (\|x\|^2+c^2)^{\frac{1}{2}}$ satisfies (i)–(v) for

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all real c. Note that $\phi(x) = \exp(-c^2\|x\|^2)$ and other integrable radial functions are ruled out because they require $\mu = 0$ on (iii). Some of our theory may nevertheless be applied to these radial functions and we will deal with them separately whenever appropriate.

Approximation orders such as we shall find for our discrete Sobolev least squares approach have been found in Ron [**25**] for the continuous (and 'non-Sobolev') $L^2(\mathbb{R}^n)$ norm and in de Boor and Ron [**8**] for the best approximation with respect to the uniform norm. In those papers, a more general point of view is taken, in that general principal (i.e. generated by just one radial function, but see Section 5 in our paper) shift-invariant spaces and their least squares or uniform distance to linear spaces of sufficiently smooth functions are regarded. The *Ansatz* differs in that no orthonormal basis functions for the radial function space are generated, as the results are derived directly from the form of the radial functions. We, however, take the work by the author [**10**] on cardinal interpolation as point of departure and identify orthonormal functions explicitly which can be useful for practical computations as well. They are similar to the cardinal functions found in Buhmann [**10**] and can be computed with a Fast Fourier Transform approach (Jetter and Stöckler [**18**]). We remark too that our assumptions on the function ϕ are weaker than in those papers.

The paper is organized as follows. In Section 2, we construct orthonormal functions (1·1) from the linear space spanned by the radial functions, thus in particular showing their existence. Then we give $L^2(\mathbb{R}^n)$ convergence results and convergence results with respect to the semi-norm associated with our discrete Sobolev inner product in Section 3 and a uniform convergence result in Section 4. The latter interests us because we view the least squares approach partly as a useful substitute for interpolation, for which uniform norm error estimates are usual. In Section 5 we will study least squares approximations from spaces spanned by several different radial functions, call them ϕ_i, with the two examples of using several multiquadric functions with different parameters $c = c_i$, or using the multiquadric function and its derivatives, at hand. One reason for studying this is that Hermite interpolation with the multiquadric function and its derivatives on an integer grid is singular (Jetter, private communication), so it is useful to replace this process by Sobolev least squares approximations with the multiquadric function and its derivatives.

Section 6 will then be concerned with the stability of the least squares approximations problem. Finally, we shall note in Section 7 that, as a consequence of our previous work, prewavelets may be constructed with respect to the inner product (1·2), where we shall follow closely the approach of Buhmann and Micchelli [**14**]. For an introduction to the subject of (pre)wavelets, see that paper, the 1993 paper [**7**] by de Boor, DeVore and Ron, or the recent book [**16**] by Chui, which contains work that is related to ours and to the work in Buhmann and Micchelli [**14**] as well.

2. *Construction of an orthonormal basis for a Sobolev inner product*

We begin our work by constructing an orthonormal basis for the space spanned by integer translates of ϕ, with respect to a given discrete inner product which we call a Sobolev inner product, because we also admit that derivatives be incorporated. The

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inner product has already been defined in the introduction, equation (1·2). We state here only the important condition that

$$Q := \max_i \deg q_i < \tilde{\mu},$$

where $\tilde{\mu}$ is a constant from condition (ii) and q_i are the components of q. Note that for the multiquadric function, Q can be chosen arbitrarily large because $\hat{\phi}$ decays exponentially (Buhmann [**10**], for instance). We observe that the orthogonality of the functions (1·1) with respect to (1·2) is tantamount to the orthogonality of the functions $\{L_{h'}(\cdot - k)\}_{k\in\mathbb{Z}^n}$ with respect to

$$(f,g)_q^* := h'^n \sum_{j\in\mathbb{Z}^n} \{q(D_t)f(t)\}_{t=jh'}^T \overline{\{q(D_t)g(t)\}}_{t=jh'}. \tag{2·1}$$

In the sequel we use the notation $\mathbb{T} := [-\pi,\pi]$. We assume throughout that ϕ satisfies the conditions set out in the introduction.

THEOREM 1. *The function*

$$L_{h'}(x) = \sum_{m\in\mathbb{Z}^n} c_m^{h'} \phi(x-m), \quad x\in\mathbb{R}^n, h' > 0, \tag{2·2}$$

where

$$c_m^{h'} = \frac{1}{(2\pi\sqrt{h'})^n} \int_{\mathbb{T}^n} \frac{e^{im^T t}\,dt}{\sqrt{(\sum^* \| \sum_{k\in\mathbb{Z}^n} e^{2\pi i l^T k/L} \hat{\phi}(t+2\pi k)\, q(t+2\pi k)\|^2)}}, \quad m\in\mathbb{Z}^n, \tag{2·3}$$

with

$$\sum^* := \sum_{\{l\in\mathbb{Z}^n_+ \,|\, \|l\|_\infty < L\}},$$

is continuous if h' is small enough, as in fact then $|c_m^{h'}| = O(\|m\|^{-n-\mu})$. It satisfies (1·4) *and*

$$(L_{h'}, L_{h'}(\cdot - k))_q^* = \delta_{0k}, \quad k\in\mathbb{Z}^n, h' > 0. \tag{2·4}$$

Remark. The restriction 'for small enough h'' is not needed if $\hat{\phi}$ is one-signed and $|\hat{\phi}(t)| > \sum' |\hat{\phi}(t+2\pi k)|$ for all $t\in\mathbb{T}^n\backslash\{0\}$, where the dashed sum indicates that we are leaving out the $k = 0$ term.

Proof. We expand condition (2·4) using (2·1) and take the discrete Fourier transform. At this stage we shall do the expansions formally, i.e. disregarding the question whether the infinite sums converge or whether the Fourier transforms employed below exist. Then the assertion is equivalent to

$$\sum_{j\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} e^{-i\vartheta^T k} \{q(D_t)\, L_{h'}(t)\}_{t=jh'}^T \{q(D_t)\, L_{h'}(t-k)\}_{t=jh'} = h'^{-n}, \quad \vartheta\in\mathbb{T}^n. \tag{2·5}$$

We shall now manipulate (2·5) into a form which will make it convenient to show that (2·2) satisfies (2·4). To this end, we apply the Poisson summation formula (Stein and Weiss [**28**], p. 252), still formally, and obtain

$$\sum_{j\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} e^{-ih'(\vartheta+2\pi k)^T j} \{q(D_t)\, L_{h'}(t)\}_{t=jh'}^T q(-i(\vartheta+2\pi k))\, \hat{L}_{h'}(-\vartheta-2\pi k) = h'^{-n}, \quad \vartheta\in\mathbb{T}^n. \tag{2·6}$$

Splitting up the sum over j in (2·6), we get

$$\sum^* \sum_{j\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} e^{-tj^T\vartheta - tl^T(\vartheta+2\pi k)/L} \{q(D_t)\, L_{h'}(t)\}_{t=j+l/L}^T \times q(-i(\vartheta+2\pi k))\, \hat{L}_{h'}(-\vartheta-2\pi k) = h'^{-n}, \quad \vartheta\in\mathbb{T}^n.$$

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Another application of the Poisson summation formula leads to

$$\sum\nolimits^{*} \sum_{j\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} e^{il^T(\vartheta+2\pi j)/L - il^T(\vartheta+2\pi k)/L} q(i(\vartheta+2\pi j))^T q(-i(\vartheta+2\pi k)) \times \hat{L}_{h'}(\vartheta+2\pi j)\hat{L}_{h'}(-\vartheta-2\pi k) = h'^{-n}, \quad \vartheta\in\mathbb{T}^n,$$

or

$$\sum\nolimits^{*} \sum_{j\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} e^{il^T(2\pi j)/L - il^T(2\pi k)/L} q(i(\vartheta+2\pi j))^T q(-i(\vartheta+2\pi k)) \times \hat{L}_{h'}(\vartheta+2\pi j)\hat{L}_{h'}(-\vartheta-2\pi k) = h'^{-n}, \quad \vartheta\in\mathbb{T}^n. \quad (2{\cdot}7)$$

We first claim that (2·2) is well-defined and satisfies (2·7), and we shall then show that all series in (2·5)–(2·7) are absolutely convergent, so that (2·7) implies (2·4). This will settle the theorem. In order to begin this, note first that (2·3) is well-defined owing to the conditions on $\hat{\phi}$ set out in the introduction. In particular, the denominator of the integrand cannot have a zero if h' is small because of (iv) and the following

LEMMA 2. *It is true for each $m\in\mathbb{Z}^n$ that*

$$\lim_{h'\to 0} c_m^{h'} = c_m := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{e^{im^T t}\,dt}{\sqrt{\left(\sum_{k\in\mathbb{Z}^n} \|\hat{\phi}(t+2\pi k)\, q(t+2\pi k)\|^2\right)}},$$

and the convergence is uniform in m.

We remark that it will be a consequence of Lemma 2 and Theorem 1 that

$$\lim_{h'\to 0} L_{h'} = L_0$$

uniformly on compacta in $\mathbb{R}^n$, where L_0 is the function

$$L_0(x) = \sum_{m\in\mathbb{Z}^n} c_m\,\phi(x-m), \quad x\in\mathbb{R}^n,$$

whose translates are mutually orthonormal with respect to the continuous Sobolev inner product

$$\langle f, g\rangle_q := \int_{\mathbb{R}^n} q(D_t) f(t)^T \overline{q(D_t)\, g(t)}\,dt,$$

where $f, g\in \mathcal{W}_2^Q(\mathbb{R}^n)$, the space of functions whose partial derivatives of order up to and including Q are square-integrable. We now prove Lemma 2.

Proof. We note the following uniform convergence result on compacta in $\mathbb{T}^n\backslash\{0\}$:

$$\begin{aligned}
&\lim_{h'\to 0} h'^n \sum\nolimits^{*} \Big\| \sum_{j\in\mathbb{Z}^n} e^{2\pi i l^T j/L} q(t+2\pi j)\,\hat{\phi}(t+2\pi j)\Big\|^2 \\
&= \int_{[0,1]^n} \Big\| \sum_{j\in\mathbb{Z}^n} e^{2\pi i \tau^T j} q(t+2\pi j)\,\hat{\phi}(t+2\pi j)\Big\|^2 d\tau \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \Big\| \sum_{j\in\mathbb{Z}^n} e^{i\tau^T j} q(t+2\pi j)\,\hat{\phi}(t+2\pi j)\Big\|^2 d\tau \\
&= \sum_{j\in\mathbb{Z}^n} \|q(t+2\pi j)\,\hat{\phi}(t+2\pi j)\|^2,
\end{aligned}$$

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according to Theorem 1·7 on p. 248 of Stein and Weiss[**28**] and condition (ii). Now the lemma follows from the dominated convergence theorem and from the fact that the last display is positive owing to condition (iv). ∎

Lemma 2·7 in the paper by Buhmann and Micchelli[**15**], which can be reformulated and proved for Fourier coefficients instead of Fourier transforms in a straightforward way, implies that (2·3) gives

$$|c_m^{h'}| = O(\|m\|^{-n-\mu}), \quad m \in \mathbb{Z}^n. \tag{2·8}$$

Here we are making use of the fact that the denominator of the integrand in (2·3) can be written as

$$|\hat{\phi}(\vartheta)| \sqrt{(\sum\nolimits^* \|q(\vartheta) + \hat{\phi}(\vartheta)^{-1} \sum\nolimits' e^{2\pi i l^T k/L} \hat{\phi}(\vartheta + 2\pi k)\, q(\vartheta + 2\pi k)\|^2)},$$

which gives, together with assumption (iii), the singularity at zero required by the lemma in Buhmann and Micchelli's paper in order for (2·8) to hold.

