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Denjoy systems and dimension groups

KENICHI MASUI†, FUMIAKI SUGISAKI‡ and MASAMICHI YOSHIDA†

† *Department of Mathematics, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585 Japan*
(e-mail: masuikenichi@yahoo.co.jp, yoshida@sci.osaka-cu.ac.jp)

‡ *Department of Mathematics, Kumamoto University, Kurokami, Kumamoto, 860-8555 Japan*
(e-mail: sugisaki@sci.kumamoto-u.ac.jp)

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Abstract. We construct an adic model for the Denjoy system (X, T) of finite double orbit number. It is based on the dual Ostrowski numeration system. By using this construction, we can determine the K-theory of $C(X) \rtimes_T \mathbb{Z}$ associated with (X, T) in an effective way.

1. *Introduction*

Putnam *et al* [**13**] studied the C^*-crossed product $C(X) \rtimes_T \mathbb{Z}$ associated with a Denjoy minimal dynamical system (X, T), and its K-theory (essentially by using a six-term cyclic exact sequence). Here the system (X, T) arises from Poincaré's *rotation number theorem* as follows. Let $S^1 = \mathbb{R}/\mathbb{Z}$. Suppose that $\varphi : S^1 \to S^1$ is an orientation-preserving homeomorphism. Letting $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be a lift of φ and $t \in \mathbb{R}$, $\lim_{n\to\infty} (\widetilde{\varphi}^n(t)/n)$ exists (where $\widetilde{\varphi}^n$ is the nth iteration of $\widetilde{\varphi}$), and moreover its fractional part $\alpha \in [0, 1)$ is independent of the choice of $\widetilde{\varphi}$ and t. So, we say $\rho(\varphi) := \alpha$ is the *rotation number* of φ. One can show that $\rho(\varphi)$ is irrational if and only if φ has no periodic points. For each $\omega \in [0, 1)$, denote by $R_\omega : S^1 \to S^1$ the rotation of angle $2\pi\omega$ (i.e. $R_\omega(z) = z + \omega$ (mod 1)). Now, we can state Poincaré's theorem.

Suppose that the rotation number α of φ is irrational. Then there is a degree-one map $F : S^1 \to S^1$ such that $F \circ \varphi = R_\alpha \circ F$ (such an F is called a *factor map* of the dynamical system (S^1, φ)). Furthermore, we have the following three properties.

(1) F is unique up to rotation (i.e. when G is a factor map, $G = R_\omega \circ F$ for some ω). Define $A = \{z \in S^1 \mid \#F^{-1}F(z) = 1\}$ (so $\varphi(A) = A$ and A is independent of the choice of factor maps), and write $X = \mathrm{cl}\, A$ (the closure of A).

(2) The following dichotomy holds: $A = S^1$; otherwise X is a Cantor set.
We say that φ is a *Denjoy homeomorphism* if the second case holds (i.e. $A \neq S^1$). In this case, denote the restriction of φ to X by

$$T : X \to X.$$

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The subsystem (X, T) is called a *Denjoy system of rotation number* α, and a connected component of $S^1 \backslash X$ is called a *cutout interval* (indeed, an open arc).

(3) Suppose that φ is a Denjoy homeomorphism. Then

$$X \backslash A = \{z \in S^1 \mid z \text{ is an endpoint of some cutout interval}\},$$

$\#F(\mathrm{cl}\, I) = 1$ for each cutout interval I (so the restriction $F_X : X \to S^1$ of F to X is surjective and $\#F_X^{-1} F_X(z) = 2$ for each $z \notin A$), and X is the unique minimal set under φ.

(Here X is *minimal* if a closed φ-invariant subset of X is $\emptyset$ or X. It is clear that the minimality of X is equivalent to the condition that each φ-orbit of X is dense in X.)

From now on, let φ be a Denjoy homeomorphism of rotation number α. By the above property, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle F_X}\downarrow & & \downarrow{\scriptstyle F_X} \\ S^1 & \xrightarrow{R_\alpha} & S^1 \end{array}$$

Moreover, F_X is surjective and at most two-to-one; more precisely, $z \in X \backslash A$ if and only if z is an endpoint of some cutout interval I, and in this case $F_X^{-1} F_X(z)$ is the set of endpoints of I. By choosing an appropriate F_X, we can assume that $\alpha = \rho(\varphi) \in F_X(X \backslash A)$. Now, we write $\mathcal{Q}(T) := F_X(X \backslash A)$, and call it the *double point set* (relative to F_X). Since there are countably many cutout intervals, the double point set is also countable. Let $\mathcal{O}_\omega = \{R_\alpha^n(\omega) \mid n \in \mathbb{Z}\}$. There exists $d \in \mathbb{N} \cup \{\infty\}$ such that, for some set $\Lambda \subset S^1$ of cardinal d,

$$\mathcal{Q}(T) = \bigcup_{\omega \in \Lambda} \mathcal{O}_\omega \quad \text{(disjoint)} \quad \text{and} \quad \alpha \in \Lambda$$

(in other words, the double point set is split into at most countably many R_α-orbits). We call such a Λ a *transversal* of a double point set. Note that the number d is independent of the choice of F_X. So we call d the *double orbit number* of (X, T). Markley [**11**] proved that, for two Denjoy homeomorphisms φ_1 and φ_2, (S^1, φ_1) is conjugate to (S^1, φ_2) via an orientation-preserving conjugacy if and only if $\rho(\varphi_1) = \rho(\varphi_2)$ and $\mathcal{Q}(\varphi_1) = R_\omega(\mathcal{Q}(\varphi_2))$ for some ω.

In general, there is a well-known C^*-crossed product construction of $C(Y) \rtimes_S \mathbb{Z}$ from a topological dynamical system (Y, S) where Y is compact Hausdorff and $S : Y \to Y$ is a homeomorphism (see, for example, [**8, 13**]). However, there is no general tool to determine the K-theory of $C(Y) \rtimes_S \mathbb{Z}$. When Y is a Cantor set and S is minimal (briefly, (Y, S) is a *Cantor system*), in [**9**], Herman *et al* constructed, by using the Kakutani–Rokhlin stack method, a suitable *Bratteli diagram* $B = (V, E)$ and a '*Bratteli–Vershik system* (X_B, θ_B)', which is conjugate to (Y, S) (we call (X_B, θ_B) an *HPS-adic presentation* of (Y, S), see the next section for details) and the associated AF-subalgebra $AF(B)$ of $C(Y) \rtimes_S \mathbb{Z}$ such that $K_0(AF(B)) = K_0(B)$ is isomorphic to $K_0(C(Y) \rtimes_S \mathbb{Z})$. In general, each AF-algebra is constructed by a Bratteli diagram (but is non-unique). Therefore, when a diagram B corresponding to $AF(B)$ is known in some effective way, the structure of $K_0(AF(B))$ is more explicit than that of the corresponding C^*-crossed product $C(Y) \rtimes_S \mathbb{Z}$.

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In this paper, for a Denjoy system (X, T) of finite double orbit number, we can obtain an HPS-adic presentation of (X, T), and determine the K-theory of $C(X) \rtimes_T \mathbb{Z}$. In fact, Putnam *et al* [**13**] have determined the K-theory of $C(X) \rtimes_T \mathbb{Z}$ in the case when Λ are rationally independent. Our main strategy is to obtain an adic model (X_B, θ_B) of a Denjoy system, and then by using it determine its K-theory for any Λ. We hope that this model will be useful for the study of the *Sturmian sequence* (for example, refer to [**1, 2, 7**]), and some applications will appear in the future.

Section 2 is devoted to the fundamental preparation for this paper. In particular, the Kakutani–Rokhlin tower construction is introduced. In §3, to obtain the tower construction for the Denjoy system (X, T), we consider the 'counterpart' for its factor (S^1, R_α). In §§4 and 5, we obtain the counterpart and discuss the arithmetical aspects. In §6, we introduce the notion of the '*section* (ξ, g)', which plays a vital role in our approach when obtaining the adic model (X_B, θ_B) for the Denjoy system. In §7, the main result (Corollary 7.2) is stated. Section 8 is devoted to the study of the matrix $A(\xi, g)$ associated with (ξ, g). In §9, we compute the dimension group for the Denjoy system. Some of the proofs are given in Appendix A.

2. *Preliminaries*

Now, we will recall some basic concepts, results and constructions. These will later be applied to investigate Denjoy systems. For example, we refer to [**3, 8, 9**] for further details.

2.1. *The Bratteli–Vershik system.* Denote by $\mathbb{N}$ (respectively $\mathbb{Z}_+$) the set of positive (respectively non-negative) integers. A *Bratteli diagram* is an infinite directed graph $B = (V, E)$, such that the vertex set V and the edge set E can be partitioned into finite sets

$$V = \bigcup_{n\in\mathbb{Z}_+} V_n \quad \text{and} \quad E = \bigcup_{n\in\mathbb{N}} E_n$$

with the following properties:

(1) V_0 is a singleton;

(2) $s(E_n) = V_{n-1}$ and $r(E_n) = V_n$ for all n, where s is the associated *source map* and r is the associated *range map*.

Writing $V_n = \{v_n(0), v_n(1), \ldots, v_n(c_n - 1)\}$, we let A_n be the $c_n \times c_{n-1}$ matrix defined by $(A_n)_{ij} = \#(s^{-1}(v_{n-1}(j)) \cap r^{-1}(v_n(i)))$ and call A_n the nth *incidence matrix* of B. (See Figure 1.)

Let B be a Bratteli diagram with incidence matrices A_n. Consider the following inductive sequence:

$$\mathbb{Z} \xrightarrow{A_1} \mathbb{Z}^{c_1} \xrightarrow{A_2} \mathbb{Z}^{c_2} \xrightarrow{A_3} \mathbb{Z}^{c_3} \xrightarrow{A_4} \cdots.$$

Since each A_n is a non-negative integer matrix, $A_n((\mathbb{Z}_+)^{c_{n-1}}) \subset (\mathbb{Z}_+)^{c_n}$. Thus, we have an ordered inductive sequence

$$(\mathbb{Z}, \mathbb{Z}_+) \xrightarrow{A_1} (\mathbb{Z}^{c_1}, (\mathbb{Z}_+)^{c_1}) \xrightarrow{A_2} (\mathbb{Z}^{c_2}, (\mathbb{Z}_+)^{c_2}) \xrightarrow{A_3} (\mathbb{Z}^{c_3}, (\mathbb{Z}_+)^{c_3}) \xrightarrow{A_4} \cdots.$$

Denote by $(K_0(B), K_0(B)_+)$ the inductive limit of the above sequence (in the category of ordered abelian groups). As an 'order unit', we chose the image $[1]$ of $1 \in \mathbb{Z}$ in $K_0(B)$. $(K_0(B), K_0(B)_+, [1])$ is called the *dimension group* of B (refer to [**4–6**]).

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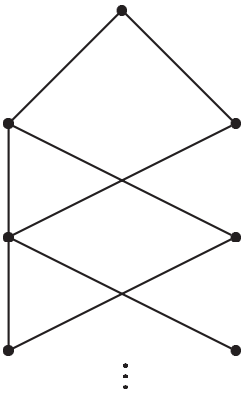


FIGURE 1. A Bratteli diagram.

Define the *infinite path space* X_B of the Bratteli diagram B by

$$X_B = \left\{ e_* = (e_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n \;\middle|\; r(e_n) = s(e_{n+1}) \text{ for each } n \in \mathbb{N} \right\}.$$

An *ordered Bratteli diagram* $B = (V, E, \leq)$ is a Bratteli diagram (V, E) together with a partial order on E so that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. Then we introduce the *adic order* (partial order) on X_B by $e_* < f_*$ if there exists $N \in \mathbb{N}$ such that $e_N < f_N$ and $e_n = f_n$ for all $n > N$. If there exist a unique maximal path $e_*^{\max}$ and a unique minimal path $e_*^{\min}$ in X_B then B is said to be *properly ordered*. Given a properly ordered Bratteli diagram B, we can define the *adic transformation* $\theta_B : X_B \to X_B$ as follows: if $e_* \neq e_*^{\max}$ then $\theta_B(e_*) = \min\{f_* \in X_B \mid f_* > e_*\}$, and $\theta_B(e_*^{\max}) = e_*^{\min}$. Vershik and colleagues introduced these ideas concerning the adic system [**14–16**]. We call (X_B, θ_B) a *Bratteli–Vershik system*.

2.2. *HPS-construction.* Let (Y, S) be a Cantor system. A *Kakutani–Rokhlin (KR) partition* of (Y, S) is a partition of the form

$$\mathcal{P} = \{S^k(z(i)) \mid 0 \leq i < c,\ 0 \leq k < h(i)\} \quad \text{where } z(i) \text{ is clopen and } c,\ h(i) \in \mathbb{N}.$$

An element of $\mathcal{P}$ is called an *atom* of $\mathcal{P}$. A *tower* of $\mathcal{P}$ is $\{S^k(z(i)) \mid 0 \leq k < h(i)\}$, and its *bottom* is $z(i)$. So, obviously, the number of bottoms is the number c of towers. The *base* of $\mathcal{P}$ is the union $\bigcup_{i=0}^{c-1} z(i)$ of bottoms.

Given a Cantor system (Y, S), there exists a sequence of KR partitions $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ where $\mathcal{P}_0 = \{Y\}$ and $\mathcal{P}_n$ consists of c_n towers, i.e.

$$\mathcal{P}_n = \{S^k(z_n(i)) \mid 0 \leq i < c_n,\ 0 \leq k < h_n(i)\} \quad (n \in \mathbb{N}),$$

with the zeroth base $B_0 = z_0(0) = Y$, $c_0 = h_0(0) = 1$ and the nth base $B_n = \bigcup_{i=0}^{c_n-1} z_n(i)$ such that the following conditions hold [**3, 9**]:

(1) $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$;
(2) $\bigcup_n \mathcal{P}_n$ generates the topology of Y;
(3) $B_{n+1} \subset B_n$ and $\bigcap_n B_n$ is a singleton.

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We say that a sequence $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ of KR partitions satisfies the *HPS-condition* if it satisfies the above three conditions. Then, by condition (1), for each $n \in \mathbb{N}$, $0 \le i < c_n$ and $0 \le j < c_{n-1}$, there exists $(H_n)_{ij} \subset [0, h_n(i)) \cap \mathbb{Z}$ such that

$$z_{n-1}(j) = \bigcup_{i=0}^{c_n-1} \bigcup_{\rho\in(H_n)_{ij}} S^\rho(z_n(i)).$$

We call such an $\{(H_n)_{ij}\}$ the *hitting time sets* of $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$. Moreover, we can associate $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ with a properly ordered Bratteli diagram $B = (V, E, \le)$ defined by

$$\begin{aligned}
&V_n = \{v_n(0), \dots, v_n(c_n - 1)\} \quad (n \in \mathbb{Z}_+),\\
&E_n = \{(v_{n-1}(j), \rho, v_n(i)) \mid \rho \in (H_n)_{ij}\} \quad (n \in \mathbb{N}),\\
&s(v_{n-1}(j), \rho, v_n(i)) = v_{n-1}(j),\\
&r(v_{n-1}(j), \rho, v_n(i)) = v_n(i),\\
&(v_{n-1}(j), \rho, v_n(i)) \le (v_{n-1}(j'), \rho', v_n(i')) \quad \text{if } i = i' \text{ and } \rho \le \rho'.
\end{aligned}$$

So the incidence matrices of B are given by

$$(A_n)_{ij} = \#(H_n)_{ij}$$

and the infinite path space X_B is

$$X_B = \{((v_{n-1}(i_{n-1}), \rho_n, v_n(i_n)))_{n\in\mathbb{N}} \mid \rho_n \in (H_n)_{i_n i_{n-1}}\}.$$

Here we have the basic facts [**9**]: $K_0(B) \cong K_0(C(Y) \rtimes_S \mathbb{Z})$ as ordered groups, the dynamical system (X_B, θ_B) (where X_B is endowed with the usual topology) is topologically conjugate to (Y, S), and a conjugacy map $\phi : X_B \to Y$ is given by

$$\{\phi((v_{n-1}(i_{n-1}), \rho_n, v_n(i_n))_{n\in\mathbb{N}})\} = \bigcap_{n\in\mathbb{N}} z_n\left(i_n, \sum_{l=1}^{n} \rho_l\right)$$

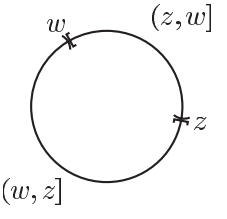
where $z_n(i_n, k) = S^k(z_n(i_n))$

(moreover, we see that $\{\phi(e_*^{\min})\} = \bigcap_n B_n$). We call this Bratteli–Vershik system (X_B, θ_B) an *HPS-adic presentation* of (Y, S). Moreover, in [**8**], it is shown that the orbit structure of (Y, S) is intimately related to $K_0(C(Y) \rtimes_S \mathbb{Z})$.

