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EXPLICIT RATIONAL FUNCTIONS WHOSE JULIA SETS ARE JORDAN ARCS

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ABSTRACT

A class of rational functions and sufficient conditions on it such that the Julia set is a Jordan arc, respectively no Jordan arc, are introduced. Further, it is shown that any rational function which has a Jordan arc as Julia set belongs up to Möbius conjugation to this class.

1. *Introduction*

Let $R = P/Q$ be a rational function where the coefficients of the two polynomials P and Q are in $\mathbb{C}$ and $\deg R := \max\{\deg P, \deg Q\} \geqslant 2$. As usual, we denote the Fatou set of R by $\mathcal{F}_R$ and the Julia set by $\mathcal{J}_R$. For basic properties of Julia and Fatou sets we refer to [**1**, **2**, **8**].

For an attracting fixed point z_0 of R, the immediate basin of attraction is denoted by $A^*(z_0)$ and the basin of attraction is denoted by $A(z_0)$.

Two rational functions R and S are said to be conjugate if there is some Möbius map $M(z) = (az+b)/(cz+d)$, $ad-bc \neq 0$, such that $S = M \circ R \circ M^{-1}$.

DEFINITION 1.1. A *Jordan arc* $\gamma \subset \bar{\mathbb{C}}$ is the image of the interval $[0,1]$ under a homeomorphism ϕ. If ϕ has a meromorphic extension in a neighbourhood of $[0,1]$, then γ is called an *analytic Jordan arc*.

In this paper we shall study the problem of how to describe explicitly the class of rational functions which have Jordan arcs as their Julia sets. Note that the Julia set of a rational function cannot consist of several disjoint Jordan arcs (see [**14**, Theorem A]). For analytic Jordan arcs, this problem was first studied by Fatou [**4**, p. 225] in 1920, who showed that if the Julia set $\mathcal{J}_R$ of a rational function R is an analytic Jordan arc, then $\mathcal{J}_R$ is an arc of a circle or an interval. For polynomials, $\mathcal{J}_P$ is an analytic Jordan arc if and only if $\mathcal{J}_P$ is a Jordan arc, since a polynomial P of degree n whose Julia set is a Jordan arc is conjugate to $\pm T_n$, where $T_n(z) = \cos(n \arccos z)$ denotes the Chebyshev polynomial of the first kind of degree n (see [**16**, p. 143]). Recently, Steinmetz [**17**] gave a description of the rational functions whose Julia sets are Jordan arcs, using quasiconformal mappings. More precisely, he proved that if $\mathcal{J}_R$ is a Jordan arc, then $\mathcal{J}_R$ is the quasiconformal image of an interval and moreover R is quasiconformally conjugate to $\pm T_n$ or to $(1/n^2)(M^{-1} \circ T_n \circ M)$, where $M(z) = (1-z)/(1+z)$. Since one knows the existence of the conjugating quasiconformal map only, an explicit description, respectively

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representation, of the rational functions whose Julia sets are Jordan arcs are still open, and will be determined to a great extent in this paper.

This paper is organised as follows. In Section 2, we introduce a class of rational functions and give sufficient conditions on it such that the Julia set is a non-analytic Jordan arc, respectively no Jordan arc. It turns out that there are even real rational functions whose Julia sets are non-analytic Jordan arcs. In Section 3, we show that any rational function which has a Jordan arc as Julia set is up to conjugation of the form considered in Section 2. In Section 4, the rationally indifferent case, which will turn out to be rather delicate, is studied. We show how to perturb in a suitable and constructive way a rational function which has a rationally indifferent fixed point, and whose Julia set is an analytic Jordan arc, to obtain explicitly a rational function with a rationally indifferent fixed point and with a non-analytic Jordan arc as Julia set.

A first, much weaker version of this paper appeared in the preprint series [**6**] and a sequel was also included in the unrefereed thesis [**5**] of the first author supervised by the second author. In the first version [**6**], a weaker variant of Theorem 2.3 below is stated (the given bound on the zeros of ρ_ν does not depend on n), the condition sufficient for the Julia set not to be a Jordan arc (that is Theorem 2.5 below) is missing, and therefore so is the rather precise description of the class of rational functions whose Julia sets are Jordan arcs. Furthermore, a much simpler approach to the derivation of the necessary conditions for the Julia set to be a Jordan arc is given below in Section 3. Finally, for the parabolic case, a general description of the rational functions whose Julia sets are Jordan arcs with an endpoint which is a rationally indifferent fixed point has also been found (see Corollary 4.9).

We would like to point out that our joint paper [**7**] was submitted before the approval of the thesis [**5**] (in which another version of [**7**] was included with the permission of the second author), and it was extended during the refereeing process by the section entitled 'New polynomials whose Julia sets are dendrites'.

2. *Sufficient conditions for the Julia set to be a Jordan arc*

NOTATION 2.1. Let $\rho_\nu(z)=\prod_{k=1}^{\nu}(z-z_k)$ be a monic complex polynomial of degree $\nu\in\mathbb{N}\cup\{0\}$, put $\rho_\nu^*(z)=z^\nu\rho_\nu(1/z)$, and let $\rho_\nu^2(z)=\sum_{k=0}^{2\nu}c_{2\nu-k}z^{2\nu-k}$, where $c_{2\nu}=1$. For $n\in\mathbb{N}\cup\{0\}$, we define

$$\mathcal{R}_n^{\pm}(y,\rho_\nu)=\pm\frac{\sum_{k=0}^{2\nu}c_{2\nu-k}T_{|n-k|}(y)}{\prod_{k=1}^{\nu}\left(1+z_k^2-2z_ky\right)}$$
$$=\pm\frac{\sum_{k=0}^{2\nu}c_{2\nu-k}T_{|n-k|}(y)}{\rho_\nu(y-\sqrt{y^2-1})\rho_\nu(y+\sqrt{y^2-1})}. \tag{1}$$

Note that $\mathcal{R}_n^{\pm}$ is of degree $d=\max\{n,2\nu-n\}$; in particular, $d=n$ if $n\geqslant\nu$. If we put

$$z=y-\sqrt{y^2-1},\quad \text{that is, } y=\frac{1}{2}\left(z+\frac{1}{z}\right)=:J(z), \tag{2}$$

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where J is the well known Joukowski map, and recall that

$$T_n(y) = \tfrac{1}{2}(z^n + z^{-n})$$

(many other properties of Chebyshev polynomials can be found in [**15**], then it follows by straightforward calculations that $\mathcal{R}_n^{\pm}$ can be represented in the form

$$\mathcal{R}_n^{\pm}(y, \rho_\nu) = J \circ g_{n,\nu}^{\pm} \circ J^{-1}(y), \tag{3}$$

where

$$\begin{aligned} g_{n,\nu}^{\pm}(z) &= \pm z^{n-\nu} \frac{\rho_\nu(z)}{\rho_\nu^*(z)} \\ &= \pm z^{n-\nu} \prod_{k=1}^{\nu} \frac{z - z_k}{1 - z z_k}. \end{aligned} \tag{4}$$

Because $g_{n,\nu}^{\pm}(1/z) = 1/g_{n,\nu}^{\pm}(z)$, it does not matter which branch of J^{-1} is taken in (3). Note that if $n \geqslant \nu$ and all zeros z_k are contained in the unit disc $\mathbb{D}$, then $g_{n,\nu}^{\pm}$ becomes a Blaschke product if and only if ρ_ν is a real polynomial.

To state the results in full generality, it is more convenient to consider the slightly more general class

$$\begin{aligned} \mathcal{R}_n^{\pm}(y, \rho_\nu, a, b) &:= \frac{a+b}{2} + \frac{a-b}{2} \mathcal{R}_n^{\pm}\left(\frac{2y}{a-b} - \frac{a+b}{a-b}, \rho_\nu \right) \\ &= l \circ \mathcal{R}_n^{\pm}(\cdot, \rho_\nu) \circ l^{-1}(y) \\ &= \tilde{J} \circ g_{n,\nu}^{\pm} \circ \tilde{J}^{-1}(y), \end{aligned} \tag{5}$$

where $a, b \in \mathbb{C}$, $a \neq b$, and

$$l(y) = \frac{a-b}{2} y + \frac{a+b}{2}, \quad \text{and } \tilde{J} = l \circ J. \tag{6}$$

Since $\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot, \rho_\nu, a, b)} = \tilde{J}(\mathcal{J}_{g_{n,\nu}^{\pm}})$, it suffices to study the dynamical behaviour of $g_{n,\nu}^{\pm}$. Furthermore, it is no restriction to investigate only the case $n \geqslant \nu$, because if $n < \nu$, then $g_{n,\nu}^{\pm} \circ g_{n,\nu}^{\pm} = \tilde{g}_{\tilde{n},\tilde{\nu}}^{\pm}$, where $\tilde{n} > \tilde{\nu}$.

