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Circle expanding maps and symmetric structures

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Abstract. A conjugacy between two uniformly asymptotic affine circle expanding maps is symmetric if they have the same eigenvalues along corresponding periodic points.

1. *Introduction*

Let f be a C^1 covering map of the unit circle $S^1 = \{z : |z| = 1\}$ with degree $\deg(f) \geq 2$. f is called expanding if there exist constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(z)| \geq C\lambda^n$ for all $z \in S^1$ and $n \geq 1$. It is well known that two expanding maps $f, g : S^1 \to S^1$ with the same degree are topologically conjugated [**Sh**]. When f and g are $C^{1+\alpha}$ $(1 > \alpha > 0)$, the conjugacy is quasisymmetric. Moreover, the conjugacy is $C^{1+\alpha}$ if it is symmetric or f and g have the same eigenvalues along corresponding periodic points (see [**Sul1**], [**Sul2**] and [**SS**]).

The concept of a circle expanding map has been extended to the category of symmetric structures—it is called a uniformly asymptotic affine (UAA) circle expanding map. Roughly speaking, a UAA circle expanding map is a covering of the unit circle with degree $d \geq 2$ and all the branches of the inverse of its iteration are uniformly asymptotic affine. It has been asked whether a conjugacy between two UAA maps is symmetric if they have same eigenvalues along corresponding periodic points [**Sul2**]. We will give an affirmative answer in this paper.

The main tool of this work is a linear model, which was introduced in [**DH**] and will be further investigated in this paper. It is closely related to another beautiful invariant—the solenoid function (see [**PS**] for more information about this function). Associated to a UAA circle expanding map, there is a Riemann surface lamination, which is the space of inverse limit orbits with product topology and affine structure on each leaf (see [**Sul2**] for the definition). With this terminology, the main result of this work says that two Riemann surface laminations are conformally equivalent if the moduli of corresponding annuli are equal.

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2. *Preliminaries*

In this section, we recall some basic definitions and results about quasiconformal maps and quasisymmetric maps which will be used in this paper (see [**Ah**] and [**GS**] for more information).

A sense-preserving homeomorphism ϕ from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$ is called *quasisymmetric* (q.s.) if there is a constant $\rho > 1$ such that for any $x, t \in \mathbb{R}$ with $t \neq 0$ and $x - t, x, x + t \in I$,

$$\frac{1}{\rho} < \rho_\phi(x, t) = \frac{\phi(x + t) - \phi(x)}{\phi(x) - \phi(x - t)} < \rho.$$

If $\rho_\phi(x, t) \to 1$ as $t \to 0$ uniformly for $x \in I$, ϕ is called *symmetric*. In the case when ϕ is a homeomorphism from an interval $I \subset S^1$ to S^1, we define $\rho_\phi(x, t)$ by the circular Lebesgue measure, i.e. the measure induced by the universal covering $\Pi : \mathbb{R} \to S^1$, $\Pi(t) = \exp(2\pi i t)$.

A homeomorphism f from a domain $U \subset \mathbb{C}$ into $\mathbb{C}$ is called *K-quasiconformal* (q.c.) if f is an L^2-solution of an equation $f_{\bar{z}} = \mu f_z$, where $\mu(z) \in L^\infty(U)$ and satisfies $\|\mu\|_\infty \leq (K - 1)/(K + 1) < 1$. $\mu(z)$ is the complex dilatation of f and $K(f) = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$ is the maximal dilatation of f.

The following statements are well known (see [**Ah**]): a 1-q.c. map is conformal. The composition of two q.c. maps w_1 and w_2 is also q.c. with maximal dilatation $K(w_1 \circ w_2) \leq K(w_1)K(w_2)$. A K-q.c. map is $1/K$-Hölder continuous on any compact subset of the domain. A q.c. homeomorphism of the upper half plane $\mathbb{H}$ which preserves the real axis $\mathbb{R}$ is q.s. on $\mathbb{R}$.

MEASURABLE RIEMANN MAPPING THEOREM. *For any $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$, there is a unique q.c. map from $\mathbb{C}$ to itself with $0, 1$ fixed so that it has complex dilatation μ.*

BEURLING–AHLFORS EXTENSION. *Let ϕ be a sense-preserving homeomorphism of $\mathbb{R}$ to itself. Then $\Phi(x, y) = (u(x, y), v(x, y))$ defined by*

$$u(x, y) = \frac{1}{2} \int_0^1 [\phi(x + ty) + \phi(x - ty)]\, dt$$

$$v(x, y) = \int_0^1 [\phi(x + ty) - \phi(x - ty)]\, dt$$

is a diffeomorphism of $\mathbb{H}$. Moreover, there exists an increasing function $C(\rho) > 1$ with $C(\rho) \to 1$ as $\rho \to 1$, so that Φ is $C(\rho)$-q.c. when ϕ is ρ-q.s.

Let ϕ be a q.s. homeomorphism of S^1. The *local dilatation* $H_p(\phi)$ of ϕ at a point $p \in S^1$ is the infimum of the maximal dilatation of the possible extensions of ϕ to a neighbourhood of p. The *boundary dilatation* $H(\phi)$ is the infimum of the maximal dilatation of the possible extensions of ϕ to neighborhoods of S^1 in the plane. The boundary dilatation is related to the local dilatation by

$$H(\phi) = \max_{p \in S^1} H_p(\phi)$$

(see [**F**]). Denote by $\mathcal{E}$ the set of bounded increasing functions $\epsilon(t)$ from $(0, +\infty)$ to itself so that $\epsilon(t) \to 0$ as $t \to 0$.