Notation. We shall introduce the main notation. Denote by $\mathbb{N}$ (respectively $\mathbb{Z}_+$) the set of positive (respectively non-negative) integers. For each $t \in \mathbb{R}$, let $\lfloor t \rfloor = \max\{i \in \mathbb{Z} \mid i \le t\}$. For each $n \in \mathbb{Z}$, let

$$n_+ = \max\{n, 0\}.$$

For distinct $z, w \in S^1$, denote by $(z, w]$ the left-open right-closed arc between z and w which lies in the positive direction from z, and let $(z, z] = S^1$ for convenience:



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For left-open right-closed arc $I = (z, w]$, let

$$\inf I = z \quad \text{and} \quad \max I = w.$$

Denote by $\operatorname{int} I$ the interior of I (so $\operatorname{int} I$ is an open arc $I \backslash \{w\}$), and by $|I|$ the length of arc I. We identify $S^1 = \mathbb{R}/\mathbb{Z}$ with $(0, 1]$ naturally. Let $\alpha \in (0, 1)$. Under the identification of S^1 and $(0, 1]$, the rotation $R_\alpha : S^1 \to S^1$ is the transformation of $(0, 1]$:

$$R_\alpha(\omega) = \begin{cases} \omega + \alpha & \text{if } \omega \in (0, 1 - \alpha], \\ \omega + \alpha - 1 & \text{if } \omega \in (1 - \alpha, 1]. \end{cases}$$

3. *Factor-HPS-condition*

We shall consider a Denjoy system (X, T) of finite double orbit number whose rotation number is α. To construct a sequence of KR partitions of (X, T) satisfying the HPS-condition, we will consider a 'counterpart' of (S^1, R_α).

Let $R = R_\alpha$. A partition $\mathcal{P}$ of S^1 is called a KR partition of (S^1, R) if $\mathcal{P}$ is of the form

$$\mathcal{P} = \{R^k(t(i)) \mid 0 \le i < c, 0 \le k < h(i)\}$$

where any element of $\mathcal{P}$ (an *atom* of $\mathcal{P}$) is either S^1 or a left-open right-closed arc.

Suppose that $(\mathcal{P}_n)_{n \in \mathbb{Z}_+}$ is a sequence of KR partitions of (S^1, R) of the form

$$\mathcal{P}_0 = \{S^1\}, \quad \mathcal{P}_n = \{R^k(t_n(i)) \mid 0 \le i < c_n, 0 \le k < h_n(i)\} \quad (n \in \mathbb{N}).$$

We say that a sequence $(\mathcal{P}_n)_{n \in \mathbb{Z}_+}$ of KR partitions of (S^1, R) satisfies the *factor-HPS-condition* if the following conditions hold:

(1) $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$;
(2) $\{z \in S^1 \mid z \text{ is an endpoint of some atom}\} = \mathcal{Q}(T)$;
(3) letting $B_n = \bigcup_{i=0}^{c_n - 1} t_n(i)$, $\{B_n\}_{n \in \mathbb{N}}$ satisfies $B_{n+1} \subset B_n$, B_n is either S^1 or an arc with $\max B_n = \max B_{n+1}$, and $\bigcap_n B_n$ is a singleton.

By condition (1), we can define the hitting time sets of $(\mathcal{P}_n)_{n \in \mathbb{Z}_+}$ similarly. Furthermore, define $(\mathcal{P}_n^X)_{n \in \mathbb{Z}_+}$ by $\mathcal{P}_0^X = \{X\}$,

$$z_n(i) = \operatorname{cl} F_X^{-1}(\operatorname{int} t_n(i)) \quad \text{and} \quad \mathcal{P}_n^X = \{T^k(z_n(i)) \mid 0 \le i < c_n, 0 \le k < h_n(i)\}.$$

Let $B_n^X = \operatorname{cl} F_X^{-1}(\operatorname{int} B_n)$.

EXTENSION LEMMA. *If $(\mathcal{P}_n)_{n \in \mathbb{Z}_+}$ satisfies the factor-HPS-condition, then $(\mathcal{P}_n^X)_{n \in \mathbb{Z}_+}$ is a sequence of KR partitions of (X, T) satisfying the HPS-condition with the same hitting time sets and B_n^X is the base of $\mathcal{P}_n^X$, $\bigcap_n B_n^X = \inf F^{-1}(\bigcap_n B_n)$.*

Proof. For any $z \in \mathcal{Q}(T)$, there exists a cutout interval I_z such that $F_X^{-1}(z)$ is the set of endpoints of I_z. This is a one-to-one correspondence between $\mathcal{Q}(T)$ and the family of cutout intervals. For each $z \in \mathcal{Q}(T)$, we pick a point

$$\widetilde{z} \in I_z.$$

If $z, w \in \mathcal{Q}(T)$ then $\operatorname{cl} F_X^{-1}(\operatorname{int}(z, w]) = (\widetilde{z}, \widetilde{w}] \cap X$ is clopen (Figure 2). Since $F_X \circ T = R \circ F_X$, $(\mathcal{P}_n^X)_{n \in \mathbb{Z}_+}$ is a sequence of KR partitions of (X, T).

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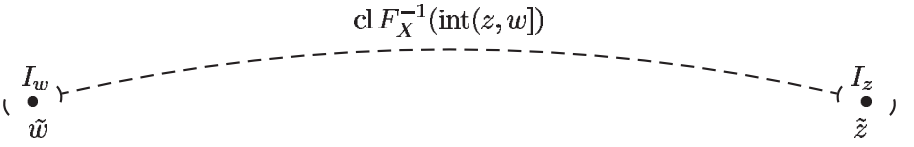


FIGURE 2. cl $F_X^{-1}(\mathrm{int}(z, w])$.

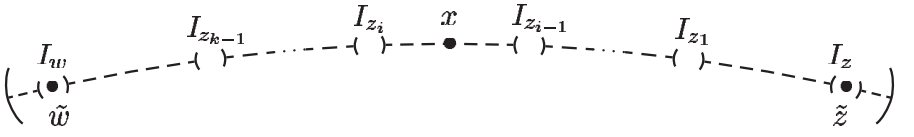


FIGURE 3. $U \cap X$.

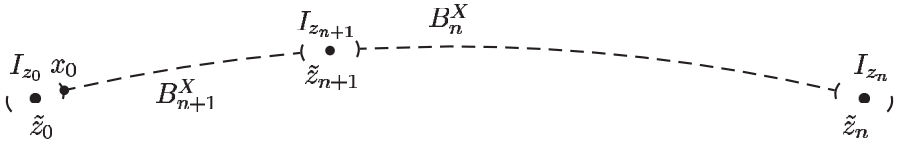


FIGURE 4. B_n^X.

(i) $\mathcal{P}_{n+1}^X$ is finer than $\mathcal{P}_n^X$ since $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$. By $F_X \circ T = R \circ F_X$, $(\mathcal{P}_n^X)_{n\in\mathbb{Z}_+}$ has the same hitting time sets of $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$.

(ii) Let $x \in X$ and $U \ni x$ be an open arc. To show that the family $\bigcup_n \mathcal{P}_n^X$ generates the topology of X, it suffices to prove that there exists $J \in \bigcup_n \mathcal{P}_n^X$ such that $x \in J \subset U \cap X$. Since X is totally disconnected, there exist cutout intervals I_z, I_w ($z, w \in \mathcal{Q}(T)$) such that $x \in (\widetilde{z}, \widetilde{w}] \cap X \subset U \cap X$. By (1) and (2) of the factor-HPS-condition, there exists $N \in \mathbb{N}$ and $z_0 = z, z_1, \ldots, z_{k-1}, z_k = w$ such that $(z_{i-1}, z_i] \in \mathcal{P}_N$ (Figure 3). Then

$$(\widetilde{z}, \widetilde{w}] \cap X = \bigcup_i \mathrm{cl}\, F_X^{-1}(\mathrm{int}(z_{i-1}, z_i]).$$

Therefore, $x \in \mathrm{cl}\, F_X^{-1}(\mathrm{int}(z_{i-1}, z_i]) \subset U \cap X$ for some i, and $\mathrm{cl}\, F_X^{-1}(\mathrm{int}(z_{i-1}, z_i])$ is an atom of $\mathcal{P}_N^X$.

(iii) Note that B_n^X is the nth base of $\mathcal{P}_n^X$. Then $B_{n+1}^X \subset B_n^X$ since $B_{n+1} \subset B_n$. Let $B_n = (z_n, z_0]$ with $B_n \neq S^1$. Then $z_n, z_0 \in \mathcal{Q}(T)$ and $B_n^X = (\widetilde{z}_n, \widetilde{z}_0] \cap X$. By (3) of the factor-HPS-condition, $z_0 \in \bigcap_n B_n$. Let $x_0 \in B_n^X$ be the endpoint of I_{z_0} such that $\{x_0\} = B_n^X \cap \mathrm{cl}\, I_{z_0}$ (Figure 4). Then $x_0 \in \bigcap_n B_n^X$ and $F_X(x_0) = z_0 \in \bigcap_n B_n$. Notice that if $x \in B_n^X \backslash \{x_0\}$ then $F_X(x) \neq F_X(x_0)$ and $F_X(x) \in \mathrm{cl}\, B_n$. Assume that $x \in \bigcap_n B_n^X \backslash \{x_0\}$. Then $F_X(x) \neq F_X(x_0)$ and $F_X(x) \in \bigcap_n \mathrm{cl}\, B_n = \bigcap_n B_n$. This contradicts the fact that $\bigcap_n B_n$ is a singleton. Thus $\bigcap_n B_n^X$ is also a singleton. □

4. *Principal sequence of KR partitions*

In this section, we will prepare a certain sequence of KR partitions of (S^1, R_α) (the *principal sequence*), and discuss its arithmetical aspect. In the next section, we will construct a sequence of KR partitions satisfying the factor-HPS-condition which is based on the principal sequence.

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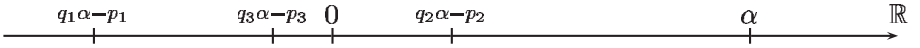


FIGURE 5. Arrangement of $\{q_n\alpha - p_n\}_{n=0}^{\infty}$.

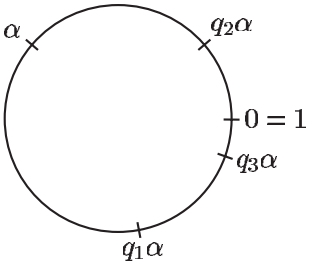


FIGURE 6. Closest returns in S^1.

4.1. *Principal sequence.* Let an irrational $\alpha \in (0, 1)$ be given, and let p_n/q_n be the nth (principal) convergent in the simple continued fraction expansion of α, that is,

$$\frac{p_1}{q_1} = \frac{1}{a_1 + 1}, \quad \frac{p_n}{q_n} = \cfrac{1}{a_1 + 1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} \ (n \geq 2),$$

where $a_1 \in \mathbb{Z}_+$ and $a_n \in \mathbb{N}$ $(n \geq 2)$. Let

$$\begin{bmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then it is well known that the following relations hold:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned} \quad \text{and} \quad q_n\alpha - p_n = (-1)^n |q_n\alpha - p_n|.$$

Here, by basic properties of continued fractions,

$$|q_{n-2}\alpha - p_{n-2}| = a_n |q_{n-1}\alpha - p_{n-1}| + |q_n\alpha - p_n| \quad (n \in \mathbb{N}).$$

(See Figure 5.)

Let R be the rotation of angle α. Denote $R^n(1)$ by '$n\alpha$' if there is no confusion. So, we have the arrangement of $\{q_n\alpha\}_{n=0}^{\infty} \subset S^1$ as in Figure 6.

Hence $q_n\alpha$ converges to 0 $(n \to \infty)$; in detail,

$$q_n = \min\{k \in \mathbb{N} \mid k\alpha \in \mathrm{int}((-1)^n q_{n-1}\alpha, (-1)^{n-1} q_{n-1}\alpha]\} \quad (n \geq 2).$$

Let $d(\omega, \omega') = \min\{|(\omega, \omega']|, |(\omega', \omega]|\}$ be the usual distance on S^1. The above implies that, if $1 \leq k < q_n$, then $d(1, k\alpha) \geq |q_{n-1}\alpha - p_{n-1}|$. Hence for any $\omega \in S^1$, if $1 \leq k < q_n$, then $d(\omega, R^k(\omega)) \geq |q_{n-1}\alpha - p_{n-1}|$.

In order to find the configuration of $\{l\alpha, m\alpha, n\alpha\}$ on S^1, it suffices to know that of $\{(l - m)\alpha, 0, (n - m)\alpha\}$ (Figure 7).

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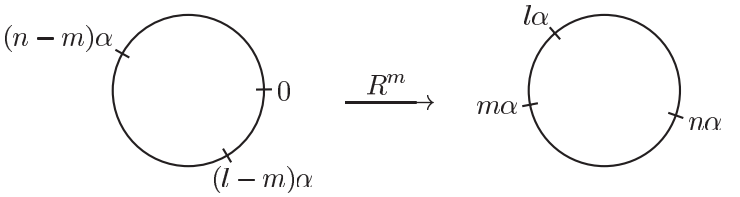


FIGURE 7. Configuration of $\{l\alpha, m\alpha, n\alpha\}$.

In particular, we have the following configurations (Figure 8).

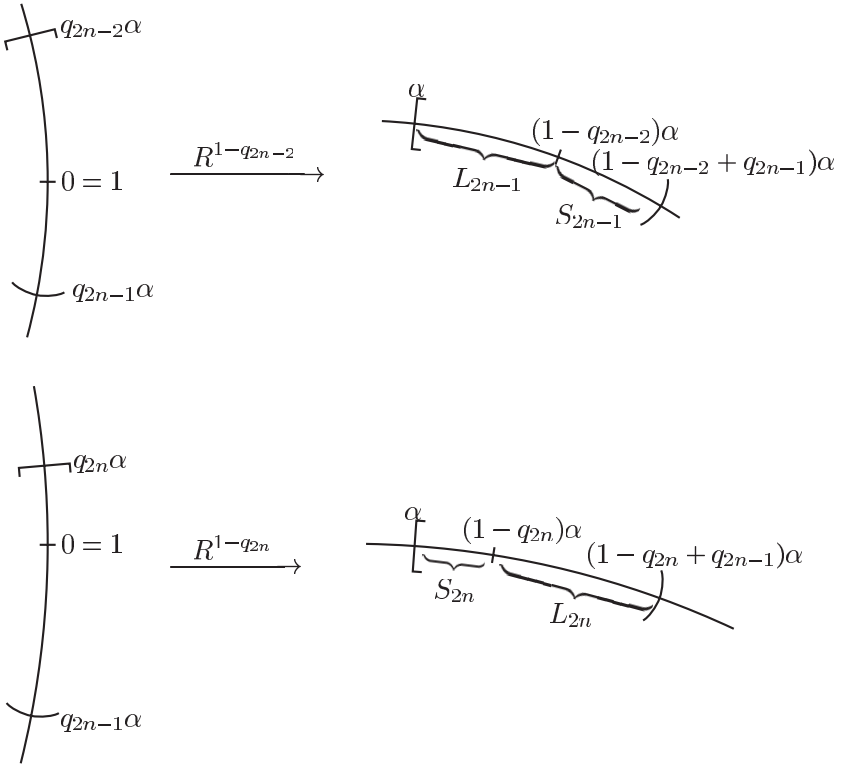


FIGURE 8. L_n and S_n.

Definition. For each $n \in \mathbb{Z}_+$, let $B_n = ((1-(-1)^n(q_n - q_{n-1}))\alpha, \alpha]$ and

$$S_n = \begin{cases} ((1-q_{n-1}+q_n)\alpha, (1-q_{n-1})\alpha] & \text{if } n \text{ is odd} \\ ((1-q_n)\alpha, \alpha] & \text{if } n \text{ is even,} \end{cases}$$

$$L_n = B_n \backslash S_n.$$

(See Figures 8 and 9.)

Observe that $B_{n+1} \subset B_n$ and $\bigcap_n B_n = \{\alpha\}$. Note that $|S_n|$ and $|L_n|$ are respectively equal to $|q_n\alpha - p_n|$ and $|q_{n-1}\alpha - p_{n-1}|$ for each $n \in \mathbb{Z}_+$.