Next we state a fact that will be very helpful in what follows.

LEMMA 2.2. *$\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot, \rho_\nu, a, b)}$ is a Jordan arc with endpoints a and b if and only if $\mathcal{J}_{g_{n,\nu}^{\pm}}$ is a simple closed curve which contains the points ± 1.*

Proof. The assertion follows by representation (5). □

By Lemma 2.2, we are able to show, even in a simple way, that under some restrictions on the location of the zeros of ρ_ν, rational functions of the type (1) and their linear conjugates (5) have Jordan arcs as Julia sets.

THEOREM 2.3. *Let $a, b \in \mathbb{C}$, $a \neq b$, $n, \nu \in \mathbb{N} \cup \{0\}$, $n \geqslant 2$ and $n > \nu$, and suppose that all zeros of ρ_ν lie in the disc $\{|z| \leqslant r\}$, where*

$$0 < r^2 \leqslant \max\left\{ 3 - 2\sqrt{2}, 1 - \sqrt{\frac{2\nu}{n-1}} \right\}.$$

Then $\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot, \rho_\nu, a, b)}$ is a Jordan arc with endpoints a and b.

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Furthermore, $\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot,\rho_\nu,a,b)}$ is an analytic Jordan arc if and only if ρ_ν is a real polynomial.

Proof. By Lemma 2.2, to prove the first statement, it suffices to show that

$$|g_{n,\nu}^{\pm}(z)| < |z| \qquad \text{for } |z| \leqslant r. \tag{7}$$

Indeed, since by assumption all zeros of ρ_ν lie in the disc $|z| \leqslant r$, relation (7) implies that all preimages of the attracting fixed point $z=0$ are in $A^*(0)$. Thus $A^*(0) = A(0)$ and $A^*(0)$ is completely invariant, which implies by the $(1/z)$-symmetry that $A^*(\infty) = A(\infty)$ is completely invariant too. Therefore (see for example [**1**, p. 94]), $A^*(0)$ and $A^*(\infty)$ are simply connected, and there are no other components of $\mathcal{F}_{g_{n,\nu}^{\pm}}$. By [**3**, Theorem VI.2.1], $\mathcal{J}_{g_{n,\nu}^{\pm}}$ is a simple closed curve, even a quasicircle. Because

$$g_{n,\nu}^{\pm}(\{-1,1\}) \subset \{-1,1\}$$

and $|(g_{n,\nu}^{\pm})'(\pm 1)| > 1$, this Jordan curve contains the points ± 1. We now prove inequality (7). By the representation (4) of $g_{n,\nu}^{\pm}$, relation (7) is satisfied if

$$r^{n-\nu-1}\left(\frac{2r}{1-r^2}\right)^{\nu} < 1. \tag{8}$$

Note that inequality (8) holds for each $n, \nu \in \mathbb{N} \cup \{0\},\ n > \nu$, if $2r/(1-r^2) \leqslant 1$, that is, if $r^2 \leqslant 3 - 2\sqrt{2}$. To show that the second bound on the zeros is sufficient, we put $r^2 = 1 - x,\ x \in (0,1)$, and $\lambda = (n-1)/2\nu$. Then (8) becomes

$$(1-x)^{\lambda} < \frac{x}{2}. \tag{9}$$

Since $(1-x)^{\lambda} < 1/(1+\lambda x)$ for $\lambda > 0,\ x \in (0,1)$, which can be obtained from the well known inequality

$$\frac{\beta^{\lambda+1} - \alpha^{\lambda+1}}{\beta - \alpha} > (\lambda+1)\alpha^{\lambda} \qquad \text{for } \beta > \alpha > 0,\ \lambda > 0,$$

if we put $\beta = 1$ and $\alpha = 1 - x$, (9) is certainly fulfilled if

$$\frac{1}{1+\lambda x} \leqslant \frac{x}{2},$$

that is, for $x \geqslant (-1 + \sqrt{1+8\lambda})/2\lambda \geqslant 1/\sqrt{\lambda}$, which is the assertion.

By Lemma 2.2 and by [**4**, p. 225], $\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot,\rho_\nu,a,b)}$ is an analytic Jordan arc if and only if $\mathcal{J}_{g_{n,\nu}^{\pm}}$ is a circle, that is, if and only if $g_{n,\nu}^{\pm}$ is a Blaschke product whose Julia set $\mathcal{J}_{g_{n,\nu}^{\pm}}$ is the unit circle $\partial\mathbb{D}$, or if $g_{n,\nu}^{\pm}$ is conjugate to a Blaschke product B with $\mathcal{J}_B = \partial\mathbb{D}$. Using $\{-1,1\} \subset \mathcal{J}_{g_{n,\nu}^{\pm}}$, the $(1/z)$-symmetry and the fact that the origin is an attracting fixed point, it follows that the only possible conjugation is the identity. Hence $g_{n,\nu}^{\pm}$ must be a Blaschke product, or equivalently ρ_ν is a real polynomial. □

We mention that if in Theorem 2.3 ρ_ν is real, then, since ρ_ν has all zeros in the open unit disc, $\mathcal{J}_{g_{n,\nu}^{\pm}}$ is the unit circle $\partial\mathbb{D}$ and hence $\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot,\rho_\nu,a,b)}$ is the interval with endpoints a and b. From Theorem 2.3, we obtain immediately a class of real rational functions which have non-analytic Jordan arcs as Julia sets.

COROLLARY 2.4. *Let $a \in \mathbb{C}\backslash\mathbb{R}$ and let $n \geqslant 3$ be odd. Suppose that the assumptions of Theorem* 2.3 *are satisfied and that ρ_ν is a non-real monic polynomial*

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which is symmetric with respect to the imaginary axis, that is, $\rho_\nu(-\bar{z}) = -\overline{\rho_\nu(z)}$. *Then* $\mathcal{R}_n^\pm(\cdot, \rho_\nu, a, \bar{a})$ *is a real rational function whose Julia set is a non-analytic Jordan arc with endpoints* a *and* $\bar{a}$.

Proof. By Theorem 2.3, what remains to be shown is that $\mathcal{R}_n^\pm$ is a real function. Put $M(z) = iz$. Since $g_{n,\nu}^\pm$ is symmetric with respect to the imaginary axis, $\hat{g}_{n,\nu}^\pm := M^{-1} \circ g_{n,\nu}^\pm \circ M$ is symmetric with respect to the real line, that is, $\hat{g}_{n,\nu}^\pm$ has real coefficients. Using

$$\hat{J}(z) := \tilde{J}(M(z)) = \operatorname{Re} a - (\operatorname{Im} a/2)(z - (1/z)),$$

we obtain $\mathcal{R}_n^\pm(\cdot, \rho_\nu, a, \bar{a}) = \tilde{J} \circ g_{n,\nu}^\pm \circ \tilde{J}^{-1} = \hat{J} \circ \hat{g}_{n,\nu}^\pm \circ \hat{J}^{-1}$, that is, $\mathcal{R}_n^\pm(\cdot, \rho_\nu, a, \bar{a})$ is the composition of three real functions. □

Naturally, any monic polynomial ρ_ν with $\rho_\nu(-\bar{z}) = -\overline{\rho_\nu(z)}$ must be of odd degree $\nu = 2\mu + 1$, and it can be obtained by setting $\rho_{2\mu+1}(z) = i\hat{\rho}_{2\mu+1}(-iz)$, where $\hat{\rho}_{2\mu+1}$ is a real polynomial with leading coefficient $(-1)^\mu$.

Next we show that the sufficient conditions of Theorem 2.3 are, in a certain sense, almost sharp.