Because of condition (v), (2·8) implies that (2·2) is defined as a pointwise absolutely convergent series, hence it is continuous, and of at most polynomial growth by Lemma 4 in Buhmann and Dyn[**12**]. Hence we can form its distributional Fourier transform (see, e.g. Stein and Weiss[**28**], pp. 19ff.) which is

$$\hat{L}_{h'}(\vartheta) = \frac{h'^{-n/2} \hat{\phi}(\vartheta)}{\sqrt{(\sum\nolimits^* \| \sum\limits_{k \in \mathbb{Z}^n} e^{2\pi i l^T k/L} \hat{\phi}(\vartheta + 2\pi k)\, q(\vartheta + 2\pi k)\|^2)}}, \quad \vartheta \in \mathbb{R}^n. \tag{2·9}$$

Expression (2·9) is, by the assumptions on $\hat{\phi}$, a continuous and absolutely integrable function. Its (classical) inverse Fourier transform

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix^T\vartheta} \hat{L}_{h'}(\vartheta)\, d\vartheta, \quad x \in \mathbb{R}^n, \tag{2·10}$$

is $L_{h'}$. Thus it follows from Lemma 2·7 in Buhmann and Micchelli[**15**] that

$$|L_{h'}(x)| = O((1 + \|x\|)^{-n-\mu}), \quad x \in \mathbb{R}^n. \tag{2·11}$$

Estimate (2·11) and $\mu > 0$ imply that $L_{h'}$ is absolutely integrable and that (2·9) is its (classical) Fourier transform. The conditions on $\hat{\phi}$ and $Q < \tilde{\mu}$ imply that the series in (2·7) are absolutely convergent. Identity (2·7) which implies (2·4) is satisfied by (2·9). Estimate (2·11) also holds for all derivatives of $L_{h'}$ which exist and are continuous at least up to total order $< \tilde{\mu}$, as one can multiply $\hat{L}_{h'}(\vartheta)$ with any polynomial $p(i\vartheta)$ of degree less than $\tilde{\mu}$ in (2·10), so that the derivative $p(D_x) L_{h'}(x)$ is well-defined (see Theorem 1·8 on p. 4 of Stein and Weiss [**28**]). In fact, for the partial derivatives of order $r < \tilde{\mu}$, say, it is true that

$$|D_x^\alpha L_{h'}(x)| = O(\|x\|^{-n-\mu-r}), \quad |\alpha| = r, \quad r < \tilde{\mu}, \quad x \in \mathbb{R}^n, \tag{2·12}$$

as already hinted at in (1·4). Showing this only requires integration by parts and another application of the same lemma in the Buhmann and Micchelli paper. Thus, all series in (2·5)–(2·7) are absolutely convergent and thus the applications of the Poisson summation formula are justified too. Hence the theorem is true. ∎

We note that (2·9) explains why we can disregard any δ-function terms and their derivatives in the distributional Fourier transform of ϕ: the high order singularity at

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zero in the denominator in (2·9) means that we can add δ-functions and their derivatives of order less than μ to the numerator without changing $L_{h'}$ as they only change $L_{h'}$ by zero. Specifically, the integration in (2·10) means that an additional derivative $p(D_\vartheta)$ of a δ-function in the numerator of (2·9) adds to $L_{h'}(x)$ a term

$$p(D_\vartheta)\left[\frac{h'^{-n/2}}{\sqrt{(\sum^* \| \sum_{k\in\mathbb{Z}^n} e^{2\pi i l^T k/L}\hat{\phi}(\vartheta+2\pi k)\, q(\vartheta+2\pi k)\|^2)}}\right]_{\vartheta=0} = 0,$$

so long as the degree of p is less than μ. Alternatively, we can say that the high order zero at the origin in the integrand of (2·3) lets the Fourier coefficients satisfy sufficiently high order moment conditions, so that polynomials of total degree $< \mu$ added to ϕ are annihilated in the expression (2·2).

THEOREM 3. *Let f be a polynomial in n unknowns of degree less than μ. Then the best least squares approximation with respect to* (1·2) *reproduces f, i.e.*

$$f(x) = \sum_{k\in\mathbb{Z}^n} \left(f, L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)} L_{h'}\left(\frac{x}{h}-k\right), \quad x\in\mathbb{R}^n. \tag{2·13}$$

Proof. The sum is absolutely convergent by (2·11) and by Lemma 4 in Buhmann and Dyn [**12**]. Without loss of generality we let f be a monomial of degree less than μ. We apply the Poisson summation formula to the inner product on the right-hand side of (2·13), but we neglect those terms that are of lower order than f which come from the derivatives in (1·2), i.e. from the components of q other than the first one. We obtain

$$\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n} f(ihD_x)\{e^{-ix^Tk}\hat{L}_{h'}(x)\}_{x=2\pi l/h'} L_{h'}\left(\frac{x}{h}-k\right) = \sum_{k\in\mathbb{Z}^n} f(kh)\, L_{h'}\left(\frac{x}{h}-k\right), \quad x\in\mathbb{R}^n. \tag{2·14}$$

This follows from the admissibility of ϕ and from (2·9). Hence it follows from Theorem 3 in Jackson [**17**] that (2·14) is $f(x)$ plus a lower order polynomial. Hence the theorem follows from the fact that the right-hand side of (2·13) is the best least squares approximation to f. ∎

Next we present a theorem that admits the construction of orthonormal functions for a class of ϕ that contains $\phi(x) = e^{-c^2\|x\|^2}$ and $\phi(x) = e^{-c\|x\|}$, $c > 0$ (see Stein and Weiss [**28**], p. 6), although condition (iii) rules out such functions from the class to which Theorem 1 applies.

THEOREM 4. *Let ϕ be such that, for positive constants ϵ, $\tilde{\mu}$ and ρ, the latter being an integer $> n$,*

(i) $|\phi(x)| = O((1+\|x\|)^{-n-\epsilon})$,

(ii) $\hat{\phi}\in\mathscr{C}^\rho(\mathbb{R}^n)$ *and*

(iii) $|D_t^\alpha \hat{\phi}(t)| = O((1+\|t\|)^{-n-\tilde{\mu}})$, $\alpha\in\mathbb{Z}_+^n$ *and* $|\alpha| \leqslant \rho$.

Then the conclusions of Theorem 1 *hold with $\mu := \rho - n$.*

Proof. The proof of Theorem 1 carries through without alteration except that (2·8) and (2·11)–(2·12) are now direct consequences of the differentiability of $\hat{\phi}$. ∎

We make several final remarks in this section. Firstly, we note that all this can be applied in order to orthonormalize functions $L_{h'}$ with respect to a different function ψ in the sense of

$$(L_{h'}(\cdot - k), \psi)_q^* = \delta_{0k}, \quad k\in\mathbb{Z}^n. \tag{2·15}$$

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In the case (2·15), (2·3) has to be replaced by

$$c_m^{h'} = \frac{1}{(2\pi\sqrt{h'})^n} \times \int \frac{e^{im^T t}\,dt}{\pi^n \sqrt{[\sum^* (\sum_{k\in\mathbb{Z}^n} e^{2\pi i l^T k/L} \hat{\phi}(t+2\pi k)\, q(t+2\pi k))^T (\sum_{k\in\mathbb{Z}^n} e^{-2\pi i l^T k/L} \vec{\hat{\psi}}(t+2\pi k)\, q(t+2\pi k))]}}.$$

Of course suitable assumptions have to be made on ψ for this to be well-defined.

The second remark is that one can replace differentiation D_t in the definition of the inner product (1·2) by differences $E_t = (\Delta_{t_1}, \Delta_{t_2}, \ldots, \Delta_{t_n})$, where $\Delta_t f(t) = f(t) - f(t-1)$. In this event, one replaces the $q(\vartheta + 2\pi k)$ terms that appear in (2·3) and (2·9) etc. by $q(1 - \exp(i\vartheta))$.

Thirdly, it is straightforward to replace (1·2) with a weighted inner product

$$(f, g)_q^* := h'^n \sum_{j\in\mathbb{Z}^n} \omega(jh') \{q(D_t) f(t)\}_{t=jh'}^T \{\overline{q(D_t)\, g(t)}\}_{t=jh'},$$

where ω is a positive function and bounded, say. If $\omega \in \mathscr{C}^Q(\mathbb{R}^n)$, then one can find the functions whose translates along $\mathbb{Z}^n$ are orthonormal by replacing $q(\vartheta + 2\pi k)$ by $q(-iD_t)\{e^{i(\vartheta+2\pi k)^T t}\omega(t)\}_{t=0}$ in (2·3) and (2·9).

Finally, we remark that, if $h' = 2^{-H}$, then it is easy to compute the denominators in (2·3) and (2·9) recursively for $H = 1, 2, \ldots$, but of course a suitable starting value has to be found first.

3. L^2 *convergence of discrete least squares approximation*

The following theorem gives a convergence result in the $L^2(\mathbb{R}^n)$ norm, i.e.

$$\|f\|_2^2 = \int_{\mathbb{R}^n} |f(x)|^2\,dx.$$

THEOREM 5. *Let $f \in \mathscr{C}^Q(\mathbb{R}^n)$ be a function which satisfies $|D_x^\alpha f(x)| = O((1 + \|x\|)^{-n/2-\epsilon})$ for positive ϵ and $|\alpha| \leqslant Q$, so that the best least squares approximation* (1·3) *with respect to $(\cdot, \cdot)_{q(h\cdot)}$ is well-defined. Suppose also that $|\hat{f}(t)\,\|t\|^\mu| = O((1 + \|t\|)^{-n/2-\epsilon})$, where μ is the positive constant from requirement* (iii). *Then we have the error estimate*

$$\|s_h - f\|_2 = O(h^\mu), \quad h \to 0, \tag{3·1}$$

uniformly in h'.