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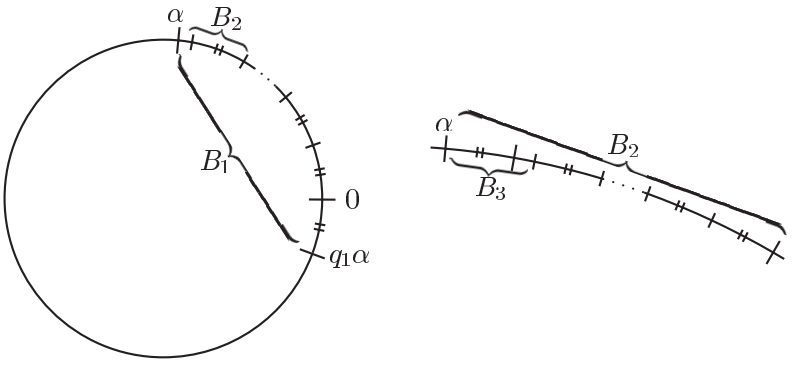


FIGURE 9. B_1 and B_2.

For each $n \in \mathbb{Z}_+$, let R_n be the induced transformation of R on B_n, that is, $R_n : B_n \to B_n$ is the first return map: $R_n(\omega) = R^{r_n(\omega)}(\omega)$ where $r_n(\omega) = \min\{n \in \mathbb{N} \mid R^n(\omega) \in B_n\}$.

Observation 4.1.

$$r_n(\omega) = \begin{cases} q_{n-1} & \text{if } \omega \in S_n, \\ q_n & \text{if } \omega \in L_n. \end{cases}$$

That is, (B_n, R_n) is the exchange of two intervals S_n and L_n.

Proof. It is easily seen that the claim holds for $n = 0, 1$.

By using $R^k(n\alpha, m\alpha] = ((n+k)\alpha, (m+k)\alpha]$, one can see that

$$B_n = R^{q_{n-1}}(S_n) \cup R^{q_n}(L_n) \quad \text{(disjoint)}.$$

This implies that

$$r_n(\omega) \le \begin{cases} q_{n-1} & \text{if } \omega \in S_n \\ q_n & \text{if } \omega \in L_n. \end{cases}$$

Let $n \ge 2$ and $\omega \in B_n$. If $1 \le k < q_{n-1}$, then

$$d(\omega, R^k(\omega)) \ge |q_{n-2}\alpha - p_{n-2}| \ge |q_{n-1}\alpha - p_{n-1}| + |q_n\alpha - p_n| = |B_n|.$$

Therefore, $R^k(\omega) \notin B_n$, that is, $r_n(\omega) \ge q_{n-1}$. Hence $r_n(\omega) = q_{n-1}$ if $\omega \in S_n$.

In order to prove that $r_n(\omega) = q_n$ ($\omega \in L_n$), it suffices to check that, if $1 \le k < q_n$, then $R^k(L_n) \cap B_n = \emptyset$. Note that $R^k(L_n) = (R^k(\inf L_n), R^k(\max L_n)]$ and $|R^k(L_n)| = |L_n| < |B_n|$. It suffices to show that, if $1 \le k < q_n$, then $R^k(\max L_n) \notin B_n$ and $R^k(\inf L_n) \notin B_n \backslash \{\alpha\}$. Let $1 \le k < q_n$. Notice that

$$d(\max L_n, R^k(\max L_n)) \ge |q_{n-1}\alpha - p_{n-1}| = |L_n|, \tag{a}$$

$$d(\inf L_n, R^k(\inf L_n)) \ge |q_{n-1}\alpha - p_{n-1}| = |L_n|. \tag{b}$$

If n is odd, then $\max L_n = R^{q_{n-1}}(\inf L_n)$. Since $1 \le k + q_{n-1} < q_{n+1}$,

$$d(\inf L_n, R^k(\max L_n)) = d(\inf L_n, R^{k+q_{n-1}}(\inf L_n)) \ge |q_n\alpha - p_n| = |S_n|. \tag{c}$$

By (a), $R^k(\max L_n) \notin L_n$; by (b), $R^k(\inf L_n) \notin B_n \backslash \{\alpha\}$; and by (c), $R^k(\max L_n) \notin S_n$ (Figure 10).

If n is even, then $\inf L_n = R^{q_{n-1}}(\max L_n)$. Since $1 \le k + q_{n-1} < q_{n+1}$,

$$d(\max L_n, R^k(\inf L_n)) = d(\max L_n, R^{k+q_{n-1}}(\max L_n)) \ge |q_n\alpha - p_n| = |S_n|. \tag{d}$$

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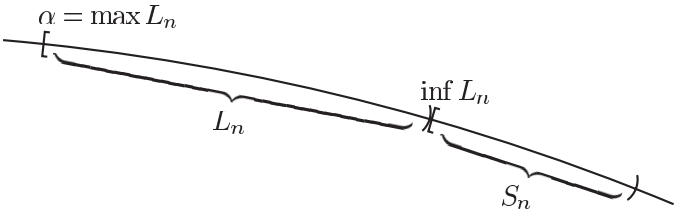


FIGURE 10. B_n when n is odd.

By (a), $R^k(\max L_n) \notin B_n$; by (b), $R^k(\inf L_n) \notin L_n \backslash \{\max L_n\}$; and by (d), $R^k(\inf L_n) \notin \{\max L_n\} \cup S_n \backslash \{\alpha\}$ (Figure 11).

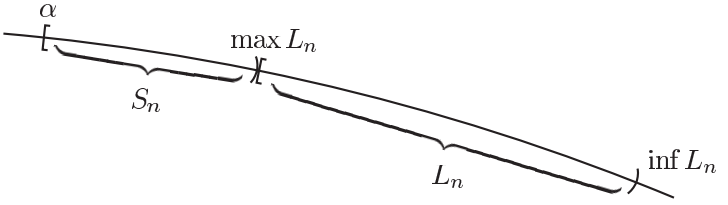


FIGURE 11. B_n when n is even.

□

From now on, we will draw pictures on $(0, 1]$ instead of S^1. Let

$$\epsilon_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Definition. For each $n \in \mathbb{N}$ and $x \in \{-1, 0, 1, \ldots, a_n\}$, let

$$s_n(x) = \begin{cases} (-\epsilon_{n+1}(-1)^n q_n + (x - a_n + \epsilon_n)(-1)^{n-1} q_{n-1})_+ & \text{if } x \neq a_n, \\ 0 & \text{if } x = a_n. \end{cases}$$

For left-open right-closed arcs I_1, I_2, we write '$I_1 \overset{\bullet}{<} I_2$' if $\max I_1 = \inf I_2$.

REFINING LEMMA. *Let $n \in \mathbb{N}$.*

(i) $$S_{n-1} = R^{s_n(-1)}(L_n), \quad L_{n-1} = \bigcup_{x=0}^{a_n-1} R^{s_n(x)}(L_n) \cup S_n \quad (\textit{disjoint}).$$

(ii) $$R^{s_n(0)}(L_n) \overset{\bullet}{<} R^{s_n(1)}(L_n) \overset{\bullet}{<} \cdots \overset{\bullet}{<} R^{s_n(a_n-1)}(L_n) \overset{\bullet}{<} R^{s_n(a_n)}(S_n) \ (= S_n).$$

Proof. For each $n \in \mathbb{N}$, notice that $|S_{n-1}| = |L_n|$ and $|L_{n-1}| = a_n|L_n| + |S_n|$. Notice that, if $-1 \leq x < a_n$, then

$$s_n(x) = \begin{cases} (q_{n-2} + x q_{n-1})_+ & \text{if } n \text{ is odd} \\ (a_n - x - 1) q_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

By Observation 4.1, we can get Figures 12 and 13. These imply that the claim holds. □

We obtain the KR partition of the $(n-1)$th induced system (B_{n-1}, R_{n-1}) with base B_n, and call this partition the *induced tower partition* (Figure 14).

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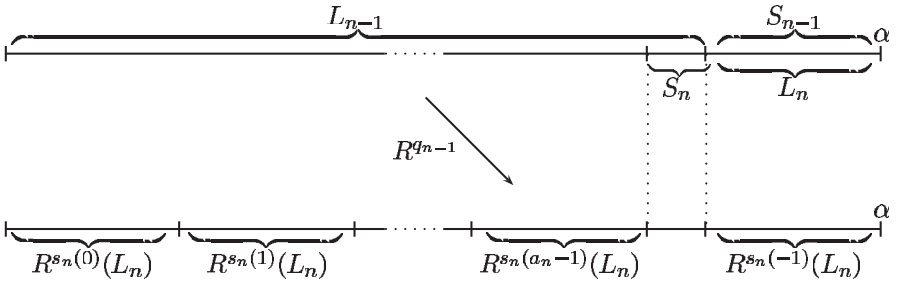


FIGURE 12. Refinement of B_{n-1} when n is odd: note that $R^{s_n(0)}(L_n) = R^{q_{n-2}}(S_{n-1})$.

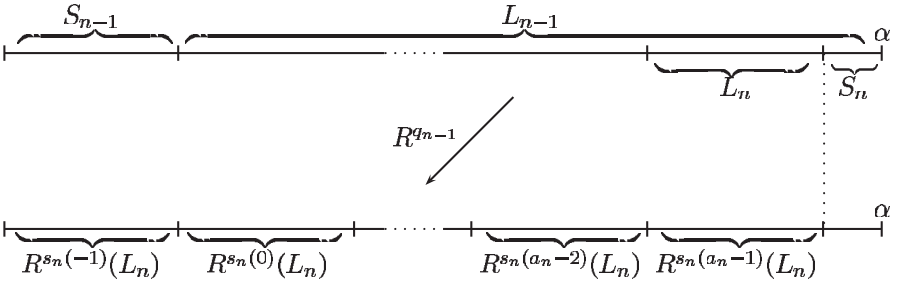


FIGURE 13. Refinement of B_{n-1} when n is even.

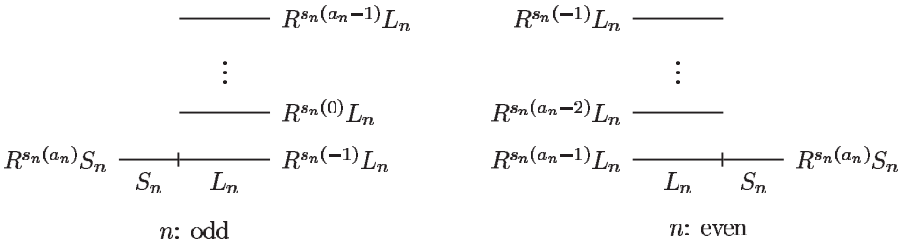


FIGURE 14. The induced tower partition of B_{n-1}.

From now on, we will draw pictures of the induced tower partition as in Figure 15, where $R^{s_n(x)}(\cdot)$ is denoted by $\boxed{x}$.

TOWER LEMMA. *For each* $n \in \mathbb{N}$,

$$(0, 1] = \bigcup_{k=0}^{q_{n-1}-1} R^k(S_n) \cup \bigcup_{k=0}^{q_n-1} R^k(L_n) \quad (\textit{disjoint}).$$

Proof. We will use induction. For $n = 1$, by the Refining lemma,

$$(0, 1] = S_0 \cup L_0 = R^{s_1(-1)}(L_1) \cup \bigcup_{x=0}^{a_1-1} R^{s_1(x)}(L_1) \cup S_1 = S_1 \cup \bigcup_{k=0}^{q_1-1} R^k(L_1).$$

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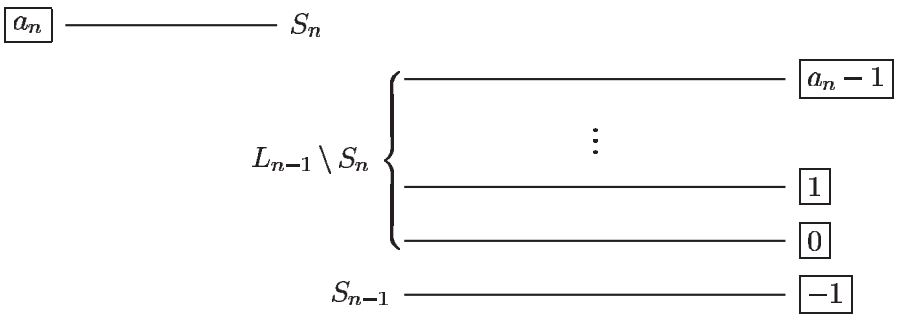


FIGURE 15. The induced tower partition of B_{n-1}.

Suppose the Tower lemma holds for $n-1 \in \mathbb{N}$. Notice that

$$[0, q_n) \cap \mathbb{Z} = [s_n(-1), s_n(-1) + q_{n-2}) \cap \mathbb{Z} \cup \bigcup_{x=0}^{a_n-1} [s_n(x), s_n(x) + q_{n-1}) \cap \mathbb{Z} \quad \text{(disjoint)}.$$

By the Refining lemma,

$$\begin{aligned}(0, 1] &= \bigcup_{k=0}^{q_{n-2}-1} R^k(S_{n-1}) \cup \bigcup_{k=0}^{q_{n-1}-1} R^k(L_{n-1}) \\ &= \bigcup_{k=0}^{q_{n-2}-1} R^{s_n(-1)+k}(L_n) \cup \bigcup_{k=0}^{q_{n-1}-1} \bigcup_{x=0}^{a_n-1} R^{s_n(x)+k}(L_n) \cup \bigcup_{k=0}^{q_{n-1}-1} R^k(S_n) \\ &= \bigcup_{k=0}^{q_{n-1}-1} R^k(S_n) \cup \bigcup_{k=0}^{q_n-1} R^k(L_n). \end{aligned}$$

□

Let

$$\mathcal{P}'_0 = \{S^1\}, \quad \mathcal{P}'_n = \{R^k(S_n) \mid 0 \le k < q_{n-1}\} \cup \{R^k(L_n) \mid 0 \le k < q_n\} \quad (n \in \mathbb{N}).$$

We call $(\mathcal{P}'_n)_{n\in\mathbb{Z}_+}$ the principal sequence of KR partitions. Notice that $(\mathcal{P}'_n)_{n\in\mathbb{Z}_+}$ satisfies the facts that:

- $\mathcal{P}'_{n+1}$ is finer than $\mathcal{P}'_n$;
- $B_{n+1} \subset B_n$, B_n is either S^1 or an arc with $\max B_n = \max B_{n+1}$, and $\bigcap_n B_n = \{\alpha\}$.

4.2. *Backward hits and signed expansion.*

Definition. For each $n \in \mathbb{Z}_+$, define $\delta_n : (0, 1] \to B_n$ by

$$\delta_n(\omega) = R^{-l_n(\omega)}(\omega)$$

where $l_n(\omega) = \min\{k \ge 0 \mid R^{-k}(\omega) \in B_n\}$. (See Figure 16.)

Remark 4.1. Notice that

$$(\max S_n, \max L_n) = \begin{cases} (\delta_n(1), \delta_n(\alpha)) & \text{if } n \text{ is odd,} \\ (\delta_n(\alpha), \delta_n(1)) & \text{if } n \text{ is even.} \end{cases}$$

So $\{\max S_n, \max L_n\} = \delta_n(\{\alpha, 1\})$.

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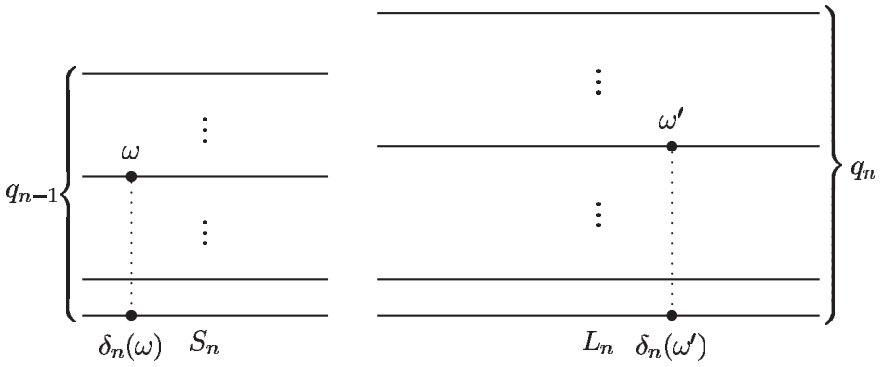


FIGURE 16. The nth principal KR partition and $\delta_n : S^1 \to B_n$.

In fact, clearly $\delta_n(\alpha) = \alpha = \max B_n$, and by the Tower lemma, we can see that

$$\delta_n(1) = \begin{cases} (1 - q_{n-1})\alpha = \max S_n & \text{if } n \text{ is odd,} \\ (1 - q_n)\alpha = \max L_n & \text{if } n \text{ is even.} \end{cases}$$

Based on the Refining lemma, we can define the 'digits' of $\omega \in (0, 1]$ as follows.

