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DIRECTED SIMPLICES IN HIGHER ORDER TOURNAMENTS

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Abstract. It is well known that a tournament (complete oriented graph) on n vertices has at most $\frac{1}{4}\binom{n}{3}$ directed triangles, and that the constant $\frac{1}{4}$ is best possible. Motivated by some geometric considerations, our aim in this paper is to consider some "higher order" versions of this statement. For example, if we give each 3-set from an n-set a cyclic ordering, then what is the greatest number of "directed 4-sets" we can have? We give an asymptotically best possible answer to this question, and give bounds in the general case when we orient each d-set from an n-set.

§1. *Introduction.* A *tournament* is a complete graph in which each edge is assigned a direction. It is well known (see, for example, [8]) that there are at most $\frac{1}{4}\binom{n}{3} + O(n^2)$ directed triangles in a tournament on n vertices. The constant $\frac{1}{4}$ is easily seen to be best possible, since for example the random tournament (where the direction of each edge is assigned randomly and independently with probability $\frac{1}{2}$) has expected number of directed triangles precisely $\frac{1}{4}\binom{n}{3}$. Actually, any tournament in which all degrees are close to $n/2$ will have about this number of directed triangles (see, for example, [8]).

Our aim in this paper is to investigate some "higher order" analogues of this result. Before we set out our definitions, we give some geometric background, to explain how the question arose. However, our question is natural even without any motivation, so the reader who is not interested in geometric considerations should feel free to skip the next few paragraphs.

Let $T \subset \mathbb{R}^d$ be a set of n points in general position. What is the greatest possible number of d-simplices of T that contain (say, in their interior) a given point of $\mathbb{R}^d$? In two dimensions, this question was asked by Kárteszi [6] and answered by Boros and Füredi [2, 3], who showed that for any set T of n points in the plane in general position and any point x the number of triangles of T containing x is at most $\frac{1}{4}\binom{n}{3} + O(n^2)$. (Note that this can be attained, for example, by taking T to be a regular n-gon and x its centre.) Their elegant proof was to note that there is a natural way to make T into a tournament: given a and b in T, direct the edge ab from a to b (respectively from b to a) in such a way that the triple abx (respectively bax) is clockwise. Then the triangles of T containing x correspond precisely to the directed triangles of this tournament.

In this paper we are usually interested in asymptotic bounds, but we remark in passing that Boros and Füredi actually proved the exact best possible bound on the number of triangles, because the exact tournament bound (namely

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$\frac{1}{24}(n^3 - n)$ if n is odd and $\frac{1}{24}(n^3 - 4n)$ if n is even) can in fact be realized geometrically. Indeed, the above construction, with x moved slightly so as not to be collinear with any pair from T, achieves this value.

The general question (in d dimensions) was asked by Boros and Füredi, and answered by Bárány [1]. He showed that if $T \subset \mathbb{R}^d$ is a set of n points in general position and x is any point then the number of d-simplices of T containing x is at most $(1/2^d)\binom{n}{d+1} + O(n^d)$. The constant $1/2^d$ is best possible, as may be seen in [1].

(The reader might like to note that, while the above is in some sense about the "best" sets T, it is also very natural to ask about the "worst" sets. Thus, for example, Boros and Füredi [3] (see also Bukh [4]) showed that for any set T of n points in general position there is a point in at least $\frac{2}{9}$ of its triangles, and the constant $\frac{2}{9}$ cannot be improved. In d dimensions, the right constant is not known: the best current bounds are a lower bound of $1/(d+1)^d$ by Bárány [1], improved slightly to $(d^2+1)/(d+1)^{d+1}$ by Wagner [9], and an upper bound of $d!/(d+1)^d$ by Bukh *et al* [5].)

