COUNTABLE STATIONARY TOWERS AND REFLECTION PRINCIPLES

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Abstract. In this article, we prove that the precipitousness of $\mathbb{Q}_\gamma$, the countable stationary tower of height γ, where γ is a limit ordinal $> \omega$, is a consequence of a certain stationary reflection principle for $\mathcal{P}_{\omega_1} H_{|V_\gamma|^+}$. Using this, we prove the consistency of the precipitousness of $\mathbb{Q}_\gamma$ for some singular γ assuming a sufficiently strong large cardinal axiom. Furthermore, assuming the existence of proper class many supercompact cardinals, we build a model where $\mathbb{Q}_\gamma$ is precipitous for every limit ordinal $> \omega$.

§1. Introduction. In the wake of work of Foreman, Magidor, and Shelah [3], H. Woodin introduced stationary tower forcing [10]. Assuming certain large cardinal axioms, he used generic elementary embeddings induced by stationary tower forcing to prove various properties of $L(\mathbb{R})$, the smallest class model of ZF containing all reals and all ordinals. Although Woodin's theory of stationary towers is more general, we will concentrate our attention on countable stationary towers in this article. We present a brief sketch of the notion of countable stationary tower forcing. For a thorough treatment of stationary tower forcing, we refer the reader to Larson's book [5].

Definition 1.1. Let γ be a limit ordinal $> \omega$. By $\mathbb{Q}_\gamma$, we denote the set of all $p \in V_\gamma$ such that p is a stationary subset of $[\bigcup p]^\omega$. For $p, q \in \mathbb{Q}_\gamma$, we say $p \leq_{\mathbb{Q}_\gamma} q$ if $\bigcup p \supseteq \bigcup q$ and $\forall x \in p(x \cap \bigcup q \in q)$.

The partial order $\langle \mathbb{Q}_\gamma, \leq_{\mathbb{Q}_\gamma} \rangle$ is known as a *countable stationary tower of height* γ. Throughout this article, whenever we mention a stationary tower, we assume that its height is a limit ordinal $> \omega$.

If G is a $\mathbb{Q}_\gamma$-generic filter over a ground model V, then we can build the generic ultrapower $\mathrm{Ult}(V; G)$ of V. We refer the reader to Larson's book [5] for the definition and basic facts concerning generic ultrapowers induced by stationary tower forcing.

In general, there is no guarantee that the generic ultrapower $\mathrm{Ult}(V; G)$ is well-founded in $V[G]$.

Definition 1.2. $\mathbb{Q}_\gamma$ is said to be *precipitous* if $\mathbb{Q}_\gamma$ forces that $\mathrm{Ult}(V; G)$ is well-founded in $V[G]$.

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It is easy to see that if $\mathrm{Ult}(V;G)$ is closed under taking ω-sequences in $V[G]$, then $\mathrm{Ult}(V;G)$ is well-founded in $V[G]$.

Woodin proved that, if γ is a Woodin cardinal, then $\mathbb{Q}_\gamma$ is precipitous. In fact, Woodin proved that $\mathbb{Q}_\gamma$ forces that $\mathrm{Ult}(V;G)$ is closed under taking ω-sequences in $V[G]$ if γ is a Woodin cardinal. This property of forcing generic ultrapowers to be closed under taking ω-sequences has its own name.

DEFINITION 1.3. $\mathbb{Q}_\gamma$ is said to be *presaturated* if $\mathbb{Q}_\gamma$ forces $\mathrm{Ult}(V;G)$ is closed under taking ω-sequences in generic extensions.

By the work of Woodin, we know that, if γ is a sufficiently strong large cardinal, then $\mathbb{Q}_\gamma$ is presaturated, hence precipitous. In this article we investigate the following questions.

QUESTION 1.4. *Can $\mathbb{Q}_\gamma$ be precipitous for γ which is not a large cardinal? Furthermore, can $\mathbb{Q}_\gamma$ be precipitous for a singular cardinal γ?*

QUESTION 1.5. *Can $\mathbb{Q}_\gamma$ be precipitous without being presaturated?*

In this article, we present positive answers to the above two questions.

REMARK 1.6. We should mention the reason why we are particularly interested in the question of whether $\mathbb{Q}_\gamma$ can be precipitous for a singular cardinal γ. Independent of stationary tower forcing, the precipitousness of the non-stationary ideal $\mathrm{NS}_{\kappa\lambda}$ over $\mathcal{P}_\kappa\lambda$ where $\mathrm{NS}_{\kappa\lambda}$ denotes the set of all non-stationary subsets of $\mathcal{P}_\kappa\lambda$ with an uncountable regular cardinal κ and λ a cardinal $> \kappa$, has been investigated by various researchers. For the definitions and basic facts concerning $\mathrm{NS}_{\kappa\lambda}$, we refer the reader to [4]. If λ is a regular cardinal, then, by using Levy collapse making a supercompact cardinal $> \lambda$ to be λ^+ in generic extensions, we can obtain a model where $\mathrm{NS}_{\kappa\lambda}$ is precipitous. But the situation with singular λ is completely different. For example, if λ is a singular strong limit cardinal, $\mathrm{NS}_{\kappa\lambda}$ cannot be precipitous [6–8]. It is not known whether $\mathrm{NS}_{\kappa\lambda}$ can be precipitous for singular λ. For this reason, we are particularly interested in asking whether $\mathbb{Q}_\gamma$ can be precipitous for a singular cardinal γ.

§2. Presaturation of $\mathbb{Q}_\gamma$. In this section, we present one theorem concerning presaturation of $\mathbb{Q}_\gamma$. This result has appeared in previous literature, for example, [2]. But we present a proof for the sake of completeness.

THEOREM 2.1. *Let γ be a limit ordinal $> \omega$. The following are equivalent*:

(1) $\Vdash_{\mathbb{Q}_\gamma}$"$\mathrm{cf}(\gamma) > \omega$."
(2) $\mathbb{Q}_\gamma$ *is presaturated.*

PROOF. (1) $\Rightarrow$ (2). Assume (1) holds. Suppose $\Vdash_{\mathbb{Q}_\gamma}$"$\langle \dot{z}_n \mid n < \omega\rangle$ is an ω-sequence such that each $\dot{z}_n \in \mathrm{Ult}(\check{V};\dot{G})$." We define $D \subseteq \mathbb{Q}_\gamma$ as follows:

$$D = \{p \in \mathbb{Q}_\gamma \mid \exists \text{a function } H \in V \text{ such that } \mathrm{dom}(H) = p \text{ and } p \Vdash_{\mathbb{Q}_\gamma} [\check{H}] = \langle \dot{z}_n \mid n < \omega\rangle\}.$$

By showing D is dense in $\mathbb{Q}_\gamma$, we prove that $\mathbb{Q}_\gamma$ is presaturated. Suppose $r \in \mathbb{Q}_\gamma$. We will find $p^* \in D$ satisfying $p^* \leq_{\mathbb{Q}_\gamma} r$.