Definition. For each $n \in \mathbb{N}$, define $x_n : (0, 1] \to \{-1, 0, \ldots, a_n\}$ by

$$x_n(\omega) = \begin{cases} x & \text{if } \delta_{n-1}(\omega) \in R^{s_n(x)}(L_n) \text{ with } -1 \le x < a_n, \\ a_n & \text{if } \delta_{n-1}(\omega) \in R^{s_n(a_n)}(S_n) = S_n. \end{cases}$$

We call x_n the *signed digital map*.

Since $B_{n-1} = \{\omega \in S^1 \mid \delta_{n-1}(\omega) = \omega\}$, we have the following observation by the definition of δ_n (see Figure 17).

Observation 4.2. For any $\omega \in B_{n-1}$, $\omega = R^{s_n(x_n(\omega))}(\delta_n(\omega))$.

Here note that $\delta_n = \delta_n\delta_{n-1}$ (since $B_n \subset B_{n-1}$) and $x_n = x_n\delta_{n-1}$ (by definition). So by Observation 4.2, for any $\omega \in (0, 1]$, $\delta_{n-1}(\omega) = R^{s_n(x_n(\omega))}(\delta_n(\omega))$.

The next observation states the basic property of (x_n, δ_n).

Observation 4.3.

(1) $x_n(\omega) = a_n \iff \delta_{n-1}(\omega) \in S_n \iff \delta_n(\omega) \in S_n \iff x_{n+1}(\omega) = -1$. (Note that $R^{s_{n+1}(-1)}(L_{n+1}) = S_n = \delta_n|^{-1}_{B_{n-1}}(S_n)$ by the Refining and Tower lemmas.)

(2) If $\{\delta_{n-1}(\omega), \delta_{n-1}(\omega')\} \subset L_{n-1}$ with $\delta_{n-1}(\omega) \le \delta_{n-1}(\omega')$, then $x_n(\omega) \le x_n(\omega')$.

(3) If $x_n(\omega) = x_n(\omega')$, then $\delta_n(\omega') - \delta_n(\omega) = \delta_{n-1}(\omega') - \delta_{n-1}(\omega)$.

Now, we shall introduce our arithmetic in the following way. Define

$$M_\alpha = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \{-1, 0, \ldots, a_n\} \;\middle|\; x_n = a_n \iff x_{n-1} = -1 \right\}.$$

By Observation 4.3(1), we can consider the following map:

$$x_* : \omega \mapsto (x_n(\omega))_{n\in\mathbb{N}} : (0, 1] \to M_\alpha.$$

We call $x_*(\omega)$ the *signed expansion* of ω.

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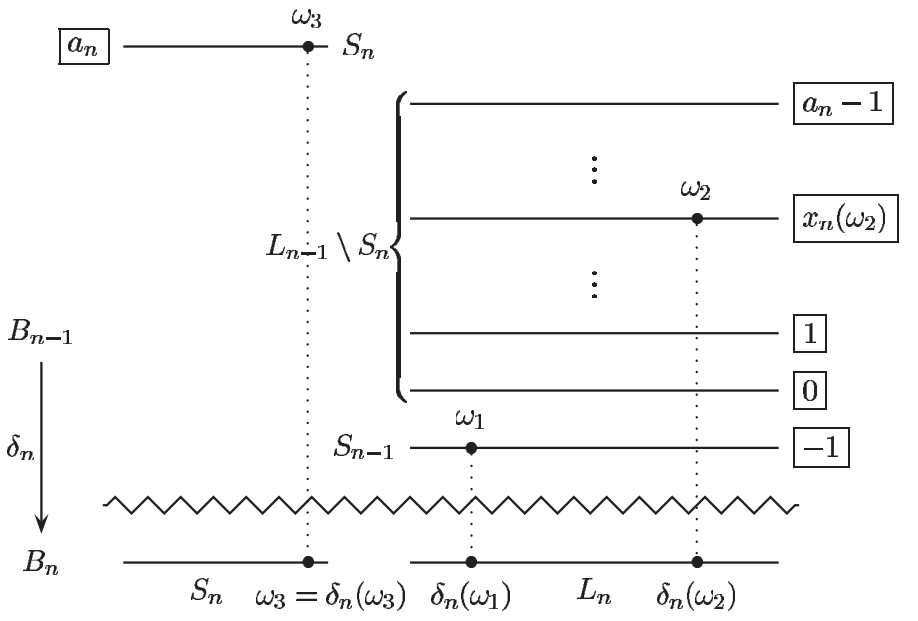


FIGURE 17. The induced tower partition of B_{n-1} and $\delta_n : B_{n-1} \to B_n$.

Remark 4.2. In particular, by Remark 4.1 and Observation 4.3(1), we have

$$x_*(\alpha) = (-1)a_2(-1)a_4 \ldots,$$
$$x_*(1) = a_1(-1)a_3(-1) \ldots.$$

4.3. *Ostrowski-type arithmetic.* Let

$$x_n^+(\omega) = \begin{cases} x_1(\omega) + 1 & \text{if } n = 1, \\ x_n(\omega)_+ & \text{otherwise,} \end{cases}$$
$$x_*^+(\omega) = (x_n^+(\omega))_{n\in\mathbb{N}},$$

and define

$$M_\alpha^+ = \left\{ (y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \{0, 1, \ldots, b_n\} \;\middle|\; y_n = b_n \Longrightarrow y_{n+1} = 0 \right\}$$

where $b_1 = a_1 + 1$, $b_n = a_n$ $(n \geq 2)$. Then we can regard x_*^+ as a map from $(0, 1]$ to M_α^+.

Definition. For each $\omega \in (0, 1]$, let

$$u_n(\omega) = \sum_{k=1}^{n} x_k^+(\omega)(-1)^{k-1}(q_{k-1}p_n - q_n p_{k-1}) \quad (n \in \mathbb{N}) \quad \text{and} \quad u_0(\omega) = u_{-1}(\omega) = 0.$$

PROPOSITION 4.1.
(1) *For each* $\omega \in (0, 1]$,

$$\omega = \sum_{n=1}^{\infty} x_n^+(\omega)|q_{n-1}\alpha - p_{n-1}|.$$

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(2) *We have the following (inhomogeneous) recurrence equation:*

$$u_n(\omega) = x_n^+(\omega) + a_n u_{n-1}(\omega) + u_{n-2}(\omega).$$

In particular, $u_n(1) = q_n - \epsilon_n$ *where*

$$\epsilon_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

(3) $(p_n/q_n, u_n(\omega)/q_n)$ *is a simultaneous approximation for* (α, ω)*, that is,*

$$\lim_{n\to\infty} \frac{u_n(\omega)}{q_n} = \omega.$$

(4) *If* $\mathcal{O}_\omega \cap \mathcal{O}_{\omega'} = \emptyset$*, then* $x_n(\omega) \neq x_n(\omega')$ *for infinitely many* n.

(For its proof, see Appendix A.)

Remark. It is known that for what is called a *dual Ostrowski numeration system*, for each $\omega \in [0, 1]$, there exists $(y_n)_{n\in\mathbb{N}} \in M_\alpha^+$ such that

$$\omega = \sum_{n=1}^{\infty} y_n |q_{n-1}\alpha - p_{n-1}|.$$

(Refer to [**1, 7, 10, 12, 17**].) The above formula in (1) gives this expansion. In particular, if $\omega \notin \mathcal{O}_\alpha$, then such a $(y_n)_{n\in\mathbb{N}}$ is unique. The map $\Delta : M_\alpha \to M_\alpha^+ : x_1x_2x_3\cdots \mapsto (x_1+1)(x_2)_+(x_3)_+\cdots$ is a bijection, and this gives the correspondence between $x_*(\omega)$ and $x_*^+(\omega)$. Therefore, obtaining the signed expansion of $\omega \notin \mathcal{O}_\alpha$ is equivalent to obtaining its dual Ostrowski expansion.

5. *KR partitions satisfying the factor-HPS-condition*

Let (X, T) be a Denjoy system of finite double orbit number d and rotation number α, and let Λ be a transversal of its double point set. We will construct a sequence of KR partitions satisfying the factor-HPS-condition.

Definition. Let $\Lambda_1 = \Lambda \cup \{1\}$. For each $n \in \mathbb{Z}_+$, the set $\delta_n(\Lambda_1)$ divides B_n into $d+1$ intervals (by Remark 4.1 and $\delta_n(\omega) \in \mathcal{O}_\omega$). We name these intervals $t_n(i)$ (left-open right-closed) as follows:

$$t_n(0) \overset{\bullet}{<} t_n(1) \overset{\bullet}{<} \cdots \overset{\bullet}{<} t_n(m_n), \quad t_n(m_n+1) \overset{\bullet}{<} t_n(m_n+2) \overset{\bullet}{<} \cdots \overset{\bullet}{<} t_n(d),$$

$$S_n = \bigcup_{i=0}^{m_n} t_n(i) \quad \text{(disjoint)}, \quad L_n = \bigcup_{i=m_n+1}^{d} t_n(i) \quad \text{(disjoint)},$$

where $m_n = \#(\delta_n(\Lambda_1) \cap S_n) - 1$. (See Figure 18.)

Remark 5.1. Let $n \in \mathbb{Z}_+$.

(1) $\delta_n(\Lambda_1) = \{\max t_n(i) \mid 0 \le i \le d\}$.

(2) Let $M_n = \min\{i \mid \max t_n(i) \in S_{n+1}\}$. Then $i \ge M_n$ if and only if $\max t_n(i) \in S_{n+1}$ (by the Refining lemma (ii)).

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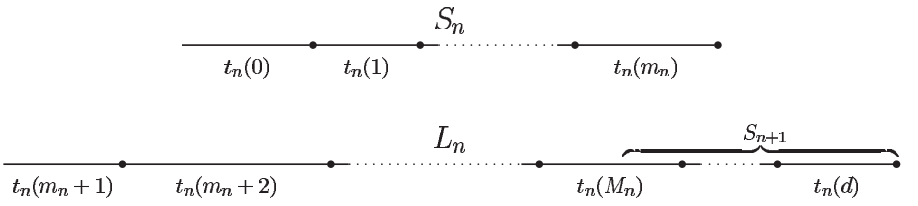


FIGURE 18. $\delta_n(\Lambda_1)$ divides B_n where each '•' denotes an element of $\delta_n(\Lambda_1)$.

Define

$$h_n(i) = \begin{cases} q_{n-1} & \text{if } 0 \le i \le m_n, \\ q_n & \text{if } m_n < i \le d, \end{cases}$$

and

$$\mathcal{P}_0 = \{S^1\}, \quad \mathcal{P}_n = \{R^k(t_n(i)) \mid 0 \le i \le d,\ 0 \le k < h_n(i)\} \quad (n \in \mathbb{N}).$$

THEOREM 5.1. *$(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ is a sequence of KR partitions of (S^1, R) satisfying the factor-HPS-condition. (By the Extension lemma, $(\mathcal{P}_n^X)_{n\in\mathbb{Z}_+}$ satisfies the HPS-condition.)*

Proof. By the Tower lemma, $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ is a sequence of KR partitions. Clearly, $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ satisfies condition (3) of the factor-HPS-condition. Then, we will check conditions (1) and (2). First, notice that

$$R^{q_{n-1}-1}(S_n) \cup R^{q_n-1}(L_n) = R^{-1}(B_n).$$

Therefore, for each $n \in \mathbb{N}$, by the Tower lemma,

$$\{z \in S^1 \mid z \text{ is an endpoint of some atom of } \mathcal{P}_n\} = \bigcup_{\omega\in\Lambda_1} \{R^k(\omega) \mid -l_n(\omega) \le k \le l^n(\omega)\}$$

where $l_n(\omega) = \min\{k \ge 0 \mid R^{-k}(\omega) \in B_n\}$ and $l^n(\omega) = \min\{k \ge 0 \mid R^k(\omega) \in R^{-1}(B_n)\}$.

(1) Since $B_{n+1} \subset B_n$, $l_{n+1}(\omega) \ge l_n(\omega)$ and $l^{n+1}(\omega) \ge l^n(\omega)$. Hence

$$\begin{aligned} &\{z \in S^1 \mid z \text{ is an endpoint of some atom of } \mathcal{P}_{n+1}\} \\ &\quad \supset \{z \in S^1 \mid z \text{ is an endpoint of some atom of } \mathcal{P}_n\}, \end{aligned}$$

so $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$.

(2) Since $\bigcap_n B_n = \{\alpha\}$ and $\bigcap_n R^{-1}(B_n) = R^{-1}\{\alpha\} = \{1\}$, we see that if $\omega \in S^1\backslash\{R^k(\alpha) \mid k \ge 0\}$ then $l_n(\omega) \to \infty$, and that if $\omega \in S^1\backslash\{R^k(1) \mid k \le 0\}$ then $l^n(\omega) \to \infty$. So

$$\bigcup_{n\in\mathbb{N}} \{z \in S^1 \mid z \text{ is an endpoint of some atom of } \mathcal{P}_n\} = \bigcup_{\omega\in\Lambda} \mathcal{O}_\omega. \qquad \square$$

Now, to obtain an HPS-adic presentation of the Denjoy system, we will investigate the hitting time sets of $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ (see §7).

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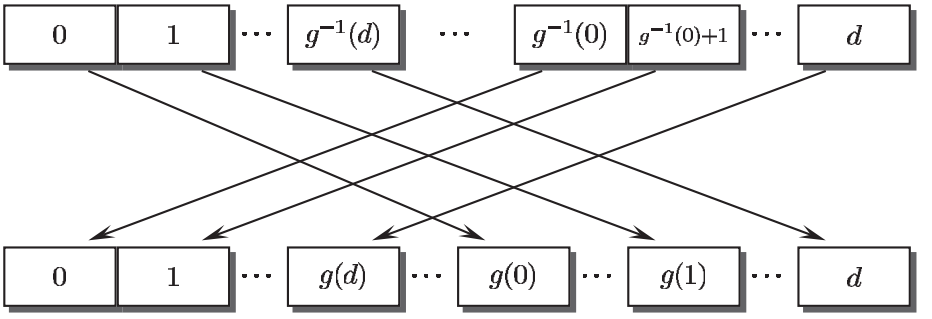


FIGURE 19. Partially mixing permutation.

6. *Section* (ξ, g)

In this section, we shall prepare some notion in order to determine an adic presentation. Here $(x, i) <_{\mathrm{lex}} (y, j)$ means the lexicographic order, that is, $(x, i) <_{\mathrm{lex}} (y, j)$ if and only if $x < y$, or $x = y$ and $i < j$.

Definition. Let ξ be a non-constant map from $\{0, 1, \ldots, d\}$ to $\{-1\} \cup \mathbb{Z}_+$ and let g be a permutation on $\{0, 1, \ldots, d\}$. Define $(\xi, g)(j) = (\xi(j), g(j))$. We call (ξ, g) a *section* if it satisfies the following:

(1) $(\xi, g)(j) <_{\mathrm{lex}} (\xi, g)(j+1)$;
(2) $\xi(g^{-1}(d)) = -1$ and $\xi(g^{-1}(0)) = \max \xi$;
(3) $g(d) = d - g^{-1}(0)$.

We say that g is *partially mixing* if there exists ξ such that (ξ, g) is a section.

Example 6.1. (ξ, g) given below is a section:

$$\xi: \begin{array}{ccc} 0 & \mapsto & -1 \\ 1 & \mapsto & -1 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 1 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 3 \\ 6 & \mapsto & 3 \\ 7 & \mapsto & 3 \end{array} \qquad g: \begin{array}{ccc} 0 & \mapsto & 4 \\ 1 & \mapsto & 7 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 6 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 0 \\ 6 & \mapsto & 1 \\ 7 & \mapsto & 2 \end{array}$$

Remark 6.1. Permutation g is partially mixing if and only if the following hold.

(1) $g \neq \mathrm{id}$.
(2) $g(j-1) < g(j)$ $(0 < j < g^{-1}(d))$.
(3) $g(j) = j - g^{-1}(0)$ $(g^{-1}(0) < j \leq d)$.

(For its proof, see Appendix A.)

Remark 6.2. We can count the number of partially mixing permutations:

$$\#\{g \mid g \text{ is partially mixing on } \{0, 1, \ldots, d\}\} = \sum_{N=0}^{d-1} \sum_{r=0}^{N} {}_N P_r.$$

(For its proof, see Appendix A.)