THEOREM 2.5. *Let* $(\nu_n)_{n\in\mathbb{N}}$ *be a sequence of natural numbers with* $\lim_{n\to\infty}(\nu_n/n) = 0$ *and suppose that there are* $c, \delta > 0$ *such that* $\rho_{\nu_n} = \prod_{k=1}^{\nu_n}(z - z_{k,n})$ *has no zero in* $1 \leqslant |z| < 1 + c + \delta$ *and at least one zero in* $|z| \geqslant 1 + c + \delta$, *and that* $Z_n := \max_{1\leqslant k\leqslant \nu_n} |z_{k,n}| = o((1 + c)^{n/2\nu_n})$ *for* $n \to \infty$. *Then there is an* $n_0 \in \mathbb{N}$ *such that for all* $n \geqslant n_0$, *the Julia set* $\mathcal{J}_{\mathcal{R}_n^\pm(\cdot,\rho_{\nu_n},a,b)}$ *is not connected and in particular no Jordan arc.*

Proof. To prove the non-connectedness of $\mathcal{J}_{\mathcal{R}_n^\pm(\cdot,\rho_{\nu_n},a,b)}$, that is, of $\mathcal{J}_{g_{n,\nu_n}^\pm}$, it suffices to show that one component of the Fatou set is not simply connected. We will demonstrate that one component of $A(\infty)$ is not simply connected. Indeed, we will prove below that for sufficiently large n,

$$|g_{n,\nu_n}^\pm(z)| > |z| \qquad \text{for } |z| > r_{e_n}, \tag{10}$$

or in other words that $\{|z| > r_{e_n}\} \subset A^*(\infty)$ ($r_{e_n} > 1 + c$ is defined in (12)). Further, we will show that for n sufficiently large,

$$|g_{n,\nu_n}^\pm(z)| > r_{e_n} \qquad \text{for } z \in C := \{z \in \mathbb{C},\ |z| = 1 + c\}, \tag{11}$$

that is, $C \subset A(\infty)$. Note that by assumption, there exists at least one preimage $z_{k,n}$, $k \in \{1, \ldots, \nu_n\}$, of $z = 0$ with $|z_{k,n}| > 1 + c$, and thus there is a component $B(z_{k,n})$ of $A(0)$ which contains $z_{k,n}$ and which is disjoint from $A^*(0)$. Now, if $C \subset A^*(\infty)$, then $A^*(\infty)$ cannot be simply connected because the complement contains the two disjoint components $B(z_{k,n})$ and $A^*(0)$. If C lies in a component different from $A^*(\infty)$, then this component has at least one hole around the attracting fixed point zero and thus is not simply connected.

What remains to be shown are the inequalities (10) and (11). We put

$$r_{e_n} := Z_n + Z_n^{2+m_n-((n-1)/\nu_n)}, \tag{12}$$

where $(m_n)_{n\in\mathbb{N}}$ is a bounded sequence of natural numbers chosen such that

$$Z_n^{m_n} \geqslant 1 + Z_n. \tag{13}$$

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First we show that (10) holds for sufficiently large n. Recall that $Z_n \geqslant 1+c+\delta$. By the representation (4) of $g^{\pm}_{n,\nu_n}$, it is sufficient to show that

$$r_{e_n}^{((n-1)/\nu_n)-2}\big(r_{e_n} - Z_n\big) > 1 + Z_n,$$

that is,

$$Z_n^{m_n}\left(1 + Z_n^{m_n-((n-1)/\nu_n)+1}\right)^{((n-1)/\nu_n)-2} > 1 + Z_n,$$

which holds by (13).

Finally we show that inequality (11) holds for sufficiently large n. Using the fact that by assumption,

$$\frac{(1+c)^{(n-\nu_n)/\nu_n}}{Z_n(1+(1+c)Z_n)} \xrightarrow[n\to\infty]{} \infty,$$

(11) follows with the help of (4) using the rough estimates

$$|1 - zz_{k,n}| \leqslant 1 + (1+c)Z_n, \quad |z - z_{k,n}| \geqslant \min\{c,\delta\} > 0, \quad Z_n \geqslant 1+c+\delta. \quad \square$$

Note that in general the zeros of ρ_ν need not lie in the open unit disc if $\mathcal{J}_{\mathcal{R}_n^{\pm}(\cdot,\rho_\nu,a,b)}$ is a Jordan arc, as the following example shows.

EXAMPLE 2.6. Let $\alpha \in \mathbb{R}$,

$$\mathcal{R}_3^+(y,\rho_1) = \frac{T_3(y) - 2i\alpha T_2(y) - \alpha^2 T_1(y)}{1 - \alpha^2 - 2i\alpha y},$$

that is, $\rho_1(z) = z - i\alpha$ and $g^+_{3,1}(z) = z^2((z-i\alpha)/(1-zi\alpha))$. Then 0 and ∞ are superattracting, ± 1 are repelling fixed points of $g^+_{3,1}$, the imaginary axis is forward invariant under $g^+_{3,1}$ and $[0, i\alpha] \subset A^*(0)$. As in the proof of Theorem 2.3, we conclude that $\mathcal{J}_{g^+_{3,1}}$ is a quasicircle. Hence $\mathcal{J}_{\mathcal{R}_3^+(\cdot,\rho_1)}$ is a Jordan arc with endpoints ± 1 for every $\alpha \in \mathbb{R}$.

3. *Necessary conditions for the Julia set to be a Jordan arc*

In this section we show that any rational function which has a Jordan arc as Julia set is up to conjugation of the form (1). The crucial point is that half of the critical points of such rational functions R are preimages of the endpoints a, b of the Jordan arc and that $R(\{a,b\}) \subset \{a,b\}$ (see [**16**, p. 141]).

PROPOSITION 3.1. *Let R be a rational function of degree $n \geqslant 2$ whose Julia set is a Jordan arc which lies in $\mathbb{C}$ and has the endpoints ± 1. Then there exist $m, \nu \in \mathbb{N} \cup \{0\}$ and a monic polynomial ρ_ν of degree ν such that $R = \mathcal{R}_m^{\pm}(\cdot, \rho_\nu)$.*

Proof. Suppose first that ± 1 are fixed points. Then there are $n-1$ distinct critical points ($\neq \pm 1$) in $\mathcal{J}_R$ with critical values ± 1; hence

$$\frac{R(y)-1}{R(y)+1} = \frac{y-1}{y+1}\left(\frac{P(y)}{Q(y)}\right)^2, \quad \text{that is, } R(y) = \frac{1 + \dfrac{y-1}{y+1}\left(\dfrac{P(y)}{Q(y)}\right)^2}{1 - \dfrac{y-1}{y+1}\left(\dfrac{P(y)}{Q(y)}\right)^2},$$

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where P and Q are relatively prime polynomials with simple zeros ($\neq \pm 1$). Straightforward calculations show that $R \circ J = J \circ g$, where J is defined in (2) and

$$g(z) = \left(1 + \frac{z-1}{z+1}\frac{P(J(z))}{Q(J(z))}\right) \Big/ \left(1 - \frac{z-1}{z+1}\frac{P(J(z))}{Q(J(z))}\right).$$

Because $g(1/z) = 1/g(z)$, we conclude that g is of the form (4), that is, $g = g^{\pm}_{m,\nu}$ for appropriate m and ν and a suitable monic polynomial ρ_ν, or, equivalently, $R = \mathcal{R}^{\pm}_m(\cdot, \rho_\nu)$. The other cases when the endpoints ± 1 are not two fixed points but a fixed point and its preimage or a 2-cycle are treated in a similar way. □

Naturally, any rational function whose Julia set is a Jordan arc is conjugate to a function R whose Julia set is a Jordan arc which lies in $\mathbb{C}$ and has the endpoints ± 1.

REMARK 3.2. (1) Let $g^{\pm}_{m,\nu}$ denote the rational function which is defined according to (3) by $\mathcal{R}^{\pm}_m(\cdot, \rho_\nu)$ from Proposition 3.1. By [**17**, Section 3] and the $(1/z)$-symmetry of $g^{\pm}_{m,\nu}$, it follows that $\mathcal{J}_{g^{\pm}_{m,\nu}}$ is a quasicircle which contains the points ± 1.

(2) It is stated in [**17**, last paragraph, p. 535] that any rational function whose Julia set is a Jordan arc is conjugate to $(S(\sqrt{y}))^2$, where S is an odd rational function, which can be considered as another version of Proposition 3.1.

COROLLARY 3.3. *Let R be a real rational function of degree $n \geqslant 2$. If $\mathcal{J}_R$ is a non-analytic Jordan arc which lies in $\mathbb{C}$ and has the endpoints $\pm i$, then $R = \mathcal{R}^{\pm}_m(\cdot, \rho_\nu, i, -i)$, where ρ_ν is a non-real polynomial of degree ν with $\rho_\nu(-\bar{z}) = -\overline{\rho_\nu(z)}$ and n and ν are odd.*

Proof. Using the facts that the coefficients of R are real and that the $n-1$ critical points from $\mathcal{J}_R$ are mapped onto $\{\pm i\}$, we obtain n is odd. By Proposition 3.1, we obtain $-iR(iz) = \mathcal{R}^{\pm}_m(z, \rho_\nu, i, -i)$, that is, $R = \mathcal{R}^{\pm}_m(\cdot, \rho_\nu, i, -i)$, where ρ_ν is a non-real polynomial of degree ν. The rest follows with the help of (5) and the remark after the proof of Corollary 2.4. □

With respect to the converse of Theorem 2.3, we mention that if R is a rational function of degree $n \geqslant 2$ with $R(\infty) = \infty$ whose Julia set is a Jordan arc which lies in $\mathbb{C}$ and has the endpoints a, b, then $R = \mathcal{R}^{\pm}_n(\cdot, \rho_\nu, a, b)$ with $n > \nu$. This fact follows by the proof of Proposition 3.1.