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For each $n < \omega$, let A_n be a maximal antichain in $\mathbb{Q}_\gamma$ satisfying the following condition:

$$\forall p \in A_n \exists F_n^p : p \to V \text{ such that } p \Vdash_{\mathbb{Q}_\gamma} \text{“}[\check{F}_n^p] = \dot{z}_n\text{.”}$$

Let $\dot{f}$ be a $\mathbb{Q}_\gamma$-name for the function $f : \omega \to \gamma$ such that

$$f(n) = \min\{\alpha < \gamma \mid A_n \cap V_\alpha \cap G \neq \emptyset\}.$$

Then by (1), there is some $p_0 \leq_{\mathbb{Q}_\gamma} r$ and $\alpha_0 < \gamma$ such that $p_0 \Vdash_{\mathbb{Q}_\gamma}$ “$\mathrm{range}(\dot{f}) \subseteq \alpha_0$.” We now prove the next claim.

Claim 2.2. *For each $n < \omega$, $A_n \cap V_{\alpha_0}$ is predense below p_0.*

Proof of Claim. Fix $n < \omega$. Suppose $q \leq_{\mathbb{Q}_\gamma} p_0$. Since A_n is a maximal antichain, there is some $s \in A_n$ with $s \parallel q$. Find $q^* \in \mathbb{Q}_\gamma$ with $q^* \leq_{\mathbb{Q}_\gamma} s, q$. So we have $q^* \Vdash_{\mathbb{Q}_\gamma} \check{s} \in \check{A}_n \cap \dot{G}$.

Since $q^* \leq_{\mathbb{Q}_\gamma} p_0, q^* \Vdash_{\mathbb{Q}_\gamma} \forall p(p \in \check{A}_n \cap \dot{G} \to p \in V_{\alpha_0})$. This implies $s \in A_n \cap V_{\alpha_0}$, showing $A_n \cap V_{\alpha_0}$ is predense below p_0. ⊣

For each $n < \omega$, we fix a club subset $D_{p,q}^n$ of $[V_{\alpha_0}]^\omega$ for every $p, q \in A_n \cap V_{\alpha_0}$ satisfying the following condition: For $p, q \in A_n \cap V_{\alpha_0}$ with $p \neq q$, $\{x \in [V_{\alpha_0}]^\omega \mid x \cap \bigcup p \in p\} \cap \{x \in [V_{\alpha_0}]^\omega \mid x \cap \bigcup q \in q\} \cap D_{p,q}^n = \emptyset$. If $p = q$, then we let $D_{p,q}^n = [V_{\alpha_0}]^\omega$. Let ΔD^n denote the set

$$\{x \in [V_{\alpha_0}]^\omega \mid \forall p, q \in x \cap A_n \cap V_{\alpha_0} (x \in D_{p,q}^n)\}.$$

It is easy to see that ΔD^n is club in $[V_{\alpha_0}]^\omega$.

Let $p^* \equiv \{x \in [V_{\alpha_0}]^\omega \mid x \cap \bigcup p_0 \in p_0\} \cap \bigcap_{n<\omega} \Delta D^n$. So we have $p^* \leq_{\mathbb{Q}_\gamma} p_0 \leq_{\mathbb{Q}_\gamma} r$. We will show that $p^* \in D$ implying D is dense in $\mathbb{Q}_\gamma$. Note that, by the definition of p^*, if $x \in p^*$, then we cannot have $p, q \in x \cap A_n \cap V_{\alpha_0}$ with $p \neq q$ satisfying $x \cap \bigcup p \in p$ and $x \cap \bigcup q \in q$. Using this property, for each $n \in \omega$, we define $h_n \colon p^* \to V$ as follows:

$$h_n(x) = \begin{cases} F_n^p(x \cap \bigcup p) & \text{if } \exists p \in x \cap A_n \cap V_{\alpha_0}(x \cap \bigcup p \in p), \\ \emptyset & \text{otherwise.} \end{cases}$$

Claim 2.3. *For $n \in \omega$, $p^* \Vdash_{\mathbb{Q}_\gamma}$ “$[\check{h}_n] = \dot{z}_n$.”*

Proof of Claim. Suppose G is $\mathbb{Q}_\gamma$-generic filter over V with $p^* \in G$. Then $p_0 \in G$. Since $A_n \cap V_{\alpha_0}$ is predense below p_0, $\exists! \, q \in A_n \cap V_{\alpha_0} \cap G$.

Let $p^{**} \equiv \{x \in p^* \mid q \in x\}$. Since p^* is stationary in $[V_{\alpha_0}]^\omega$ and $q \in V_{\alpha_0}$, we know $p^{**} \in G$ as well. Note that $p^{**} \Vdash_{\mathbb{Q}_\gamma}$ “$[\check{h}_n] = [\check{F}_n^q]$” and $q \Vdash_{\mathbb{Q}_\gamma}$ “$[\check{F}_n^q] = \dot{z}_n$.” From $p^{**}, q \in G$, we conclude $[h_n]_G = (\dot{z}_n)_G$. This shows that $p^* \Vdash_{\mathbb{Q}_\gamma}$ “$[\check{h}_n] = \dot{z}_n$.” ⊣

Now define $H : p^* \to V$ by $H(x) = \langle h_n(x) \mid n < \omega\rangle$. By Claim 2.3, we have $p^* \Vdash_{\mathbb{Q}_\gamma}$ “$[\check{H}] = \langle \dot{z}_n \mid n < \omega\rangle$.” Hence $p^* \in D$, completing the proof of D being dense in $\mathbb{Q}_\gamma$.