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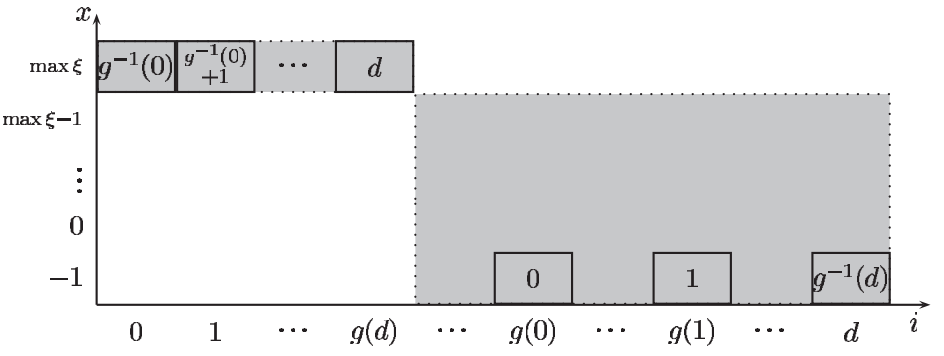


FIGURE 20. $D(\xi, g)$ where $\boxed{j}$ denotes the place of $(\xi, g)(j)$.

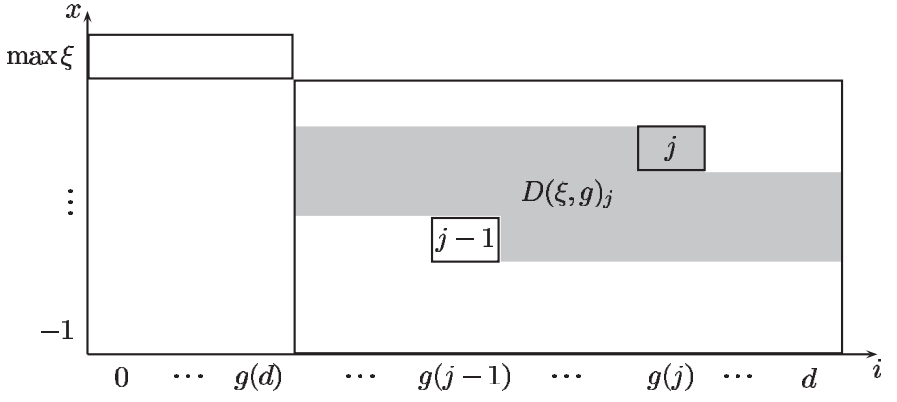


FIGURE 21. $D(\xi, g)_j$ (shaded region) when $g(j) > g(j-1)$ and $g^{-1}(d) < j < g^{-1}(0)$.

For each section (ξ, g), define $D(\xi, g)$ by

$$D(\xi, g) = \{-1, 0, \ldots, \max \xi - 1\} \times \{g(d) + 1, g(d) + 2, \ldots, d\} \cup \{\max \xi\} \times \{0, 1, \ldots, g(d)\}.$$

Notice that

$$\{(-1, d), (\max \xi, 0), (\max \xi, 1), \ldots, (\max \xi, g(d))\} \subset \mathrm{Im}(\xi, g) \subset D(\xi, g).$$

(See Figure 20.)

Definition. Let (ξ, g) be a section and $0 \le j \le d$. Define $D(\xi, g)_j$ by

$$D(\xi, g)_j = \begin{cases} \{(x, i) \in D(\xi, g) \mid (x, i) \le_{\mathrm{lex}} (\xi, g)(0)\} & \text{if } j = 0, \\ \{(x, i) \in D(\xi, g) \mid (\xi, g)(j-1) <_{\mathrm{lex}} (x, i) \le_{\mathrm{lex}} (\xi, g)(j)\} & \text{otherwise.} \end{cases}$$

(See Figures 21 and 22.)

Definition. For each section (ξ, g), define the $(d+1) \times (d+1)$ matrix $A(\xi, g)$ by

$$A(\xi, g)_{ij} = \#\{x \mid (x, i) \in D(\xi, g)_j\}.$$

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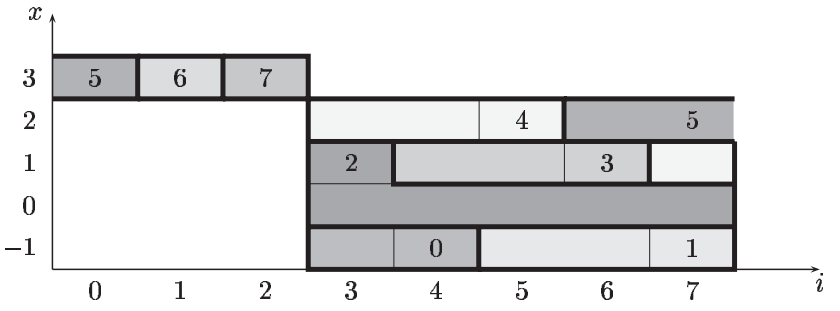


FIGURE 22. $\mathrm{Im}(\xi, g)$ and $D(\xi, g)_j$ where (ξ, g) is given in Example 6.1.

7. *An HPS-adic presentation of* (X, T)

We will return to the Denjoy system (X, T), and describe the hitting time sets of $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$. Recall that $\delta_n(\Lambda_1) = \{\max t_n(i) \mid 0 \le i \le d\}$ and $\delta_n : \delta_{n-1}(\Lambda_1) \to \delta_n(\Lambda_1)$ is bijective. Moreover, by the definition of $t_n(i)$, we can consider the following map:

$$\max t_n : \{0, 1, \ldots, d\} \to \delta_n(\Lambda_1) : i \mapsto \max t_n(i).$$

Then $\max t_n$ is a bijection: more precisely,

$$\max t_n : \{0, \ldots, m_n\} \to S_n \cap \delta_n(\Lambda_1),$$
$$\max t_n : \{m_n + 1, \ldots, d\} \to L_n \cap \delta_n(\Lambda_1)$$

are increasing bijections.

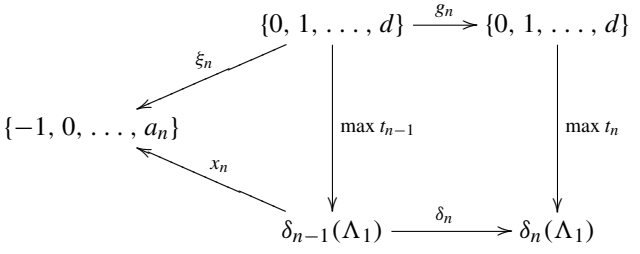
Definition. For each $n \in \mathbb{N}$, define the permutation g_n on $\{0, 1, \ldots, d\}$ by

$$\delta_n(\max t_{n-1}(j)) = \max t_n(g_n(j)),$$

and define the map $\xi_n : \{0, 1, \ldots, d\} \to \{-1, 0, \ldots, a_n\}$ by

$$\xi_n(j) = x_n(\max t_{n-1}(j))$$

where $x_n : (0, 1] \to \{-1, \ldots, a_n\}$ is the nth signed digital map:



Consider the map

$$(\xi_n, g_n) : \{0, 1, \ldots, d\} \to \{-1, 0, 1, \ldots, a_n\} \times \{0, 1, \ldots, d\} : j \mapsto (\xi_n(j), g_n(j)).$$

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PROPOSITION 7.1. *Let* $n \in \mathbb{N}$.

(1) *For each* $0 \le j \le d$,

$$\max t_{n-1}(j) = R^{s_n(\xi_n(j))}(\max t_n(g_n(j))).$$

(2) $g_n(m_{n-1}) = d$ *and* $g_n(d) = m_n$. *So* $g_{n+1}g_n(d) = d$.

(3) (ξ_n, g_n) *is a section with* $\max \xi_n = a_n$.

Proof. (1) By applying Observation 4.2 to $\omega = \max t_{n-1}(j)$, (1) holds.

(2) As

$$\max t_n(g_n(m_{n-1})) = \delta_n(\max t_{n-1}(m_{n-1})) = \delta_n(\max S_{n-1}) = \max L_n = \max t_n(d),$$

then

$$\max t_n(g_n(d)) = \delta_n(\max t_{n-1}(d)) = \delta_n(\max L_{n-1}) = \max S_n = \max t_n(m_n).$$

(3) Recall the induced tower partition of B_{n-1}:

$$\begin{aligned} B_{n-1} &= R^{s_n(-1)}(L_n) \cup \bigcup_{x=0}^{a_n-1} R^{s_n(x)}(L_n) \cup S_n \\ &= S_{n-1} \cup (L_{n-1} \setminus S_n) \cup S_n. \end{aligned}$$

Let $M_{n-1} = \min\{j \mid \max t_{n-1}(j) \in S_n\}$.

First, we shall show the following claim:

$$\begin{aligned} &\text{(I)} \quad j \le m_{n-1} && \iff \max t_{n-1}(j) \in S_{n-1} && \iff \xi_n(j) = -1 \\ &\text{(II)} \quad m_{n-1} < j < M_{n-1} && \iff \max t_{n-1}(j) \in L_{n-1} \setminus S_n && \iff -1 < \xi_n(j) < a_n \\ &\text{(III)} \quad j \ge M_{n-1} && \iff \max t_{n-1}(j) \in S_n && \iff \xi_n(j) = a_n \end{aligned}$$

and

$$\max t_n(g_n(j)) \in S_n \iff j \ge M_{n-1}. \qquad (*)$$

By the definition of $t_{n-1}(j)$ and Remark 5.1, each first equivalence in (I), (II), (III) holds. Let $\omega = \max t_{n-1}(j)$, so $\delta_{n-1}(\omega) = \omega$. By Observation 4.3(1),

$$\begin{aligned} \omega \in S_n &\iff \delta_{n-1}(\omega) \in S_n \iff \delta_n(\omega) \in S_n \iff x_n(\omega) = a_n, \\ \omega \in S_{n-1} &\iff \delta_{n-1}(\omega) \in S_{n-1} \iff x_n(\omega) = -1. \end{aligned}$$

Hence each second equivalence also holds. Since $\max t_n(g_n(j)) = \delta_n(\omega)$, $(*)$ holds.

Now, we shall consider the three cases:

$$\begin{aligned} &\text{Case I:} \quad j \le m_{n-1} \\ &\text{Case II:} \quad m_{n-1} < j < M_{n-1} \\ &\text{Case III:} \quad j \ge M_{n-1}. \end{aligned}$$

Case I. Let $j < m_{n-1}$, so $\{\max t_{n-1}(j), \max t_{n-1}(j+1)\} \subset S_{n-1}$. Then $\max t_{n-1}(j) < \max t_{n-1}(j+1)$ with $\{\max t_n(g_n(j)), \max t_n(g_n(j+1))\} \subset L_n$ (by $(*)$). By Observation 4.3(3), $\max t_n(g_n(j)) < \max t_n(g_n(j+1))$. So $g_n(j) < g_n(j+1)$. (See Figure 23.)

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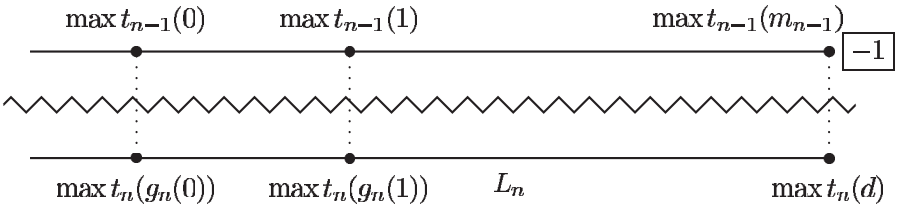


FIGURE 23. From S_{n-1} to L_n.

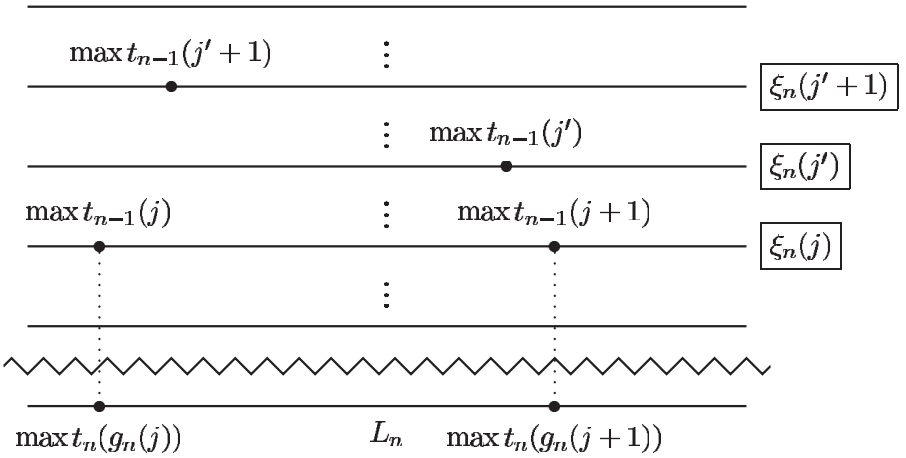


FIGURE 24. From $L_{n-1}\backslash S_n$ to L_n: here $\xi_n(j)=\xi_n(j+1)$, $g_n(j)<g_n(j+1)$ and $\xi_n(j')<\xi_n(j'+1)$.

Case II. Let $m_{n-1}<j<M_{n-1}-1$, so $\{\max t_{n-1}(j), \max t_{n-1}(j+1)\}\subset L_{n-1}\backslash S_n$. Then $\max t_{n-1}(j)<\max t_{n-1}(j+1)$ with $\{\max t_n(g_n(j)), \max t_n(g_n(j+1))\}\subset L_n$ (by $(*)$). By Observation 4.3(2), $\xi_n(j)\le\xi_n(j+1)$. Moreover, if $\xi_n(j)=\xi_n(j+1)$, then by Observation 4.3(3), $\max t_n(g_n(j))<\max t_n(g_n(j+1))$, so $g_n(j)<g_n(j+1)$. Anyway $(\xi_n, g_n)(j)<_{\mathrm{lex}}(\xi_n, g_n)(j+1)$. (See Figure 24.)

Case III. Let $M_{n-1}\le j<d$, so $\{\max t_{n-1}(j), \max t_{n-1}(j+1)\}\subset S_n$. Then $\max t_{n-1}(j)<\max t_{n-1}(j+1)$. By Observation 4.3(3), $\max t_n(g_n(j))<\max t_n(g_n(j+1))$. So $g_n(j)<g_n(j+1)$. By $(*)$,

$$\{\max t_n(g_n(j)) \mid j\ge M_{n-1}\}=\{\max t_n(i)\mid i\le m_n\}.$$

By Observation 4.3(3),

$$\begin{aligned}\max t_n(g_n(M_{n-1}))&=\min\{\max t_n(g_n(j))\mid j\ge M_{n-1}\}\\&=\min\{\max t_n(i)\mid i\le m_n\}=\max t_n(0).\end{aligned}$$

Hence $M_{n-1}=g_n^{-1}(0)$ and

$$\begin{aligned}g_n(d)=m_n&=\#\{\max t_n(i)\mid i\le m_n\}-1\\&=\#\{\max t_n(g_n(j))\mid j\ge M_{n-1}\}-1=d-g_n^{-1}(0).\end{aligned}$$

(See Figure 25.)

Combining these, we obtain the desired result. □

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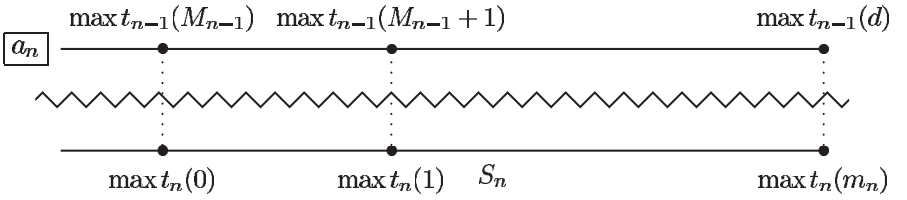


FIGURE 25. From $S_n(\subset L_{n-1})$ to S_n.

Now, the next lemma follows from Proposition 7.1.