4. *Jordan arcs with a rationally indifferent fixed point*

In this section we will describe rational functions with a rationally indifferent fixed point whose Julia set is a Jordan arc. Similarly to the above, this task can be transformed to a problem on $(1/z)$-symmetric functions with a rationally indifferent fixed point whose Julia set is a quasicircle. Now let us recall that a real Blaschke product B with $B(1) = B'(1) = 1$ has the unit circle $\partial\mathbb{D}$ as Julia set. Hence perturbing real Blaschke products may lead to functions whose Julia sets are non-analytic quasicircles. However, there are two main difficulties. First, we have to perturb in such a way that the resulting function again has exactly one rationally indifferent fixed point. Second, we want to have concrete bounds on the perturbation

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parameter such that the Julia set is a quasicircle. It turns out that these problems are overcome more easily when we make another transformation from the unit disc to the right half-plane by the Möbius map $M(z)=(1+z)/(1-z)$. In particular, real Blaschke products B with all zeros in the unit disc are transformed into so-called positive odd real function $\tilde{R}_0$ (also called Foster functions), and circular attracting petals at the rationally indifferent fixed point $z=1$ of the Blaschke product B are transformed into half-planes of the form $\{\operatorname{Re} w>c\}$ and $\{\operatorname{Re} w<-c\}$ which are attracting petals at the rationally indifferent fixed point $w=\infty$ of the transformed function $\tilde{R}_0$. Of course $\mathcal{J}_B=\partial\mathbb{D}$ corresponds to $\mathcal{J}_{\tilde{R}_0}=i\mathbb{R}\cup\{\infty\}$.

Suppose first that R is a rational function of degree 2 which has a rationally indifferent fixed point. It is easy to prove that $\mathcal{J}_R$ is a Jordan arc if and only if R is conjugate to the function $S(z)=z+(1/z)-2$ with $\mathcal{J}_S=[0,\infty]$. More generally, it can be shown that a rational function of degree n with a rationally indifferent fixed point and a critical point of multiplicity $n-1$ can only have a Jordan arc as Julia set if this arc is analytic and hence, by Fatou [**4**], is an interval or an arc of a circle.

NOTATION 4.1. We denote by f_{e} and f_{o} the even and odd part of a function f, that is, $f_{\mathrm{e}}(w)=\frac{1}{2}(f(w)+f(-w))$ and $f_{\mathrm{o}}(w)=\frac{1}{2}(f(w)-f(-w))$. Furthermore, we call a function f *positive* if $\operatorname{Re} f(w)>0$ for $\operatorname{Re} w>0$.

NOTATION 4.2. (a) Let $B(z)=\prod_{j=1}^{n}((z-z_j)/(1-z\overline{z_j}))$, $n\geqslant 2$, be a real Blaschke product with $z_j\in\mathbb{D}$ for $j=1,\dots,n$ and $\mathcal{J}_B=\partial\mathbb{D}$. We write $d=\sum_{j=1}^{n}((1+|z_j|)/(1-|z_j|))$. Then there is a unique $\tilde{\varphi}\in[2\pi/d,\pi]$ such that $B(e^{i\tilde{\varphi}})=1$ and $\arg B(e^{i\varphi})$ increases strictly from 0 to 2π on $[0,\tilde{\varphi}]$. Indeed, $\mathcal{J}_B=\partial\mathbb{D}$ guarantees that there are no critical points of B on $\partial\mathbb{D}$. Furthermore, $\tilde{\varphi}$ is the limit of the increasing sequence $(\varphi_j)_{j\in\mathbb{N}}$ defined recursively by

$$\varphi_j=\varphi_{j-1}+\frac{2\pi-\arg B(e^{i\varphi_{j-1}})}{d},\quad \text{for } j\geqslant 2 \text{ with starting value } \varphi_1=\frac{2\pi}{d}. \tag{14}$$

(b) For a real polynomial $h(w)=\prod_{j=1}^{n}(w-w_j)$ with $\operatorname{Re} w_j>0$ for $j=1,\dots,n$, we set $r_{\max}=\max\{|w_j|\}$ and define an associated Blaschke product $B_h(z)=\prod_{j=1}^{n}((z-z_j)/(1-z\overline{z_j}))$, where $z_j=(w_j-1)/(w_j+1)\in\mathbb{D}$. Furthermore, we put

$$c_p=\cot\frac{\tilde{\varphi}}{2}, \tag{15}$$

where $\tilde{\varphi}$ with respect to B_h is given in (a).

Assume that $w_j\in\mathbb{R}$ for $j\geqslant 2l+1$, $l\in\mathbb{N}_0$, and that the other $2l$ zeros of h are pairs of complex conjugate numbers. Then we put

$$c_b=\max\left\{\frac{\prod_{j=1}^{n}\operatorname{Re} w_j}{2c_p^{n-2}},\ \frac{1}{\sum_{j=2l+1}^{n}\frac{1}{w_j}+2\sum_{j=1}^{l}\frac{c_p^2+|w_j|^2}{(\operatorname{Re} w_j)^3}}\right\}. \tag{16}$$

LEMMA 4.3. *Let $h(w)=\prod_{j=1}^{n}(w-w_j)$ be a real polynomial of degree $n\geqslant 2$ with $\operatorname{Re} w_j>0$ for $j=1,\dots,n$ and $\sum_{j=1}^{n}w_j=1$. For the definition of the rational function $\tilde{R}_0$ of degree n, we distinguish between the two cases n odd and n even.*

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(i) *If n is odd, then we put*

$$\tilde{R}_0(w) := -\frac{h_{\mathrm{o}}(w)}{h_{\mathrm{e}}(w)} = w + \sum_{j=1}^{(n-1)/2} \frac{2b_j w}{w^2 + p_j^2}. \tag{17}$$

(ii) *If n is even, then we put*

$$\tilde{R}_0(w) := -\frac{h_{\mathrm{e}}(w)}{h_{\mathrm{o}}(w)} = w + \frac{b_0}{w} + \sum_{j=1}^{(n-2)/2} \frac{2b_j w}{w^2 + p_j^2}. \tag{18}$$

The following statements hold for both case (i) *and case* (ii). *$\tilde{R}_0$ is a positive odd real rational function which has simple poles only and a rationally indifferent fixed point at ∞. Furthermore, the p_j and b_j in the above partial fraction expansions satisfy the following equalities and inequalities.*

$$0 < p_j \leqslant c_p \qquad \text{for } \ j \geqslant 1 \tag{19}$$

and

$$0 < c_b \leqslant b_j = \frac{1}{\sum_{k=1}^{n} \frac{w_k}{w_k^2 + p_j^2}} \qquad \text{for } \ j \geqslant 1, b_0 = \frac{1}{\sum_{k=1}^{n} \frac{1}{w_k}}, \tag{20}$$

where c_p and c_b are defined in (15) *and* (16).

Proof. By the definition of $\tilde{R}_0$ and the fact that $\operatorname{Re} w_j > 0$ for $j = 1, \ldots, n$, it follows (see [**9**, Theorem 5.1, Theorem 5.2]) that $\tilde{R}_0$ is a positive odd real rational function which has a series expansion of the form (17), respectively (18), with $b_j > 0$, $p_j > 0$ and $p_i \neq p_j$ for $i \neq j$, where the coefficient of w is equal to 1 since $\sum_{j=1}^{n} w_j = 1$.

Next we prove that $p_j \leqslant c_p$. By the definition of $\tilde{\varphi}$, no point of the form $z = e^{i\varphi}$, $0 < |\varphi| < \tilde{\varphi}$, satisfies $B_h(z) = 1$. Since $M(e^{i\tilde{\varphi}}) = i(\sin\tilde{\varphi}/(1 - \cos\tilde{\varphi})) = ic_p$, where $M(z) = (1+z)/(1-z)$, it follows by the properties of M that no point $w = it$ on $i\mathbb{R}$ with $|t| > c_p$ satisfies $\tilde{R}_0(w) = \infty$. This proves that $p_j \leqslant c_p$.