Proof. Because the Fourier transform is an $L^2(\mathbb{R}^n)$ isometry, we may prove (3·1) by considering

$$\|\hat{s}_h - \hat{f}\|_2 \tag{3·2}$$

instead of the left-hand side of (3·1). Here we are using the fact that our conditions imply

$$\|f\|_2^2 = \frac{1}{(2\pi)^n} \|\hat{f}\|_2^2 < \infty.$$

We are also using in (3·2) the Fourier transform of (1·3)

$$\hat{s}_h(\vartheta) = h^n \sum_{k\in\mathbb{Z}^n} \left(f, L_{h'}\left(\frac{\cdot}{h} - k\right)\right)_{q(h\cdot)} e^{-ih\vartheta^T k} \hat{L}_{h'}(h\vartheta), \quad \vartheta \in \mathbb{R}^n, \tag{3·3}$$

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which is in $L^2(\mathbb{R}^n)$ because Lemma 4 in Buhmann and Dyn[**12**], (2·12) and the assumptions about the rate of decay of f imply that the trigonometric series above is square-summable, hence in $L^2(\mathbb{T}^n)$, and because $\hat{L}_{h'}$ is continuous and integrable.

Employing the Poisson summation formula, we find that the square of the error (3·2) is

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\hat{f}(\vartheta)-h^n\sum_{k\in\mathbb{Z}^n}\left(f,L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)}e^{-ih\vartheta^Tk}\hat{L}_{h'}(h\vartheta)\right|^2 d\vartheta\\
&\quad=\int_{\mathbb{R}^n}\left|\hat{f}(\vartheta)-h^nh'^n\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\{q(hD_t)f(t)\}^T_{t=lhh'}\left\{q(hD_t)L_{h'}\left(\frac{t}{h}-k\right)\right\}_{t=lhh'}\right.\\
&\qquad\left.\times e^{-ih\vartheta^Tk}\hat{L}_{h'}(h\vartheta)\right|^2 d\vartheta\\
&\quad=\int_{\mathbb{R}^n}\left|\hat{f}(\vartheta)-h^nh'^n\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\{q(hD_t)f(t)\}^T_{t=lhh'}q(-i(h\vartheta+2\pi k))\right.\\
&\qquad\left.\times\hat{L}_{h'}(h\vartheta)\,e^{-ih'l^T(h\vartheta+2\pi k)}\hat{L}_{h'}(-h\vartheta-2\pi k)\right|^2 d\vartheta\\
&\quad=\int_{h^{-1}\mathbb{T}^n}\sum_{m\in\mathbb{Z}^n}\left|\hat{f}(\vartheta+h^{-1}2\pi m)-\hat{L}_{h'}(h\vartheta+2\pi m)\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}q\left(i\left(h\vartheta+2\pi k+\frac{2\pi l}{h'}\right)\right)^T\right.\\
&\qquad\left.\times\hat{f}\left(\vartheta+\frac{2\pi l}{hh'}+\frac{2\pi k}{h}\right)q(-i(h\vartheta+2\pi k))\,\hat{L}_{h'}(-h\vartheta-2\pi k)\right|^2 d\vartheta. \qquad (3\cdot4)
\end{aligned}$$

We may interchange summation over m and integration in (3·4) by the Fubini theorem.

Next, we restrict the support of $\hat{f}$ to $h^{-1}\mathbb{T}^n$. This changes (3·4) only by a term of order $O(h^{2\mu})$ (cf. also Ron[**25**]). Indeed, if $\operatorname{supp}\hat{f}$ is restricted to $h^{-1}\mathbb{T}^n$, (3·4) becomes

$$\begin{aligned}
\int_{h^{-1}\mathbb{T}^n}\sum_{m\in\mathbb{Z}^n}\left|\delta_{0m}\hat{f}(\vartheta)-\hat{L}_{h'}(h\vartheta+2\pi m)\,q(i(h\vartheta))^T\hat{f}(\vartheta)\right.\\
\left.\times\sum_{k\in\mathbb{Z}^n}q\left(-i\left(h\vartheta+\frac{2\pi k}{h'}\right)\right)\hat{L}_{h'}\left(-h\vartheta-\frac{2\pi k}{h'}\right)\right|^2 d\vartheta. \qquad (3\cdot5)
\end{aligned}$$

Comparing (3·4) and (3·5) and observing that

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus h^{-1}\mathbb{T}^n}|\hat{f}(\vartheta)|^2\,d\vartheta &\leqslant \int_{\mathbb{R}^n\setminus h^{-1}\mathbb{T}^n}\|\vartheta\|^{2\mu}|\hat{f}(\vartheta)|^2\,d\vartheta\max_{\|\vartheta\|_\infty\geqslant h^{-1}\pi}\|\vartheta\|^{-2\mu}\\
&= O(h^{2\mu})
\end{aligned}$$

confirms that an error of only $O(h^{2\mu})$ is introduced when $\operatorname{supp}\hat{f}$ is cut off outside $h^{-1}\mathbb{T}^n$.

It is highly relevant to the proof of (3·1) that, when h is small and $k=m=0$,

$$|\,\|\hat{L}_{h'}(h\vartheta)\,q(h\vartheta)\|^2-1-C\|h\vartheta\|^\mu| = O(\|h\vartheta\|^{\mu+\eta}), \qquad (3\cdot6)$$

and we are going to show this below. In (3·6), η and C denote suitable positive constants, the latter standing for a generic positive number that can be different each time it occurs in our manuscript. In fact, depending on the sign of $\hat{C}$, which is a constant from (iii), it may be required to take a negative constant C in (3·6), but there

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is no loss of generality if we assume that it is positive. Further, for small h and $m \neq 0$ or $k \neq 0$, there exists $C > 0$ such that

$$\left| \hat{L}_{h'}(h\vartheta + 2\pi m)\, \hat{L}_{h'}\left(-h\vartheta - \frac{2\pi k}{h'}\right) q(h\vartheta)^T q\left(h\vartheta + \frac{2\pi k}{h'}\right) - C\|h\vartheta\|^{\mu} \right| = O(\|h\vartheta\|^{\mu+\eta}) \quad (3{\cdot}7)$$

as well, but C may depend on k and m. The latter short asymptotic expansion still holds when the sums over m and over k are taken owing to the summability of the translates of $\hat{L}_{h'}$, but we dispense with writing the sums into (3·7) and (3·9) below in order to keep our expressions simpler.

Before we establish (3·6) and (3·7), we remark that (3·5)–(3·7) imply the error estimate (3·1) on account of the assumption $|\hat{f}(t)\, \|t\|^{\mu}| = O((1 + \|t\|)^{-n/2-e})$, because that assumption implies

$$\begin{aligned} \int_{h^{-1}\mathbb{T}^n} \|h\vartheta\|^{2(\mu+\eta)} |\hat{f}(\vartheta)|^2\, d\vartheta &\leqslant \int_{h^{-1}\mathbb{T}^n} \|h\vartheta\|^{2\mu} |\hat{f}(\vartheta)|^2\, d\vartheta \max_{\|\vartheta\| \leqslant h^{-1}\pi} \|h\vartheta\|^{2\eta} \\ &\leqslant \pi^{2\eta} h^{2\mu} \int_{\mathbf{R}^n} \|\vartheta\|^{2\mu} |\hat{f}(\vartheta)|^2\, d\vartheta \\ &= O(h^{2\mu}), \end{aligned}$$

whenever η is a non-negative quantity.

We observe that the first term of the left-hand side of (3·6) is, because (2·9),

$$\begin{aligned} &\frac{h'^{-n} \hat{\phi}(h\vartheta)^2\, \|q(h\vartheta)\|^2}{\sum^* \| \sum_{k \in \mathbb{Z}^n} e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k)\|^2} \\ &\quad = \frac{1}{h'^n \sum^* \|[\hat{\phi}(h\vartheta)\|q(h\vartheta)\|]^{-1} \{\hat{\phi}(h\vartheta)\, q(h\vartheta) + \sum' e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k)\}\|^2}. \end{aligned} \quad (3{\cdot}8)$$

The denominator in (3·8) can be re-written as

$$\begin{aligned} 1 + h'^n \sum^* \left(2 \left[\frac{q(h\vartheta)}{\|q(h\vartheta)\|} \right]^T \left[\frac{1}{\|q(h\vartheta)\|\, \hat{\phi}(h\vartheta)} \sum' e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k) \right] \right. \\ \left. + \frac{1}{\|q(h\vartheta)\|^2\, \hat{\phi}(h\vartheta)^2} \| \sum' e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k)\|^2 \right). \end{aligned}$$

Using the facts that $\|q(0)\| \neq 0$, that the sums $\sum'$ converge absolutely and are bounded at zero owing to conditions (i)–(ii) and that $\hat{\phi}$ has a singularity at zero prescribed by condition (iii), we see from this that (3·6) holds. Estimate (3·7) is established by noticing that the first term of the left-hand side of (3·7) is

$$\begin{aligned} &\frac{h'^{-n} \hat{\phi}(h\vartheta + 2\pi m)\, \hat{\phi}\left(-h\vartheta - \frac{2\pi k}{h'}\right) q(h\vartheta)^T q\left(h\vartheta + \frac{2\pi k}{h'}\right)}{\sum^* \| \sum_{k \in \mathbb{Z}^n} e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k)\|^2} \\ &\quad = \frac{\hat{\phi}(h\vartheta + 2\pi m)\, \hat{\phi}(h\vartheta)^{-2}\, \hat{\phi}\left(-h\vartheta - \frac{2\pi k}{h'}\right) q(h\vartheta)^T q\left(h\vartheta + \frac{2\pi k}{h'}\right) \|q(h\vartheta)\|^{-2}}{h'^n \sum^* \|[\hat{\phi}(h\vartheta)\|q(h\vartheta)\|]^{-1} \{\hat{\phi}(h\vartheta)\, q(h\vartheta) + \sum' e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k)\}\|^2} \end{aligned} \quad (3{\cdot}9)$$

which readily affirms (3·7). ∎

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Remark. The error bound (3·1) multiplied by h^{-s} on the right-hand side holds for all partial derivatives of the error function of total degree $s < \min[\mu, \tilde{\mu}]$, if f is such that $|\hat{f}(t)\,\|t\|^{\mu+s}| = O((1+\|t\|)^{-n/2-\epsilon})$ instead of the analogous condition in the statement of the theorem, because in that event the integrand in (3·4) is just multiplied by a monomial in ϑ of total degree s.

This remark means that the same error estimate as (3·1) also holds for the norm associated with the inner product $\langle \cdot, \cdot \rangle_{q(h\cdot)}$ if f is such that $|\hat{f}(t)\,\|t\|^{\mu+Q}| = O((1+\|t\|)^{-n/2-\epsilon})$.