LEMMA 7.1. *For each $n \in \mathbb{N}$ and $0 \le j \le d$,*

$$t_{n-1}(j) = \bigcup_{(x,i)\in D(\xi_n,g_n)_j} R^{s_n(x)}(t_n(i)) \quad (disjoint).$$

Proof. Define $\mathcal{S}_n = \{R^{s_n(x)}(t_n(i)) \mid (x, i) \in D(\xi_n, g_n)\}$, so then the correspondence

$$(x, i) \mapsto R^{s_n(x)}(t_n(i)) : D(\xi_n, g_n) \to \mathcal{S}_n$$

is bijective. By the Refining lemma, $\mathcal{S}_n$ is a partition of B_{n-1}. Moreover, if $(x_k, i_k) \in D(\xi_n, g_n)$ $(1 \le k \le 4)$ satisfy

$$(x_1, i_1) <_{\text{lex}} (x_2, i_2) \le_{\text{lex}} (-1, d) <_{\text{lex}} (x_3, i_3) <_{\text{lex}} (x_4, i_4),$$

then

$$\begin{aligned}
&R^{s_n(x_1)}(t_n(i_1)),\ R^{s_n(x_2)}(t_n(i_2)) \subset S_{n-1},\\
&R^{s_n(x_1)}(t_n(i_1)) \text{ lies in the negative direction from } R^{s_n(x_2)}(t_n(i_2)),\\
&R^{s_n(x_3)}(t_n(i_3)),\ R^{s_n(x_4)}(t_n(i_4)) \subset L_{n-1},\\
&R^{s_n(x_3)}(t_n(i_3)) \text{ lies in the negative direction from } R^{s_n(x_4)}(t_n(i_4)).
\end{aligned}$$

Therefore, when $(x, i) \in D(\xi_n, g_n)$, by Proposition 7.1(1), we have

$$(x, i) \in D(\xi_n, g_n)_j \text{ if and only if } R^{s_n(x)}(t_n(i)) \subset t_{n-1}(j).$$

(Compare Figures 26 and 27.) Hence

$$t_{n-1}(j) = \bigcup_{(x,i)\in D(\xi_n,g_n)_j} R^{s_n(x)}(t_n(i)). \qquad \square$$

COROLLARY 7.1. *The hitting time sets of $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ are given by*

$$\begin{aligned}
(H_1)_{i0} &= \{s_1(x) \mid (x, i) \in D(\xi_1, g_1)\},\\
(H_n)_{ij} &= \{s_n(x) \mid (x, i) \in D(\xi_n, g_n)_j\} \quad (n \ge 2).
\end{aligned}$$

By Theorem 5.1 and Corollary 7.1, we have an HPS-adic presentation of (X, T).

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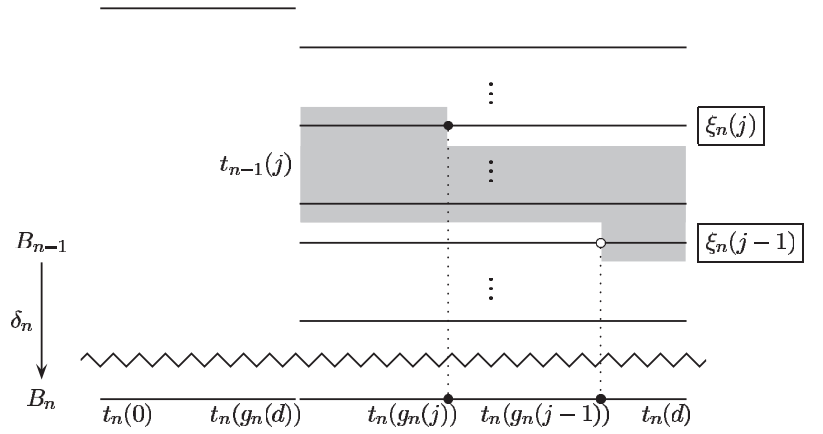


FIGURE 26. $t_{n-1}(j)$ when $\max t_n(g_n(j)) < \max t_n(g_n(j-1))$.

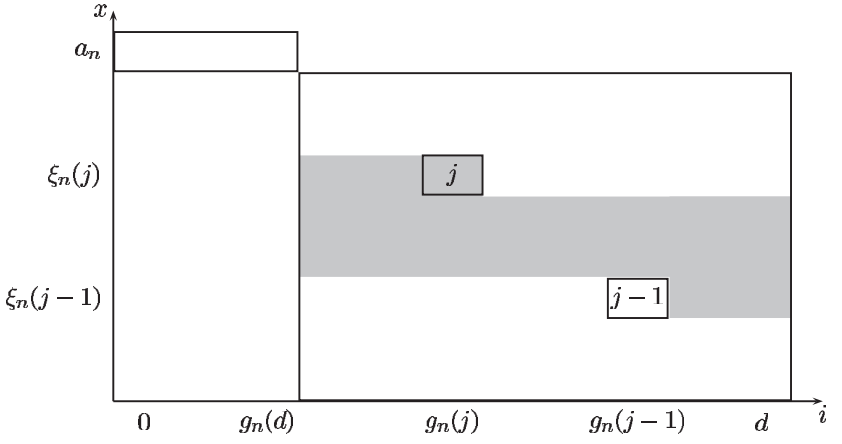


FIGURE 27. $D(\xi_n, g_n)_j$ when $g_n(j) < g_n(j-1)$: $(x, i) \in D(\xi_n, g_n)_j \Leftrightarrow R^{s_n(x)}(t_n(i)) \subset t_{n-1}(j)$.

COROLLARY 7.2. *Define* $B = (V, E, \leq)$ *by*

$$V_0 = \{v_0(0)\}, \quad V_n = \{v_n(0), v_n(1), \ldots, v_n(d)\} \quad (n \in \mathbb{N}),$$
$$E_n = \{(v_{n-1}(j), \rho, v_n(i)) \mid \rho \in (H_n)_{ij}\},$$
$$(v_{n-1}(j), \rho, v_n(i)) \leq (v_{n-1}(j'), \rho', v_n(i')) \quad \text{if } i = i' \text{ and } \rho \leq \rho'.$$

Then the incidence matrices of B *are given by*

$$(A_1)_{i0} = \sum_{j=0}^{d} A(\xi_1, g_1)_{ij}, \quad A_n = A(\xi_n, g_n) \quad (n \geq 2)$$

and the Bratteli–Vershik system (X_B, θ_B) *is topologically conjugate to* (X, T).

Notes. Let $(x, i) \in D(\xi_n, g_n)$. Then $R^{s_n(x)}(t_n(i)) \subset B_{n-1}$ and

$$(x, i) \in D(\xi_n, g_n)_j \Longleftrightarrow (v_{n-1}(j), s_n(x), v_n(i)) \in E_n.$$

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8. *Matrix $A(\xi, g)$*

8.1. *Basic formula of $A(\xi, g)$.* In this subsection, we will study the matrix $A(\xi, g)$ to calculate the dimension groups.

For each $0 \le j, k \le d$, define the column vector $\mathbf{1}(k, j) = [\mathbf{1}(k, j)_i]_{0 \le i \le d}$ by

$$\mathbf{1}(k, j)_i = \begin{cases} 1 & \text{if } i \le j \le k \text{ or } k < i \le j, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we denote $\mathbf{1}(d, d)$ by $\mathbf{1}$. Denote by I_n the $n \times n$ identity matrix and by $\mathbb{O}$ the zero matrix.

Remark 8.1. Notice that, for each k,

$$\det[\mathbf{1}(k, j)]_{0 \le j \le d} = \det I_{d+1} = 1$$

because for usual base $\{\mathbf{e}_j \mid 0 \le j \le d\}$ (i.e. $(\mathbf{e}_j)_i = \delta_{ij}$),

$$\mathbf{e}_j = \begin{cases} \mathbf{1}(k, j) & \text{if } j = 0 \text{ or } k+1, \\ \mathbf{1}(k, j) - \mathbf{1}(k, j-1) & \text{otherwise.} \end{cases}$$

So, in particular, for each k, the set $\{\mathbf{1}(k, j) \mid 0 \le j \le d\}$ is a base of $\mathbb{Z}^{d+1}$.

FORMULA 8.1. *For each section (ξ, g), $A(\xi, g)$ has the following equivalent relations.*

(1) $A(\xi, g)\mathbf{1}(d, j) = (\xi(j) + 1)\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j))$ $(0 \le j \le d)$.

(2) $A(\xi, g)\mathbf{1}(g^{-1}(d), j) = \xi(j)_+\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j))$ $(0 \le j \le d)$.

So $A(\xi, g)$ is uniquely determined by these relations (because $\{\mathbf{1}(k, j)\}_{j=0}^{d}$ is a base).

Proof. (1) We shall consider three cases:

Case 1: $j \le g^{-1}(d)$

Case 2: $g^{-1}(d) < j < g^{-1}(0)$

Case 3: $j \ge g^{-1}(0)$.

Notice that

$$(A(\xi, g)\mathbf{1}(d, j))_i = \sum_{k=0}^{j} A(\xi, g)_{ik} = \#\left\{x \mid (x, i) \in \bigcup_{k=0}^{j} D(\xi, g)_k\right\}.$$

Case 1. If $j \le g^{-1}(d)$, then $\xi(j) = -1$ and

$$(A(\xi, g)\mathbf{1}(d, j))_i = \begin{cases} 1 & \text{if } g(d) < i \le g(j) \\ 0 & \text{otherwise} \end{cases} \qquad \text{(see Figure 28).}$$

Therefore,

$$A(\xi, g)\mathbf{1}(d, j) = \mathbf{1}(g(d), g(j)) = (\xi(j) + 1)\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j)).$$

Case 2. If $g^{-1}(d) < j < g^{-1}(0)$, then $\xi(j) \ge 0$ and

$$(A(\xi, g)\mathbf{1}(d, j))_i = \begin{cases} 0 & \text{if } i \le g(d) \\ \xi(j) + 2 & \text{if } g(d) < i \le g(j) \\ \xi(j) + 1 & \text{if } i > g(j) \end{cases} \qquad \text{(see Figure 29).}$$

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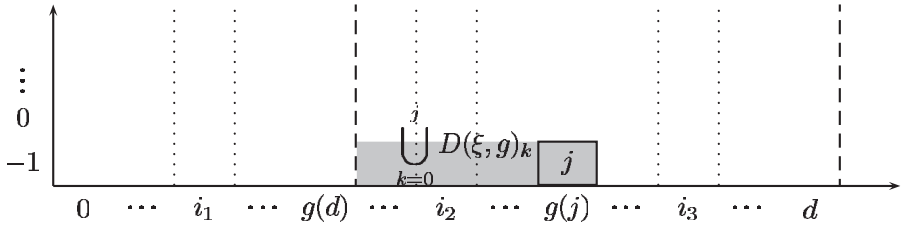


FIGURE 28. Case 1.

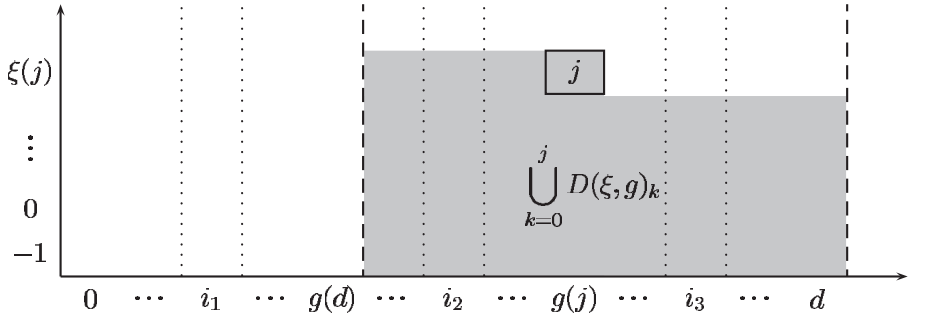


FIGURE 29. Case 2.

Therefore,

$$A(\xi, g)\mathbf{1}(d, j) = (\xi(j) + 1)\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j)).$$

Case 3. If $j \geq g^{-1}(0)$, then $\xi(j) = m$ (write $m = \max \xi$) and

$$(A(\xi, g)\mathbf{1}(d, j))_i = \begin{cases} 1 & \text{if } i \leq g(j) \\ 0 & \text{if } g(j) < i \leq g(d) \\ m+1 & \text{if } i > g(d) \end{cases} \quad \text{(see Figure 30).}$$

Therefore,

$$A(\xi, g)\mathbf{1}(d, j) = (m + 1)\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j)).$$

(2) By the above, $\xi(j) = -1$ if and only if $j \leq g^{-1}(d)$. By the definition of $\mathbf{1}(k, j)$,

$$\mathbf{1}(g^{-1}(d), j) = \begin{cases} \mathbf{1}(d, j) & \text{if } j \leq g^{-1}(d), \\ \mathbf{1}(d, j) - \mathbf{1}(d, g^{-1}(d)) & \text{if } j > g^{-1}(d). \end{cases}$$

One can see that (2) is equivalent to (1). □

COROLLARY 8.1. $\det A(\xi, g) = \operatorname{sgn} g$.

Proof. By Formula 8.1(1),

$$\det A(\xi, g) = \det[(\xi(j) + 1)\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j))]_{0 \leq j \leq d}.$$

Since $\xi(g^{-1}(d)) + 1 = 0$,

$$\det[(\xi(j) + 1)\mathbf{1}(g(d), d) + \mathbf{1}(g(d), g(j))]_{0 \leq j \leq d} = \det[\mathbf{1}(g(d), g(j))]_{0 \leq j \leq d} = \operatorname{sgn} g. \quad \square$$

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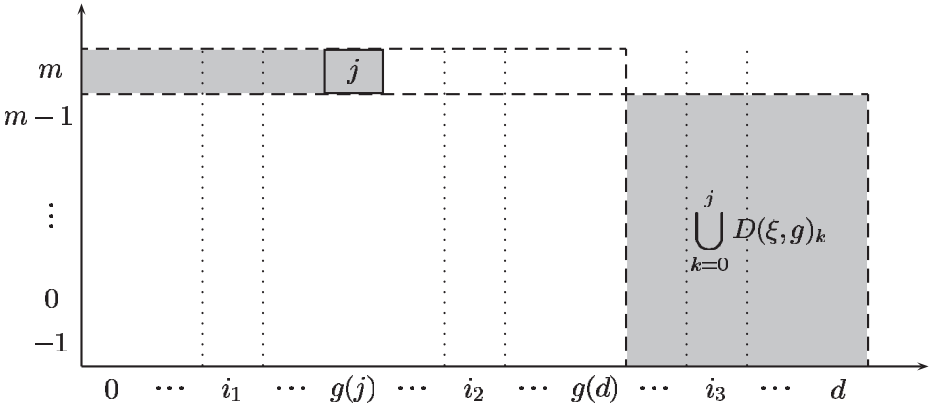


FIGURE 30. Case 3.

8.2. *Determination of* (ξ_n, g_n) *and* $A(\xi_n, g_n)$ *from arithmetic.* The argument in this subsection is unnecessary to compute dimension groups, but is useful from a practical viewpoint.

8.2.1. *Recursive computation of* $A(\xi, g)$ *from* (ξ, g). We use the row vector presentation for the $(d+1) \times (d+1)$ matrix M, i.e.

$$M = \begin{bmatrix} M_{0\bullet} \\ M_{1\bullet} \\ \vdots \\ M_{d\bullet} \end{bmatrix}.$$

We can determine the matrix $A(\xi, g)$ by the following recursive procedure.

PROPOSITION 8.1. *For each section* (ξ, g) *and* $g(d) < i \le d$, *let*

$$\nabla_\xi = \big(0, \quad \xi(1) - \xi(0), \quad \xi(2) - \xi(1), \quad \ldots, \quad \xi(d) - \xi(d-1)\big),$$

and

$$f_i = f(g,i) = \big(\overset{0}{0}, \quad \overset{1}{0}, \quad \overset{\cdots}{\ldots}, \quad \overset{g^{-1}(i)-1}{0}, \quad \overset{g^{-1}(i)}{1}, \quad \overset{g^{-1}(i)+1}{-1}, \quad \overset{g^{-1}(i)+2}{0}, \quad \overset{\cdots}{\ldots}, \quad \overset{d}{0}\big).$$

Then the matrix $A = A(\xi, g)$ *is given by the following.*

(a)

$$\begin{bmatrix} A_{0\bullet} \\ A_{1\bullet} \\ \vdots \\ A_{g(d)\bullet} \end{bmatrix} = \begin{bmatrix} \mathbb{O} & I_{g(d)+1} \end{bmatrix} \quad ((g(d)+1) \times (d+1) \textit{ matrix}).$$

(b)

$$A_{d\bullet} = \nabla_\xi + f_d,$$
$$A_{i\bullet} = A_{i+1\bullet} + f_i \quad (i = d-1, d-2, \ldots, g(d)+1).$$

(For its proof, see Appendix A.)