Now we prove (2) ⇒ (1). Suppose that $\mathbb{Q}_\gamma$ is presaturated. From $\nVdash_{\mathbb{Q}_\gamma}$ “$\mathrm{cf}(\gamma) > \omega$,” we will derive a contradiction. Assume that there exists $p \in \mathbb{Q}_\gamma$ such that

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$p \Vdash_{\mathbb{Q}_\gamma}$ "$\mathrm{cf}(\gamma) = \omega$." Let G be a $\mathbb{Q}_\gamma$-generic filter with $p \in G$ and work in $V[G]$. Since $\mathbb{Q}_\gamma$ is presaturated, it is precipitous. Let $j : V \to M$ be the corresponding generic elementary embedding and $\langle \delta_n \mid n < \omega \rangle$ be a sequence of ordinals cofinal in γ. For each $n < \omega$, let g_n be the identity function on $[V_{\delta_n}]^\omega$. Therefore $[g_n]_G = j``V_{\delta_n}$ and $M \models$"$|j``V_{\delta_n}| = \omega$." Now, by presaturation of $\mathbb{Q}_\gamma$, we know $\langle j``V_{\delta_n} \mid n < \omega \rangle \in M$. Therefore $j``V_\gamma \in M$ and $M \models$"$|j``V_\gamma| = \omega$." Note that $M \models j``V_\gamma \in j([V_\gamma]^\omega)$. Since $j``V_\gamma \in M$, there exists a function $F \in V$ such that $\mathrm{dom}(F) \in \mathbb{Q}_\gamma$ and $[F]_G = j``V_\gamma$. Without loss of generality, we can assume $\mathrm{range}(F) \subseteq ([V_\gamma]^\omega)^V$. From $\mathrm{dom}(F) \in V_\gamma$, we know $|\mathrm{dom}(F)|^V < |V_\gamma|^V$. Therefore $V_\gamma \setminus \bigcup \mathrm{range}(F) \neq \emptyset$. Pick an element $a^* \in V_\gamma \setminus \bigcup \mathrm{range}(F)$. Then $a^* \notin F(x)$ for every $x \in \mathrm{dom}(F)$. Hence $j(a^*) \notin [F]_G$ contradicting $[F] = j``V_\gamma$.

Since $\mathbb{Q}_\gamma$ makes every ordinal $< \gamma$ to be countable, the next corollary is an immediate consequence of Theorem 2.1.

COROLLARY 2.4. *If γ is a singular ordinal, then $\mathbb{Q}_\gamma$ cannot be presaturated.*

§3. Reflection principles. In this section we show that certain reflection principles imply the precipitousness of countable stationary towers. The following types of reflection principles have been investigated by various researchers.

DEFINITION 3.1. Let κ and λ be cardinals with $\omega_1 \leq \kappa \leq \lambda$. We say that $\mathrm{RP}(\kappa, \lambda)$ holds if, for every stationary $S \subseteq [\lambda]^\omega$ and $\alpha < \kappa$, there is some $A \in \mathcal{P}_\kappa \lambda$ such that $S \cap [A]^\omega$ is stationary in $[A]^\omega$ and $\alpha \subseteq A$.

By using a suitable bijection between H_{κ^+} and 2^κ, we can find the following version of a reflection principle that is equivalent to $\mathrm{RP}(\kappa, 2^\kappa)$.

LEMMA 3.2. *Assume $\kappa = |V_\gamma|$, where γ is a limit ordinal $> \omega$. Then the following are equivalent*:

(1) $\mathrm{RP}(\kappa, 2^\kappa)$.
(2) *For every stationary $T \subseteq [H_{\kappa^+}]^\omega$ and $\beta < \gamma$, there is some $B \prec H_{\kappa^+}$ such that $|B| < \kappa$, $V_\beta \subseteq B$, and $T \cap [B]^\omega$ is stationary in $[B]^\omega$.*

Later we will show that the reflection principle in the last lemma implies $\mathbb{Q}_\gamma$ is precipitous. Before presenting that result, we mention two facts concerning $\mathrm{RP}(\kappa, \lambda)$ whose proofs are straightforward.

LEMMA 3.3. *If κ is a supercompact cardinal, then $\mathrm{RP}(\kappa, \lambda)$ holds for every cardinal $\lambda \geq \kappa$.*

The next lemma is an immediate consequence of the definition of $\mathrm{RP}(\kappa, \lambda)$.

LEMMA 3.4. *Let $\lambda \geq \kappa$ be uncountable cardinals. If $\{\mu < \kappa \mid \mathrm{RP}(\mu, \lambda)$ holds$\}$ is unbounded in κ, then $\mathrm{RP}(\kappa, \lambda)$ holds as well.*

Now we state the main result of this section.

THEOREM 3.5. *Let γ be a limit ordinal $> \omega$ and $\kappa = |V_\gamma|$. If $\mathrm{RP}(\kappa, 2^\kappa)$ holds, then $\mathbb{Q}_\gamma$ is precipitous.*

We prove this theorem using the following definition and two lemmas.

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DEFINITION 3.6. Let γ be a limit ordinal $> \omega$, $\kappa = |V_\gamma|$, and fix Δ a well-ordering on H_{κ^+}. For $D \subseteq \mathbb{Q}_\gamma$, let $\mathrm{sp}(D)$ be the set of all $X \in [H_{\kappa^+}]^\omega$ such that:

(1) $\langle X, \in, \Delta \restriction X \times X\rangle$ is an elementary submodel of $\langle H_{\kappa^+}, \in, \Delta\rangle$ with $D \in X$.
(2) For every $p \in X \cap \mathbb{Q}_\gamma$, if $X \cap \bigcup p \in p$, then there exists a countable $Y \subseteq H_{\kappa^+}$ and $q \in Y \cap D$ such that $\langle Y, \in, \Delta \restriction Y \times Y\rangle \preccurlyeq \langle H_{\kappa^+}, \in, \Delta\rangle$, $q \leq_{\mathbb{Q}_\gamma} p$, $X \subseteq Y$, $Y \cap \bigcup q \in q$, and $X \cap \bigcup p = Y \cap \bigcup p$.

For ease of notation, for $X \subseteq H_{\kappa^+}$, we abbreviate $\langle X, \in, \Delta \restriction N \times N, ...\rangle \preccurlyeq \langle H_{\kappa^+}, \in, \Delta, ...\rangle$ by $X \preccurlyeq \langle H_{\kappa^+}, \in, \Delta, ...\rangle$.

The next lemma relates $\mathrm{RP}(\kappa, 2^\kappa)$ and the nature of $\mathrm{sp}(D)$ for every dense $D \subseteq \mathbb{Q}_\gamma$.

LEMMA 3.7. *If* $\mathrm{RP}(\kappa, 2^\kappa)$ *holds, then, for every dense set* $D \subseteq \mathbb{Q}_\gamma$, $\mathrm{sp}(D)$ *contains a club subset of* $[H_{\kappa^+}]^\omega$.

The next lemma shows that the consequence of $\mathrm{RP}(\kappa, 2^\kappa)$ mentioned in Lemma 3.7 implies the precipitousness of $\mathbb{Q}_\gamma$.

LEMMA 3.8. *Let* $\kappa = |V_\gamma|$ *with* γ *a limit ordinal* $> \omega$. *If* $\mathrm{sp}(D)$ *contains a club subset of* $[H_{\kappa^+}]^\omega$ *for every dense* $D \subseteq \mathbb{Q}_\gamma$, *then* $\mathbb{Q}_\gamma$ *is precipitous.*

We now present a proof of Lemma 3.7.