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LEMMA 2.1. *Suppose ϕ is a q.s. homeomorphism of $\mathbb{R}$. Let*

$$\rho_\phi(x,t,k)=\frac{\phi(x+kt)-\phi(x)}{\phi(x)-\phi(x-t)}$$

for $x,t\in\mathbb{R}$, $t\neq 0$ and $0<k\leq 1$. The following conditions are equivalent:

(1) *$\exists\varepsilon_1\in\mathcal{E}$ so that $|\rho_\phi(x,t)-1|<\varepsilon_1(|t|)$, i.e. ϕ is symmetric;*

(2) *$\exists\varepsilon_2\in\mathcal{E}$ so that $|\mu(z)|<\varepsilon_2(\Im z)$, where μ is the complex dilatation of the Beurling–Ahlfors extension of ϕ;*

(3) *$\exists\varepsilon_3\in\mathcal{E}$ so that $|\rho_\phi(x,t,k)-k|<\varepsilon_3(|t|)$.*

Proof. (1) $\Rightarrow$ (2) comes from [**F**]. (3) $\Rightarrow$ (1) is obvious. So we only need to prove (2) $\Rightarrow$ (3).

Let f be the Beurling–Ahlfors extension of ϕ, it can be defined on the complex plane by symmetric extension. For any $t\in\mathbb{R}$ with $1/2>|t|>0$, let g be a q.c. homeomorphism of the complex plane with complex dilatation $\mu_g=\mu_f$ in the strip $B=\{z:-\sqrt{|t|}<\Im z<\sqrt{|t|}\}$ and zero outside. By the Teichmüller distortion theorem (see [**Ah**]), there exists $\delta(t)\in\mathcal{E}$, which depends only on $\varepsilon_2(t)$, so that

$$|\rho_g(x,t,k)-k|<\delta(|t|)$$

for any $x\in\mathbb{R}$ and $0<k\leq 1$. Since $g\circ f^{-1}$ is conformal on $f(B)$, by the Koëbe distortion theorem, there exists $\theta(t)\in\mathcal{E}$ so that

$$|\rho_g(x,t,k)-\rho_\phi(x,t,k)|<\theta(|t|)$$

for any $x\in\mathbb{R}$ and $0<k\leq 1$. The existence of $\varepsilon_3(t)$ follows immediately. □

3. *$C^{1+\alpha}$ circle expanding maps*

Let f be a $C^{1+\alpha}$ circle expanding map (we always assume that the degree $d=2$ in this paper when we refer to an expanding map or a covering map of the circle. In fact, all the statements are still true for higher degree by a small modification). Let $\Pi_f(t)=\exp(2\pi ti+\theta i)$ be the universal covering from $\mathbb{R}$ to S^1 such that $\Pi_f(0)=\exp(\theta i)$ is the fixed point of f. The (normalized) lift F of f is a $C^{1+\alpha}$ homeomorphism of $\mathbb{R}$ with $F(0)=0$, $FT=T^2F$ and $\Pi_f F=f\Pi_f$, where $T(t)=t+1$ is the deck transformation.

For any pair $(i,j)\in\{\mathbb{N}\times\mathbb{N}:i>0,0\leq j\leq 2^i-1\}$, denote $I(i,j)=I_{ij}=F^{-i}T^j(I)$, $I=[0,1]$. Then $\{\Pi_f(I_{ij})\}$ forms a Markov partition of f for any fixed $i>0$. There is only one $t_{ij}\in I_{ij}$ such that $F^i(t_{ij})=T^j(t_{ij})$, so $\Pi_f(t_{ij})$ is a periodic point of f. Denote $e_{ij}=e_{ij}(f)=(F^i)'(t_{ij})=(f^i)'(\Pi_f(t_{ij}))$, $\{e_{ij}\}$ are the eigenvalues of f at periodic points.

LEMMA 3.1. (Naive distortion) *With the above assumptions, there is a constant $C>0$ such that*

$$|\log(F^{-n})'(t_1)-\log(F^{-n})'(t_2)|\leq C|t_1-t_2|^\alpha.$$

This is a well known lemma and can be verified by the chain rule of derivatives of composed functions and the expanding property of F. The next two lemmas are simple corollaries of this lemma.

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LEMMA 3.2. (Eigenvalues and Markov partition) *There is a constant $C > 0$ such that*

$$-C \leq \log(e_{ij}|I_{ij}|) \leq C,$$

where $|I_{ij}|$ is the length of interval I_{ij}.

LEMMA 3.3. (Distortion of eigenvalues) *Let f and g be $C^{1+\alpha}$ circle expanding maps. The conjugacy between them is β-Hölder continuous if and only if their eigenvalues have the relation*

$$[e_{ij}(f)]^{\beta} \leq e_{ij}(g) \leq [e_{ij}(f)]^{1/\beta}.$$

THEOREM 3.4. (Q.s. conjugacy) *A $C^{1+\alpha}$ circle covering map f is expanding if and only if it is q.s. conjugated to $P_2(z) = z^2$.*

Proof. If f is expanding, the conjugacy ϕ is a homeomorphism. Its lift Φ satisfies $\Phi(0) = 0$, $\Phi T = T\Phi$ and $F\Phi(t) = \Phi(2t)$. For any symmetric triple $(x - t, x, x + t)$ in $\mathbb{R}$, denoting $L = [x - t, x]$ and $R = [x, x + t]$, there is some $n \in \mathbb{Z}$ such that $1 \leq 2^n|R| < 2$. So the lengths of $F^n\Phi(L)$ and $F^n\Phi(R)$ are also in $[1, 2)$. From Lemma 3.1, $1/C \leq |\Phi(L)|/|\Phi(R)| \leq C$. Hence Φ is q.s.