Now, Bárány's result uses the upper bound theorem [7] (about facet counts in polytopes). In other words, it uses a geometric theorem, as opposed to the abstract tournament theorem used by Boros and Füredi. But what would the corresponding abstract result be? Just as in the case $d = 2$, for a general d we would give an orientation to each d-set ($(d-1)$-simplex) in T, according to "on which side of it" the point x lies. And then the d-simplices containing x would correspond exactly to the $(d+1)$-sets in T whose d-sets were "oriented compatibly" (in other words, whose d-sets had orientations that could be induced from a fixed orientation of the $(d+1)$-set—this will be made more precise in a moment). Hence our abstract question is as follows: suppose that we orient (in some sense) every d-set of an n-set; what is the greatest number of directed $(d+1)$-sets that arise? In particular, do we get as small a bound as $(1/2^d)\binom{n}{d+1} + O(n^d)$?

We now give the precise (and non-geometric) definitions. We define an *orientation* of a d-set inductively. An orientation of a 1-set $\{x\}$ is just an assignment of ± 1 to x, and an orientation of a 2-set $\{a, b\}$ is a directed edge from a to b or vice versa. (We may, if we wish, think of a directed edge from a to b as assigning $+1$ to b and -1 to a.) And for $d \geq 3$, an orientation of a d-set consists of an orientation for each of its $(d-1)$-subsets in such a way that these orientations are *compatible*, meaning that any two give different orientations to their common $(d-2)$-subset. Then, for $d \geq 2$, a *tournament of order* d, or d*-tournament*, consists of a set together with an orientation of each of its d-sets. Finally, in a d-tournament a d*-simplex* is a $(d+1)$-set, and we say that it is *directed* if its d-subsets are pairwise compatible.

For example, a 2-tournament is just a tournament, and its directed 3-sets are precisely its directed triples in the usual sense. And a 3-tournament is specified by giving each 3-set (from a given set) a cyclic ordering: then a 4-set is directed if any two of its 3-sets have cyclic orderings that go in opposite directions on their common 2-set.

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Our question is then: what is the greatest number of directed $(d+1)$-sets for a d-tournament on n vertices? For $d=2$ this is $\frac{1}{4}\binom{n}{3}+O(n^2)$; what can we say in general? And how does this bound compare with the "geometric" version (when the d-tournament is induced from a set T in $\mathbb{R}^d$), where the bound is $(1/2^d)\binom{n}{d+1}+O(n^d)$?

To put it another way, define the constant c_d to be the limit, as $n\to\infty$, of this greatest number as a fraction of $\binom{n}{d+1}$—an easy averaging argument shows that the limit does exist. In this language, the $d=2$ result is that $c_2=\frac{1}{4}$, and the geometric construction shows that $c_d\geq(1/2^d)$. In fact, another reason why it is obvious that $c_d\geq(1/2^d)$ is that a random d-tournament has expected number of directed $(d+1)$-sets exactly $(1/2^d)\binom{n}{d+1}$. How does c_d behave, for fixed small d and also as d gets large?

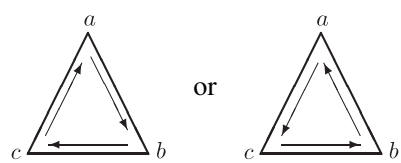
The plan of the paper is as follows. We start by considering the case $d=3$. Here it turns out that $\frac{1}{8}$ is not the right answer. We give an upper bound of $\frac{1}{4}$, by a simple counting argument. And then we show that in fact $c_3=\frac{1}{4}$, by a slightly unexpected random argument. This is the content of §2.

Then in §3 we turn our attention to general d. Here we do not know what the exact value of c_d is. We give an upper bound of $1/(d+1)$, again by a simple counting argument. For the lower bound, the method for $d=3$ seems unfortunately not to generalize, and indeed we do not know how to use any random methods to improve significantly on $1/2^d$. However, we give an explicit construction to show that $c_d\geq 1/d^2$. Thus the abstract version of the problem exhibits genuinely different behaviour than the geometric version.

In §4, we give some remarks and open questions.

§2. *Directed tetrahedra in tournaments of order 3.* In this section, we determine c_3. In a 3-tournament, we call a 3-set a *triangle* and a 3-simplex (or a 4-set) a *tetrahedron*. So a directed tetrahedron is simply a directed 4-set.