THEOREM 6. *If $2\mu > n$, the error bound*

$$\|f - s_h\|^2_{q(h\cdot)} = O(h^{2\mu-n}), \quad h \to 0, \tag{3·10}$$

holds for any f that satisfies the assumptions of Theorem 5. *The estimate is uniform in h'.*

Proof. We can evaluate the left-hand side of (3·10) as follows, using the Poisson summation formula:

$$h^{-n} \sum_{j \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \left(\hat{f}\left(\frac{2\pi j}{hh'} - t\right) q\left(i\left(\frac{2\pi j}{h'} - th\right)\right) - \hat{s}_h\left(\frac{2\pi j}{hh'} - t\right) q\left(i\left(\frac{2\pi j}{h'} - th\right)\right)\right)^T \times \left(\overline{\hat{f}}(t)\, q(-iht) - \overline{\hat{s}_h}(t)\, q(-iht)\right) dt.$$

Using the form of $\hat{s}_h$ found in the previous proof, truncating $\hat{f}$ outside $h^{-1}\mathbb{T}^n$ and employing the Poisson summation formula, we get, after some manipulation which is by now standard, that this is, up to an error of $O(h^{2\mu-n})$, the same as

$$h^{-n} \int_{h^{-1}\mathbb{T}^n} |\hat{f}(t)|^2\, \|q(ht)\|^2 \left[1 - \left\| \sum_{k \in \mathbb{Z}^n} \hat{L}_{h'}\left(th + \frac{2\pi k}{h'}\right) q\left(th + \frac{2\pi k}{h'}\right) \right\|^2\right]^2 dt.$$

Now it is shown precisely in the same way as in the previous proof that the required error estimate is true. ∎

We note that it follows from (3·8)–(3·9) that the errors of the least squares approximations for the multiquadric function $\phi(x) = (\|x\|^2 + c^2)^{\frac{1}{2}}$ decrease as $O(h^{\hat{\mu}})$ for any $\hat{\mu}$ for which $|\hat{f}(t)\,\|t\|^{\hat{\mu}}| = O((1+\|t\|)^{-n/2-\epsilon})$, when $h \to 0$ and, for instance, $c = O(h^{-1})$ (slower algebraic growth rates and certain logarithmic growth rates of c would also be acceptable). In order to justify this remark, consider for example (3·8). If expanded as a geometric series, each term in the resulting expansion contains a power of

$$\sum{}' e^{2\pi i l^T k/L} \hat{\phi}(h\vartheta + 2\pi k)\, q(h\vartheta + 2\pi k) \tag{3·11}$$

for $\vartheta \in h^{-1}\mathbb{T}^n$. Expression (3·11) can be bounded above by a term that decays exponentially with increasing c, on account of the explicit form of $\hat{\phi}$ given for instance in Buhmann[**10**], where it is stated that $\hat{\phi}(t)$ is a constant multiple of

$$K_{(n+1)/2}(c\|t\|)\,(\|t\|/c)^{-(n+1)/2},$$

and on account of the fact that $K_{(n+1)/2}(z)$ can be bounded above by a constant multiple of $e^{-z}/\sqrt{z}$ for $z \in (1, \infty)$, say (Abramowitz and Stegun[**1**]).

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We have already seen in Theorem 4 that the orthonormal functions (1·1) also exist for a class of ϕ which is a subclass of those called 'pseudo-admissible' in Buhmann[**10**], in spite of the fact that they do not satisfy condition (iii) for any positive μ. For this, the Gauss-kernel $\phi(x) = e^{-c^2\|x\|^2}$ (c a positive constant) that has been mentioned in the introduction is an example. Another example is the Poisson-kernel $\phi(x) = e^{-c\|x\|}$. We can in fact also obtain convergence estimates for the best least squares approximations using these radial basis functions by varying their parameters c with h, as is done in Beatson and Light[**3**] and is reminiscent of the work in Buhmann and Dyn[**12**].

THEOREM 7. *Let ϕ be the Gauss-kernel and assume $c = O(h)$. Let $f \in \mathscr{C}^Q(\mathbb{R}^n)$ be a function which satisfies $|D_x^\alpha f(x)| = O((1+\|x\|)^{-n/2-\epsilon})$ for positive ϵ and $|\alpha| \leqslant Q$, so that the best least squares approximation* (1·3) *with respect to $(\cdot,\cdot)_{q(h\cdot)}$ is well-defined. Suppose also that $|\hat{f}(t)\,\|t\|^n| = O((1+\|t\|)^{-n/2-\epsilon})$. Then* (1·3) *gives, for the inner product* (1·2) *with respect to $q(h\cdot)$,*

$$\|s_h - f\|_2 = O(h^n), \quad h \to 0. \tag{3·12}$$

For the Poisson-kernel and $c = O(h)$ we get the error estimate

$$\|s_h - f\|_2 = O(h^{n+1}), \quad h \to 0, \tag{3·13}$$

if $\hat{f}$ satisfies the above growth condition with n replaced by $n+1$. In both instances, the estimates are uniform in h'.

Proof. In view of the penultimate proof we have, in order to prove (3·12), to consider

$$\frac{h'^{-n}\, e^{-2h^2\|\vartheta\|^2/(4c^2)}\|q(h\vartheta)\|^2}{\sum^* \| \sum_{k\in\mathbb{Z}^n} e^{2\pi i l^T k/L} e^{-\|h\vartheta+2\pi k\|^2/(4c^2)} q(h\vartheta+2\pi k)\|^2}$$

$$= \frac{1}{h'^n \sum^* \|[\|q(h\vartheta)\|]^{-1}\{q(h\vartheta) + \sum' e^{2\pi i l^T k/L} e^{-\|h\vartheta+2\pi k\|^2/(4c^2)+h^2\|\vartheta\|^2/(4c^2)} q(h\vartheta+2\pi k)\}\|^2}. \tag{3·14}$$

Whenever $\|k\|_\infty > 1$ in the dashed sum in the denominator in (3·14), the exponential decay of the summand as $h \to 0$ (and therefore $c \to 0$) provides exponentially decaying contributions to the error which we therefore need not be concerned about. When $\|k\|_\infty = 1$, the terms contribute expressions of the following size to the error:

$$\int_{h^{-1}\mathbb{T}^n} e^{-\pi^2\|k\|^2/c^2 - h\pi\vartheta^T k/c^2}\, d\vartheta = O(h^n),$$

as can be shown by direct computation. The other required estimate (cf. (3·7)) is dealt with in the same way. Estimate (3·13) is a simple consequence of the fact that $\hat{\phi}(h\vartheta)$ is a constant multiple of $((h\vartheta)^2+c^2)^{-(n+1)/2}$ and of the arguments in the proof of Theorem 5. ∎

An analogous theorem for convergence in the $\|\cdot\|_{q(h\cdot)}$ semi-norm can of course also be derived, in the same way as in the proof of Theorem 6, but we omit this simple result.

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4. *Uniform convergence result*

THEOREM 8. *Let K be the largest integer that is less than μ and let $f \in \mathscr{C}^{K+1}(\mathbb{R}^n)$ have ultimate and penultimate total order partial derivatives bounded. Let $Q \leqslant K$. Then the uniform error of the best least squares approximation with respect to the inner product $(\cdot,\cdot)_{q(h\cdot)}$ to f is bounded above by a constant multiple of $h^\mu|\log h|$ if h is small enough. This estimate is uniform in h' and the $|\log h|$ factor can be omitted when $\mu \neq K+1$.*

A result of this kind can also be derived from the fact that the interpolation operator on the space of integer translates of ϕ provides approximation orders such as stated in the theorem and from the fact that the interpolation operator is bounded and a projection (Buhmann[**10**]), but we choose to give a more direct proof for the convenience of the reader.

Proof. Let p be the Kth order Taylor polynomial to f about x. Then we have, by Theorem 3,

$$|s_h(x)-f(x)| = \left| \sum_{k\in\mathbb{Z}^n} \left(f-p, L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)} L_{h'}\left(\frac{x}{h}-k\right)\right|$$

$$\leqslant \sum_{\|x-kh\|<h} \left|\left(f-p, L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)} L_{h'}\left(\frac{x}{h}-k\right)\right| \tag{4·1}$$

$$+ \sum_{h\leqslant\|x-kh\|\leqslant 1} \left|\left(f-p, L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)} L_{h'}\left(\frac{x}{h}-k\right)\right| \tag{4·2}$$

$$+ \sum_{\|x-kh\|>1} \left|\left(f-p, L_{h'}\left(\frac{\cdot}{h}-k\right)\right)_{q(h\cdot)} L_{h'}\left(\frac{x}{h}-k\right)\right|. \tag{4·3}$$

We shall estimate (4·1)–(4·3) separately. As a consequence of Lemma 2, (4·1) can be bounded uniformly in h' by a constant multiple of

$$h'^n \sum_{\|x-kh\|<h} \left| \sum_{j\in\mathbb{Z}^n} \{q(hD_t)[f(t)-p(t)]\}^T_{t=jhh'} \left\{ q(hD_t) L_{h'}\left(\frac{t}{h}-k\right)\right\}_{t=jhh'}\right|$$

$$\leqslant Ch'^n \sum_{\|x-kh\|<h} \sum_{\{j\,|\,\|jh'-k\|>2\}} \sum_{r=1}^{Q} \|x-jhh'\|^{K+1-r} h^r \|jh'-k\|^{-n-\mu-r} \tag{4·4}$$

$$+ Ch'^n \sum_{\|x-kh\|<h} \sum_{\{j\,|\,\|jh'-k\|\leqslant 2\}} \sum_{r=0}^{Q} \|x-jhh'\|^{K+1-r} h^r \tag{4·5}$$

$$+ Ch'^n \sum_{\|x-kh\|<h} \sum_{\{j\,|\,2<\|jh'-k\|<2/h\}} \|x-jhh'\|^{K+1} \|jh'-k\|^{-n-\mu} \tag{4·6}$$

$$+ Ch'^n \sum_{\|x-kh\|<h} \sum_{\{j\,|\,\|jh'-k\|\geqslant 2/h\}} \|x-jhh'\|^{K} \|jh'-k\|^{-n-\mu}, \tag{4·7}$$

where we have made use of (2·12) as well. It is immediate that (4·5) is $O(h^{K+1})$. Expression (4·4) can be bounded by a fixed constant multiple of

$$h'^n \sum_{\|x-kh\|<h} \sum_{\{j\,|\,\|jh'-k\|>2\}} \sum_{r=1}^{Q} (\|x-kh\|+h\|jh'-k\|)^{K+1-r} h^r \|jh'-k\|^{-n-\mu-r}.$$

If we expand the term in parentheses, each term in the resulting expression is a multiple of

$$h'^n \|x-kh\|^m h^{K+1-m} \|jh'-k\|^{-n-\mu+K+1-2r-m}, \quad m=0,1,\ldots,K+1-r, \quad r=1,2,\ldots,Q.$$

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We may sum this over j subject to $\|jh'-k\| > 2$ and obtain that this is $O(h^{K+1})$ as required, since $\|x-kh\| < h$. Here we are making use of the fact that for any positive ϵ

$$h'^n \sum_{\{j \mid \|jh'-k\|>2\}} \|jh'-k\|^{-n-\epsilon} = O(1), \quad h \to 0,$$

as follows from the work in the convergence proof on p. 329 of Buhmann and Micchelli [**15**].