PROOF OF LEMMA 3.7. We assume that $\mathrm{RP}(\kappa, 2^\kappa)$ and $[H_{\kappa^+}]^\omega \setminus \mathrm{sp}(D)$ is stationary in $[H_{\kappa^+}]^\omega$ for some dense $D \subseteq \mathbb{Q}_\gamma$ and derive a contradiction. Since $\{X \in [H_{\kappa^+}]^\omega \mid X \preccurlyeq \langle H_{\kappa^+}, \in, \Delta\rangle\}$ is a club subset of $[H_{\kappa^+}]^\omega$, by applying the Fodor-like theorem due to Jech [4], there exists some $p^* \in \mathbb{Q}_\gamma$ such that the following set S is stationary in $[H_{\kappa^+}]^\omega$: $S \equiv \{X \in [H_{\kappa^+}]^\omega \mid X \preccurlyeq \langle H_{\kappa^+}, \in, \Delta\rangle, X \cap \bigcup p^* \in p^*, p^*, D \in X$, and $\neg\exists Y \in [H_{\kappa^+}]^\omega (Y \preccurlyeq \langle H_{\kappa^+}, \in, \Delta\rangle \wedge X \subseteq Y \wedge \exists q \in Y \cap D (q \leq_{\mathbb{Q}_\gamma} p^* \wedge Y \cap \bigcup q \in q \wedge X \cap \bigcup p^* = Y \cap \bigcup p^*))\}$.

Let $\alpha < \gamma$ be large enough that $p^*, \bigcup p^* \in V_\alpha$. By $\mathrm{RP}(\kappa, 2^\kappa)$, there exists $A \in [H_{\kappa^+}]^{<\kappa}$ such that $A \preccurlyeq H_{\kappa^+}$, $V_\alpha \subseteq A$, and $S \cap [A]^\omega$ is stationary in $[A]^\omega$.

Since $|A| < \kappa = |V_\gamma|$, there are $\overline{A} \in V_\gamma$ and a bijection $\pi : A \to \overline{A}$ such that $V_\alpha \subseteq \overline{A}$ and $\pi \restriction V_\alpha$ is the identity map. Then $\overline{S} \equiv \{\pi``X \mid X \in S \cap [A]^\omega\}$ is stationary in $[\overline{A}]^\omega$, implying $\overline{S} \in \mathbb{Q}_\gamma$ and $\bigcup \overline{S} = \overline{A}$.

Since D is dense in $\mathbb{Q}_\gamma$, there exists some $\overline{q} \in D$ such that $\overline{q} \leq_{\mathbb{Q}_\gamma} \overline{S}$. Without loss of generality, we may assume that $\overline{q} \leq p^*$ or $\{X \in [\bigcup p^* \cup \bigcup \overline{q}]^\omega \mid X \cap \bigcup p^* \in p^* \wedge X \cap \bigcup \overline{q} \in \overline{q}\} = \emptyset$. For convenience, we denote $\{X \in [\bigcup p \cup \bigcup q]^\omega \mid X \cap \bigcup p \in p \wedge X \cap \bigcup q \in q\} = \emptyset$ by $p \perp^* q$.

Now take a countable $Y^* \subseteq H_{\kappa^+}$ such that $Y^* \preccurlyeq \langle H_{\kappa^+}, \in, \Delta\rangle$, $Y^* \cap \bigcup \overline{q} \in \overline{q}$, and $\pi, A, \overline{A}, S \cap [A]^\omega, p^*, \overline{q} \in Y^*$. Since $Y^* \cap \bigcup \overline{q} \in \overline{q}$ and $\overline{q} \leq_{\mathbb{Q}_\gamma} \overline{S}$, we have $Y^* \cap \overline{A} = Y^* \cap \bigcup \overline{S} \in \overline{S}$. So there exists some $X^* \in S \cap [A]^\omega$ such that $\pi``X^* = Y^* \cap \bigcup \overline{S}$. Since $\pi^{-1} \in Y^*$, $X^* \subseteq Y^*$. From $\bigcup p^* \in V_\alpha$ and $\pi \restriction V_\alpha = \mathrm{id} \restriction V_\alpha$, we know $\pi``\bigcup p^* = \bigcup p^*$. Note that $Y^* \cap \bigcup p^* = \pi``X^* \cap \pi``\bigcup p^* = \pi``(X^* \cap \bigcup p^*) = X^* \cap \bigcup p^* \in p^*$. From $Y^* \cap \bigcup \overline{q} \in \overline{q}$ and $Y^* \cap \bigcup p^* \in p^*$, we cannot have $p^* \perp^* \overline{q}$. So we conclude $\overline{q} \leq_{\mathbb{Q}_\gamma} p^*$.

We have $X^* \in S$ and $Y^* \in [H_{\kappa^+}]^\omega$ such that $Y^* \preccurlyeq \langle H_{\kappa^+}, \in, \Delta\rangle$, $X^* \subseteq Y^*$, $\overline{q} \in Y^* \cap D$, $\overline{q} \leq_{\mathbb{Q}_\gamma} p^*$, $Y^* \cap \bigcup \overline{q} \in \overline{q}$, and $X^* \cap \bigcup p^* = Y^* \cap \bigcup p^*$. But this contradicts the definition of S. ⊣

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We now present a proof of Lemma 3.8. To prove this lemma, we use the game characterization of precipitousness of a stationary tower due to Burke [1].

For a limit ordinal $\gamma > \omega$, let Γ_γ denote the following two player game of length ω: At each inning, Players I and II choose $p_n, q_n \in \mathbb{Q}_\gamma$ alternately with $p_0 \geq_{\mathbb{Q}_\gamma} q_0 \geq_{\mathbb{Q}_\gamma} p_1 \geq_{\mathbb{Q}_\gamma} q_1 \geq_{\mathbb{Q}_\gamma} \ldots$. For the play $\langle p_n, q_n \mid n < \omega \rangle$, player II wins if there is some $X \in [V_\gamma]^\omega$ such that $X \cap \bigcup p_n \in p_n$ for every $n < \omega$. Otherwise I wins. Burke proved the following theorem.

THEOREM 3.9. *Let γ be a limit ordinal $> \omega$. Then the following are equivalent*:

(1) *$\mathbb{Q}_\gamma$ is precipitous.*
(2) *Player I does not have a winning strategy in the game Γ_γ.*

PROOF OF LEMMA 3.8. Under the hypothesis of our lemma, we will show that Player I does not have a winning strategy in the game Γ_γ.