Thus (20) remains to be shown. We first consider case (i). $\tilde{R}_0(w) - w$ from (17) can be rewritten in the following way.

$$\tilde{R}_0(w) - w = \frac{-h(w) + h(-w) - w(h(w) + h(-w))}{h(w) + h(-w)}. \tag{21}$$

Since by (17) $h(w) + h(-w)$ has the zeros $\pm ip_j$, we obtain by (21) and the residue theorem for $j \geqslant 1$

$$\begin{aligned} \frac{1}{b_j} &= \frac{-h'(\mp ip_j) + h'(\pm ip_j)}{(-2)h(\pm ip_j)} = \frac{h'(-ip_j)}{-2h(-ip_j)} - \frac{h'(ip_j)}{2h(ip_j)} \\ &= \sum_{k=1}^{n} \frac{w_k}{w_k^2 + p_j^2} = \sum_{k=2l+1}^{n} \frac{w_k}{w_k^2 + p_j^2} + 2\sum_{k=1}^{l} \operatorname{Re}\left(\frac{w_k}{w_k^2 + p_j^2}\right) \\ &\leqslant \sum_{k=2l+1}^{n} \frac{1}{w_k} + 2\sum_{k=1}^{l} \frac{|w_k|^2 + c_p^2}{(\operatorname{Re} w_k)^3}, \end{aligned} \tag{22}$$

where we have assumed that the zeros of h are numbered as in the preceding notation.

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On the other hand, writing b_j in the form

$$b_j = -\frac{h(ip_j)}{h'_{\mathrm{e}}(ip_j)} = \frac{\prod_{k=1}^{n}(ip_j - w_k)}{2ip_j \prod_{\substack{k=1\\k\neq j}}^{(n-1)/2}(p_k^2 - p_j^2)}, \tag{23}$$

we obtain

$$b_j = |b_j| \geqslant \frac{\prod_{k=1}^{n} \operatorname{Re} w_k}{2c_p^{n-2}} \qquad \text{for } j \geqslant 1. \tag{24}$$

In view of (22) and (24), relation (20) is proved for case (i), that is, for n odd.

For case (ii), by (18), $\tilde{R}_0(w) - w$ can be written in the form

$$\tilde{R}_0(w) - w = \frac{h(-w) + h(w) - w(h(-w) - h(w))}{h(-w) - h(w)}.$$

Taking a look at (18), we see that $h(-w) - h(w)$ has the zeros $\pm ip_j, 0$, and proceeding in the same way as in case (i), we obtain

$$\begin{aligned}\frac{1}{b_j} &= \frac{h'(\pm ip_j) + h'(\mp ip_j)}{(-2)h(\pm ip_j)} = \frac{-h'(-ip_j)}{2h(-ip_j)} - \frac{h'(ip_j)}{2h(ip_j)} \\ &= \sum_{k=1}^{n} \frac{w_k}{w_k^2 + p_j^2} \leqslant \sum_{k=2l+1}^{n} \frac{1}{w_k} + 2\sum_{k=1}^{l} \frac{|w_k|^2 + c_p^2}{(\operatorname{Re} w_k)^3}.\end{aligned} \tag{25}$$

Again, on the other hand, writing b_j in the form

$$b_j = -\frac{h(ip_j)}{h'_{\mathrm{o}}(ip_j)} = \frac{-\prod_{k=1}^{n}(ip_j - w_k)}{2p_j^2 \prod_{\substack{k=1\\k\neq j}}^{(n-2)/2}(p_k^2 - p_j^2)},$$

and observing that

$$b_0 = \prod_{k=1}^{n} w_k \Big/ \prod_{j=1}^{(n-2)/2} p_j^2,$$

we find that

$$b_j = |b_j| \geqslant \frac{\prod_{k=1}^{n} \operatorname{Re} w_k}{2c_p^{n-2}}. \tag{26}$$

By (25) and (26), relation (20) is proved. □

REMARK 4.4. The inequalities (19) and (20) remain true if in the definition of the constant c_p the angle $\tilde{\varphi}$ is replaced by any $\varphi \in (0, \tilde{\varphi})$, and in particular, by any φ_j, $j \in \mathbb{N}$, from (14).

THEOREM 4.5. *Let $n \in \mathbb{N}$, $n \geqslant 4$, be odd. Let $h(w) = \prod_{j=1}^{n}(w - w_j)$ be a real polynomial with $\operatorname{Re} w_j > 0$ for $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$ and let c be such that $0 \leqslant c < (n/(n+1)) \min\{\operatorname{Re} w_j\}$. Furthermore, let $S_{\mathrm{o}}(w) = \sum_{j=0}^{n-2m} a_j w^j$ be an odd*

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polynomial of degree $n-2m$, $4 \leqslant 2m \leqslant n$, with $\sum |a_j| \leqslant 1$. We define the rational function $\tilde{R}_\eta$, $\eta \in \mathbb{C}$, of odd degree n by

$$\tilde{R}_\eta(w) = (-h_{\mathrm{o}}(w) + \eta S_{\mathrm{o}}(w))/h_{\mathrm{e}}(w). \tag{27}$$

Then for $|\eta| < \eta^*(c)$, the Julia set $\mathcal{J}_{\tilde{R}_\eta}$ is a quasicircle symmetric with respect to the origin, and any half-plane of the form $\{\operatorname{Re} w > \tilde{c}\}$ and $\{\operatorname{Re} w < -\tilde{c}\}$, $\tilde{c} \geqslant c$, is an attracting petal at the rationally indifferent fixed point ∞.

$\eta^*(c)$ is given by

$$\min\left\{ \frac{c^{n-2}(n-2)c_b}{\left(c^2+c_p^2\right)^{(n-4)/2}}, c^{n-2}(n-2)c_b, \frac{\mu^n}{(r_{\max}+\mu)^{n-4}}, \mu^n \right\},$$

where $r_{\max}$, c_p and c_b are defined in Notation 4.2 and

$$\mu = \prod_{j=1}^{n} (\operatorname{Re} w_j - c((n+1)/n))^{1/n}.$$

Proof. It is easy to see that $\tilde{R}_\eta$ is an odd function with $\tilde{R}_\eta(\infty) = \infty$ and $\tilde{R}'_\eta(\infty) = 1$. First we will prove that any half-plane of the form $\mathcal{P} = \{\operatorname{Re} w > \tilde{c}\}$, $\tilde{c} \geqslant c$, is an attracting petal at the rationally indifferent fixed point ∞, that is, $\operatorname{Re}(\tilde{R}_\eta(w)) > \tilde{c}$ for $\operatorname{Re} w > \tilde{c}$. Without loss of generality, we may assume that $\eta \geqslant 0$.

Since $\tilde{R}_\eta$ can be written in the form

$$\tilde{R}_\eta(w) = w + \frac{\eta S_{\mathrm{o}}(w)\overline{h_{\mathrm{e}}(w)} - h_{\mathrm{o}}(w)\overline{h_{\mathrm{e}}(w)} - w|h_{\mathrm{e}}(w)|^2}{|h_{\mathrm{e}}(w)|^2}, \tag{28}$$

and since, by (17) and (20),

$$\begin{aligned}
\operatorname{Re}(-h_{\mathrm{o}}(w)\overline{h_{\mathrm{e}}(w)} - w|h_{\mathrm{e}}(w)|^2) &= |h_{\mathrm{e}}(w)|^2(-\operatorname{Re} w + \operatorname{Re} \tilde{R}_0(w)) \\
&\geqslant 2\operatorname{Re} w|\overline{h_{\mathrm{e}}(w)}| \sum_{j=1}^{(n-1)/2} b_j \frac{|h_{\mathrm{e}}(w)|}{|w^2+p_j^2|} \\
&= 2\operatorname{Re} w|\overline{h_{\mathrm{e}}(w)}| \sum_{j=1}^{(n-1)/2} b_j \prod_{\substack{k=1 \\ k\neq j}}^{(n-1)/2} |w^2+p_k^2|,
\end{aligned}$$

it suffices to show that for $\operatorname{Re} w > c$,

$$\eta|S_{\mathrm{o}}(w)| \leqslant 2\operatorname{Re} w \sum_{j=1}^{(n-1)/2} b_j \prod_{\substack{k=1 \\ k\neq j}}^{(n-1)/2} |w^2+p_k^2|. \tag{29}$$

Put $\lambda_1 = 1 + (c^2/c_p^2) > 1$. To prove (29), we distinguish between the cases $|w| \geqslant 1$ and $|w| < 1$.