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Example 8.1. When (ξ, g) is given in Example 6.1, we will compute $A(\xi, g)$ by the procedure in Proposition 8.1:

$$\begin{aligned}
\nabla_\xi &= (0, \quad 0, \quad 2, \quad 0, \quad 1, \quad 1, \quad 0, \quad 0), \\
f_7 &= (0, \quad 1, \quad -1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0), \\
f_6 &= (0, \quad 0, \quad 0, \quad 1, \quad -1, \quad 0, \quad 0, \quad 0), \\
f_5 &= (0, \quad 0, \quad 0, \quad 0, \quad 1, \quad -1, \quad 0, \quad 0), \\
f_4 &= (1, \quad -1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0), \\
f_3 &= (0, \quad 0, \quad 1, \quad -1, \quad 0, \quad 0, \quad 0, \quad 0).
\end{aligned}$$

We get

$$A(\xi, g) = \left[\begin{array}{ccccc|ccc} \multicolumn{5}{c|}{\mathbb{O}} & \multicolumn{3}{c}{I_3} \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

8.2.2. *Determination of* (ξ_n, g_n) *from arithmetics.* By Corollary 7.1, in order to describe the hitting time sets, it is sufficient to determine the (ξ_n, g_n). Here we shall introduce a procedure determining (ξ_n, g_n) from signed expansions of Λ_1.

For each $n \in \mathbb{Z}_+$, define

$$x_{(n,\infty)}(\omega) = x_{n+1}(\omega)x_{n+2}(\omega)\cdots$$

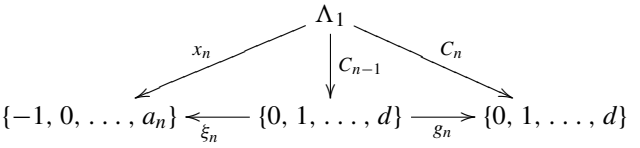
and the lexicographic order $x_{(n,\infty)}(\omega) <_{\mathrm{lex}} x_{(n,\infty)}(\omega')$ if $x_{k_0}(\omega) < x_{k_0}(\omega')$ where $k_0 = \min\{k > n \mid x_k(\omega) \neq x_k(\omega')\}$.

Now we can determine (ξ_n, g_n) from $x_*(\Lambda_1)$ by the following formula. For each $\omega_0 \in \Lambda_1$ and $n \in \mathbb{Z}_+$, let

$$C_n(\omega_0) = \#\{\omega \in \Lambda_1 \mid x_{(n,\infty)}(\omega) <_{\mathrm{lex}} x_{(n,\infty)}(\omega_0)\}.$$

(So $C_n : \Lambda_1 \to \{0, 1, \ldots, d\}$ is injective by Remark 4.2 and Proposition 4.1(4).)

PROPOSITION 8.2. $(\xi_n, g_n) \circ C_{n-1}(\omega) = (x_n(\omega), C_n(\omega))$.



(For its proof, see Appendix A.)

The nth incidence matrix of B in Corollary 7.2 is $A_n = A(\xi_n, g_n)$ $(n \geq 2)$. We will obtain (ξ_n, g_n) and A_n for some examples. For a map q defined on $\{0, 1, \ldots, d\}$, we use the notation

$$q = (q(0), q(1), \ldots, q(d)).$$

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Example 8.2. If $d = 1$ then $\Lambda_1 = \{\alpha, 1\}$ and

$$(\xi_n, g_n) = ((-1, 1), (a_n, 0)), \quad A_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}.$$

Example 8.3. Let $d = 2$, $\Lambda_1 = \{\alpha, \omega, 1\}$ and $x_n := x_n(\omega)$. So

$$(\xi_n, g_n) = \begin{cases} ((-1, 1), (-1, 2), (a_n, 0)) & \text{if } x_n = -1, \\ ((-1, 2), (x_n, 1), (a_n, 0)) & \text{if } 0 \le x_n < a_n, \\ ((-1, 2), (a_n, 0), (a_n, 1)) & \text{if } x_n = a_n, \end{cases}$$

$$A_n = \begin{cases} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & a_n \\ 0 & 1 & a_n \end{bmatrix} & \text{if } x_n = -1, \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & x_n + 1 & a_n - x_n - 1 \\ 1 & x_n & a_n - x_n \end{bmatrix} & \text{if } 0 \le x_n < a_n, \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a_n & 0 \end{bmatrix} & \text{if } x_n = a_n. \end{cases}$$

9. *Dimension group for the Denjoy system*

We shall calculate the dimension group for the Denjoy system (X, T). Consider the Bratteli diagram B and its incidence matrices in Corollary 7.2. Let $A_n = A(\xi_n, g_n)$ and $M_n = A_n A_{n-1} \cdots A_1$ for each $n \in \mathbb{N}$. Then $\bar{A}_1 = A_1 \mathbf{1}$ is the first incidence matrix of B, and A_n the nth one ($n \ge 2$).

Number

$$\Lambda_1 = \{\omega_0, \omega_1, \ldots, \omega_d\} \quad \text{where } 0 < \omega_0 < \omega_1 < \cdots < \omega_d = 1.$$

Define a row vector μ by

$$\mu = \begin{pmatrix} \omega_0, & \omega_1 - \omega_0, & \ldots, & \omega_d - \omega_{d-1} \end{pmatrix}.$$

By the definition of $t_0(i)$ (see §5; recall that $B_0 = (0, 1]$, $\delta_0 = \mathrm{id}$), $\max t_0(i) = \omega_i$. So $\xi_{n+1}(g_n g_{n-1} \cdots g_1(j)) = x_{n+1}(\omega_j)$ (see §7; recall that $x_{n+1}\delta_n = x_{n+1}$). Note that

$$\mu = \begin{pmatrix} |t_0(0)|, & |t_0(1)|, & \ldots, & |t_0(d)| \end{pmatrix}.$$

LEMMA 9.1. *For each $n \in \mathbb{N}$,*

$$\mu M_n^{-1} = \begin{pmatrix} |t_n(0)|, & |t_n(1)|, & \ldots, & |t_n(d)| \end{pmatrix}.$$

Proof. By Lemma 7.1, we have the equation

$$\begin{pmatrix} |t_n(0)|, & |t_n(1)|, & \ldots, & |t_n(d)| \end{pmatrix} A_n = \begin{pmatrix} |t_{n-1}(0)|, & |t_{n-1}(1)|, & \ldots, & |t_{n-1}(d)| \end{pmatrix},$$

from which Lemma 9.1 follows. □

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FORMULA 9.1. *For each $n \in \mathbb{N}$ and $0 \leq j \leq d$,*

$$M_n\mathbf{1}(d,\, j) = u_n(\omega_j)\mathbf{1}(g_n(d),\, d) + u_{n-1}(\omega_j)\mathbf{1}(g_n(d),\, g_n(d)) + \mathbf{1}(g_n(d),\, g_n g_{n-1} \cdots g_1(j)).$$

In particular,

$$M_n\mathbf{1} = q_n\mathbf{1}(g_n(d),\, d) + q_{n-1}\mathbf{1}(g_n(d),\, g_n(d)).$$

Proof. First, we use induction on n to show the first assertion. By Formula 8.1(1),

$$\begin{aligned} M_1\mathbf{1}(d,\, j) &= (\xi_1(j) + 1)\mathbf{1}(g_1(d),\, d) + \mathbf{1}(g_1(d),\, g_1(j)) \\ &= u_1(\omega_j)\mathbf{1}(g_1(d),\, d) + u_0(\omega_j)\mathbf{1}(g_n(d),\, g_n(d)) + \mathbf{1}(g_1(d),\, g_1(j)). \end{aligned}$$

Suppose the claim holds for n:

$$\begin{aligned} M_{n+1}\mathbf{1}(d,\, j) &= u_n(\omega_j)A_{n+1}\mathbf{1}(g_n(d),\, d) + u_{n-1}(\omega_j)A_{n+1}\mathbf{1}(g_n(d),\, g_n(d)) \\ &\quad + A_{n+1}\mathbf{1}(g_n(d),\, g_n g_{n-1} \cdots g_1(j)) && \text{(by the assumption)} \\ &= u_n(\omega_j)A_{n+1}\mathbf{1}(g_{n+1}^{-1}(d),\, d) + u_{n-1}(\omega_j)A_{n+1}\mathbf{1}(g_{n+1}^{-1}(d),\, g_{n+1}^{-1}(d)) \\ &\quad + A_{n+1}\mathbf{1}(g_{n+1}^{-1}(d),\, g_n g_{n-1} \cdots g_1(j)) && \text{(by Proposition 7.1(2))} \\ &= u_n(\omega_j)(\xi_{n+1}(d)_+\mathbf{1}(g_{n+1}(d),\, d) + \mathbf{1}(g_{n+1}(d),\, g_{n+1}(d))) \\ &\quad + u_{n-1}(\omega_j)(\xi_{n+1}(g_{n+1}^{-1}(d))_+\mathbf{1}(g_{n+1}(d),\, d) + \mathbf{1}(g_{n+1}(d),\, d)) \\ &\quad + \xi_{n+1}(g_n g_{n-1} \cdots g_1(j))_+\mathbf{1}(g_{n+1}(d),\, d) \\ &\quad + \mathbf{1}(g_{n+1}(d),\, g_{n+1}g_n \cdots g_1(j)) && \text{(by Formula 8.1(2))} \\ &= u_n(\omega_j)(a_{n+1}\mathbf{1}(g_{n+1}(d),\, d) + \mathbf{1}(g_{n+1}(d),\, g_{n+1}(d))) \\ &\quad + u_{n-1}(\omega_j)\mathbf{1}(g_{n+1}(d),\, d) + x_{n+1}^+(\omega_j)\mathbf{1}(g_{n+1}(d),\, d) \\ &\quad + \mathbf{1}(g_{n+1}(d),\, g_{n+1}g_n \cdots g_1(j)) \\ &\quad \text{(because } \xi_{n+1}(d) = a_{n+1},\, \xi_n(g_n^{-1}(d)) = -1,\, \xi_{n+1}(g_n \cdots g_1(j)) = x_{n+1}(\omega_j)\text{)} \\ &= u_{n+1}(\omega_j)\mathbf{1}(g_{n+1}(d),\, d) + u_n(\omega_j)\mathbf{1}(g_{n+1}(d),\, g_{n+1}(d)) \\ &\quad + \mathbf{1}(g_{n+1}(d),\, g_{n+1}g_n \cdots g_1(j)). && \text{(by Proposition 4.1(2))} \end{aligned}$$

This completes the induction.

Next, notice, by Proposition 7.1(2), that

$$g_n g_{n-1} \cdots g_1(d) = \begin{cases} g_n(d) & \text{if } n \text{ is odd,} \\ d & \text{if } n \text{ is even.} \end{cases}$$

By Proposition 4.1(2),

$$\begin{aligned} M_n\mathbf{1} &= u_n(1)\mathbf{1}(g_n(d),\, d) + u_{n-1}(1)\mathbf{1}(g_n(d),\, g_n(d)) + \mathbf{1}(g_n(d),\, g_n g_{n-1} \cdots g_1(d)) \\ &= q_n\mathbf{1}(g_n(d),\, d) + q_{n-1}\mathbf{1}(g_n(d),\, g_n(d)). \end{aligned}$$

□

We use the row (respectively column) vector presentation for the $(d+1) \times (d+1)$ matrix M, i.e.

$$M = \begin{bmatrix} M_{0\bullet} \\ M_{1\bullet} \\ \vdots \\ M_{d\bullet} \end{bmatrix} \quad \left(\text{respectively } M = \begin{bmatrix} M_{\bullet 0}, & M_{\bullet 1}, & \ldots, & M_{\bullet d} \end{bmatrix}\right).$$

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LEMMA 9.2. *For each* $0 \le i \le d$,

$$\lim_{n\to\infty} \frac{1}{\sum_{k=0}^{d}(M_n)_{ik}}(M_n)_{i\bullet} = \mu.$$

Proof. By Remark 8.1,

$$(M_n)_{\bullet j} = M_n \mathbf{e}_j = \begin{cases} M_n\mathbf{1}(d,\, j) & \text{if } j = 0, \\ M_n\mathbf{1}(d,\, j) - M_n\mathbf{1}(d,\, j-1) & \text{otherwise,} \end{cases}$$

and

$$\sum_{j=0}^{d}(M_n)_{\bullet j} = M_n\mathbf{1}.$$

By Formula 9.1, if $j = 0$ then

$$\frac{u_{n(i)}(\omega_0)}{q_{n(i)}} \le \frac{(M_n)_{ij}}{\sum_{k=0}^{d}(M_n)_{ik}} \le \frac{u_{n(i)}(\omega_0)+1}{q_{n(i)}},$$

and if $j \ne 0$ then

$$\frac{u_{n(i)}(\omega_j) - u_{n(i)}(\omega_{j-1}) - 1}{q_{n(i)}} \le \frac{(M_n)_{ij}}{\sum_{k=0}^{d}(M_n)_{ik}} \le \frac{u_{n(i)}(\omega_j) - u_{n(i)}(\omega_{j-1}) + 1}{q_{n(i)}}$$

where

$$n(i) = \begin{cases} n-1 & \text{if } i \le g_n(d), \\ n & \text{otherwise.} \end{cases}$$

Therefore, the claim follows from Proposition 4.1(3). □

Finally, by using the previous HPS-construction, we shall calculate the dimension group for the Denjoy system (X, T). Define

$$P_\mu = \{v \in \mathbb{Z}^{d+1} \mid \mu v > 0\} \cup \{0\}.$$

THEOREM 9.1. $(K_0(B), K_0(B)_+, [1])$ *is order isomorphic to* $(\mathbb{Z}^{d+1}, P_\mu, \mathbf{1})$.

Proof. Set $G_0 = (\mathbb{Z}, \mathbb{Z}_+)$, $G_n = (\mathbb{Z}^{d+1}, (\mathbb{Z}_+)^{d+1})$ for each $n \in \mathbb{N}$ and $G_\mu = (\mathbb{Z}^{d+1}, P_\mu)$. Let $\mathbf{1} : \mathbb{Z} \to \mathbb{Z}^{d+1} : v \mapsto v\mathbf{1}$. Since $\bar{A}_1 = A_1\mathbf{1}$, the following commutative diagram holds:

$$\begin{array}{ccccccc} G_0 & \xrightarrow{\bar{A}_1} & G_1 & \xrightarrow{A_2} & G_2 & \xrightarrow{A_3} & \cdots \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle M_1^{-1}} & & \downarrow{\scriptstyle M_2^{-1}} & & \\ G_\mu & \xrightarrow{\mathrm{id}} & G_\mu & \xrightarrow{\mathrm{id}} & G_\mu & \xrightarrow{\mathrm{id}} & \cdots \end{array}$$

It suffices to show that

$$P_\mu = \mathbf{1}(\mathbb{Z}_+) \cup \bigcup_{n\in\mathbb{N}} M_n^{-1}((\mathbb{Z}_+)^{d+1}).$$

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($\supset$) Clearly, $P_\mu \supset \mathbf{1}(\mathbb{Z}_+)$. Let $v \in (\mathbb{Z}_+)^{d+1}\backslash\{0\}$. Notice that $M_n^{-1}v \in P_\mu\backslash\{0\}$ if and only if $\mu M_n^{-1}v > 0$. By Lemma 9.1,

$$\mu M_n^{-1}v = |t_n(0)|v_0 + |t_n(1)|v_1 + \cdots + |t_n(d)|v_d > 0.$$

($\subset$) Let $v \in P_\mu\backslash\{0\}$. It suffices to show that there exists $n \in \mathbb{N}$ such that $v \in M_n^{-1}((\mathbb{Z}_+)^{d+1})$, that is, $M_n v \in (\mathbb{Z}_+)^{d+1}$. By Lemma 9.2, for each $0 \le i \le d$,

$$\lim_{n\to\infty} \frac{(M_n v)_i}{\sum_{j=0}^{d}(M_n)_{ij}} = \mu\, v > 0.$$

Hence for sufficiently large n, $M_n v \in (\mathbb{Z}_+)^{d+1}$. □

Acknowledgement. The authors appreciate the valuable suggestions of the referee.

A. *Appendix*

To prove Proposition 4.1, we will prepare some arithmetic lemmas. For each $\omega \in (0, 1]$ and $n \in \mathbb{N}$, let

$$\mathrm{r}_n(\omega) = \begin{cases} \delta_n(\omega) - \inf S_n & \text{if } \delta_n(\omega) \in S_n, \\ \delta_n(\omega) - \inf L_n & \text{if } \delta_n(\omega) \in L_n. \end{cases}$$

LEMMA A.1.

$$\mathrm{r}_n(\omega) = \omega - \sum_{k=1}^{n} x_k^+(\omega)|q_{k-1}\alpha - p_{k-1}|.$$

Proof. We will use induction on n.