Given a triangle $\triangle=\{a,b,c\}$, it can be oriented (in a 3-tournament) either as



We write $\overrightarrow{abc}$ $(=\overrightarrow{bca}$ or $\overrightarrow{cab})$ for the former and $\overrightarrow{acb}$ $(=\overrightarrow{bac}$ or $\overrightarrow{cba})$ for the latter. Thus, in this language, a directed tetrahedron $\{a,b,c,d\}$ in a 3-tournament has the orientations of its triangles as $\{\overrightarrow{abc},\overrightarrow{adb},\overrightarrow{acd},\overrightarrow{bdc}\}$ or $\{\overrightarrow{acb},\overrightarrow{abd},\overrightarrow{adc},\overrightarrow{bcd}\}$.

It is easy to check (by hand) that in a tetrahedron there are at least two compatible pairs of triangles.

THEOREM 2.1. *Let T_3 be a 3-tournament on n vertices. Then the number of directed tetrahedra in T_3 is at most $\frac{1}{4}\binom{n}{4}+O(n^3)$.*

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Proof. Let X be the number of directed tetrahedra in T_3. For each directed tetrahedron, there are six compatible pairs of triangles. We also know that there are at least two compatible pairs of triangles in each of the non-directed tetrahedra. Therefore, there are at least $6X + 2(\binom{n}{4} - X)$ compatible pairs of triangles in T_3.

On the other hand, consider any 2-set $\{a, b\}$ and count the number of compatible pairs of triangles having $\{a, b\}$ as their common 2-subset. By considering the orientation of $\{a, b, x\}$ for each $x \notin \{a, b\}$, it is easy to see that this number is at most $\lfloor (n-2)/2 \rfloor \lceil (n-2)/2 \rceil$. Therefore, there are at most $\lfloor (n-2)/2 \rfloor \lceil (n-2)/2 \rceil \binom{n}{2}$ compatible pairs of triangles in T_3.

Putting the two bounds together, we have

$$6X + 2\left(\binom{n}{4} - X\right) \leq \left\lfloor \frac{n-2}{2} \right\rfloor \left\lceil \frac{n-2}{2} \right\rceil \binom{n}{2}$$

so

$$X \leq \frac{1}{4}\binom{n}{4} + O(n^3). \qquad \square$$

A natural guess for c_3 would be $\frac{1}{8}$. This is from looking at the geometric version of the problem or by assigning the orientation of each triangle randomly and independently with probability $\frac{1}{2}$. But surprisingly this is not the case, as shown in the following construction.

The idea is to construct a 3-tournament T_3 by inducing it in a certain way from a random 2-tournament. We will show that there is a way of inducing it which gives T_3 many directed tetrahedra.

Suppose T_2 is a 2-tournament on n vertices. Let T_3 be a 3-tournament with vertex set $V(T_2)$. Note that there are only two types of 2-tournament on three vertices; either a directed triangle or a transitive 3-set. Let $\triangle = \{a, b, c\}$ be a triangle. If $\triangle$ is a directed triangle (in T_2) with directed edges $\{a \to b, b \to c, c \to a\}$, we will, in T_3, orient $\triangle$ the "same" way, $\overrightarrow{abc}$. If $\triangle$ is a transitive 3-set with directed edges $\{a \to b, b \to c, a \to c\}$, we could choose to orient $\triangle$ following the "minority", $\overrightarrow{acb}$, or the "majority", $\overrightarrow{abc}$. It turns out that it is better to orient the transitive 3-sets following the "minority".

THEOREM 2.2. *For $n \geq 4$, there is a 3-tournament on n vertices that has at least $\frac{1}{4}\binom{n}{4}$ directed tetrahedra.*

Proof. Let T_2 be a random tournament on n vertices where the direction of each edge is assigned independently with probability $\frac{1}{2}$. Let T_3 be the 3-tournament induced from T_2 where the orientation of each triangle is assigned according to the preceding discussion (that is, orient a directed triangle the "same" way and a transitive 3-set following the "minority").

Given a set of four vertices $D = \{a, b, c, d\}$, it is easy to check that D is a directed tetrahedron in T_3 if and only if D is one of the following two types of tournaments in T_2:

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(i) a vertex dominating a directed triangle,

$$\{t \to x, t \to y, t \to z, x \to y, y \to z, z \to x\}, \quad \text{or}$$

(ii) a vertex dominated by a directed triangle,

$$\{x \to t, y \to t, z \to t, x \to y, y \to z, z \to x\},$$

for $\{t, x, y, z\} = \{a, b, c, d\}$.