Next, we deal with (4·6) and (4·7). The summands in the former term are, when treated as we did with (4·4), bounded above by constant multiples of terms

$$h'^n \|x-kh\|^m h^{K+1-m} \|jh'-k\|^{-n-\mu+K+1-m}, \quad m = 0, 1, \ldots, K+1. \tag{4·8}$$

When summed over j subject to $\|jh'-k\| \in (2, 2/h)$, this is $O(h^\mu |\log h|)$, as follows from the work in the aforementioned paper, but the $|\log h|$ factor only occurs when $\mu = K+1$. Expression (4·7) is split up into a sum of constant multiples of

$$h'^n \|x-kh\|^m h^{K-m} \|jh'-k\|^{-n-\mu+K-m}, \quad m = 0, 1, \ldots, K. \tag{4·9}$$

When summed over j subject to $\|jh'-k\| \geqslant 2/h$, this is $O(h^\mu)$.

We continue with (4·3). Using (2·12) we can bound it by a fixed constant multiple of

$$h'^n \sum_{\|x-kh\|>1} \sum_{j \in \mathbf{Z}^n} \sum_{r=0}^{Q} \|x-jhh'\|^{K-r} h^r (1+\|jh'-k\|)^{-n-\mu-r} \left\| \frac{x}{h} - k \right\|^{-n-\mu} \tag{4·10}$$

$$\leqslant \sum_{\|x-kh\|>1} \sum_{j \in \mathbf{Z}^n} \sum_{r=0}^{Q} (\|x-kh\| + h\|jh'-k\|)^{K-r} \times h^{r+n+\mu} h'^n (1+\|jh'-k\|)^{-n-\mu-r} \|x-kh\|^{-n-\mu}. \tag{4·11}$$

If we expand the parentheses in (4·11), each resulting term can be bounded above by a multiple of

$$\|x-kh\|^{K-r-m-n-\mu} h^{m+r+n+\mu} h'^n (1+\|jh'-k\|)^{m-n-\mu-r}, \quad m = 0, 1, \ldots, K-r, \quad r = 0, 1, \ldots, Q.$$

When summed over j, this is at most a constant multiple of

$$h^{m+r+n+\mu} \|x-kh\|^{K-r-m-n-\mu}, \quad m = 0, 1, \ldots, K-r, \quad r = 0, 1, \ldots, Q.$$

It follows once more from the work in the convergence proof of Buhmann and Micchelli that this is always $O(h^\mu)$ when summed over k subject to $\|x-kh\| > 1$.

We finish by considering (4·2). This is dealt with in just the same way as (4·1) except that the $\|x/h-k\|$ terms coming from $L_{h'}(x/h-k)$ are accounted for. The expression that replaces (4·4) is bounded above by a constant multiple of a sum of terms

$$h^{m+r+n+\mu} \|x-kh\|^{K+1-r-m-n-\mu}, \quad m = 0, 1, \ldots, K+1-r, \quad r = 1, 2, \ldots, Q,$$

which is to be summed over k subject to $h \leqslant \|x-kh\| \leqslant 1$, and which gives at most an $O(h^{K+1}|\log h|)$ term by invoking the work in the aforementioned paper again (the log term is only present if $\mu = K+1$). Instead of (4·5) we have a constant multiple of

$$h'^n \sum_{h \leqslant \|x-kh\| \leqslant 1} \sum_{\{j \mid \|jh'-k\| \leqslant 2\}} \sum_{r=0}^{Q} \|x-jhh'\|^{K+1-r} h^{n+\mu+r} \|x-kh\|^{-n-\mu}$$

$$\leqslant C \sum_{h \leqslant \|x-kh\| \leqslant 1} \sum_{r=0}^{Q} (\|x-kh\| + 2h)^{K+1-r} h^{n+\mu+r} \|x-kh\|^{-n-\mu}$$

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which expands into a collection of constant multiples of

$$h^{r+m+n+\mu}\|x-kh\|^{K+1-r-m-n-\mu}, \quad m=0,1,\ldots,K+1-r, \quad r=0,1,\ldots,Q.$$

They are $O(h^{K+1}|\log h|+h^{\mu})$ when summed over k subject to $h \leqslant \|x-kh\| \leqslant 1$.

The terms that correspond to the sum of (4·6) plus (4·7) are now

$$Ch'^{n} \sum_{h\leqslant\|x-kh\|\leqslant 1} \sum_{\{j\,|\,2<\|jh'-k\|<2/h\}} \|x-jhh'\|^{K+1}\,\|jh'-k\|^{-n-\mu}h^{n+\mu}\|x-kh\|^{-n-\mu} \quad (4\cdot12)$$

$$+Ch'^{n} \sum_{h\leqslant\|x-kh\|\leqslant 1} \sum_{\{j\,|\,\|jh'-k\|\geqslant 2/h\}} \|x-jhh'\|^{K}\,\|jh'-k\|^{-n-\mu}h^{n+\mu}\|x-kh\|^{-n-\mu}. \quad (4\cdot13)$$

Expression (4·12) can be bounded above by a collection of terms, each multiplied by a positive constant factor, of the form

$$\|x-kh\|^{K+1-m-n-\mu}h^{m+n+\mu}h'^{n}\|jh'-k\|^{m-n-\mu}, \quad m=0,1,\ldots,K+1,$$

which are $O(h^{\mu}|\log h|)$ when summed over j under the constraint $2 \leqslant \|jh'-k\| \leqslant 2/h$ and summed over k subject to $h \leqslant \|x-kh\| \leqslant 1$. Expression (4·13) receives the same treatment and gives $O(h^{\mu}|\log h|)$. Our earlier remarks about the $\log h$ factor apply here and in our previous estimate as well. ∎

5. *Orthonormality between different multiquadric type functions*

In this section, it is proposed that approximation spaces be considered which involve multi-integer translates of several different functions of the form

$$\phi_i(x) = (\|x\|^2+\gamma_i^2)^{\beta/2}, \quad \beta > -n, \quad \tfrac{1}{2}\beta \notin \mathbb{Z}_+, \quad x\in\mathbb{R}^n, \tag{5·1}$$

where γ_i, $i=1,2,\ldots,M$, are different positive parameters. Hence we consider best least squares approximations from the space (and its scales)

$$\text{span}\,\{L_{h,i}(\cdot-k)\}_{i=1,\,k\in\mathbb{Z}^n}^{M}, \tag{5·2}$$

where $L_{h,i}$ are the orthonormal functions (2·2) with respect to ϕ_i. By 'span' we mean in (5·2) the set of all, especially infinite, linear combinations of the generating functions that constitute pointwise absolutely convergent series. Note that we dropped the distinction between h and h' here because we are no longer dealing with both quantities simultaneously. We shall do so from now on and always assume that $h=L^{-1}$.

We require to find generating functions $\tilde{L}_{h,i}$ for the space (5·2) that are orthonormal:

$$(\tilde{L}_{h,i}(\cdot-k),\tilde{L}_{h,j})_q^* = \delta_{0k}\,\delta_{ij}, \quad k\in\mathbb{Z}^n, \quad i,j=1,2,\ldots,M. \tag{5·3}$$

A suitable approach to this is an extension of the final section of the work in Buhmann and Micchelli [**15**] where an orthonormal set of generating functions from the span of $\{\phi_i(\cdot-k)\}_{i=1,\,k\in\mathbb{Z}^n}^{M}$ was found for $\phi_i=\phi(\cdot-\alpha_i)$, $\alpha_i\in[0,1)^n$, ϕ being any of the functions that satisfy the required properties (i)–(v). There, the orthonormality was with respect to the continuous inner product $\langle\cdot,\cdot\rangle_{q=1}$. In that case, the M-vector $\tilde{L}:=\{\tilde{L}_i\}_{i=1}^{M}$ of orthonormal functions is defined componentwise by its Fourier transform

$$\bar{B}^{-1}\hat{\boldsymbol{\phi}}, \tag{5·4}$$

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where $\hat{\boldsymbol{\phi}} = (\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_M)$ and $B\bar{B}^T$ is the Cholesky factorization of the positive definite $M \times M$ matrix

$$A^+ = \{ \sum_{l \in \mathbb{Z}^n} e^{i(\alpha_j - \alpha_k)^T(\cdot + 2\pi l)} |\hat{\phi}(\cdot + 2\pi l)|^2 \}_{j,k=1}^M.$$

Now let a set of functions $\phi_1, \phi_2, \ldots, \phi_M$ be given that all satisfy the assumptions (i)–(v) in the introduction for the same parameters, except that we allow the $\mu = \mu_i$ and $\hat{C} = \hat{C}_i$ to depend on $i = 1, 2, \ldots, M$. We also assume that the $\{\hat{\phi}_i(t + 2\pi \cdot)\}_{i=1}^M$ are linearly independent over $\mathbb{Z}^n$ for all $t \in \mathbb{R}^n \backslash 2\pi\mathbb{Z}^n$, i.e. the only vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M) \in \mathbb{R}^M$ for which there is a $t \in \mathbb{R}^n \backslash 2\pi\mathbb{Z}^n$ such that

$$\sum_{i=1}^M \lambda_i \hat{\phi}_i(t + 2\pi l) = 0, \quad \forall l \in \mathbb{Z}^n, \tag{5·5}$$

is the zero vector. Note for example that the aforementioned set of multiquadric-type functions with different parameters satisfies this condition: inserting the explicit form of $\hat{\phi}_i$, see for instance Buhmann [**10**], and absorbing constant non-zero multipliers into the coefficients, we get from (5·5)

$$\sum_{i=1}^M \lambda_i K_{(n+\beta)/2}(\gamma_i \|t + 2\pi l\|) \, \|t + 2\pi l\|^{-(n+\beta)/2} = 0, \quad \forall l \in \mathbb{Z}^n. \tag{5·6}$$

It follows from (5·6) that $\lambda = 0$, because when we substitute the infinite expansions of the modified Bessel functions (Abramowitz and Stegun [**1**]) into (5·6), we obtain that necessarily

$$\sum_{i=1}^M \lambda_i \gamma_i^\nu = 0,$$

for infinitely many different ν – which can only be true if $\lambda = 0$.

We remark further that this construction can be carried through for many other collections of $\{\phi_i\}_{i=1}^M$, e.g. the multiquadric function and some of it derivatives. This may be particularly useful, as cardinal Hermite interpolation with those functions is always singular (Jetter, private communication), so it may be suitable to replace it by the best least squares approximation with respect to a Sobolev inner product by the multiquadric function and its derivatives. In that event, the $\hat{\boldsymbol{\phi}}$ is replaced by the vector $\{q_j(-i\vartheta)\hat{\phi}(\vartheta)\}$ for the $\{q_j(D_x)\}$ derivative vector of ϕ.