Let σ be a strategy for Player I in Γ_γ. We will show that σ is not a winning strategy for Player I. We now define the set C^* as follows:

$$C^* \equiv \{N \in [H_{\kappa^+}]^\omega \mid N \prec \langle H_{\kappa^+}, \in, \Delta \rangle \wedge \sigma, \mathbb{Q}_\gamma \in N \wedge \forall D \in N (D \text{ dense in } \mathbb{Q}_\gamma \to N \in \mathrm{sp}(D))\}.$$

It is easy to see that C^* contains a club subset of $[H_{\kappa^+}]^\omega$. Hence there is some $R^* \subseteq [H_{\kappa^+}]^\omega$ such that, for every $N \in [H_{\kappa^+}]^\omega$, we have $N \in C^*$ if $N \prec \langle H_{\kappa^+}, \in, \Delta, R^* \rangle$. From now on let us denote the structure $\langle H_{\kappa^+}, \in, \Delta, R^* \rangle$ by $\mathcal{M}^*$.

First let $p_0 \equiv \sigma(\emptyset)$. Take a countable set M_0 such that $M_0 \prec \mathcal{M}^*$, $p_0 \in M_0$, $\mathbb{Q}_\gamma \in M_0$, and $M_0 \cap \bigcup p_0 \in p_0$. Then let

$$D_0 \equiv \{p \in \mathbb{Q}_\gamma \mid \exists q \leq_{\mathbb{Q}_\gamma} p_0 (\sigma(\langle q \rangle) = p) \vee p \perp^* p_0\}.$$

Then D_0 is dense in $\mathbb{Q}_\gamma$ and $D_0 \in M_0$, implying $M_0 \in \mathrm{sp}(D_0)$.

Since $M_0 \in \mathrm{sp}(D_0)$, $p_0 \in M_0$, and $M_0 \cap \bigcup p_0 \in p_0$, there exist a countable set $N \prec \langle H_{\kappa^+}, \in, \Delta \rangle$ and $p_1 \in N \cap D_0$ such that $p_1 \leq_{\mathbb{Q}_\gamma} p_0$, $M_0 \subseteq N$, $N \cap \bigcup p_1 \in p_1$, and $N \cap \bigcup p_0 = M_0 \cap \bigcup p_0 \in p_0$. From $N \cap \bigcup p_1 \in p_1$ and $N \cap \bigcup p_0 \in p_0$, we cannot have $p_1 \perp^* p_0$. Thus there is some $q \leq_{\mathbb{Q}_\gamma} p_0$ with $\sigma(\langle q \rangle) = p_1$. Let q_0 be the Δ-least element such that $q_0 \leq_{\mathbb{Q}_\gamma} p_0$ and $\sigma(\langle q_0 \rangle) = p_1$. Note that N might not be an elementary submodel of $\mathcal{M}^*$. Now we let $M_1 \equiv$ Skolem hull of $M_0 \cup (N \cap \bigcup p_1) \cup \{p_1\}$ making $M_0 \subseteq M_1 \prec \mathcal{M}^*$ and $\{p_0, q_0, p_1\} \subseteq M_1$.

CLAIM 3.10. *$M_1 \cap \bigcup p_1 = N \cap \bigcup p_1$. Hence $M_1 \cap \bigcup p_0 = N \cap \bigcup p_0 = M_0 \cap \bigcup p_0$.*

PROOF OF CLAIM. Since $N \cap \bigcup p_1 \subseteq M_1$, we just need to show that $M_1 \cap \bigcup p_1 \subseteq N$.

Take $a \in M_1 \cap \bigcup p_1$. Then there is a Skolem term t and some $\vec{x} \in M_0^n$, $\vec{y} \in (N \cap \bigcup p_1)^m$ such that $a = t(\vec{x}, \vec{y}, p_1)$. Now define a function f on $V_\gamma^m \times \mathbb{Q}_\gamma$ by $f(\vec{z}, r) = t(\vec{x}, \vec{z}, r)$ for $\vec{z} \in V_\gamma^m$ and $r \in \mathbb{Q}_\gamma$. So $f(\vec{y}, p_1) = a$. Since t is definable in M_0 and $\vec{x} \in M_0$, we have $f \in M_0 \subseteq N$. From $\vec{y}, p_1 \in N$, we conclude that $a = f(\vec{y}, p_1) \in N$. ⊣

Note that we have $M_0, M_1 \in [H_{\kappa^+}]^\omega$ satisfying the following conditions:

(1) $M_0 \prec M_1 \prec \mathcal{M}^*$.
(2) $p_0 \in M_0$, $\{q_0, p_1\} \subseteq M_1$.

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(3) $\sigma(\emptyset) = p_0 \geq_{\mathbb{Q}_\gamma} q_0 \geq_{\mathbb{Q}_\gamma} \sigma(\langle q_0 \rangle) = p_1$.
(4) $M_0 \cap \bigcup p_0 \in p_0$, $M_1 \cap \bigcup p_1 \in p_1$, $M_1 \cap \bigcup p_0 = M_0 \cap \bigcup p_0$.

Now we repeat the above procedure using the following dense set $D_1 \in M_1$:

$$D_1 \equiv \{p \in \mathbb{Q}_\gamma \mid \exists q \leq_{\mathbb{Q}_\gamma} p_1(\sigma(\langle q_0, q \rangle)) = p \vee p \perp^* p_1\}.$$

By the above argument, we obtain $M_2 \in [H_{\kappa^+}]^\omega$ satisfying:

(1) $M_1 \preccurlyeq M_2 \preccurlyeq \mathcal{M}^*$.
(2) $\{q_1, p_2\} \subseteq M_2$.
(3) $\sigma(\emptyset) = p_0 \geq_{\mathbb{Q}_\gamma} q_0 \geq_{\mathbb{Q}_\gamma} \sigma(\langle q_0 \rangle) = p_1 \geq_{\mathbb{Q}_\gamma} q_1 \geq_{\mathbb{Q}_\gamma} \sigma(\langle q_0, q_1 \rangle) = p_2$.
(4) $M_2 \cap \bigcup p_2 \in p_2$, $M_2 \cap \bigcup p_1 = M_1 \cap \bigcup p_1$.