Letting X be the total number of type (i) and type (ii) tournaments in T_2, it follows that

$$\mathbb{E}(\text{number of directed tetrahedra in } T_3) = \mathbb{E}(X).$$

Now, there are $2^6 = 64$ different 2-tournaments on $\{1, 2, 3, 4\}$, of which there are 8 type (i) tournaments and 8 type (ii) tournaments. So,

$$\mathbb{E}(X) = \frac{8+8}{64}\binom{n}{4} = \frac{1}{4}\binom{n}{4}.$$

Therefore, there exists a 2-tournament T_2 with at least $\frac{1}{4}\binom{n}{4}$ type (i) and type (ii) tournaments. Hence, there is a 3-tournament that has at least $\frac{1}{4}\binom{n}{4}$ directed tetrahedra. □

Putting Theorems 2.1 and 2.2 together, we have the following corollary.

COROLLARY 2.3. $c_3 = \frac{1}{4}$. □

§3. *Tournaments of higher order.* In this section, we consider tournaments of order d for a general $d \geq 2$.

We first show that in a d-tournament, at most a fraction $1/d + 1$ of the d-simplices are directed. This uses a very similar counting argument to the case $d = 3$.

The following lemma gives a lower bound on the number of compatible pairs of d-sets in a d-simplex.

LEMMA 3.1. *Fix $d \geq 2$. Let S be a d-simplex. Then S has at least $s(d)$ compatible pairs of d-sets, where*

$$s(d) = \binom{\lfloor \frac{d+1}{2} \rfloor}{2} + \binom{\lceil \frac{d+1}{2} \rceil}{2}.$$

Proof. The d-simplex S has $d + 1$ oriented d-sets. Suppose that switching the orientations of x d-sets turns S into a directed d-simplex. Then we can partition the d-sets of S as $S' \cup S''$, where S' is the set of d-sets for which we need to switch orientations. Consider two d-sets in S'; switching the orientation of both the d-sets results in them being compatible by the definition of a directed simplex. Hence they must be compatible before switching. For a similar reason,

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a d-set in S' and a d-set in S'' cannot be a compatible pair. Also, any pair of d-sets in S'' are compatible. So the number of compatible pairs of d-sets in S is

$$\binom{x}{2} + \binom{d+1-x}{2}$$

which is minimized at $x = \lfloor (d+1)/2 \rfloor$. □

Based on this, we present an upper bound for c_d.

THEOREM 3.2. *Fix $d \geq 2$. Let T_d be a d-tournament on n vertices. Then the number of directed d-simplices in T_d is at most*

$$\begin{cases} \dfrac{1}{d+1}\binom{n}{d+1} + O(n^d) & \text{if } d \text{ is odd} \\ \dfrac{1}{d+2}\binom{n}{d+1} + O(n^d) & \text{if } d \text{ is even.} \end{cases}$$

In particular,

$$c_d \leq \begin{cases} \dfrac{1}{d+1} & \text{if } d \text{ is odd} \\ \dfrac{1}{d+2} & \text{if } d \text{ is even.} \end{cases}$$

Proof. Let X be the number of directed d-simplices in T_d. For each directed d-simplex, there are $\binom{d+1}{2}$ compatible pairs of d-sets. By Lemma 3.1, the non-directed d-simplices each have at least $s(d)$ compatible pairs of d-sets. Therefore, there are at least

$$\binom{d+1}{2} X + s(d)\left(\binom{n}{d+1} - X\right) \tag{1}$$

compatible pairs of d-sets in T_d.