Conversely, if the circle covering map f is q.s. conjugated to $P(z) = z^2$, by the Hölder continuity of q.s. maps, there is a constant $C > 0$ and $K \geq 1$ such that f has eigenvalues e_{ij} satisfying $2^{i/K} \leq e_{ij} \leq 2^{iK}$, and $C2^{-i/K} \geq |I_{ij}| \geq 2^{-iK}/C$. Noticing that Lemma 3.1 still holds in this situation, there is some $n > 0$ such that $(F^n)'(t) > 1$ for all $t \in \mathbb{R}$, i.e. f is expanding. □

COROLLARY. (Smoothness of conjugacy) *Let f and g be $C^{1+\alpha}$ circle expanding maps. The following conditions are equivalent:*

(1) *f and g are $C^{1+\alpha}$ conjugated;*
(2) *f and g are symmetric conjugated;*
(3) *the corresponding eigenvalues of f and g are equal;*
(4) *f and g are Lipchitz conjugated.*

(1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are from above lemmas. (2) $\Rightarrow$ (3) holds because a symmetric conjugacy is α-Hölder continuous for any $1 > \alpha > 0$ (or refer [**Sc**]). (4) $\Rightarrow$ (1) follows from the well known blow-up and blow-down argument applied to a point where the conjugacy has positive derivative (refer to [**Sul2**] and [**SS**]).

4. *UAA circle expanding maps*

Definition. A circle covering map f with degree $d \geq 2$ is called *uniformly asymptotic affine (UAA)* if there exists $\varepsilon(t) \in \mathcal{E}$ such that for all the inverses F^{-n}, where F is a lift of f, the distortions of every symmetric triple satisfy

$$|\log \rho_{F^{-n}}(x, t)| < \varepsilon(|t|).$$

Notice that $C^{1+\alpha}$ circle expanding maps are always UAA by the naive distortion lemma with $\varepsilon(t) = Ct^{\alpha}$. The next theorem illustrates why we call the UAA map 'expanding'.

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THEOREM 4.1. (Q.s. conjugacy) *Every UAA circle covering map is q.s. conjugated to* $P_2(z) = z^2$.

Proof. Let f be a UAA circle covering map and F be a lift of f. We assume $F(0) = 0$ since F has at least one fixed point. Let $I(i, j) = F^{-i}T^j(I)$ be a partition of $I = [0, 1]$, $I(i, j) = I(i + 1, 2j) \cup I(i + 1, 2j + 1)$. By definition,

$$\exp(-\varepsilon(1)) \leq \frac{|I(i + 1, 2j)|}{|I(i + 1, 2j + 1)|} \leq \exp(\varepsilon(1)).$$

So both $|I(i + 1, 2j)|$ and $|I(i + 1, 2j + 1)|$ are less than or equal to $|I(i, j)|(1 + \exp(-\varepsilon(1)))^{-1}$. Thus $|I(i, j)| \leq (1 + \exp(-\epsilon(1)))^{-i}$. This shows that there is a homeomorphism Φ of $\mathbb{R}$ with $0, 1$ fixed, such that $\Phi T = T\Phi$ and $F\Phi(t) = \Phi(2t)$.

For any $x, t \in \mathbb{R}$ with $t \neq 0$, let $n \in \mathbb{Z}$ so that $1 \leq 2^n|t| < 2$. Then $1 \leq |F^n\Phi(x - t, x)| < 2$ and $1 \leq |F^n\Phi(x, x + t)| < 2$. By definition and Lemma 2.1, there exists a constant $C > 0$ which depends only on $\varepsilon(t)$, so that

$$\left|\log \frac{|\Phi(x - t, x)|}{|\Phi(x, x + t)|}\right| < C.$$

Thus Φ is quasisymmetric. □

LEMMA 4.2. *The eigenvalues of f are well defined, i.e. f^n is differentiable at each fixed point t_n and its derivative $(f^n)'(t_n) > 1$.*

Proof. Let Π_n be the universal covering from $\mathbb{R}$ to S^1 with $\Pi_n(0) = t_n$ and F_n be the lift of f^n with $F_n(0) = 0$. We will prove that F_n^{-1} is differentiable at zero and $0 < (F_n^{-1})'(0) < 1$.

First note that $F_n^{-1}(s)/s$ is contained in a closed interval in $(0, 1)$ for all s from Theorem 4.1. For a fixed s, since F_n^{-1} are uniformly asymptotic affine, by Lemma 2.1,

$$0 < \lim_{m \to +\infty} \frac{F_n^{-1}(F_n^{-m}(s))}{F_n^{-m}(s)} < 1.$$

Also, because of the UAA property, the above limits are the same for all s. So F_n^{-1} is differentiable at zero. □

Remark. This lemma is also proved in [**Sc**].