In order to generalize the approach in the Buhmann and Micchelli paper suitably, we now define the following matrices:

$$\mathbf{A}_0 = \{ \sum_{l \in \mathbb{Z}^n} \|q(\cdot + 2\pi l)\|^2 \overline{\hat{\phi}_j(\cdot + 2\pi l)} \, \hat{\phi}_k(\cdot + 2\pi l) \}_{j,k=1}^M$$

and

$$\mathbf{A}_h = \{h^n \sum\nolimits^* [\sum_{k \in \mathbb{Z}^n} e^{-2\pi i l^T k/L} q(\cdot + 2\pi k) \overline{\hat{\phi}_{j_1}(\cdot + 2\pi k)}]^T [\sum_{k \in \mathbb{Z}^n} e^{2\pi i l^T k/L} q(\cdot + 2\pi k) \hat{\phi}_{j_2}(\cdot + 2\pi k)] \}_{j_1, j_2 = 1}^M.$$

THEOREM 9. *$\mathbf{A}_0$ is a positive definite matrix for arguments on $\mathbb{R}^n \backslash 2\pi\mathbb{Z}^n$ and so is $\mathbf{A}_h$ for small enough h. We define vectors*

$$F_j := \bar{B}_j^{-1} \hat{\boldsymbol{\phi}}, \quad j \in \{0, h\}, \tag{5·7}$$

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where $B_j \bar{B}_j^{-T}$ is the Cholesky factorization of $\mathbf{A}_j$. The vectors

$$\tilde{\mathbf{L}}_j = \{\tilde{L}_{j,k}\}_{k=1}^M = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it \cdot T_t} F_j(t)\, dt, \quad j \in \{0, h\}, \tag{5·8}$$

are well-defined, uniformly bounded and in $(\mathscr{C}(\mathbb{R}^n))^M$. Their components satisfy

$$|D_x^\alpha \tilde{L}_{j,k}(x)| = O(\|x\|^{-n-|\alpha|-\max_l \mu_l}), \quad |\alpha| \leqslant Q, \quad k = 1, 2, \ldots, M, \quad j \in \{0, h\}, \tag{5·9}$$

and the orthonormality conditions

$$(\tilde{L}_{j,i}(\cdot - l), \tilde{L}_{j,k})_q^* = \delta_{0l}\, \delta_{ik}, \quad l \in \mathbb{Z}^n, \quad i, k = 1, 2, \ldots, M, \quad j \in \{0, h\}, \tag{5·10}$$

where for $j = 0$ the inner product in (5·10) is replaced by $\langle \cdot, \cdot \rangle q$.

Proof. First, we have to show that $\mathbf{A}_0$ is positive definite. It then follows that also $\mathbf{A}_h$ is positive definite for small enough h by arguments as those employed in the proof of Lemma 2, which show that $\mathbf{A}_h \to \mathbf{A}_0$ uniformly on compact sets in $\mathbb{T}^n \backslash \{0\}$ and entrywise as $h \to 0$.

According to the Hurwitz criterion, we only have to show that all of $\mathbf{A}_0$'s principal minors have positive determinant. Hence we consider

$$\det\{\sum_{l \in \mathbb{Z}^n} \|q(\cdot + 2\pi l)\|^2 \overline{\hat{\phi}_j}(\cdot + 2\pi l)\, \hat{\phi}_k(\cdot + 2\pi l)\}_{j,k=1}^{\hat{M}}, \quad \hat{M} = 1, 2, \ldots, M.$$

According to the Cauchy–Binet formula (Karlin[**20**], p. 1), this is the same as

$$\sum_{l_1 < l_2 < \ldots < l_{\hat{M}}} |\det\{\hat{\phi}_i(\cdot + 2\pi l_j)\, \|q(\cdot + 2\pi l_j)\|\}_{i,j=1}^{\hat{M}}|^2, \quad \hat{M} = 1, 2, \ldots, M. \tag{5·11}$$

Here '$<$' means '$\leqslant$' componentwise but '$\neq$'. Clearly, (5·11) is non-negative. In order to show that it is positive we recall that the $\hat{\phi}_i(t + 2\pi \cdot)$ are linearly independent over $\mathbb{Z}^n$ for each $t \notin 2\pi\mathbb{Z}^n$. Therefore the matrices B_j, $j \in \{0, h\}$, and their inverses are well-defined.

The functions that are the components of the vectors F_j, $j \in \{0, h\}$, are absolutely integrable for the following reasons. First note that it suffices to show that they are uniformly bounded on $\mathbb{R}^n$, because then they must be integrable due to the decay of each $\hat{\phi}_i$ and the 2π-periodicity of the entries of B_j^{-1}. In fact, we only have to show boundedness at zero on account of the assumption (i). Indeed, consider for $j = 0$

$$\begin{aligned} \sum_{l \in \mathbb{Z}^n} (\bar{F}_0^T F_0)(t + 2\pi l)\, \|q(t + 2\pi l)\|^2 &= \sum_{k_1=1}^M \sum_{k_2=1}^M \sum_{l \in \mathbb{Z}^n} \vec{\phi}_{k_1}(t + 2\pi l)\, \{\bar{\mathbf{A}}_0^{-1}\}_{k_1, k_2}(t) \\ &\quad \times \hat{\phi}_{k_2}(t + 2\pi l) \|q(t + 2\pi l)\|^2 \\ &= \sum_{k_1=1}^M \sum_{k_2=1}^M \{\bar{\mathbf{A}}_0^{-1}\}_{k_1, k_2}(t) \{\bar{\mathbf{A}}_0\}_{k_2, k_1}(t) \\ &= M. \end{aligned} \tag{5·12}$$

Hence our claim is true for $j = 0$ and the argument generalizes for $j = h$. The absolute integrability implies boundedness of $\tilde{L}_{j,k}$ and $\tilde{L}_{j,k} \in \mathscr{C}(\mathbb{R}^n)$. The decay estimates (5·9) and the orthonormality conditions (5·10) are established in the same way as in Buhmann and Micchelli[**15**], where it is shown that the latter are a straightforward consequence of the definition of F_j. The former are derived using the arguments of the

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proof of Theorem 1. In order that Lemma 2·7 of Buhmann and Micchelli [**15**] can be applied in this context to the components of each F_j, $j \in \{0, h\}$, one has to note that the entries of $\bar{B}_j^{-1}\hat{\boldsymbol{\phi}}$ are

$$\sum_{l=1}^{M} \frac{\det(\overline{B_{jkl}^*})}{\sqrt{\det \mathbf{A}_j}} \hat{\phi}_l, \quad k = 1, 2, \ldots, M, \tag{5·13}$$

where B_{jkl}^* is formed by replacing the kth column of B_j by the lth standard unit vector. It follows from (iii), (5·11) (for $\hat{M} = M$), and (5·13), that the components of F_j have the singularities at zero which are required by Lemma 2·7 for (5·9) to hold. ∎

Convergence results for these approximants can also be obtained. They are, in the $L^2(\mathbb{R}^n)$, $\|\cdot\|_{q(h\cdot)}$ and in the uniform norm, the same results as in our three convergence results, where the convergence order is defined by $\mu := \max_i \mu_i$. The derivation of these statements is carried out in the same way as the respective proofs in the previous two sections and therefore we dispense with them.

6. *Stability with respect to the discrete Sobolev semi-norm*

Here we shall consider stability estimates for the least squares problem with respect to the inner product (1·2), as in Sivakumar and Ward [**27**], where condition numbers of ('non Sobolev') inner product matrices are considered for a certain, smaller class of radial basis functions. We have the following result. By $l_0(\mathbb{Z}^n)$ we shall denote the sequence space of sequences indexed over $\mathbb{Z}^n$ which have only a finite number of non-zero entries. We are still assuming that ϕ satisfies the conditions set out in the introduction.

THEOREM 10. *Let $y \in l_0(\mathbb{Z}^n)$ be such that*

$$\left| \sum_{j \in \mathbb{Z}^n} y_j e^{-ix^T j} \right| = O(\|x\|^{\mu - n/2 + \epsilon}), \quad \|x\| \to 0, \tag{6·1}$$

where $\epsilon > 0$. Then we have the estimate

$$\Big(\sum_{j \in \mathbb{Z}^n} y_j \phi(\cdot - j), \sum_{k \in \mathbb{Z}^n} y_k \phi(\cdot - k) \Big)_q^* \geqslant \lambda \sum_{j \in \mathbb{Z}^n} |y_j|^2, \tag{6·2}$$

where

$$\lambda := \min_{t \in \mathbb{T}^n} \sum_{j \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \hat{\phi}(t + 2\pi l)\, \hat{\phi}\left(\frac{2\pi j}{h} - t - 2\pi l\right) q(t + 2\pi l)^T q\left(\frac{2\pi j}{h} - t - 2\pi l\right) > 0$$

for small enough h. Indeed, if $\langle \cdot, \cdot \rangle_q$ replaces $(\cdot, \cdot)_q^*$ in (6·2), the estimate holds for

$$\lambda = \min_{t \in \mathbb{T}^n} \sum_{l \in \mathbb{Z}^n} \|q(t + 2\pi l)\|^2 |\hat{\phi}(t + 2\pi l)|^2 > 0. \tag{6·3}$$

Remark. If $z \in l^2(\mathbb{Z}^n)$ is a discrete convolution of $y \in l_0(\mathbb{Z}^n)$ with $t \in l^2(\mathbb{Z}^n)$, y satisfying (6·1), then (6·1) and (6·2) are both true with z replacing y.

Proof [of Theorem 10]. Estimates of this type are considered in Baxter [**2**], but here we follow the ideas of Buhmann and Micchelli [**13**]. By the definition of the inner product we get that the left-hand side of (6·2) is

$$h^n \sum_{j \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} y_k y_l \{q(D_t)\, \phi(t - k)\}_{t - jh}^T \{q(D_t)\, \phi(t - l)\}_{t - jh}.$$

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Because of the moment conditions (6·1), this can be transformed by an application of the Parseval–Plancherel Formula (Stein and Weiss[**28**], p. 17) to

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\sum_{j\in\mathbf{Z}^n}\sum_{k\in\mathbf{Z}^n}\sum_{l\in\mathbf{Z}^n} y_k y_l\,\hat{\phi}(t)\,e^{it^Tk}q(-it)^Tq\left(i\left(\frac{2\pi j}{h}-t\right)\right)\hat{\phi}\left(\frac{2\pi j}{h}-t\right)e^{i(2\pi j/h-t)^Tl}\,dt. \quad (6\cdot4)$$

This integral and the infinite series are absolutely convergent, from Lemma 4 in Buhmann and Dyn[**12**], from (6·1) and from (i)–(iii), thus also confirming the well-definedness of (6·2). Because $h=L^{-1}$, L a positive integer, and because $y\in l_0(\mathbb{Z}^n)$, (6·4) is the same as

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\left|\sum_{k\in\mathbb{Z}^n} y_k\,e^{it^Tk}\right|^2\sum_{j\in\mathbb{Z}^n} q(-it)^Tq\left(i\left(\frac{2\pi j}{h}-t\right)\right)\hat{\phi}(t)\,\hat{\phi}\left(\frac{2\pi j}{h}-t\right)dt$$

or

$$\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\left|\sum_{k\in\mathbb{Z}^n} y_k\,e^{it^Tk}\right|^2\sum_{l\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n} q(-i(t+2\pi l))^Tq\left(i\left(\frac{2\pi j}{h}-t-2\pi l\right)\right)$$

$$\times\,\hat{\phi}(t+2\pi l)\,\hat{\phi}\left(\frac{2\pi j}{h}-t-2\pi l\right)dt. \quad (6\cdot5)$$

We are allowed to take the infinite sum over l inside the integral by the Fubini theorem. The assertion of the theorem now follows easily from Theorem 1·7 on p. 248 of Stein and Weiss[**28**]. The limiting form (6·3) follows from the assumption (ii) on $\hat{\phi}$. The theorem is proved. ∎

Remark. Theorem 10 can be extended to the least squares approximation in the setting of the previous section in a straightforward way. We then get the lower bound

$$\sum_{l=1}^{M}\sum_{m=1}^{M}\Big(\sum_{j\in\mathbb{Z}^n} y_j^l\,\phi_l(\cdot-j),\ \sum_{k\in\mathbb{Z}^n} y_k^m\,\phi_m(\cdot-k)\Big)_q^* \geqslant \min_{t\in\mathbb{T}^n}\lambda(t)\sum_{j=1}^{M}\sum_{j\in\mathbb{Z}^n}|y_j^l|^2, \quad (6\cdot6)$$

where $\lambda(t)$ is the reciprocal of the determinant of the $M\times M$ matrix with the entries

$$\sum_{j\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\hat{\phi}_k(t+2\pi l)\,\hat{\phi}_m\left(\frac{2\pi j}{h}-t-2\pi l\right)q(t+2\pi l)^Tq\left(\frac{2\pi j}{h}-t-2\pi l\right),\quad k,m=1,2,\ldots,M. \quad (6\cdot7)$$

This matrix has positive determinant for small enough h as asserted by Theorem 9. The y^l, $l=1,2,\ldots,M$, are required to be in $l_0(\mathbb{Z}^n)$ and to satisfy the estimate (6·1).