By repeating the above procedure, we build a sequence $\langle M_n, p_n, q_n \mid n < \omega \rangle$ satisfying the following conditions:

(1) $M_0 \preccurlyeq M_1 \preccurlyeq M_2 \preccurlyeq M_3 \preccurlyeq \ldots \preccurlyeq \mathcal{M}^*$.
(2) $\{q_n, p_{n+1}\} \subseteq M_{n+1}$ for each $n < \omega$.
(3) $\sigma(\emptyset) = p_0 \geq_{\mathbb{Q}_\gamma} q_0 \geq_{\mathbb{Q}_\gamma} \sigma(\langle q_0 \rangle) = p_1 \geq_{\mathbb{Q}_\gamma} q_1 \geq_{\mathbb{Q}_\gamma} \sigma(\langle q_0, q_1 \rangle) = p_2 \geq_{\mathbb{Q}_\gamma} \ldots \geq_{\mathbb{Q}_\gamma} p_n \geq_{\mathbb{Q}_\gamma} q_n \geq_{\mathbb{Q}_\gamma} \sigma(\langle q_0, \ldots, q_n \rangle) = p_{n+1} \geq_{\mathbb{Q}_\gamma} q_{n+1} \geq_{\mathbb{Q}_\gamma} \ldots$.
(4) For each $n \in \omega$, $M_n \cap \bigcup p_n \in p_n$ and $M_m \cap \bigcup p_m = M_n \cap \bigcup p_m$ for every $m < n$.

We now let $X = \bigcup_{n<\omega} M_n$. Then $X \cap \bigcup p_n \in p_n$ for every $n < \omega$. This means that σ is not a winning strategy for Player I.

In this section, we related a certain reflection principle and the precipitousness of a countable stationary tower. In the next section we present some consequences of large cardinals using the results from this section.

§4. Consequences of large cardinals. Now using the results from the last section, we present a theorem and its corollary that give positive answers to Questions 1.4 and 1.5.

THEOREM 4.1. *If κ is a limit of supercompact cardinals, then $\mathbb{Q}_\kappa$ is precipitous.*

PROOF. If κ is a limit of supercompact cardinals, then, by Lemmas 3.3 and 3.4, $\mathrm{RP}(\kappa, 2^\kappa)$ holds. Since $|V_\kappa| = \kappa$, by Theorem 3.5, we conclude that $\mathbb{Q}_\kappa$ is precipitous. ⊣

In our previous paper [9], we proved that if $\mathbb{Q}_\kappa$ is semiprecipitous, that is, $\Vdash_{\mathbb{Q}_\kappa}$ "$\mathrm{Ult}(\check{V}_\kappa; \dot{G})$ is well-founded," for a cardinal κ such that $|V_\kappa| = \kappa$ and $\omega < \mathrm{cf}(\kappa) < \kappa$, then $\mathbb{Q}_\kappa$ is precipitous. The above theorem shows that the hypothesis of the above mentioned result from the previous paper is not vacuous.

The following result is an immediate consequence of the last theorem, Theorem 2.1, and Corollary 2.4.

COROLLARY 4.2. *If κ is a singular limit of supercompact cardinals, then $\mathbb{Q}_\kappa$ is precipitous but not presaturated.*

It turns out that $\mathbb{Q}_\kappa$ can be precipitous when κ is the least weakly compact cardinal. In order to prove that, we need the following lemma.

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LEMMA 4.3. *Let κ be an inaccessible cardinal, and suppose* $\mathrm{Col}(\kappa, 2^\kappa)$ *forces that "κ is weakly compact" where* $\mathrm{Col}(\kappa, 2^\kappa)$ *is a standard κ-closed Levy collapse that adds a surjection from κ onto 2^κ. Then* $\mathrm{RP}(\kappa, 2^\kappa)$ *holds.*

PROOF. Let $\lambda = 2^\kappa$. Fix a stationary set $S \subseteq [\lambda]^\omega$ and $\alpha < \kappa$. Take a $\mathrm{Col}(\kappa, \lambda)$-generic filter G over V, and work in $V[G]$. S remains a stationary subset of $[\lambda]^\omega$ in $V[G]$. Fix a bijection $f : \lambda \to \kappa$, and let $T = \{f``x \mid x \in S\}$. T is a stationary subset of $[\kappa]^\omega$.

In $V[G]$, since κ is a weakly compact cardinal, there is $\delta < \kappa$ such that $f``\alpha \subseteq \delta$ and $T \cap [\delta]^\omega$ is stationary in $[\delta]^\omega$. Let $A = f^{-1}``\delta$. Then we have that $\alpha \subseteq A$, $|A| < \kappa$, and $S \cap [A]^\omega$ is stationary in $[A]^\omega$. From the κ-closedness of $\mathrm{Col}(\kappa, 2^\kappa)$, we have $A \in V$. ⊣

THEOREM 4.4. *Suppose κ is supercompact. Then there is a generic extension where there is a weakly compact cardinal, and, if δ is the least weakly compact cardinal, then the weak compactness of δ is preserved by* $\mathrm{Col}(\delta, 2^\delta)$.

PROOF. We define a $\leq \kappa$-stage Easton support iteration as follows.

For $\xi < \kappa$, suppose $\mathbb{P}_\xi$ is defined and $\mathbb{P}_\xi$ forces that "there is no weakly compact cardinal $< \xi$."

Case 1: There is some condition $p \in \mathbb{P}_\xi$ that forces the following statement: "ξ is a weakly compact cardinal, and the weak compactness of ξ is preserved by $\mathrm{Col}(\xi, 2^\xi)^{V[\dot{G}_\xi]}$" where $\dot{G}_\xi$ is a canonical name for a $\mathbb{P}_\xi$-generic filter. Then take a $\mathbb{P}_\xi$-generic filter G_ξ with $p \in G_\xi$. Clearly $V[G_\xi]$ is a required model.

Case 2: There is no such $p \in \mathbb{P}_\xi$. Then let $\dot{\mathbb{Q}}_\xi$ be a $\mathbb{P}_\xi$-name such that:

$$\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi = \begin{cases} \text{trivial poset} & \text{if } \xi \text{ is not weakly compact,} \\ \mathrm{Col}(\xi, 2^\xi)^{V[\dot{G}_\xi]} & \text{if } \xi \text{ is weakly compact but } \mathrm{Col}(\xi, 2^\xi)^{V[\dot{G}_\xi]} \\ & \text{destroys the weak compactness of } \xi. \end{cases}$$