On the other hand, let $A = \{a_1, a_2, \ldots, a_{d-1}\}$ be a set of $d-1$ vertices in T_d. We want to count the number of compatible pairs of d-sets that have A as their common $(d-1)$-set. For $x \in T_d \backslash A$, the d-set $A \cup \{x\}$ is oriented one way or the other way. Let $h^+(A)$ be the number of vertices x in $T_d \backslash A$ such that $A \cup \{x\}$ is oriented in one orientation and $h^-(A)$ is the number of vertices y in $T_d \backslash A$ such that $A \cup \{y\}$ is oriented in the other orientation. The number of compatible pairs of d-sets that have A as their common $(d-1)$-set is then $h^+(A)h^-(A)$, and as $h^+(A) + h^-(A) = n - d + 1$ this value is maximized when $h^+(A) = \lfloor (n-d+1)/2 \rfloor$. Therefore, there are at most

$$\binom{n}{d-1} \left\lfloor \frac{n-d+1}{2} \right\rfloor \left\lceil \frac{n-d+1}{2} \right\rceil \tag{2}$$

compatible pairs of d-sets in T_d.

Comparing (1) and (2), we have

$$\binom{d+1}{2} X + s(d)\left(\binom{n}{d+1} - X\right) \leq \binom{n}{d-1} \left\lfloor \frac{n-d+1}{2} \right\rfloor \left\lceil \frac{n-d+1}{2} \right\rceil$$

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so

$$X\left(\binom{d+1}{2} - s(d)\right) \leq \left(\frac{d(d+1)}{4} - s(d)\right)\binom{n}{d+1} + O(n^d)$$

so

$$X \leq \left(\frac{((d(d+1))/4) - s(d)}{((d(d+1))/2) - s(d)}\right)\binom{n}{d+1} + O(n^d).$$

When d is odd, $s(d) = ((d+1)/2)((d-1)/2)$ and

$$X \leq \frac{1}{d+1}\binom{n}{d+1} + O(n^d).$$

When d is even, $s(d) = d^2/4$ and

$$X \leq \frac{1}{d+2}\binom{n}{d+1} + O(n^d).$$

This completes the proof. □

Remark. From the proof above, the bound is tight only if there exists a d-tournament T_d with the following properties.

(i) Almost all d-simplices in T_d are either directed or have the minimum number of compatible pairs of d-sets.

(ii) For almost all $(d-1)$-sets A, the number of compatible pairs of d-sets that have A as their common $(d-1)$-set is about $n^2/4$.

Now we consider a lower bound on c_d.

It is easy to see that a d-tournament where the orientation of each d-set is assigned randomly and independently with probability $\frac{1}{2}$ has expected number of directed d-simplices equal to $(1/2^d)\binom{n}{d+1}$. In other words, $c_d \geq 1/2^d$. Unfortunately, we do not see how to generalize the construction for $d = 3$ (inducing from a random 2-tournament) to obtain a significant improvement on $(1/2^d)$. In fact, we do not see how to use random methods to give a non-exponential lower bound on c_d. Instead, we give an explicit construction. Curiously, this construction gives $c_3 \geq \frac{1}{7}$, which is worse than our previous method for $d = 3$, and yet is much better in general.

So, for the remainder of this section, we will give an explicit construction of a d-tournament that contains many directed d-simplices.

THEOREM 3.3. *Let $n = (d+1)^m$ where $m \in \mathbb{N}$ is sufficiently large. Then there is a d-tournament T_d on n vertices with at least*

$$\frac{1}{1 + \binom{d+1}{2}}\binom{n}{d+1} - o(n^{d+1})$$

directed d-simplices. In particular, $c_d \geq 1/(1 + \binom{d+1}{2})$.

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Proof. Let $D = \{1, 2, \dots, d+1\}$ be a fixed directed d-simplex: thus the d-sets of D are oriented in a pairwise compatible way.

Let the vertex set of T_d be $\{\mathbf{a} = (a_1, a_2, \dots, a_m) \mid 1 \le a_i \le d+1 \text{ for all } 1 \le i \le m\}$. Given a set of d vertices $F = \{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(d)}\}$, set

$$j_F = \min\{1 \le j \le m \mid a_j^{(1)}, a_j^{(2)}, \dots, a_j^{(d)} \text{ all distinct}\} \text{ if it exists.}$$

If j_F exists, orient F as the d-set $\{a_{j_F}^{(1)}, a_{j_F}^{(2)}, \dots, a_{j_F}^{(d)}\}$ is oriented in D. If j_F does not exist, orient F in any way. Note that for a randomly chosen set of d vertices F, we have $\mathbb{P}(j_F \text{ exists}) \to 1$ as $m \to \infty$. That is to say, almost all d-sets get orientations according to the orientations of d-sets in D.

