Conclusion

We have derived a sufficient condition for a complex polynomial not to have any real zeros by using geometric properties of its coefficients, and illustrated the result with a few examples.

References

1. G. V. Milovanović, D.S. Mitrinović and Th. M. Rassias, *Topics in polynomials: extremal problems, inequalities, zeros*, World Scientific Publishing Co., Inc., River Edge, NJ, (1994).
2. Q. I. Rahman and G. Schmeisser, Analytic *Theory of Polynomials*, London Mathematical Society Monographs. New Series, 26. The Clarendon Press, Oxford University Press, Oxford (2002).
3. E. J. Barbeau, *Polynomials*, Problem Books in Mathematics. Springer Verlag, New York (1989).

10.1017/mag.2021.17

AARON MELMAN

© The Mathematical Association 2021

Department of Applied Mathematics, Santa Clara University, Santa Clara, CA 95053, USA
e-mail: *amelman@scu.edu*

105.05 Simpler derivation of Bring radical

The infinite series solution of the single-parameter Bring quintic equation of the form $x^5 + x = a$ (with a real) is known as the Bring radical. The solution is derived by making use of the Lagrange inversion theorem [1, 2]. In this Note, we obtain the same result by an alternative (simpler) method, for which prior knowledge of the Lagrange inversion theorem is not required. Consider the following single-parameter Bring quintic equation,

$$x^5 + x = a, \tag{1}$$

where a is real. Since the polynomials of odd degree in x (with real coefficients) cross the x-axis at least once, we note that (1) has at least one real solution, which we plan to determine using a new method. Notice that (1) can be rearranged as $\frac{x}{a} = \frac{1}{1 + x^4}$, implying $0 < \frac{x}{a} < 1$, and further $0 < |x| < 1$ for $|a| < 1$. Let $|a| < 1$; so, we can write (1) as $x = a(1 + x^4)^{-1}$ and expand it using the binomial theorem to obtain

$$x = a\left(1 - x^4 + x^8 - x^{12} + x^{16} - x^{20} + x^{24} - \ldots\right). \tag{2}$$

Raising (1) to the power four yields

$$a^4 = x^4 + 4x^8 + 6x^{12} + 4x^{16} + x^{20}.$$

Eliminating x^4 from (2) using the above expression results in

$$x = a\left(1 - a^4 + 5x^8 + 5x^{12} + 5x^{16} + x^{24} - x^{28} + x^{32} - \ldots\right). \tag{3}$$

Raising (1) to the power eight yields

$$a^8 = x^8 + 8x^{12} + 28x^{16} + 56x^{20} + 70x^{24} + 56x^{28} + 28x^{32} + 8x^{36} + x^{40}.$$

https://doi.org/10.1017/mag.2021.18 Published online by Cambridge University Press

Using the above expression, we eliminate x^8 from (3) to get

$$x = a\left(1 - a^4 + 5a^8 - 35x^{12} - 135x^{16} - 280x^{20} - 349x^{24} - \ldots\right). \quad (4)$$

Continuing in the same fashion, we raise (1) to the power twelve to obtain,

$$a^{12} = x^{12} + 12x^{16} + 66x^{20} + \ldots,$$

and then use it to eliminate x^{12} from (4) to get

$$x = a\left(1 - a^4 + 5a^8 - 35a^{12} + 285x^{16} + 2030x^{20} + \ldots\right). \quad (5)$$

Again raising (1) to the power sixteen yields

$$a^{16} = x^{16} + 16x^{20} + \ldots$$

and using it in (5) leads to

$$x = a\left(1 - a^4 + 5a^8 - 35a^{12} + 285a^{16} - 2530a^{20} - \ldots\right). \quad (6)$$

Repeating this process, we obtain an infinite series solution to the Bring quintic equation (1) in the form

$$x = a - a^5 + 5a^9 - 35a^{13} + 285a^{17} - 2530a^{21} + \ldots, \quad (7)$$

and it is known as the Bring radical. The infinite series in (7) can also be expressed in general form as

$$x = \sum_{k=0}^{\infty} \binom{5k}{k} \frac{(-1)^k a^{4k+1}}{4k+1},$$

and it converges for $|a| < \dfrac{4}{5 \cdot \sqrt[4]{5}}$; see [3]. Consider one numerical example; let the Bring quintic equation be $x^5 + x = 0.41024$. If we truncate the infinite series in (7) after six terms and determine x, it yields $x = 0.3999960533\ldots$, while the exact solution is $x = 0.4$. Thus an error of $\approx 3.9 \times 10^{-6}$ results due to truncation.

Acknowledgements

The author thanks the administration of PES University for supporting this work. The valuable suggestions of the anonymous reviewer led to the improved manuscript.

References

1. Jorg Bewersdorff, *Galois theory for beginners: a historical perspective*, translated by David Kramer, American Mathematical Society, Providence, Rhode Island (2006).
2. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, (4th edn.) Cambridge University Press (1927) pp. 129-130.
3. Online web source: *Wikipedia*, https://en.wikipedia.org/wiki/Bring radical#Series_representation

10.1017/mag.2021.18

© The Mathematical Association 2021

RAGHAVENDRA G. KULKARNI
Department of Electronics & Communication Engineering, PES University, 100 Feet Ring Road, BSK III Stage, Bengaluru - 560085, INDIA
e-mail: *raghavendrakulkarni@pes.edu*

https://doi.org/10.1017/mag.2021.18 Published online by Cambridge University Press