We show that this procedure has to be stopped at some stage $\xi < \kappa$. Suppose to the contrary that $\mathbb{P}_\xi$ was defined for every $\xi < \kappa$. Let $\mathbb{P}_\kappa$ be a direct limit of the $\mathbb{P}_\xi$'s. By a standard argument, we know that $|\mathbb{P}_\kappa| = \kappa$, $\mathbb{P}_\kappa$ satisfies the κ-c.c., and $\mathbb{P}_\kappa$ forces that "κ is inaccessible and there is no weakly compact cardinal $< \kappa$." In V, let $\lambda = 2^\kappa$, and take a λ-supercompact elementary embedding $j : V \to M$ with the critical point κ. $j(\mathbb{P}_\kappa)$ can be factored as $\mathbb{P}_\kappa * \dot{\mathbb{Q}}^M_\kappa * \dot{\mathbb{P}}_{\mathrm{tail}}$. Take a $\mathbb{P}_\kappa$-generic filter G and we work in $V[G]$. Note that $M[G]$ is closed under taking λ-sequences in $V[G]$, in particular $\lambda = (2^\kappa)^{M[G]}$, and $\mathrm{Col}(\kappa, \lambda)^{V[G]} = \mathrm{Col}(\kappa, \lambda)^{M[G]}$. In $M[G]$, κ is not weakly compact, or is weakly compact but the weak compactness is destroyed by $\mathrm{Col}(\kappa, \lambda)^{M[G]} (= \mathrm{Col}(\kappa, \lambda)^{V[G]})$. Let $\mathbb{Q} = (\dot{\mathbb{Q}}^M_\kappa)_G$. By the construction of $\mathbb{P}_\kappa$, $\mathbb{Q}$ is either a trivial poset, or $\mathrm{Col}(\kappa, \lambda)^{V[G]}$. Note that $\mathbb{Q}$ is κ-directed closed in either cases. Take a $\mathbb{Q}$-generic filter H over $V[G]$. We know that κ is not weakly compact in $M[G][H]$. We derive a contradiction by proving the following claim.

CLAIM 4.5. *In $M[G][H]$, κ is a weakly compact cardinal.*

PROOF OF CLAIM. Again, $M[G][H]$ is closed under taking λ-sequences in $V[G][H]$, hence $\mathbb{P}_{\mathrm{tail}} (= (\dot{\mathbb{P}}_{\mathrm{tail}})_G)$ is κ^+-closed in $M[G][H]$, and so is in $V[G][H]$.

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Take a $\mathbb{P}_{\text{tail}}$-generic filter G_{tail} over $V[G][H]$. We can extend $j : V[G] \to M[G][H][G_{\text{tail}}]$ by a canonical way. Furthermore, we have $j\text{“}H \in M[G][H][G_{\text{tail}}]$. Since $j\text{“}H \subseteq j(\mathbb{Q})$ and $|j\text{“}H| \leq \kappa < j(\kappa)$, $j\text{“}H$ has a lower bound p^* in $j(\mathbb{Q})$. Take a $j(\mathbb{Q})$-generic filter $j(H)$ with $p^* \in j(H)$. $j : V[G] \to M[G][H][G_{\text{tail}}]$ can be extended to $j : V[G][H] \to M[G][H][G_{\text{tail}}][j(H)]$. To show that κ is weakly compact in $M[G][H]$, since κ is inaccessible, it is sufficient to show that κ has the tree property in $M[G][H]$. Take a κ-tree $T \in M[G][H]$, and we may assume $T = \kappa$. Then, since $j(T) \in M[G][H][G_{\text{tail}}][j(H)]$, $T = j(T) \cap \kappa$ has a cofinal branch B in $M[G][H][G_{\text{tail}}][j(H)]$. By the closedness of $\mathbb{P}_{\text{tail}} * j(\mathbb{Q})$ in $M[G][H]$, B lies in $M[G][H]$. ⊣

Now the following corollary is an immediate consequence of the last theorem and Lemma 4.3.

COROLLARY 4.6. *Relative to a certain large cardinal axiom, it is consistent that κ is the least weakly compact cardinal and $\mathbb{Q}_\kappa$ is precipitous.*

Finally we prove that $\mathbb{Q}_\gamma$ can be precipitous for every limit ordinal $\gamma > \omega$.

For a regular cardinal κ and an inaccessible cardinal $\lambda > \kappa$, let $\mathrm{Col}(\kappa, < \lambda)$ be a standard κ-closed λ-c.c. Levy collapse making λ to be κ^+.

LEMMA 4.7. *Let κ be a regular uncountable cardinal and $\lambda > \kappa$ a supercompact cardinal. Let $\dot{\mathbb{Q}}$ be a $\mathrm{Col}(\kappa, < \lambda)$-name such that $\Vdash_{\mathrm{Col}(\kappa,<\lambda)}$ “$\dot{\mathbb{Q}}$ is a λ-directed closed poset.” Then $\mathrm{Col}(\kappa, < \lambda) * \dot{\mathbb{Q}}$ forces “$\kappa^+ = \lambda$ and $\mathrm{RP}(\lambda, \mu)$ holds for every $\mu \geq \lambda$.”*

PROOF. Fix a sufficiently large cardinal $\mu > \lambda$. It enough to show that $\mathrm{Col}(\kappa, < \lambda) * \dot{\mathbb{Q}}$ forces $\mathrm{RP}(\lambda, \mu)$. Let $j : V \to M$ be a μ-supercompact embedding with the critical point λ. Let us consider a poset $\mathrm{Col}(\kappa, < j(\lambda))^M$ that is κ-closed. It is known that there is a name $\dot{\mathbb{R}}$ for a κ-closed poset such that $\mathrm{Col}(\kappa, < j(\lambda))^M \approx \mathrm{Col}(\kappa, < \lambda) * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$. Now choose a $\mathrm{Col}(\kappa, < j(\lambda))^M$-generic filter $j(G)$ over V, and let $G = j(G) \cap \mathrm{Col}(\kappa, < \lambda)$. G is a $\mathrm{Col}(\kappa, < \lambda)$-generic filter over V. In $V[j(G)]$, we can extend j to $j : V[G] \to M[j(G)]$ by a canonical way. Let $\mathbb{Q} = \dot{\mathbb{Q}}_G$, and take a $\mathbb{Q}$-generic filter H over $V[G]$ induced by $j(G)$. We have $H \in M[j(G)]$, and $j\text{“}H \in M[j(G)]$. In $M[j(G)]$, since μ was sufficiently large, the size of $j\text{“}H$ is $\leq \mu$, hence is $< j(\lambda)$. Since $j(\mathbb{Q})$ is $j(\lambda)$-directed closed in $M[j(G)]$, we can find a condition $p^* \in j(\mathbb{Q})$ with $p^* \leq j(q)$ for every $q \in H$. Then take a $j(\mathbb{Q})$-generic filter $j(H)$ over $V[j(G)]$ with $p^* \in j(H)$, and, in $V[j(G)][j(H)]$, we can extend j to $j : V[G][H] \to M[j(G)][j(H)]$.