(a) $|w| \geqslant 1$: $\eta|S_{\mathrm{o}}(w)| \leqslant \eta|w|^{n-4} \sum |a_j| \leqslant \eta|w|^{n-4}$.

If $|w| < \sqrt{\lambda_1}c_p = \sqrt{c^2+c_p^2}$, then $\eta < \eta^*(c)$ and $\sum_{j=1}^{(n-1)/2} b_j \geqslant ((n-1)/2)c_b$, which is a consequence of (20), imply that

$$\eta|w|^{n-4} \leqslant \eta\left(c^2+c_p^2\right)^{(n-4)/2} \leqslant c^{n-2}(n-2)c_b \leqslant 2c \sum_{j=1}^{(n-1)/2} b_j (\operatorname{Re} w)^{n-3}, \tag{30}$$

which is smaller than the right-hand side of (29), since, because

$$|w \pm ip_k|^2 = (\pm p_k + \operatorname{Im} w)^2 + (\operatorname{Re} w)^2, \tag{31}$$

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the following estimate holds.

$$\prod_{\substack{k=1\\k\neq j}}^{(n-1)/2} \left|w^2+p_k^2\right| \geqslant (\operatorname{Re} w)^{n-3}.$$

If $|w| \geqslant \sqrt{\lambda_1}c_p$, then, by the assumption on η, we have

$$\eta|w|^{n-4} \leqslant \frac{c^{n-2}(n-2)c_b|w|^{n-4}}{\left(c^2+c_p^2\right)^{(n-4)/2}} = \frac{c(n-2)c_b}{|w|/(c_p\sqrt{\lambda_1})}\left(\frac{|w|^2(\lambda_1-1)}{\lambda_1}\right)^{(n-3)/2}, \quad (32)$$

which is again smaller than the right-hand side of (29), taking into consideration the fact that

$$\left|w^2+p_k^2\right| \geqslant |w|^2-|c_p|^2 \geqslant |w|^2(\lambda_1-1)/\lambda_1. \quad (33)$$

(b) $|w|<1$: We have $\eta|S_o(w)|<\eta$. By the assumption on η and because $\operatorname{Re} w > c$, we obtain

$$\eta \leqslant (n-2)c_b c^{n-2} < 2c \sum_{j=1}^{(n-1)/2} b_j(\operatorname{Re} w)^{n-3},$$

which is smaller than the right-hand side of (29). Hence (29) holds for $\operatorname{Re} w > c$.

To summarize, any half-plane of the form $\mathcal{P}=\{\operatorname{Re} w > \tilde{c}\}$, $\tilde{c} \geqslant c$, is forward invariant under $\tilde{R}_\eta$.

Finally, we prove that the attracting petal $\mathcal{P}_c=\{\operatorname{Re} w > c\}$ contains the point $w=1$ and all preimages of it under $\tilde{R}_\eta$, that is, $\tilde{R}_\eta^{-1}(1)$. This implies that $\tilde{R}_\eta$ has local degree n on that component of $\mathcal{F}_{\tilde{R}_\eta}$ which contains $\mathcal{P}_c$ and because $\tilde{R}_\eta(-w)=-\tilde{R}_\eta(w)$ also on a second component of $\mathcal{F}_{\tilde{R}_\eta}$; hence the Fatou set consists of two completely invariant components which are the basin of attraction of the rationally indifferent fixed point ∞. By [**17**, Theorem 4], which is the rationally indifferent counterpart of [**3**, Theorem VI.2.1], it follows that $\mathcal{J}_{\tilde{R}_\eta}$ is a quasicircle symmetric with respect to the origin, and the theorem is proved.

Again we may assume that $\eta \geqslant 0$. Obviously, $1 \in \mathcal{P}_c$ since $c<1$. With the help of Rouche's theorem, we will prove that $\tilde{R}_\eta^{-1}(1) \subset \mathcal{P}_c$. We put $\lambda_2 = 1+(\mu/r_{\max})>1$, where $r_{\max}=\max\{|w_j|\}$, and define the closed Jordan curve $\gamma=\gamma_1\cup\gamma_2 : \mathbb{R} \longrightarrow \mathbb{C}$ by

$$\gamma_1(t) = c\frac{n+1}{n} - it, \qquad \text{for } -t_1 \leqslant t \leqslant t_1, |\gamma_1(t_1)| = \lambda_2 r_{\max},$$

$$\gamma_2(t) = \lambda_2 r_{\max} e^{it}, \qquad \text{for } -t_2 \leqslant t \leqslant t_2, \operatorname{Re}(\gamma_2(t_2)) = c\frac{n+1}{n}.$$

If there is a zero of $\tilde{R}_\eta - 1$ on γ, then we modify the Jordan curve γ such that the new Jordan curve $\tilde{\gamma}$ satisfies $\operatorname{int}(\tilde{\gamma}) \supset \operatorname{int}(\gamma)$ and $\tilde{\gamma} \subset \{\operatorname{Re} w \geqslant c\}$, and such that there is no zero of $\tilde{R}_\eta - 1$ on $\tilde{\gamma}$. Now, by (27), all poles of $\tilde{R}_\eta - 1$ lie on $i\mathbb{R}\cup\{\infty\}$ and all n zeros w_j of $\tilde{R}_0 - 1$ are contained in $\operatorname{int}(\tilde{\gamma})$. Hence, by Rouche's theroem, it suffices to show that

$$|\tilde{R}_0(w) - \tilde{R}_\eta(w)| < |\tilde{R}_0(w) - 1| \quad \text{on } \tilde{\gamma},$$

which is equivalent to

$$\eta|S_o(w)| < |h(w)| \quad \text{on } \tilde{\gamma}. \quad (34)$$

In order to prove (34), we will distinguish between two cases.

(α) $w \in \tilde{\gamma} \cap \{|w| \geqslant 1\}$: $\eta|S_o(w)| \leqslant \eta|w|^{n-4}$.

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If $|w| \geqslant \lambda_2 r_{\max} = r_{\max} + \mu$, then we have $|w - w_j| \geqslant |w| - |w_j| \geqslant |w|(\lambda_2 - 1)/\lambda_2$, and therefore

$$|h(w)| \geqslant \left(|w| \frac{\lambda_2 - 1}{\lambda_2} \right)^n > \eta |w|^{n-4},$$

where the last inequality is an immediate consequence of $\eta < \eta^*(c)$.

If $|w| < \lambda_2 r_{\max}$, then we have $\operatorname{Re} w \in [c, c((n+1)/n)]$, and therefore $|w - w_j| \geqslant |\operatorname{Re} w - \operatorname{Re} w_j| \geqslant |c((n+1)/n) - \operatorname{Re} w_j|$. Hence

$$|h(w)| \geqslant \prod_{j=1}^{n} \left(\operatorname{Re} w_j - c \frac{n+1}{n} \right) > \eta (\lambda_2 r_{\max})^{n-4} > \eta |w|^{n-4},$$

where we have used $\eta < \eta^*(c)$.

(β) $w \in \tilde{\gamma} \cap \{|w| < 1\}$: $\eta |S_{\mathrm{o}}(w)| \leqslant \eta$. Since by assumption $\eta < \mu^n$, we obtain $\eta < \mu^n < |h(w)|$. Thus the theorem is proved. □

THEOREM 4.6. *Let $n \in \mathbb{N}, n \geqslant 4$, be even. Let $h(w) = \prod_{j=1}^{n} (w - w_j)$ be a real polynomial with $\operatorname{Re} w_j > 0$ for $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$, and let c be such that*

$$0 \leqslant c < (n/(n+1)) \min\{\operatorname{Re} w_j\}.$$

Furthermore, let $S_{\mathrm{e}}(w) = \sum_{j=0}^{n-2m} a_j w^j$ be an even polynomial of degree $n - 2m$, $4 \leqslant 2m \leqslant n$, with $\sum |a_j| \leqslant 1$. We define the rational function $\tilde{R}_\eta$, $\eta \in \mathbb{C}$, of even degree n by

$$\tilde{R}_\eta(w) = (-h_{\mathrm{e}}(w) + \eta S_{\mathrm{e}}(w))/h_{\mathrm{o}}(w). \tag{35}$$

Then, for $|\eta| < \eta^*(c)$, the Julia set $\mathcal{J}_{\tilde{R}_\eta}$ is a quasicircle symmetric with respect to the origin, and any half-plane of the form $\{\operatorname{Re} w > \tilde{c}\}$ and $\{\operatorname{Re} w < -\tilde{c}\}$, $\tilde{c} \geqslant c$, is an attracting petal at the rationally indifferent fixed point ∞.