LEMMA 4.3. (Linearization) *Let F be the (normalised) lift of a UAA circle expanding map f, $F'(0) = \delta > 1$. Then there is a unique symmetric homeomorphism γ_f of $\mathbb{R}$ with $0, 1$ fixed, such that*

$$\gamma_f \circ F \circ \gamma_f^{-1}(x) = \delta x.$$

Proof. Let $\gamma_n(t) = F^{-n}(t)/F^{-n}(1)$. Then $\{\gamma_n\}$ is uniformly symmetric and converges to a symmetric map γ of $\mathbb{R}$. Moreover, $\gamma \circ F \circ \gamma^{-1}(x) = \delta x$. If there is another symmetric homeomorphism β of $\mathbb{R}$ so that $\beta \circ F \circ \beta^{-1}(x) = \delta x$, then $\gamma \circ \beta^{-1}$ commutes with $\delta : x \to \delta x$, and is thus a linear map.

It can be proved that γ_f is locally $C^{1+\alpha}$ (or analytic) when f is $C^{1+\alpha}$ (or analytic). □

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Let $L_f = \gamma_f \circ T \circ \gamma_f^{-1}$. Then $L_f(0) = 1$ and $L_f^2 \circ \delta = \delta \circ L_f$. We call L_f the *linear model* of f. From Lemma 4.3, we see that:

LEMMA 4.4. *$L_f = L_g$ if and only if f and g are symmetric conjugated.*

THEOREM 4.5. (Criterion of linear models) *Suppose L is a homeomorphism of $\mathbb{R}$ with $L(0) = 1$ and $L^2 \circ \delta = \delta \circ L$ for some constant $\delta > 1$. Then L is a linear model of some UAA circle expanding map if and only if there exists $\epsilon(t) \in \mathcal{E}$, such that for any $n, m \in \mathbb{Z}$,*

$$\left| \log \frac{|L^n(I)|}{|L^{n-1}(I)|} - \log \frac{|L^m(I)|}{|L^{m-1}(I)|} \right| < \epsilon(d_2(n, m)), \tag{4.1}$$

where $d_2(n, m) = 2^{-l}$ when $n - m = p2^l$ and $p \in \mathbb{Z}$ is an odd integer.

Before the proof of the theorem, we first give some lemmas.

LEMMA 4.6. *Let L be a homeomorphism of $\mathbb{R}$ with $L(0) = 1$ and $L^2 \circ \delta = \delta \circ L$ for some constant $\delta > 1$. Suppose that there exists $\epsilon(t) \in \mathcal{E}$ so that for any $n, m \in \mathbb{Z}$,*

$$\left| \log \frac{|L^n(I)|}{|L^{n-1}(I)|} - \log \frac{|L^m(I)|}{|L^{m-1}(I)|} \right| < \epsilon(d_2(n, m)). \tag{4.2}$$

Then there exists a unique q.s. homeomorphism w of $\mathbb{R}$, so that

$$w(2x) = \delta w(x), \quad w \circ T(x) = L \circ w(x), \tag{4.3}$$

and there exists $\theta(t) \in \mathcal{E}$, so that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$ and $0 < k \leq 1$,

$$|\rho_w(x, t, k) - \rho_w(x + n, t, k)| \leq \theta(|t|). \tag{4.4}$$

Proof. First note that $|\log |L^n(I)| - \log |L^{n-1}(I)||$ is bounded. So there is a unique q.s. homeomorphism w of $\mathbb{R}$ which satisfies (4.3). Moreover,

$$|\log \rho_w(n, 1) - \log \rho_w(m, 1)| \leq \epsilon(d_2(n, m)). \tag{4.5}$$

Given any $\theta > 0$, since w is quasisymmetric, there exists a constant $0 < k_0 < 1$ so that for any $x, t \in \mathbb{R}$ with $t \neq 0$ and $0 < k \leq k_0$, $\rho_w(x, t, k) < \theta/2$.

Moreover, there exists a constant $s_0 > 1/k_0$ so that for any $y, s \in \mathbb{R}$ with $|s| > s_0$ and $1 \geq k > k_0$,

$$|\rho_w(y, s, k) - \rho_w[y, s, k]| < \theta/3, \tag{4.6}$$

where $\rho_w[y, s, k] = (w[y + ks] - w[y])/(w[y] - w[y - s])$, and here $[y]$ denotes the maximal integer less than or equal to y.

On the other hand, from (4.5), there exists a constant $l_0 > 0$ so that for any $y, s, n, l \in \mathbb{Z}$ with $|s| \leq 2s_0$ and $l > l_0$, and $1 \geq k > 0$ with $ks \in \mathbb{Z}$,

$$|\rho_w(y, s, k) - \rho_w(y + n2^l, s, k)| < \theta/3. \tag{4.7}$$

Now, as $|t| < s_0 2^{-l_0}$, there exists a unique integer $m > l_0$ so that $s_0 < 2^m |t| \leq 2s_0$. Therefore, for any $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $1 \geq k > k_0$,

$$\begin{aligned}
&|\rho_w(x, t, k) - \rho_w(x + n, t, k)| \\
&\quad = |\rho_w(2^m x, 2^m t, k) - \rho_w(2^m x + 2^m n, 2^m t, k)| \\
&\quad \leq |\rho_w(2^m x, 2^m t, k) - \rho_w[2^m x, 2^m t, k]| \\
&\qquad + |\rho_w(2^m x + 2^m n, 2^m t, k) - \rho_w[2^m x + 2^m n, 2^m t, k]| \\
&\qquad + |\rho_w[2^m x, 2^m t, k] - \rho_w[2^m x + 2^m n, 2^m t, k]|.
\end{aligned}$$

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The first and second terms are less than $\theta/3$ by (4.6). The last term is less than $\theta/3$ by (4.7). The lemma is proved. □

A similar argument as above also applies for the next lemma.