Now we claim that T_d has many directed d-simplices.

Given a d-simplex $B = \{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \dots, \mathbf{b}^{(d+1)}\}$, set

$$k_B = \min\{1 \le k \le m : b_k^{(1)}, b_k^{(2)}, \dots, b_k^{(d+1)} \text{ all distinct}\} \text{ if it exists.}$$

Note that k_B exists for almost all d-simplices B. Note also that, for a d-simplex B for which k_B exists, B is a directed d-simplex if and only if

$$|\{b_k^{(1)}\} \cup \{b_k^{(2)}\} \cup \cdots \cup \{b_k^{(d+1)}\}| \le d-1 \quad \text{for all } k < k_B.$$

Let

$$\begin{aligned} x &= \mathbb{P}(\text{a fixed coordinate of } d+1 \text{ vectors has exactly } d+1 \text{ distinct values}) \\ &= \frac{(d+1)!}{(d+1)^{d+1}}, \end{aligned}$$

and

$$\begin{aligned} y &= 1 - x - \mathbb{P}(\text{a fixed coordinate of } d+1 \text{ vectors has exactly } d \text{ distinct values}) \\ &= 1 - \frac{(d+1)!}{(d+1)^{d+1}} - \frac{(d+1)d\binom{d+1}{2}(d-1)!}{(d+1)^{d+1}} \\ &= 1 - \frac{(d+1)!}{(d+1)^{d+1}}\left(1 + \binom{d+1}{2}\right). \end{aligned}$$

Now, choosing a random set of $d+1$ vectors is (as $m \to \infty$) asymptotically the same as choosing $d+1$ vectors with repetitions allowed. For the latter, the probability that these $d+1$ vectors form a directed d-simplex is exactly $x + xy + xy^2 + \cdots + xy^{m-1}$. Thus, the fraction of directed d-simplices tends to $x/(1-y) = 1/(1+\binom{d+1}{2})$ as m tends to infinity. The number of directed d-simplices in T_d is therefore

$$\frac{1}{1+\binom{d+1}{2}}\binom{n}{d+1} - o(n^{d+1}). \qquad \square$$

§4. *Remarks and questions.* From Theorems 3.2 and 3.3, we have $\frac{1}{11} \le c_4 \le \frac{1}{6}$. Using a similar idea to the case $d = 3$, that is, by inducing from a random 2-tournament, we can show that there is a 4-tournament on n vertices which has at least $\frac{9}{64}\binom{n}{5}$ directed 5-simplices. (This is a messy case analysis, as there are many possible ways to induce a 4-tournament from a 2-tournament.) Hence

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we have a better lower bound of $c_4 \geq \frac{9}{64}$. However, as d increases, the number of 2-tournaments on d (or $d+1$) vertices increases rapidly. This makes the problem of finding the optimal way, or indeed any sensible way, of inducing a d-tournament from a 2-tournament hard. One could also try adapting this method to induce, for example, a 4-tournament from a 3-tournament, but unfortunately this seems no better than inducing a 4-tournament from a 2-tournament.

Section 2 and the discussion above tell us that our construction in Theorem 3.3 for the lower bound of c_d is not the best for the cases $d = 3$ and $d = 4$.

Question 1. Is the construction of a d-tournament in Theorem 3.3 optimal, or at least close to optimal, for $d \geq 5$?

We have mentioned in the remark after Theorem 3.2 that for our upper bound of c_d to be tight, there must exist a very structured d-tournament. We believe the existence of such a d-tournament is fairly unlikely.

Question 2. Can one improve the upper bound for c_d, for $d \geq 4$? What is the growth speed of c_d?

Finally, we would like to see an example of an explicit 3-tournament attaining the exact value of c_3.

Question 3. Is there a simple construction of a 3-tournament on n vertices that has at least $\frac{1}{4}\binom{n}{4} + O(n^3)$ directed 3-simplices?

By "simple", we mean not allowing a random construction or a quasi-random construction (such as a quadratic residue tournament).

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