If $\delta_1(\omega) \in L_1$ then $\delta_1(\omega) = \omega - x_1^+(\omega)\alpha$. By the definitions of δ_1 and x_1^+,

$$\mathrm{r}_1(\omega) = \delta_1(\omega) - \inf L_1 = \omega - x_1^+(\omega)|q_0\alpha - p_0|.$$

If $\delta_1(\omega) \in S_1$ then $\delta_1(\omega) = \omega$ and $x_1^+(\omega) = a_1 + 1$. By the definition of δ_1,

$$\mathrm{r}_1(\omega) = \delta_1(\omega) - \inf S_1 = \omega - q_1\alpha = \omega - x_1^+(\omega)|q_0\alpha - p_0|.$$

Let $n \ge 2$. Suppose the claim holds for $n-1$ (see Figure 17).

If $\delta_{n-1}(\omega) \in S_{n-1}$ then $x_{n-1}(\omega) = a_{n-1}$ and $x_n(\omega) = -1$, and hence we have $\mathrm{r}_n(\omega) = \mathrm{r}_{n-1}(\omega)$ because $\delta_{n-1}(\omega) \in S_{n-1} = R^{s_n(x_n(\omega))}(L_n)$.

If $\delta_{n-1}(\omega) \in L_{n-1}\backslash S_n$ then $\mathrm{r}_n(\omega) = \mathrm{r}_{n-1}(\omega) - x_n^+(\omega)|L_n|$.

If $\delta_{n-1}(\omega) \in S_n$ then $x_n^+(\omega) = a_n$ and $\mathrm{r}_n(\omega) = \mathrm{r}_{n-1}(\omega) - a_n|L_n|$.

Any way, the claim holds for n. □

LEMMA A.2. *For each $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} a_{2k-1}q_{2k-2} = q_{2n-1} - 1, \quad \sum_{k=1}^{n} a_{2k}q_{2k-1} = q_{2n} - 1.$$

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Proof. By recursive relations,

$$\sum_{k=1}^{n} a_{2k-1}q_{2k-2} = q_{-1} + \sum_{k=1}^{n} a_{2k-1}q_{2k-2} - 1$$
$$= q_1 + \sum_{k=2}^{n} a_{2k-1}q_{2k-2} - 1 = \cdots = q_{2n-1} - 1.$$

Similarly,

$$\sum_{k=1}^{n} a_{2k}q_{2k-1} = q_0 + \sum_{k=1}^{n} a_{2k}q_{2k-1} - 1 = q_{2n} - 1. \qquad \square$$

For each $n \in \mathbb{Z}_+$, define

$$x_{(n,\infty)}(\omega) = x_{n+1}(\omega)x_{n+2}(\omega)\cdots$$

and the lexicographic order $x_{(n,\infty)}(\omega) <_{\text{lex}} x_{(n,\infty)}(\omega')$ if $x_{k_0}(\omega) < x_{k_0}(\omega')$ where $k_0 = \min\{k > n \mid x_k(\omega) \neq x_k(\omega')\}$.

LEMMA A.3.

(1) $x_{(n,\infty)}(\omega) = x_{(n,\infty)}(\omega')$ *if and only if* $\delta_n(\omega) = \delta_n(\omega')$.

(2) *Suppose* $\delta_n(\omega) < \delta_n(\omega')$. *If* $\{\delta_n(\omega), \delta_n(\omega')\} \subset L_n$ *or* $\{\delta_n(\omega), \delta_n(\omega')\} \subset S_n$, *then* $x_{(n,\infty)}(\omega) <_{\text{lex}} x_{(n,\infty)}(\omega')$.

Proof. (1) Clearly $\delta_n(\omega) = \delta_n(\omega')$ implies $x_{(n,\infty)}(\omega) = x_{(n,\infty)}(\omega')$ by the definition of signed digital maps. Suppose $x_{(n,\infty)}(\omega) = x_{(n,\infty)}(\omega')$. By Observation 4.3(3),

$$|\delta_n(\omega') - \delta_n(\omega)| = |\delta_{n+1}(\omega') - \delta_{n+1}(\omega)| = \cdots$$

and

$$|\delta_n(\omega') - \delta_n(\omega)| < |L_n|, \quad |\delta_{n+1}(\omega') - \delta_{n+1}(\omega)| < |L_{n+1}|, \ldots.$$

Since $|L_n| \to 0$, we have $\delta_n(\omega') = \delta_n(\omega)$.

(2) By (1), there exists $N > n$ such that $x_k(\omega) = x_k(\omega')$ $(n < k < N)$ and $x_N(\omega) \neq x_N(\omega')$. By Observation 4.3(3) and (1),

$$\delta_{N-1}(\omega') - \delta_{N-1}(\omega) = \delta_n(\omega') - \delta_n(\omega) > 0 \quad \text{and} \quad \{\delta_{N-1}(\omega'), \delta_{N-1}(\omega)\} \subset L_{N-1}.$$

By Observation 4.3(2), $x_N(\omega) < x_N(\omega')$. Therefore, $x_{(n,\infty)}(\omega) <_{\text{lex}} x_{(n,\infty)}(\omega')$. $\square$

Proof of Proposition 4.1. (1) Since $\mathrm{r}_n(\omega) \le |L_n|$ and $|L_n| \to 0$, by Lemma A.1,

$$\omega = \sum_{n=1}^{\infty} x_n^+(\omega)|q_{n-1}\alpha - p_{n-1}|.$$

(2) By recursive relations about p_n and q_n,

$$u_n(\omega) = x_n^+(\omega)$$
$$+ \sum_{k=1}^{n-1} x_k^+(\omega)(-1)^{k-1}(q_{k-1}(a_n p_{n-1} + p_{n-2}) - (a_n q_{n-1} + q_{n-2})p_{k-1})$$
$$= x_n^+(\omega) + a_n u_{n-1}(\omega) + u_{n-2}(\omega). \qquad (*)$$

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By Remark 4.2,

$$x_n^+(1) = \begin{cases} a_1 + 1 & \text{if } n = 1, \\ a_n & \text{if } n \neq 1\text{: odd}, \\ 0 & \text{otherwise.} \end{cases}$$

By using $(*)$, one can show that $u_n(1) = q_n - \epsilon_n$.

(3)

$$\begin{aligned} \frac{u_n(\omega)}{q_n} &= \sum_{k=1}^{n} x_k^+(\omega)(-1)^{k-1}\left(q_{k-1}\,\frac{p_n}{q_n} - p_{k-1}\right) \\ &= \sum_{k=1}^{n} x_k^+(\omega)|q_{k-1}\alpha - p_{k-1}| - \left(\alpha - \frac{p_n}{q_n}\right)\sum_{k=1}^{n} x_k^+(\omega)(-1)^{k-1}q_{k-1}. \end{aligned}$$

Here we have $|\sum_{k=1}^{n} x_k^+(\omega)(-1)^{k-1}q_{k-1}| \leq q_n$, because by Lemma A.2,

$$\begin{aligned} 1 - q_{2\lfloor n/2\rfloor} = -\sum_{k=1}^{\lfloor n/2\rfloor} a_{2k}q_{2k-1} &\leq \sum_{k=1}^{n} x_k^+(\omega)(-1)^{k-1}q_{k-1} \\ &\leq \sum_{k=1}^{\lfloor (n+1)/2\rfloor} a_{2k-1}q_{2k-2} + 1 = q_{2\lfloor (n+1)/2\rfloor - 1}. \end{aligned}$$

Therefore,

$$\left|\frac{u_n(\omega)}{q_n} - \sum_{k=1}^{n} x_k^+(\omega)|q_{k-1}\alpha - p_{k-1}|\right| \leq |q_n\alpha - p_n|.$$

By (1), $u_n(\omega)/q_n \to \omega$.

(4) Notice that $\delta_n(\omega) \in \mathcal{O}_\omega$. So if $\mathcal{O}_\omega \cap \mathcal{O}_{\omega'} = \emptyset$, then $\delta_n(\omega) \neq \delta_n(\omega')$. By Lemma A.3(1), $x_n(\omega) \neq x_n(\omega')$ for infinitely many n. □

Recall the conditions in Remark 6.1:

(1) $g \neq \mathrm{id}$;
(2) $g(j-1) < g(j)$ $(0 < j < g^{-1}(d))$;
(3) $g(j) = j - g^{-1}(0)$ $(g^{-1}(0) < j \leq d)$.

Proof of Remark 6.1. First, we will show that if g satisfies the above conditions, then g is partially mixing. Clearly $g(d) = d - g^{-1}(0)$ follows (3). By (1) and (3), $g^{-1}(0) > g^{-1}(d) \geq 0$. Let $\xi : \{0, 1, \ldots, d\} \to \mathbb{Z}$ be a map satisfying

$$\begin{aligned} &\xi(0) = \xi(1) = \cdots = \xi(g^{-1}(d)) = -1, \\ &0 \leq \xi(g^{-1}(d)+1) < \xi(g^{-1}(d)+2) < \cdots < \xi(g^{-1}(0)-1), \\ &\xi(g^{-1}(0)-1) < \xi(g^{-1}(0)) = \xi(g^{-1}(0)+1) = \cdots = \xi(d). \end{aligned}$$

By (2) and (3), $(\xi, g)(j) <_{\mathrm{lex}} (\xi, g)(j+1)$. Therefore, (ξ, g) is a section.

Conversely, we will show that g satisfies the above when (ξ, g) is a section. Since $(\xi, g)(j) <_{\mathrm{lex}} (\xi, g)(j+1)$ and $\xi(g^{-1}(d)) = -1 = \min \xi$, (2) holds. On the other hand, since $(\xi, g)(j) <_{\mathrm{lex}} (\xi, g)(j+1)$ and $\xi(g^{-1}(0)) = \max \xi$,

$$0 = g(g^{-1}(0)) < g(g^{-1}(0)+1) < g(g^{-1}(0)+2) < \cdots < g(d).$$

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Combining this and $g(d) = d - g^{-1}(0)$, we get (3). Moreover, since $(\xi, g)(j) <_{\text{lex}} (\xi, g)(j+1)$ and $\xi(g^{-1}(d)) < \xi(g^{-1}(0))$, $g^{-1}(d) < g^{-1}(0)$, therefore (1) holds. □

Proof of Remark 6.2. First, let g be partially mixing. By Remark 6.1(1) and (3), we see that $g^{-1}(0) \neq 0$, and hence

$$g^{-1}(d) < g^{-1}(0).$$

Let $g^{-1}(d) < j < g^{-1}(0)$. Then, by Remark 6.1(3),

$$g(j) \in \{g(d)+1, g(d)+2, \ldots, d-1\} = \{d - g^{-1}(0) + 1, d - g^{-1}(0) + 2, \ldots, d-1\}.$$

Therefore, the restriction of g on $\{j \mid g^{-1}(d) < j < g^{-1}(0)\}$ is a injection into $\{d - g^{-1}(0) + 1, d - g^{-1}(0) + 2, \ldots, d-1\}$.

Next, if $0 \leq l < m \leq d$ and

$$f : \{j \mid l < j < m\} \to \{d - m + 1, d - m + 2, \ldots, d-1\}$$

is an injective map, then there exists a unique partially mixing g such that $g^{-1}(d) = l$, $g^{-1}(0) = m$, and g is an extension of f to $\{0, \ldots, d\}$. Indeed, such a g is uniquely determined by the following conditions:

(1) $g(l) = d$, $g(m) = 0$, and $g(j) = f(j)$ $(l < j < m)$ (by the assumption);
(2) $g(j) = j - m$ $(j > m)$ (by Remark 6.1(3));
(3) $g : \{0, 1, \ldots, l-1\} \to \{d - m + 1, d - m + 2, \ldots, d-1\} \setminus \operatorname{Im} f$ is increasing (by Remark 6.1(2)).

Therefore,

$$\begin{aligned}
&\#\{g \mid g \text{ is partially mixing on } \{0, 1, \ldots, d\}\} \\
&\quad = \sum_{l=0}^{d-1} \sum_{m=l+1}^{d} \#\{f \mid f : \{j \mid l < j < m\} \\
&\qquad \to \{d - m + 1, d - m + 2, \ldots, d-1\} \text{ is injective}\} \\
&\quad = \sum_{l=0}^{d-1} \sum_{m=l+1}^{d} {}_{m-1}P_{m-l-1} = \sum_{m=1}^{d} \sum_{l=0}^{m-1} {}_{m-1}P_{m-l-1} = \sum_{N=0}^{d-1} \sum_{r=0}^{N} {}_{N}P_{r}.
\end{aligned}$$

□

Proof of Proposition 8.1. It suffices to show that the matrix A given by (a), (b) in Proposition 8.1 satisfies Formula 8.1(1). By Remark 6.1(3), if $i \leq g(d)$ then $d - g(d) + i = g^{-1}(0) + i = g^{-1}(i)$. By (a), if $i \leq g(d)$,

$$\sum_{k=0}^{j} A_{ik} = \begin{cases} 0 & \text{if } j < g^{-1}(i), \\ 1 & \text{if } j \geq g^{-1}(i). \end{cases}$$

By (b), if $i > g(d)$, since $\xi(0) = -1$,

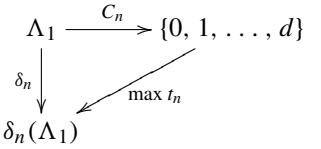
$$\sum_{k=0}^{j} A_{ik} = \begin{cases} \xi(j) + 2 & \text{if } j \leq g^{-1}(i), \\ \xi(j) + 1 & \text{if } j > g^{-1}(i). \end{cases}$$

Combining these, we get that A satisfies Formula 8.1(1). □

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Proof of Proposition 8.2. At first, we show that

$$\delta_n(\omega) = \max t_n(C_n(\omega)) \quad (\omega \in \Lambda_1).$$



By Observation 4.3(1) and Lemma A.3(2), if

$$\begin{aligned}
&\{\omega_1, \omega_2, \omega_3, \omega_4\} \subset \Lambda_1, \\
&\{\delta_n(\omega_1), \delta_n(\omega_2)\} \subset S_n \quad \text{with } \delta_n(\omega_1) < \delta_n(\omega_2), \\
&\{\delta_n(\omega_3), \delta_n(\omega_4)\} \subset L_n \quad \text{with } \delta_n(\omega_3) < \delta_n(\omega_4),
\end{aligned}$$

then

$$x_{(n,\infty)}(\omega_1) <_{\text{lex}} x_{(n,\infty)}(\omega_2) <_{\text{lex}} x_{(n,\infty)}(\omega_3) <_{\text{lex}} x_{(n,\infty)}(\omega_4).$$

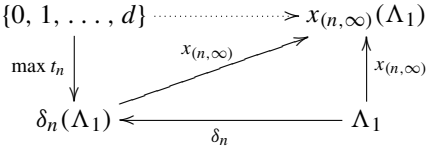
Therefore,

$$x_{(n,\infty)}(\max t_n(0)) <_{\text{lex}} x_{(n,\infty)}(\max t_n(1)) <_{\text{lex}} \cdots <_{\text{lex}} x_{(n,\infty)}(\max t_n(d)).$$

Thus, the map

$$x_{(n,\infty)} \circ \max t_n : \{0, 1, \ldots, d\} \to x_{(n,\infty)}(\Lambda_1)$$

is an increasing bijection (under lexicographic order).



Let $\omega_0 \in \Lambda_1$. Recall that $\delta_n : \Lambda_1 \to \{\max t_n(i) \mid 0 \le i \le d\}$ is a bijection. If $\delta_n(\omega_0) = \max t_n(i_0)$, then

$$\begin{aligned}
i_0 &= \#\{\omega \in \Lambda_1 \mid \delta_n(\omega) = \max t_n(i),\ i < i_0\} \\
&= \#\{\omega \in \Lambda_1 \mid x_{(n,\infty)}\delta_n(\omega) = x_{(n,\infty)}(\max t_n(i)),\ i < i_0\} \\
&= \#\{\omega \in \Lambda_1 \mid x_{(n,\infty)}(\omega) = x_{(n,\infty)}(\max t_n(i)),\ i < i_0\} \\
&= C_n(\omega_0) \quad (\text{since } x_{(n,\infty)} \circ \max t_n : \{0, 1, \ldots, d\} \to x_{(n,\infty)}(\Lambda_1) \text{ is increasing}).
\end{aligned}$$

Hence

$$\delta_n(\omega) = \max t_n(C_n(\omega)) \quad (\omega \in \Lambda_1).$$

Now, the assertion follows

$$\xi_n(C_{n-1}(\omega)) = x_n(\max t_{n-1}(C_{n-1}(\omega))) = x_n\delta_{n-1}(\omega) = x_n(\omega),$$

and

$$\begin{aligned}
\max t_n(g_nC_{n-1}(\omega)) &= \delta_n(\max t_{n-1}(C_{n-1}(\omega))) \\
&= \delta_n\delta_{n-1}(\omega) = \delta_n(\omega) = \max t_n(C_n(\omega)).
\end{aligned}$$

□

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