LEMMA 4.7. *Suppose that L_1, L_2 are two homeomorphisms of $\mathbb{R}$ which satisfy the condition in Lemma 4.6. Let w_1, w_2 be the q.s. homeomorphisms of $\mathbb{R}$ so that*

$$w_i(2x) = \delta_i w_i(x), \quad w_i \circ T(x) = L_i \circ w_i(x) \quad (i = 1, 2).$$

Then there exists $\epsilon(t) \in \mathcal{E}$, which depends only on L_1, so that for any $x, t \in \mathbb{R}$ and $0 < k \leq 1$,

$$|\rho_{w_1}(x, t, k) - \rho_{w_2}(x, t, k)| < \epsilon(s),$$

where

$$s = \sup_{n \in \mathbb{Z}} \left| \log \frac{|L_1^n(I)|}{|L_1^{n-1}(I)|} - \log \frac{|L_2^n(I)|}{|L_2^{n-1}(I)|} \right|.$$

LEMMA 4.8. *Let h_1, h_2 be two q.s. homeomorphisms of $\mathbb{R}$, and*

$$\eta(t) = \sup\{|\rho_{h_1}(x, s, k) - \rho_{h_2}(x, s, k)| : x \in \mathbb{R}, |s| < t, 0 < k \leq 1\}.$$

Then there exists $\epsilon(t) \in \mathcal{E}$, which depends only on h_1 and $\eta(t)$, so that

$$\sup_{0 < \Im z < t} |\mu_1(z) - \mu_2(z)| < \epsilon(\eta(t)),$$

where μ_1 and μ_2 is the complex dilatation of the Beurling–Ahlfors extension of h_1 and h_2, respectively.

Proof. Let $f_j(x + iy) = u_j(x, y) + i v_j(x, y)$ $(j = 1, 2)$ denote the Beurling–Ahlfors extension of h_j. Then

$$K_j(z) = \frac{1 + \mu_j(z)}{1 - \mu_j(z)} = \frac{(u_j)_x + i(v_j)_x}{(v_j)_y - i(u_j)_y} = \frac{1 + i a_j}{b_j - i c_j},$$

where $a_j = (v_j)_x/(u_j)_x$, $b_j = (v_j)_y/(u_j)_x$, $c_j = (u_j)_y/(u_j)_x$ and

$$\begin{aligned}
(u_j)_x &= \frac{1}{2y}[h_j(x + y) - h_j(x_y)], \\
(u_j)_y &= \frac{1}{2y}\left(h_j(x + y) + h_j(x - y) - \frac{1}{y}\int_{x-y}^{x+y} h_j(t)\, dt\right), \\
(v_j)_x &= \frac{1}{y}[h_j(x + y) + h_j(x - y) - 2h_j(x)], \\
(v_j)_y &= \frac{1}{y}[h_j(x + y) - h_j(x - y)] - \frac{1}{y^2}\left(\int_x^{x+y} h_j(t)\, dt - \int_{x-y}^{x} h_j(t)\, dt\right).
\end{aligned}$$

Denote $\rho_j = \rho_{h_j}$. A direct computation implies that (4.8)

$$\begin{aligned}
a_j[1 + \rho_j(x, y)] &= 2[\rho_j(x, y) - 1], \\
b_j[1 + \rho_j(x, y)] &= 2\left(\rho_j(x, y) + 1 - \int_0^1 \rho_j(x, y, t)\, dt - \int_0^1 \rho_j(x, -y, t)\, dt\right), \\
c_j[1 + \rho_j(x, y)] &= \rho_j(x, y)\left(1 + \int_0^1 \rho(x, -y, t)\, dt\right) - \left(1 + \int_0^1 \rho_j(x, y, t)\, dt\right).
\end{aligned}$$

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Now,

$$\begin{aligned}&[K_1(z) - K_2(z)](b_1 - ic_1)(b_2 - ic_2)\\ &\quad = \quad (a_1 - a_2)(c_1 + ib_1) + (b_2 - b_1)(1 + ia_1) + (c_1 - c_2)(1 - a_1).\end{aligned}$$

There exist constants $C_1, C_2 > 0$, which depend only on h_1 and $\eta(t)$, so that $b_j^2 + c_j^2 > C_1$ and $a_j^2 < C_2$. Thus there exists a constant $C > 0$, which depends only on h_1 and $\eta(t)$, so that

$$|\mu_1(z) - \mu_2(z)| < C(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2|).$$

The lemma follows by (4.8). □

Proof of Theorem 4.5. The necessary conditions are easy to check by the following identities and the symmetry of γ:

$$\frac{|F^{-l} \circ T^n(I)|}{|F^{-l} \circ T^{n-1}(I)|} = \frac{|T^p \circ F^{-l} \circ T^n(I)|}{|T^p \circ F^{-l} \circ T^{n-1}(I)|} = \frac{|F^{-l} \circ T^m(I)|}{|F^{-l} \circ T^{m-1}(I)|},$$

$$\frac{|L^n(I)|}{|L^{n-1}(I)|} = \frac{|\delta^{-l} \circ L^n(I)|}{|\delta^{-l} \circ L^{n-1}(I)|} = \frac{|\gamma \circ F^{-l} \circ T^n(I)|}{|\gamma \circ F^{-l} \circ T^{n-1}(I)|}.$$