Following Baxter[**2**] we next prove

THEOREM 11. *If $q=1$ and ϕ is radially symmetric, so that its distributional Fourier transform is as well, and satisfies*

$$\hat{\phi}(x)=\int_0^\infty e^{-\|x\|^2/t}\,d\eta(t),\quad x\in\mathbb{R}^n\backslash\{0\}, \quad (6\cdot8)$$

for some non-decreasing measure η, then

$$\lambda\geqslant\sum_{l\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\hat{\phi}(\pi+2\pi l)\,\hat{\phi}\left(\pi+2\pi l-\frac{2\pi j}{h}\right)>0,$$

where $\pi=(\pi,\pi,\ldots,\pi)$.

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We remark that (6·8) is, for instance, valid for the multiquadric function (Buhmann and Micchelli [13]).

Proof. Under the assumptions of the theorem we have

$$\sum_{l\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\hat{\phi}(t+2\pi l)\,\hat{\phi}\left(t+2\pi l-\frac{2\pi j}{h}\right)$$

$$=\sum_{l\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\int_0^\infty\int_0^\infty e^{-\|t+2\pi l\|^2/t_1-\|t+2\pi l-2\pi j/h\|^2/t_2}\,d\eta(t_1)\,d\eta(t_2)$$

$$=\int_0^\infty\int_0^\infty\sum_{l\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}e^{-\|t+2\pi l\|^2/t_1-\|t+2\pi l-2\pi j/h\|^2/t_2}\,d\eta(t_1)\,d\eta(t_2).$$

Now the assertion follows from Baxter's result that

$$\min_{t\in\mathbb{T}^n}\sum_{j\in\mathbb{Z}^n}e^{-\|t+2\pi j\|^2/t_i}=\sum_{j\in\mathbb{Z}^n}e^{-\|\pi+2\pi j\|^2/t_i},\quad i=1,2.$$

The theorem is proved. ∎

7. *Univariate prewavelets with respect to a discrete Sobolev semi-norm*

The work of the previous sections also suggests some remarks about prewavelets with radial basis functions and with respect to discrete Sobolev inner products. We exemplify this idea in the univariate setting in an approach that is tailored to fit the multiquadric radial function. It is, however, also suitable for other radial functions that satisfy the assumptions given in the introduction. For the sake of generality, we shall initially consider not just equally spaced centres but non-equally spaced ones as well and prove some useful results about spaces spanned by radial functions with non-equally spaced centres, but we shall shortly revert to the equally spaced setting.

Specifically, let two sequences of real numbers

$$\mathbf{x}=\{\ldots<x_{-1}<x_0<x_1<x_2<\ldots\}$$

and

$$\boldsymbol{\tau}=\{\ldots<\tau_{-1}<\tau_0<\tau_1<\tau_2<\ldots\}$$

with $\tau_{2i-1}=x_i$, $i\in\mathbb{Z}$, be given. We let $\{B_j^c\}_{j=-\infty}^{\infty}$ and $\{B_j^f\}_{j=-\infty}^{\infty}$ be the sequences of B-splines of degree ν on the knot-sequences $\mathbf{x}$ and $\boldsymbol{\tau}$, respectively, with supports $[x_j,x_{j+\nu+1}]$ and $[\tau_j,\tau_{j+\nu+1}]$, respectively (cf. de Boor [6] for instance). They are assumed to be normalized to form a partition of unity. Let $\Phi\in\mathscr{C}^Q(\mathbb{R})$ with

$$|\Phi^{(l)}(x)|=O(|x|^{-1-\epsilon}),\quad l=0,1,\ldots,Q, \tag{7·1}$$

and $\hat{\Phi}(t)>0$, for all $t\in\mathbb{R}$, and $\hat{\Phi}(0)=1$ be given. We pause for an example: $\Phi(x)=\frac{1}{2}\phi''(x)$, where $\phi(x)$ is the multiquadric function $\sqrt{(x^2+c^2)}$ satisfies these assumptions. We consider, in analogy to Buhmann and Micchelli [14], the functions

$$\left.\begin{aligned}C_j&:=B_j^c*\Phi,\quad j\in\mathbb{Z},\\F_j&:=B_j^f*\Phi,\quad j\in\mathbb{Z}.\end{aligned}\right\} \tag{7·2}$$

In (7·2), $*$ denotes convolution. Thus, C_j and F_j are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, because the B-splines are and because $\Phi\in L^1(\mathbb{R})$ (Stein and Weiss [28], Theorem 1·3 on p. 3). We let H_2^Q be the space of functions $f\in\mathscr{C}^Q(\mathbb{R})$ for which $\|f\|_Q^*=\sqrt{(f,f)_Q^*}$ is finite. In particular, C_j and F_j are in H_2^Q by (7·1) and Lemma 4 in Buhmann and Dyn [12].

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With the choice of Φ as suggested in our example and $\nu = 1$, C_j and F_j are second divided differences of the multiquadric function with respect to the sequences $\mathbf{x}$ and $\boldsymbol{\tau}$, respectively (see also Powell [21]). We note that if Φ is the δ-distribution, then we will get the spline prewavelets considered in Buhmann and Micchelli [14]. We now define

$$\left.\begin{aligned} V_0 &:= \left\{ \sum_{j=-\infty}^{\infty} d_j C_j \,|\, d \in l^2(\mathbb{Z}) \right\}, \\ V_1 &:= \left\{ \sum_{j=-\infty}^{\infty} d_j F_j \,|\, d \in l^2(\mathbb{Z}) \right\}, \end{aligned}\right\} \tag{7·3}$$

where $d = \{d_j\}_{j=-\infty}^{\infty}$. We shall prove some elementary facts about the spaces defined in (7·3) and we shall then proceed to construct prewavelets that span the orthogonal complement W_0 of V_0 in V_1. After that we shall analyse the asymptotic properties of these prewavelets. We begin with

PROPOSITION 12. $V_0 \subset V_1 \subset H_2^Q \cap L^2(\mathbb{R})$.

Proof. Let $d \in l^2(\mathbb{Z})$. Hence the function

$$V_0 \ni f = \sum_{j=-\infty}^{\infty} d_j \int_{-\infty}^{\infty} B_j^c(y)\, \Phi(\cdot - y)\, dy \tag{7·4}$$

is defined by an absolutely convergent series that is bounded in modulus by a multiple of $\|d\|_\infty$, because $\Phi \in L^1(\mathbb{R})$ and because the B-splines form a partition of unity. Recall that there exist positive constants λ and Λ such that

$$\lambda \|d\|_p \leqslant \left\| \sum_{j=-\infty}^{\infty} d_j B_j^c \right\|_p \leqslant \Lambda \|d\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{7·5}$$

where $\|\cdot\|_p$ denotes the $l^p(\mathbb{Z})$ or the $L^p(\mathbb{R})$ norm, as is appropriate in the context (de Boor [5]). It follows from (7·5) with $p = \infty$ that we may exchange summation and integration in (7·4) and so it is the same as

$$f = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j B_j^c(y)\, \Phi(\cdot - y)\, dy$$

which can be re-written by the Curry–Schoenberg theorem (cf. de Boor [6], p. 113) and (7·5) as

$$f = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d'_j B_j^f(y)\, \Phi(\cdot - y)\, dy$$

with $d' \in l^2(\mathbb{Z})$. Hence, reverting back to the form of f with the sum outside the integral, $f \in V_1$. As a final remark, we point out that it follows from (7·5) with $p = 2$, from (7·1) and from Theorem 1·3 on p. 3 of Stein and Weiss [28] that $V_1 \subset H_2^Q \cap L^2(\mathbb{R})$. ∎

THEOREM 13. *Let $\{\boldsymbol{\tau}^k\}_{k=-\infty}^{\infty}$ be a set of nested knot sequences in $\mathbb{R}$ (i.e.*

$$\boldsymbol{\tau}^k = \{\ldots < \tau_{-1}^k < \tau_0^k < \tau_1^k < \tau_2^k < \ldots\}$$

and $\tau_{2i-1}^{k+1} = \tau_i^k$ for all integers k and i) that satisfy

$$\sup_{-\infty < j < \infty} (\tau_{j+1}^k - \tau_j^k) \to 0, \quad k \to +\infty,$$

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and
$$\inf_{-\infty<j<\infty}(\tau_{j+1}^k-\tau_j^k)\to\infty,\quad k\to-\infty.$$

Let V_k be defined according to this setting and to V_1 in (7·2)–(7·3), *where the $\{B_j^{f,k}\}_{j=-\infty}^{\infty}$, which are the B-splines on $\boldsymbol{\tau}^k$, replace the $\{B_j^f\}_{j=-\infty}^{\infty}$. Then*

$$\left.\begin{aligned}\overline{\bigcup_{k=-\infty}^{\infty}V_k}&=L^2(\mathbb{R}),\\ \bigcap_{k=-\infty}^{\infty}V_k&=\{0\}.\end{aligned}\right\}\tag{7·6}$$

Proof. The second claim follows immediately from the analogous statement about B-splines in Buhmann and Micchelli [**14**]. As the Fourier transform is an isometrical isomorphism $L^2(\mathbb{R})\to L^2(\mathbb{R})$, we may prove the first claim in (7·6) in the Fourier domain. Let $\hat{h}\in\mathscr{C}_0(\mathbb{R})$, the space of compactly supported continuous function on the reals; $\mathscr{C}_0(\mathbb{R})\subset L^2(\mathbb{R})$ is dense. We need to show that $\hat{h}$ can be approximated arbitrarily closely in $L^2(\mathbb{R})$ by a function of the form $\sum_{j=-\infty}^{\infty}d_j\widehat{B_j^{f,k}}\hat{\Phi}$ if k is large enough, where $d\in l^2(\mathbb{Z})$. As $\hat{\Phi}(t)>0$, it follows that $\hat{h}/\hat{\Phi}\in\mathscr{C}_0(\mathbb{R})$. This can be approximated arbitrarily closely by $\sum_{j=-\infty}^{\infty}d_i\widehat{B_j^{f,k}}$ if k is large enough, because a sequence of spline-spaces of fixed degree defined on knot sequences that become dense in $\mathbb{R}$, is dense in $L^2(\mathbb{R})$. ∎

Remark. $L^2(\mathbb{R})$ contains a dense subset of H_2^Q, so (7·6) also means that the union of the V_k is dense in H_2^Q.