$\eta^*(c)$ is given by

$$\min \left\{ \frac{c^{n-2}(n-2)c_b}{\left(c^2 + c_p^2\right)^{(n-4)/2}}, c^{n-2}(n-2)c_b, \frac{\mu^n}{(r_{\max} + \mu)^{n-4}}, \mu^n \right\},$$

where $r_{\max}$, c_p and c_b are defined in Notation 4.2 and

$$\mu = \prod_{j=1}^{n} (\operatorname{Re} w_j - c((n+1)/n))^{1/n}.$$

Proof. $\tilde{R}_\eta$ is an odd function with $\tilde{R}_\eta(\infty) = \infty$ and $\tilde{R}'_\eta(\infty) = 1$. As in the proof of Theorem 4.5, we will demonstrate that any half-plane of the form $\mathcal{P} = \{\operatorname{Re} w > \tilde{c}\}$, $\tilde{c} \geqslant c$, is an attracting petal at the rationally indifferent fixed point ∞, that is, $\operatorname{Re}(\tilde{R}_\eta(w)) > \tilde{c}$ for $\operatorname{Re} w > \tilde{c}$. Without loss of generality, we may assume that $\eta \geqslant 0$.

$\tilde{R}_\eta$ can be written in the form

$$\tilde{R}_\eta(w) = w + \frac{\eta S_{\mathrm{e}}(w)\overline{h_{\mathrm{o}}(w)} - h_{\mathrm{e}}(w)\overline{h_{\mathrm{o}}(w)} - w|h_{\mathrm{o}}(w)|^2}{|h_{\mathrm{o}}(w)|^2}, \tag{36}$$

from which it follows that

$$\eta |S_{\mathrm{e}}(w)\overline{h_{\mathrm{o}}(w)}| \leqslant \operatorname{Re}(-h_{\mathrm{e}}(w)\overline{h_{\mathrm{o}}(w)} - w|h_{\mathrm{o}}(w)|^2) \qquad \text{for } \operatorname{Re} w > c$$

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is sufficient for the forward invariance of the half-plane $\mathcal{P}$. Since

$$\begin{aligned}\operatorname{Re}(-h_{\mathrm{e}}(w)\overline{h_{\mathrm{o}}(w)} - w|h_{\mathrm{o}}(w)|^2) &= |h_{\mathrm{o}}(w)|^2\left(-\operatorname{Re} w - \operatorname{Re}\frac{h_{\mathrm{e}}(w)}{h_{\mathrm{o}}(w)}\right)\\ &= |h_{\mathrm{o}}(w)|^2\left(b_0\operatorname{Re}\frac{1}{w} + \sum_{j=1}^{(n-2)/2}\frac{2b_j\operatorname{Re} w\left(|w|^2+p_j^2\right)}{\left|w^2+p_j^2\right|^2}\right)\\ &= |h_{\mathrm{o}}(w)|^2\operatorname{Re} w\left(\frac{b_0}{|w|^2} + 2\sum_{j=1}^{(n-2)/2} b_j\frac{|w|^2+p_j^2}{\left|w^2+p_j^2\right|^2}\right),\end{aligned} \tag{37}$$

By (36) and the fact that $h_{\mathrm{o}}(w) = -w\prod_{j=1}^{(n-2)/2}(w^2+p_j^2)$, it suffices to show that for $\operatorname{Re} w > c$,

$$\eta|S_{\mathrm{e}}(w)| \leqslant \frac{b_0|h_{\mathrm{o}}(w)|\operatorname{Re} w}{|w|^2} + 2\operatorname{Re} w\sum_{j=1}^{(n-2)/2} b_j|w|\prod_{\substack{k=1\\k\neq j}}^{(n-2)/2}|w^2+p_k^2|. \tag{38}$$

Next put $\lambda_1 = 1 + (c^2/c_p^2) > 1$. To prove (38), we distinguish between the cases $|w| \geqslant 1$ and $|w| < 1$.

(a) Let $|w| \geqslant 1$. Then $\eta|S_{\mathrm{e}}(w)| \leqslant \eta|w|^{n-4}\sum|a_j| \leqslant \eta|w|^{n-4}$. If $|w| < \sqrt{\lambda_1}c_p = \sqrt{c^2+c_p^2}$, then $\eta < \eta^*(c)$ and (20) imply that

$$\eta|w|^{n-4} \leqslant \eta\left(c^2+c_p^2\right)^{(n-4)/2} \leqslant \frac{b_0c^n}{c^2+c_p^2} + c^{n-2}(n-2)c_b, \tag{39}$$

which is smaller than the right-hand side of (38), since by (31),

$$\prod_{\substack{k=1\\k\neq j}}^{(n-2)/2}\left|w^2+p_k^2\right| \geqslant (\operatorname{Re} w)^{n-4}.$$

If $|w| \geqslant \sqrt{\lambda_1}c_p$, then, by the assumption on η,

$$\begin{aligned}\eta|w|^{n-4} &\leqslant \frac{c^{n-2}}{\left(c^2+c_p^2\right)^{(n-4)/2}}\left(\frac{b_0c^2}{c^2+c_p^2} + (n-2)c_b\right)\\ &\leqslant |w|^{n-3}\operatorname{Re} w\left(\frac{\lambda_1-1}{\lambda_1}\right)^{(n-4)/2}\left(b_0\frac{\lambda_1-1}{\lambda_1} + c_b(n-2)\right)\\ &= \frac{b_0|w|\operatorname{Re} w}{|w|^2}\left(|w|^2\frac{\lambda_1-1}{\lambda_1}\right)^{(n-2)/2}\\ &\quad + (n-2)c_b\operatorname{Re} w|w|\left(|w|^2\frac{\lambda_1-1}{\lambda_1}\right)^{(n-4)/2},\end{aligned} \tag{40}$$

which is, by (33), smaller than the right-hand side of (38).

(b) For $|w| < 1$, we have $\eta|S_{\mathrm{e}}(w)| < \eta$. By the assumption on η and because $\operatorname{Re} w > c$, we obtain $\eta \leqslant (n-2)c_bc^{n-2} < 2c\sum_{j=1}^{(n-2)/2} b_j(\operatorname{Re} w)^{n-3}$, which is smaller than the right-hand side of (38). Hence (38) holds for $\operatorname{Re} w > c$.

The remaining part of the proof is identical word for word with the section of the proof of Theorem 4.5 after part (b), except that relation (34) has to be replaced by

$$\eta|S_{\mathrm{e}}(w)| < |h(w)| \qquad \text{on } \tilde{\gamma}, \tag{41}$$

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and in case (α), that is, for $w \in \tilde{\gamma} \cap \{|w| \geqslant 1\}$, $\eta|S_{\mathrm{e}}(w)| \leqslant \eta|w|^{n-4}$ holds, and in case (β), that is, for $w \in \tilde{\gamma} \cap \{|w| < 1\}$, $\eta|S_{\mathrm{e}}(w)| \leqslant \eta$. □

EXAMPLE 4.7. Let $h(w) = \prod_{j=1}^{4}(w - w_j)$ with $w_1 = \frac{1}{4} + i$, $w_2 = \frac{1}{4} + 2i$, $w_3 = \overline{w_1}$ and $w_4 = \overline{w_2}$. According to Notation 4.2, we obtain the constants $r_{\max} = \frac{\sqrt{65}}{4}$, $z_1 = \frac{1}{41} + i\frac{32}{41}$, $z_2 = \frac{49}{89} + i\frac{64}{89}$, $z_3 = \overline{z_1}$, $z_4 = \overline{z_2}$, $c_p = 1.600\,8$, $c_b = 0.001\,126$ (instead of $\tilde{\varphi}$ (see Remark 4.4), we have taken $\varphi = 1.116\,75$, which is a lower bound for φ_{100}) and

$$\tilde{R}_0(w) = -\frac{h_{\mathrm{e}}(w)}{h_{\mathrm{o}}(w)} = \frac{256w^4 + 1\,376w^2 + 1\,105}{256w^3 + 656w}.$$

If we choose $c = \frac{14}{125} < \frac{4}{5}\min\{\operatorname{Re} w_j\}$, then $\mu = \frac{11}{100}$ and $\eta^*(c) = 0.000\,028\,252$. Hence, if $S_{\mathrm{e}}(w) = \frac{3+4i}{5}$ and $\eta = \frac{28}{100\,000\,0}$, then the Julia set of

$$\tilde{R}_\eta(w) = \tilde{R}_0(w) + \eta\frac{S_{\mathrm{e}}(w)}{h_{\mathrm{o}}(w)} = \frac{250\,000\,0w^2(43 + 8w^2) + 863\,277\,89 - 448i}{125\,000\,0w(41 + 16w^2)}$$

is a non-analytic quasicircle symmetric to the origin, and the point at infinity is a rationally indifferent fixed point of $\tilde{R}_\eta$.