Now we prove the sufficiency. Let w be the q.s. homeomorphism guaranteed by Lemma 4.6. By μ we denote the complex dilatation of the Beurling–Alhfors extension of w. Then $\mu(2z) = \mu(z)$ for all $z \in \mathbb{H}$. By Lemmas 4.6 and 4.8, there exists $\epsilon(t) \in \mathcal{E}$ such that for all $n \in \mathbb{Z}$,

$$\sup_{0<\Im z<t} |\mu(z) - \mu(z + n)| \le \epsilon(t).$$

Let Φ be the q.c. homeomorphism of $\mathbb{H}$ with $0, 1, \infty$ fixed, and whose complex dilatation $\mu_\Phi = \mu$ on $\{z : 0 < \Re z < 1\}$ and $\mu_\Phi(z) = \mu_\Phi(z + n)$ for any $z \in \mathbb{H}$ and $n \in \mathbb{Z}$. Then $\Phi(z + 1) = \Phi(z) + 1$. Let $\Gamma = w \circ \Phi^{-1}$. Then the complex dilatation μ_Γ of Γ satisfies $\mu_\Gamma(\Phi(z)) = \mu(z) - \mu(z + n)$, where $n \in \mathbb{Z}$ satisfies $0 \le \Re(z + n) < 1$. But Φ is quasiconformal and $\Phi(n) = n$ for $n \in \mathbb{Z}$, $\Im\Phi^{-1}(z) \to 0$ as $\Im z \to 0$ uniformly. This shows that $\Gamma|_{\mathbb{R}}$ is symmetric.

Let $F = (\Gamma|_{\mathbb{R}})^{-1} \circ \delta \circ \Gamma|_{\mathbb{R}}$. Then $F(x + 1) = F(x) + 2$ and $\{F^{-n}\}$ is uniformly symmetric. Let f be a projection of F onto S^1. Then f is the desired UAA map. □

For any linear model L, define a function $\mathcal{S}_L : \mathbb{Z} \to \mathbb{R}$ by

$$\mathcal{S}_L(n) = \log|L^n(I)| - \log|L^{n-1}(I)|.$$

There is an identity for $\mathcal{S}_L$ corresponding to $L^2 \circ \delta = \delta \circ L$.

$$\frac{1 + \exp[\mathcal{S}_L(2n + 1)]}{1 + \exp[-\mathcal{S}_L(2n - 1)]} = \frac{\exp[\mathcal{S}_L(n)]}{\exp[\mathcal{S}_L(2n)]}.$$

Theorem 4.5 says that $\mathcal{S}_L$ can be extended to a continuous function on the completion space of $\mathbb{Z}$ with metric d_2, which is a Cantor set. Conversely, for any continuous function $\mathcal{S}$ on this Cantor set satisfying the above identity, there is also a linear model L such that $\mathcal{S}_L = \mathcal{S}$. This function is called the *solenoid function* in [**PS**].

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5. *Proof of the main theorem*

By $\mathcal{T}_e$ we denote the space consisting of symmetric conjugate classes of UAA circle expanding maps with a distance defined by

$$d_{\mathcal{T}}([f_1],[f_2]) = \tfrac{1}{2}\log H(\phi),$$

where ϕ is the q.s. conjugacy between f_1 and f_2, and $H(\phi)$ is the boundary dilatation of ϕ. It may be identified with the space of linear models, or the space of solenoid functions.

Let L_1 and L_2 be linear models with $L_1^2 \circ \delta_1 = \delta_1 \circ L_1$ and $L_2^2 \circ \delta_2 = \delta_2 \circ L_2$, w be the conjugacy from L_1 to L_2, i.e. w is a q.s. homeomorphism of $\mathbb{R}$ with $0, 1$ fixed, $w \circ L_1 = L_2 \circ w$ and $w \circ \delta_1 = \delta_2 \circ w$. Suppose f_1 and f_2 are UAA circle expanding maps, whose linear models are L_1 and L_2, respectively. Let ϕ be the conjugacy from f_1 to f_2 and F_1, F_2 and Φ be the lift of f_1, f_2 and ϕ, respectively. Then $w = \gamma_2 \circ \Phi \circ \gamma_1^{-1}$, where γ_1 and γ_2 are the linearization coordinates of F_1 and F_2. Since γ_1 and γ_2 are symmetric, for any $\epsilon > 0$, we can extend w to be a q.c. map W on a neighborhood of the original, such that its maximal dilatation $K < H(\phi) + \epsilon$. Let $W_n = \delta_2^n \circ W \circ \delta_1^{-n}$. As a normal family, it has a subsequence converging to a K-q.c. map W_∞ from $\mathbb{H}$ to itself. Obviously, $W_\infty|_{\mathbb{R}} = w$.

Conversely, if w can be extended to a K-q.c. map of $\mathbb{H}$, then the local dilatation of w satisfies $H_x(w) \le K$, for every $x \in \mathbb{R}$. Thus $H_t(\phi) \le K$ for all $t \in S^1$. It follows that $H(\phi) = \max_{t \in S^1} H_t(\phi) \le K$. We have proved the following.