Proposition 14. *If $\Phi\geqslant 0$, the assumptions on Φ of the beginning of this section hold h, if $q=1$ and if*
$$\sup_{-\infty<j<\infty}(x_{j+1}-x_j)\leqslant 1,$$
then the $l^1(\mathbb{Z})$ and $l^\infty(\mathbb{Z})$ norms of $A=\{(C_j,C_k)_q^*\}_{j,k=-\infty}^{\infty}$ are both less than two.

Proof. Because A is symmetric, we only have to show this for $l^1(\mathbb{Z})$. We get, by (Schumaker [**26**], p. 129), because the C_j form a partition of unity and because all derivatives disappear from the inner product (2·1),

$$\begin{aligned}\|A\|_1&=\sup_{-\infty<j<\infty}h\sum_{l=-\infty}^{\infty}C_j(lh)\\&<2\sup_{-\infty<j<\infty}\int_{-\infty}^{\infty}C_j(x)\,dx\\&=2\sup_{-\infty<j<\infty}(x_{\nu+j+1}-x_j)/(\nu+1)\\&\leqslant 2,\end{aligned}$$

as required. ∎

This observation and the positivity of A's entries subject to $\Phi\geqslant 0$ imply in particular that A may be inverted by a Neumann series. Hence it is a homeomorphism $l^p(\mathbb{Z})\to l^p(\mathbb{Z})$ for $p=1$ and $p=\infty$ and hence for $p\in[1,\infty]$ by the Riesz convexity theorem (Bennett and Sharpley [**4**], p. 194).

From now on, we take $\mathbf{x}=\mathbb{Z}$ and $\boldsymbol{\tau}=\frac{1}{2}\mathbb{Z}$. We prove a result that shows that $A: l^2(\mathbb{Z})\to l^2(\mathbb{Z})$ is a homeomorphism uniformly when $h\to 0$ for general q.

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THEOREM 15. *Let ϕ satisfy the assumptions of the introduction for $\mu \leqslant \nu+1$, and suppose $\Phi = \phi^{(\nu+1)}$ satisfies the assumptions set out at the beginning of this section. Then, for small enough h, the Gramian matrix $A = \{(C_j, C_k)_q^*\}_{j,k=-\infty}^{\infty}$, is a homeomorphism between $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ uniformly as $h \to 0$, i.e. the norm of its inverse is uniformly bounded for $h \to 0$.*

Proof. The hypotheses of the theorem imply that each C_j is a positive multiple of a $(\nu+1)$st order divided difference of integer translates of ϕ, because one way of defining B-splines is to define them as the Peano-kernel of divided differences. So, in particular, the coefficient vector of this finite linear combination of $\{\phi(|\cdot - j|)\}_{j=-\infty}^{\infty}$, call it y, satisfies (6·1) for $\epsilon = \frac{3}{2}+\nu-\mu$, $n = 1$. Thus, the quadratic form $z^T A z$ can, for any $z \in l^2(\mathbb{Z})$, be bounded below by $\lambda z^T z$, by Theorem 10, λ being a positive constant (if h is small enough) from that theorem. Hence $\|A^{-1}\|_2 < \infty$. ∎

THEOREM 16. *Let the assumptions of Theorem 15 hold and let $P: V_1 \to V_0$ be the orthogonal projection with respect to the inner product $(\cdot,\cdot)_q^*$. Let $\delta > 1$, where δ is a constant from the introduction (requirement (iii)). Then $\psi_l := F_l - PF_l$, $l \in \mathbb{Z}$, is a prewavelet, i.e. it satisfies*

(a) $\psi_l \in V_1$, $\psi_l \perp V_0$ (with respect to the inner product $(\cdot,\cdot)_q^*$),

(b) $V_1 = V_0 \oplus W_0$ with

$$W_0 := \left\{ \sum_{j=-\infty}^{\infty} d_j \psi_j \,|\, d \in l^2(\mathbb{Z}) \right\}.$$

Proof. We observe that

$$PF_l = \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} A_{kj}^{-1} (F_l, C_j)_q^* \right\} C_k \tag{7·7}$$

if the right-hand side of (7·7) is in V_0. It follows from $\lceil\delta\rceil$-fold integration by parts and the Riemann–Lebesgue Lemma (Stein and Weiss[**28**], p. 2) applied to the formulae

$$C_j(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\hat{\Phi}(t)\hat{B}_j^c(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\hat{\Phi}(t)\,e^{-itj}\left(\frac{\sin(\frac{1}{2}t)}{(\frac{1}{2}t)}\right)^{\nu+1} dt$$

and

$$F_l(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\hat{\Phi}(t)\hat{B}_l^f(t)\,dt = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ixt}\hat{\Phi}(t)\,e^{-it(l+1)/2}\left(\frac{\sin(\frac{1}{4}t)}{(\frac{1}{4}t)}\right)^{\nu+1} dt$$

that C_j and F_l satisfy the estimates

$$|C_j(x)| = o((1+|x-j|)^{-\lceil\delta\rceil}), \quad |F_l(x)| = o((1+|x-l|)^{-\lceil\delta\rceil}).$$

Here we are using the fact, due to (ii) and (iii), that the derivatives of $\hat{\Phi}\hat{B}_l^f$ and $\hat{\Phi}\hat{B}_j^c$ up to and including order $\lceil\delta\rceil$ are absolutely integrable. The derivatives of F_l and C_j of order at most Q satisfy these decay estimates as well, as can be shown by integration by parts and the Riemann–Lebesgue Lemma. Hence

$$|(F_l, C_j)_q^*| = O((1+|j-l|)^{-\lceil\delta\rceil}), \tag{7·8}$$

by Lemma 4 of Buhmann and Dyn[**12**]. Now, $\{(F_l, C_j)_q^*\}_{j=-\infty}^{\infty} \in l^2(\mathbb{Z})$ and thus, by Theorem 15, (7·7) is indeed in V_0. Hence, Proposition 12 implies that (*a*) is true. Assertion (*b*) becomes a consequence of the definition of ψ_l as soon as we show that

$$\left\{ \sum_{l=-\infty}^{\infty} c_l \sum_{j=-\infty}^{\infty} A_{kj}^{-1}(F_l, C_j)_q^* \right\}_{k=-\infty}^{\infty} \in l^2(\mathbb{Z}), \quad \forall c \in l^2(\mathbb{Z}).$$

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Because of the Cauchy–Schwarz inequality and Theorem 15 we can exchange the sums above and thus by Theorem 15 we have, in order to settle this, only to confirm that

$$\left\{\left(\sum_{l=-\infty}^{\infty} c_l F_l, C_j\right)_q^*\right\}_{j=-\infty}^{\infty} \in l^2(\mathbb{Z}). \tag{7·9}$$

Indeed, (7·8), $\delta > 1$ and Theorem 1·3 on p. 3 of Stein and Weiss[**28**], imply (7·9). ∎

THEOREM 17. *The prewavelets of Theorem* 16 *satisfy*

$$|\psi_l(x)| = o((1+|x-l|)^{-\lceil\delta\rceil}).$$

Proof. We may take $l = 0$ because $\psi_l = \psi_0(\cdot - l)$. It is straightforward to work out – in the same vein as in the proof of Theorem 1 – that the orthogonality conditions imply that ψ_0 is the inverse Fourier transform of

$$\hat{\psi}_0(\vartheta) = \hat{C}_0(\tfrac{1}{2}\vartheta) - \hat{C}_0(\vartheta)$$

$$\times \frac{\Sigma^* \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{2\pi i l(j-k)/L}\, \hat{\phi}(\tfrac{1}{2}\vartheta+\pi j)\, \hat{\phi}(-\vartheta-2\pi k)\, q(\vartheta+2\pi j)^T q(\vartheta+2\pi k)}{\Sigma^* \left\| \sum_{k=-\infty}^{\infty} e^{-2\pi i l k} \hat{\phi}(\vartheta+2\pi k)\, q(\vartheta+2\pi k) \right\|^2}, \quad \vartheta \in \mathbb{R}. \tag{7·10}$$

It follows from this and from $\lceil\delta\rceil$-fold integration by parts and the Riemann–Lebesgue Lemma that the theorem is true. ∎

THEOREM 18. *Given the set-up of Theorem* 13, *we can form prewavelets* $\{\psi_{lk}\}_{l=-\infty}^{\infty}$, $k \in \mathbb{Z}$, *in* V_{k+1} *and spaces* W_k *spanned by sums of the prewavelets with square-summable coefficients such that*

$$H_2^Q / \{f \in H_2^Q \mid \|f\|_q^* = 0\} = \bigoplus_{k=-\infty}^{\infty} W_k. \tag{7·11}$$

Proof. This is a standard argument as already pointed out in Buhmann and Micchelli[**14**]. Let $f \in H_2^Q$ be orthogonal to all W_k and let a positive ϵ be given. By (7·6), there are $k \in \mathbb{Z}$ and $f_k \in V_k$ such that

$$\|f - f_k\|_q^{*2} \leqslant \epsilon.$$

Hence, in particular,
$$\|f\|_q^{*2} - 2(f, f_k)_q^* \leqslant \epsilon. \tag{7·12}$$

Inequality (7·12) and f's orthogonality to all W_k imply

$$\|f\|_q^{*2} - 2(f, f_l)_q^* \leqslant \epsilon, \quad \forall l < k, \tag{7·13}$$

where
$$f_{l+1} = f_l + g_l, \quad f_l \in V_l, g_l \in W_l, \quad \forall l < k. \tag{7·14}$$

It follows from (7·14) that $\|f_l\|_q^* \leqslant \|f_k\|_q^*$ for all l, so $\liminf_{l\to-\infty} \|f_l\|_q^* = 0$ by (7·6). Therefore, (7·13) and the arbitrariness of $\epsilon > 0$ imply (7·11). ∎

We remark finally that the expression (7·10) and Lemma 2·7 in the paper [**15**] of Buhmann and Micchelli also imply that the decomposition coefficients, i.e. the coefficients of the C_j in the expansion of PF_l, also decay at the same rate as the prewavelet does.

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