We now show that $\mathrm{RP}(\lambda, \mu)$ holds in $V[G][H]$. Take a stationary $S \subseteq [\mu]^\omega$ and $\alpha < \lambda$. Since $\dot{\mathbb{R}}_G$ is σ-closed in $V[G][H]$ and $j(\mathbb{Q})$ is σ-closed in $V[j(G)]$, S remains stationary in $V[j(G)][j(H)]$, hence $j\text{“}S$ is also stationary in $[j\text{“}\mu]^\omega$ as well. We know $|j\text{“}\mu| = \mu < j(\lambda)$ and $j\text{“}S = j(S) \cap [j\text{“}\mu]^\omega \in M[j(G)][j(H)]$. Thus,

$$M[j(G)][j(H)] \models \exists A(A \in \mathcal{P}_{j(\lambda)} j(\mu) \wedge j(\alpha) \subseteq A \wedge j(S) \cap [A]^\omega \text{ is stationary in } [A]^\omega).$$

By the elementarity of j,

$$V[G][H] \models \exists A(A \in \mathcal{P}_\lambda \mu \wedge \alpha \subseteq A \wedge S \cap [A]^\omega \text{ is stationary in } [A]^\omega). \quad ⊣$$

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LEMMA 4.8. *Suppose there are proper class many supercompact cardinals. Then there is a class forcing extension where* $\mathrm{RP}(\delta^+, \lambda)$ *holds for every regular uncountable cardinal* δ *and cardinal* $\lambda > \delta$.

PROOF. Let $\langle \kappa_\xi \mid \xi \in \mathrm{ON} \rangle$ be the increasing enumeration of all supercompact cardinals. To simplify our argument, we assume that there is no inaccessible cardinal which is a limit of supercompact cardinals: If such a cardinal exists, let δ be the least inaccessible cardinal which is a limit of supercompact cardinals, and we identify V_δ as our universe V. So we know that $\kappa_\xi > \sup_{\zeta<\xi} \kappa_\zeta$ for every ξ, and if ξ is a limit ordinal then $\sup_{\zeta<\xi} \kappa_\zeta$ is a singular cardinal.

We define a class iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \mid \xi \in \mathrm{ON} \rangle$ as follows. First, let $\mathbb{P}_0$ be a trivial poset, and $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for a poset $\mathrm{Col}(\omega_1, <\kappa_0)$. Put $\mathbb{P}_1 = \mathbb{P}_0 * \dot{\mathbb{Q}}_0$. Suppose $\mathbb{P}_\xi$ is defined such that $|\mathbb{P}_\xi| < \kappa_\xi$. Moreover, suppose that $\mathbb{P}_\xi$ forces that $\kappa_{\xi-1}$ to be regular if ξ is successor, and that $\kappa = \sup_{\zeta<\xi} \kappa_\zeta$ to be a cardinal if ξ is limit. Let $\dot{\mathbb{Q}}_\xi$ be a $\mathbb{P}_\xi$-name such that:

$$\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi = \begin{cases} \mathrm{Col}(\kappa_{\xi-1}, <\kappa_\xi)^{V[\dot{G}_\xi]} & \text{if } \xi \text{ is successor,} \\ \mathrm{Col}(\kappa^+, <\kappa_\xi)^{V[\dot{G}_\xi]} & \text{if } \xi \text{ is limit, where } \kappa = \sup_{\zeta<\xi} \kappa_\zeta. \end{cases}$$

Let $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$. If ξ is limit, then let $\mathbb{P}_\xi$ be an inverse limit of the $\mathbb{P}_\zeta$'s for $\zeta < \xi$. Finally, let $\mathbb{P}_\infty$ be a class poset that is a direct limit of the $\mathbb{P}_\xi$'s.

By a standard argument, we can check the following conditions hold:

(1) $\mathbb{P}_\infty$ forces ZFC.
(2) $\mathbb{P}_\infty$ forces "for every regular uncountable cardinal δ, the successor cardinal δ^+ is κ_ξ for some ξ."
(3) For a successor ξ, $\mathbb{P}_\xi$ forces that $\kappa_{\xi-1}$ to be regular, and that "κ_ξ is supercompact and $\dot{\mathbb{Q}}_\xi = \mathrm{Col}(\kappa_{\xi-1}, <\kappa_\xi)^{V[\dot{G}_\xi]}$." Furthermore $\mathbb{P}_\infty$ can be factored as $\mathbb{P}_\xi * \mathrm{Col}(\kappa_{\xi-1}, <\kappa_\xi)^{V[\dot{G}_\xi]} * \dot{\mathbb{P}}_{\xi,\infty}$ such that $\mathbb{P}_\xi * \mathrm{Col}(\kappa_{\xi-1}, <\kappa_\xi)^{V[\dot{G}_\xi]} (= \mathbb{P}_{\xi+1})$ forces "$\dot{\mathbb{P}}_{\xi,\infty}$ is κ_ξ-directed closed."
(4) For a limit ξ, let $\kappa = \sup_{\zeta<\xi} \kappa_\zeta$. Then $\mathbb{P}_\xi$ forces that κ to be a singular cardinal, and that "κ_ξ is supercompact and $\dot{\mathbb{Q}}_\xi = \mathrm{Col}(\kappa^+, <\kappa_\xi)^{V[\dot{G}_\xi]}$." Furthermore $\mathbb{P}_\infty$ can be factored as $\mathbb{P}_\xi * \mathrm{Col}(\kappa^+, <\kappa_\xi)^{V[\dot{G}_\xi]} * \dot{\mathbb{P}}_{\xi,\infty}$ such that $\mathbb{P}_\xi * \mathrm{Col}(\kappa^+, <\kappa_\xi)^{V[\dot{G}_\xi]} (= \mathbb{P}_{\xi+1})$ forces "$\dot{\mathbb{P}}_{\xi,\infty}$ is κ_ξ-directed closed."

We show that a generic extension via $\mathbb{P}_\infty$ is as required. Take a $\mathbb{P}_\infty$-generic filter G over V and work in $V[G]$. Fix a regular uncountable cardinal δ. We can find ξ such that $\delta^+ = \kappa_\xi$. By (2), (3) above, and Lemma 4.7, $\mathrm{RP}(\delta^+, \lambda)$ holds for every $\lambda > \delta$. ⊣

THEOREM 4.9. *Suppose there are proper class many supercompact cardinals. Then there is a class forcing extension where* $\mathbb{Q}_\gamma$ *is precipitous for every limit ordinal* $\gamma > \omega$.

PROOF. By Lemma 4.8, we can work in a generic extension where $\mathrm{RP}(\delta^+, \lambda)$ holds for every regular uncountable cardinal δ and cardinal $\lambda > \delta$. We now take a limit ordinal $\gamma > \omega$ and let $\kappa = |V_\gamma|$. κ must be a limit cardinal. By our assumption and Lemma 3.4, $\mathrm{RP}(\kappa, 2^\kappa)$ holds, and, from Theorem 3.5, $\mathbb{Q}_\gamma$ is precipitous. ⊣

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