LEMMA 5.1. (The boundary dilatation metric is equal to the Teichmüller metric) *Let L_1 and L_2 be the linear models of circle expanding maps f_1 and f_2, respectively. Let w be the conjugacy between L_1 and L_2, and ϕ be the conjugacy between f_1 and f_2. Then*

$$d_{\mathcal{T}}([f_1],[f_2]) = \tfrac{1}{2}\log H(\phi) = \tfrac{1}{2}\log K(w),$$

where $K(w)$ is the infimum of the maximal dilatations of the possible extensions of w to $\mathbb{H}$.

From Lemmas 4.7 and 4.8, we also have the next lemma.

LEMMA 5.2. *The distance $d_{\mathcal{T}}$ is topologically equivalent to the supreme norm of the solenoid functions.*

Now we embed $\mathcal{T}_e$ into the universal Teichmüller space. By $\mathcal{M}(T)$ we denote a Banach space consisting of all $\mu \in L^\infty(\mathbb{H})$ with $\mu(2z) = \mu(z)$ for all $z \in \mathbb{H}$ and

$$\|(\mu(z+p) - \mu(z))|_{B_t}\|_\infty \le \epsilon(t)$$

for some $\epsilon(t) \in \mathcal{E}$ and all $p \in \mathbb{Z}$, where B_t is a strip domain $\{z, 0 < \Im z < t\}$. Define an equivalence relation on the unit ball $\mathcal{M}_1(T)$ of $\mathcal{M}(T)$ by $\mu_1 \sim \mu_2 \Leftrightarrow w_{\mu_1}|_{\mathbb{R}} = w_{\mu_2}|_{\mathbb{R}}$, where w_μ is the q.c. homeomorphism from $\mathbb{H}$ to itself with Beltrami coefficient μ and normalised by fixing $0, 1$ and ∞. From the proof of Theorem 4.5, we see that this quotient space may be identified with $\mathcal{T}_e$ by the conjugacy from $T(x) = x + 1$ to a

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linear model L. This gives an isometric embedding of the space $\mathcal{T}_e$ into the universal Teichmüller space with base point $T(x) = x + 1$ by Lemma 5.1.

$\mu \in \mathcal{M}_1(T)$ is called special if there is some $t > 0$ such that $\mu(z + p) - \mu(z) = 0$ for a.e. $z \in B_t$. Such a complex dilatation corresponds to an analytic linear model, and thus there is an analytic circle expanding map corresponding to it. Since special elements are dense in $\mathcal{M}_1(T)$, we have

LEMMA 5.3. *Analytic circle expanding maps are dense in* $\mathcal{T}_e$.

LEMMA 5.4. (Eigenvalues and Markov partitions) *For any UAA circle expanding map, the lengths of Markov partitions and the eigenvalues of periodic points have the relation*

$$\log |I_{ij}| / \log e_{ij} \to -1$$

as $i \to \infty$ *uniformly for* $0 \le j \le 2^i - 1$.

Proof. Let f be a UAA circle expanding map. By Lemma 5.3, there exists a sequence of analytic circle expanding maps $\{f_n\}$, so that $d_T([f], [f_n]) \to 0$ as $n \to \infty$. Denote by ϕ_n the q.s. conjugacy between f and f_n. Then $H(\phi_n) \to 1$ as $n \to \infty$.

Given any $8 > \epsilon > 0$, choose an f_n in the sequence so that $H(\phi_n) < 1/(1 - \epsilon/8)$. Since both ϕ_n and ϕ_n^{-1} are $(1 - \epsilon/8)$-Hölder continuous, as i is big enough,

$$\left| \frac{\log |I_{ij}(f)|}{\log |I_{ij}(f_n)|} - 1 \right| < \frac{\epsilon}{7}, \quad \left| \frac{\log e_{ij}(f)}{\log e_{ij}(f_n)} - 1 \right| < \frac{\epsilon}{7}$$

for all j. On the other hand, from Lemma 3.2, as i is big enough,

$$\left| \frac{\log e_{ij}(f_n)}{\log |I_{ij}(f_n)|} + 1 \right| < \frac{\epsilon}{7}.$$

Now the Lemma follows immediately. □

THEOREM 5.5. (Eigenvalues form a complete set of invariants) *Suppose* f_1 *and* f_2 *are two UAA circle expanding maps. Then* $e_{ij}(f_1) = e_{ij}(f_2)$ *if and only if* $L_{f_1} = L_{f_2}$.

Proof. Let F_i be normalised lifts of f_i, γ_i be the symmetric homeomorphisms of $\mathbb{R}$ so that $\gamma_i \circ F_i \circ \gamma_i^{-1}(x) = \delta_i(x)$ $(i = 1, 2)$ and w be the conjugacy from L_{f_1} to L_{f_2}. Denote $J(i, j) = \delta_1^{-i} L_{f_1}^j(I) = \gamma_1(I_{ij}(f_1))$. Since γ_i is symmetric, for any $1 > \alpha > 0$, both γ_i and its inverse are α-Hölder continuous on any compact set in $\mathbb{R}$. Thus

$$\log |I_{ij}(f_1)| / \log |J(i, j)| \to 1$$

as $i \to \infty$ uniformly for j. By Lemma 5.4 and $e_{ij}(f_1) = e_{ij}(f_2)$, we have

$$\lim_{i \to \infty} \frac{\log |J(i, j)|}{\log |\omega(J(i, j))|} = 1$$

uniformly for j. If $L_{f_1} \neq L_{f_2}$, we will prove that there is a nested sequence of $J(i_k, j_k)$ such that the above formula does not hold.