REMARK 4.8. (1) In particular, it follows from Theorem 4.5 and Theorem 4.6 that, for all $\eta \in \mathbb{C}$ with $|\eta| < \eta^*(c)$, there are no attracting periodic points of $\tilde{R}_\eta$. By [**10**, Theorem 4.2 (characterizations of stability)] (see also [**11**]), this implies that the Julia set $\mathcal{J}_{\tilde{R}_\eta}$ moves holomorphically at $\eta = 0$, that is, there is a holomorphic motion $\Phi_\eta : \mathcal{J}_{\tilde{R}_0} \longrightarrow \bar{\mathbb{C}}$ such that $\Phi_\eta(\mathcal{J}_{\tilde{R}_0}) = \mathcal{J}_{\tilde{R}_\eta}$ and $\Phi_\eta \circ \tilde{R}_0 = \tilde{R}_\mu \circ \Phi_\eta$ holds on $\mathcal{J}_{\tilde{R}_0}$.

(2) The proof of Theorem 4.5, respectively Theorem 4.6, works only if the degree of S_{o}, respectively S_{e}, is less than or equal to $n - 4$, because otherwise the right-hand side of (32), respectively (40), is not smaller than the right-hand side of (29), respectively (38). For $n = 3$, it can be shown that for $\tilde{R} = -\tilde{h}_{\mathrm{o}}/\tilde{h}_{\mathrm{e}}$, where $\tilde{h}(w) = \prod_{k=1}^{3}(w - w_k)$, $w_1 = c + id_1$, $w_2 = ((1-c)/2) + id_2$, $w_3 = 1 - w_1 - w_2$, $d_2 \in \mathbb{R}\backslash\{0\}$, $d_1 = (1 - 2c - 3c^2 - 4d_2^2)/4d_2$ and $(7 - \sqrt{17})/16 \leqslant c < \frac{1}{2}$, the point at infinity is a rationally indifferent fixed point and $\mathcal{J}_{\tilde{R}}$ is a quasicircle symmetric with respect to the origin.

With the help of Theorem 4.5 and Theorem 4.6, we are able to describe rational functions of the form of (5) whose Julia sets are Jordan arcs where one endpoint of the Jordan arc is a rationally indifferent fixed point.

COROLLARY 4.9. *Let $\rho_n(z) = q(z) - \eta(1-z)^{2m}s(z)$, where q is a real monic polynomial of degree $n, n \geqslant 4$, with all zeros in $\mathbb{D}$ and $(q/q^*)'(1) = 1, s$ is a polynomial of degree $n-2m$, $4 \leqslant 2m \leqslant n$, with $s(z) = s^*(z)$ and $\eta \in \mathbb{C}$. Then there is a constant $\eta^* > 0$ such that, for $|\eta| < \eta^*$, the Julia set $\mathcal{J}_{\mathcal{R}_n^+(\cdot,\rho_n,a,b)}$ is a Jordan arc with endpoints a and b and $\mathcal{F}_{\mathcal{R}_n^+(\cdot,\rho_n,a,b)}$ is the immediate basin of attraction of the rationally indifferent fixed point a.*

Furthermore, $\mathcal{J}_{\mathcal{R}_n^+(\cdot,\rho_n,a,b)}$ is an analytic Jordan arc if and only if ρ_n has real coefficients.

Proof. Let $q(z) = \prod_{j=1}^{n}(z - z_j)$ and put $w_j = ((1+z_j)/(1-z_j))$ for $j = 1, \ldots, n$. Then $h(w) = \prod_{j=1}^{n}(w - w_j)$ is a real polynomial with $\operatorname{Re} w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$

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since the z_j are from $\mathbb{D}$ and $(q/q^*)'(1) = 1$, respectively. Furthermore, we note that by the property $s = s^*$, the polynomial s can be represented in the form

$$s(z) = \sum_{j=0}^{n-2m} d_j(1+z)^j(1-z)^{n-2m-j}$$

with $d_j = 0$ if $n-j$ is odd. Simple calculations imply that $\rho_n/\rho_n^* = M^{-1} \circ \tilde{R}_\eta \circ M$, where $M(z) = (1+z)/(1-z)$, $\tilde{R}_\eta$ is the rational function from Theorem 4.5, respectively. Theorem 4.6, and the coefficients a_j of S_{o}, respectively S_{e}, are given by $a_j = d_j \prod_{k=1}^{n}(2/(1-z_k))$, $j = 0, \ldots, n-2m$. Hence, if $|\eta|$ is sufficiently small, then $g_{n,n}^+ = \rho_n/\rho_n^*$ is a $(1/z)$-symmetric rational function whose Julia set is a quasicircle which contains the points ± 1 and which has a rationally indifferent fixed point at $z = 1$. By Lemma 2.2,

$$\mathcal{J}_{\mathcal{R}_n^+(\cdot,\rho_n,a,b)}$$

is a Jordan arc with endpoints a and b, where $a = \tilde{J}(1)$ is a rationally indifferent fixed point of $\mathcal{R}_n^+(\cdot, \rho_n, a, b)$ whose immediate basin of attraction coincides with the Fatou set $\mathcal{F}_{\mathcal{R}_n^+(\cdot,\rho_n,a,b)}$.

Let us prove the last statement of the corollary. By Lemma 2.2 and [**4**, p. 225], $\mathcal{J}_{\mathcal{R}_n^+(\cdot,\rho_n,a,b)}$ is an analytic Jordan arc if and only if $\mathcal{J}_{g_{n,n}^+}$ is a circle. Since

$$\mathcal{P} = \{\operatorname{Re} w > c\}$$

is an attracting petal at the rationally indifferent fixed point ∞ of $\tilde{R}_\eta$, the set $M^{-1}(\mathcal{P})$ is an attracting petal at the fixed point $z = 1$ of $g_{n,n}^+$. $M^{-1}(\mathcal{P})$ is a circle centered at some $x \in (0,1)$ with radius $1-x$. With the help of $\{-1,1\} \subset \mathcal{J}_{g_{n,n}^+}$, we find that $\mathcal{J}_{g_{n,n}^+}$ is a circle if and only if $\mathcal{J}_{g_{n,n}^+} = \partial\mathbb{D}$, that is, recall that $g_{n,n}^+$ is $(1/z)$-symmetric if and only if $g_{n,n}^+$ is symmetric with respect to the unit circle, or equivalently, if ρ_n has real coefficients. □

REMARK 4.10. As in the proof of Corollary 4.9, let $q(z) = \prod_{j=1}^{n}(z - z_j)$ and

$$s(z) = \sum_{j=0}^{n-2m} d_j(1+z)^j(1-z)^{n-2m-j}$$

with $d_j = 0$ if $n-j$ is odd, and let us put $a_j = d_j \prod_{k=1}^{n}(2/(1-z_k))$, $j = 0, \ldots, n-2m$. If the z_j and d_j are such that $\sum |a_j| \leqslant 1$, then the constant η^* in Corollary 4.9 can be taken as $\eta^*(c)$ from Theorem 4.5 and Theorem 4.6, respectively, where the w_j appearing in the definition of $\eta^*(c)$ are given by $w_j = (1+z_j)/(1-z_j)$ and $0 \leqslant c < (n/(n+1)) \min\{\operatorname{Re} w_j\}$.

It would be interesting to find a characterization of the polynomials ρ_ν for which $\mathcal{J}_{\mathcal{R}_n^\pm(\cdot,\rho_\nu)}$ is connected, but not necessarily a Jordan arc. The connectedness of Julia sets of rational functions has been studied recently in [**13**], (see also [**7**]). Finally, we mention that the connectedness locus of the related family $\lambda\mathcal{R}_n(\cdot, z^2 - \mu)$, $\mu = (\lambda/c)^2$, $|\mu| < 1$, $c > 2$, has been investigated in [**12**].

REMARK 4.11. As mentioned in the introduction, the results of this paper and [**7**] were obtained by the authors in joint work carried out mainly in the period during which the first author was writing his thesis under the supervision of the second author. The first author would like to thank the second author for allowing him to insert a first version of this paper and of [**7**] in his thesis [**5**].

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