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Consider the solenoid functions $\mathcal{S}_1$ and $\mathcal{S}_2$ of L_{f_1} and L_{f_2}, respectively. Since $L_{f_1} \neq L_{f_2}$, $\mathcal{S}_1 \neq \mathcal{S}_2$. Using the uniform continuity of solenoid functions, there must exist constants $\epsilon > 0$, $p \in \mathbb{N}$ and $0 \leq l_1 < l_2 \leq 2^p - 1$ such that for all $n \in \mathbb{Z}$,

$$\mathcal{S}_1(n2^p + l_i) - \mathcal{S}_2(n2^p + l_i) > \epsilon$$

for $i = 1, 2$ (or less than $-\epsilon$; without loss of generality, we assume the above holds). Let $\sigma > 0$ be a constant so that

$$1 - (1 + \sigma)\exp(-\epsilon) > (1 + \exp\|\mathcal{S}_2\|)^{2^p}\sigma,$$

where $\|\mathcal{S}_2\|$ is the supreme norm of $\mathcal{S}_2$, and denote $b_i(n) = |L_{f_i}^n(I)|$ $(i = 1, 2,\ n \in \mathbb{Z})$. We claim that for any 2^p consecutive intervals

$$L_{f_1}^n(I), L_{f_1}^{n+1}(I), \ldots, L_{f_1}^{n+2^p-1}(I),$$

there is one of them, $L_{f_1}^k(I)$, $n \leq k \leq n + 2^p - 1$, satisfying

$$\frac{b_1(k)}{b_2(k)} \geq (1 + \sigma) \sum_{i=n}^{n+2^p-1} b_1(i) \Bigg(\sum_{i=n}^{n+2_1^p} b_2(i) \Bigg)^{-1}.$$

Otherwise, let $m_1 < m_2$ be two integers in $[n, n + 2_1^p]$ so that $m_i \equiv l_i \bmod 2^p$ $(i = 1, 2)$ and take $m = m_i$ so that $m > n$. Then

$$\frac{b_1(m)}{b_1(m-1)} \frac{b_2(m-1)}{b_2(m)} = \exp(\mathcal{S}_1(m) - \mathcal{S}_2(m)) > \exp(\epsilon),$$

$$\begin{aligned} \frac{b_1(m-1)}{b_2(m-1)} &< \exp(-\epsilon)\frac{b_1(m)}{b_2(m)} \\ &< \exp(-\epsilon)(1 + \sigma) \sum b_1(i) \Bigg(\sum b_2(j) \Bigg)^{-1} \end{aligned}$$

by the assumption, where the summation is taken from n to $n + 2^p - 1$. So

$$\begin{aligned} 1 &= \sum \Bigg[b_1(i) \Bigg(\sum b_1(j) \Bigg)^{-1} \Bigg] \\ &< (1 + \sigma) \sum_{i \neq m-1} \Bigg[b_2(i) \Bigg(\sum b_2(j) \Bigg)^{-1} \Bigg] + \exp(-\epsilon)(1 + \sigma) b_2(m-1) \Bigg(\sum b_2(j) \Bigg)^{-1} \end{aligned}$$

from which we deduce that

$$\sigma \sum_{i \neq m-1} b_2(i) > [1 - (1 + \sigma)\exp(-\epsilon)] b_2(m-1).$$

But $\sum b_2(i) \leq [1 + \exp\|\mathcal{S}_2\|]^{2^p} b_2(m-1)$. A contradiction.

From the claim, we may choose an integer $k_1 \in [0, 2^p - 1]$ so that

$$\frac{b_1(k_1)}{\sum b_1(i)} \geq (1 + \sigma) \frac{b_2(k_1)}{\sum b_2(i)}.$$

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Then $J(p,k_1)=\delta^{-p}L_{f_1}^{k_1}(I)\subset I$. Inductively, let $k_{n+1}\in[k_n2^p,k_n2^p+2^p-1]$ be an integer so that

$$b_1(k_{n+1})\Big(\sum b_1(i)\Big)^{-1}\geq(1+\sigma)b_2(k_{n+1})\Big(\sum b_2(i)\Big)^{-1}.$$

Then $J(np,k_n)=\delta^{-np}L_{f_1}^{k_n}(I)\subset J((n-1)p,k_{n-1})$. But $b_1(k_n2^p)+\cdots+b_1(k_n2^p+2^p-1)=\delta^p b_1(k_n)$, thus

$$\frac{b_1(k_{n+1})}{b_2(k_{n+1})}\geq(1+\sigma)\frac{b_1(k_n)}{b_2(k_n)}\geq(1+\sigma)^{n+1}.$$

By the Hölder continuity of the q.s. conjugacy from T to L_1, there are constant $C>0$ and $0<\alpha<1$ such that $|J(np,k_n)|\geq C2^{-np/\alpha}$. Thus as $n\to\infty$,

$$\frac{\log|w(J(np,k_n))|}{\log|J(np,k_n)|}-1\geq\frac{n\log(1+\sigma)}{\frac{np}{\alpha}\log 2-\log C}\to\frac{\alpha\log(1+\sigma)}{p\log 2}>0.\qquad\square$$

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