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The entropy of solvable groups

D. V. OSIN
Department of Higher Algebra, MEHMAT, Moscow State University, 119899, Moscow, Russia
(e-mail: denis.osin@mtu-net.ru)

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Abstract. We prove that any finitely generated solvable group of zero entropy contains a nilpotent subgroup of finite index. In particular, any finitely generated solvable group of exponential growth is of uniformly exponential growth.

1. *Introduction*

Let G be a group generated by a finite set X. As usual, we denote by $\|g\|_X$ the *word length* of an element $g \in G$ with respect to X, i.e. the length of the shortest word over the alphabet $X \cup X^{-1}$ which represents g. Recall that the *growth function* $\gamma_G^X : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by putting

$$\gamma_G^X(n) = \mathrm{card}\{g \in G : \|g\|_X \leq n\}.$$

Growth considerations in group theory with motivations arising from differential geometry and theory of invariant means were introduced in the 1950s by Efremovic [**4**], Švarc [**27**], and Følner [**5**], and (independently) in the 1960s by Milnor [**23**].

The *exponential growth rate* of G with respect to X is the number

$$\omega(G, X) = \lim_{n\to\infty} \sqrt[n]{\gamma_G^X(n)}.$$

This limit exists by the submultiplicativity of γ_G^X [**32**, Theorem 4.9]. The quantity

$$\omega(G) = \inf_X \omega(G, X)$$

is called the *minimal exponential growth rate* of G (the infimum is taken over all the finite generating sets of G). Finally, the *entropy* of the group G is defined by the formula

$$h(G) = \log \omega(G).$$

This notion of entropy comes from geometry and should not be confused with the notion of entropy for a pair (G, μ), where μ is a symmetric probability measure on a group G, as defined in [**1**]. In particular, if G is the fundamental group of a compact Riemannian

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manifold of unit diameter, then $h(G)$ is the lower bound for the topological entropy of the geodesic flow of the manifold [**22**]. Exponential growth rates also appear in the study of random walks on the Cayley graphs of finitely generated groups. I refer the reader to [**9**], [**13**] and [**16**], for more detail and background.

The group G is said to be of *exponential growth* if $\omega(G, X) > 1$, of *uniformly exponential growth* if $\omega(G) > 1$ and of *subexponential growth* if $\omega(G, X) = 1$. If there exist constants $C, d > 0$ such that $\gamma_G^X(n) \leq Cn^d$ for all $n \in \mathbb{N}$, then G is said to be of polynomial growth. These depend only on G, not on the finite generating set X. An inspirational question is posed as Milnor's problem [**24**]. Is there a finitely generated group of *intermediate growth*, i.e. one with subexponential growth but not of polynomial growth? Negative answers have been obtained by Milnor [**25**], Wolf [**33**] and Tits [**29**] for some particular classes of groups but finitely generated groups of intermediate growth have been discovered more recently by Grigorchuk [**6**].

Therefore, it has become important to know whether a finitely generated group with exponential growth but not uniformly exponential growth exists. This question goes back to [**13**] and can be found in [**7**] as well as in [**9**] and [**16**]. Let us give some known results in this direction. There are many examples of classes of groups which are known to have uniformly exponential growth, for example, non-elementary hyperbolic groups [**20**], one-relator groups of exponential growth [**11**] and others. Amalgamated products and HNN-extensions were investigated in [**3**]. One possible approach for constructing groups with exponential growth but not uniformly exponential growth was suggested in [**8**] (see also [**10**]).

Let us recall that a group G is said to be *virtually nilpotent* if it contains a nilpotent subgroup of finite index. The aim of the present paper is to prove the following theorem.

THEOREM 1.1. *Let G be a finitely generated solvable group of zero entropy. Then G is virtually nilpotent.*

This extends Milnor's [**25**] and Wolf's [**33**] results by saying that any finitely generated solvable group of subexponential growth is virtually nilpotent. Recall that a finitely generated group G has polynomial growth if (and only if) G is virtually nilpotent (see [**2**], [**12**] and [**14**]). Thus we can reformulate Theorem 1.1 as follows.

COROLLARY 1.2. *Any finitely generated solvable group with exponential growth has uniformly exponential growth.*

Note that the analog of Corollary 1.2 has been proved in [**3**] for the particular case of split extensions of type $\mathbb{Z}^d \rtimes \mathbb{Z}$. Pierre de la Harpe has informed me that he has a proof of the same result for arbitrary finitely presented Abelian-by-cyclic groups based on the techniques in [**11**]. I have also learned that the same result was obtained simultaneously and independently by R. Alperin for the polycyclic groups.

The paper is organized as follows. In the next section I will prove a number of technical results which are needed for the following proofs. The particular case of polycyclic groups will be considered in §3. Finally, the proof of the main theorem will be given in §4.

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2. *Technical lemmas*

For any group H, we denote by $\gamma_i H$ the ith term of the lower central series

$$H = \gamma_1 H \rhd \gamma_2 H \rhd \cdots,$$

where $\gamma_{i+1} H = [\gamma_i H, H]$. Recall that a group N is *nilpotent of degree t* if $\gamma_{t+1} N = 1$. During this section, $\mathfrak{AN}_t$ will denote the class of all *Abelian-by-(nilpotent of degree t)* groups. Thus G lies in $\mathfrak{AN}_t$ if and only if there exists an exact sequence

$$1 \longrightarrow A \longrightarrow G \longrightarrow N \longrightarrow 1, \tag{1}$$

where A is Abelian and N is nilpotent of degree t. Similarly, G is called *Abelian-by-cyclic* if there is an exact sequence (1) such that A is Abelian and N is an infinite cyclic. Recall also that a group P is *polycyclic* if there is a subnormal series

$$1 = Pk \lhd P_{k-1} \lhd \cdots \lhd P_0 = P, \tag{2}$$

where P_{i-1}/P_i is cyclic for all $i = 1, \ldots, k$. For a polycyclic group P, we denote the Hirsch number of P by $\chi(P)$, that is the number of infinite factors in a subnormal series of type (2). Note that the Hirsch number is well defined, i.e. it does not depend on the choice of the subnormal series [**26**]. Finally, for elements u, v of a group, we denote by $[u, v]$ the commutator $u^{-1}v^{-1}uv$.

Definition 2.1. Let G be a group with a given finite generating set X. For any finite set $Y = \{y_1, \ldots, y_m\} \subseteq G$, we define its *depth* with respect to X as follows:

$$\operatorname{depth}_X(Y) = \max_{i=1,\ldots,m} \|y_i\| X.$$

If H is a finitely generated subgroup of G, then we define its depth with respect to X by putting

$$\operatorname{depth}_X(H) = \min_{H=gp(Y)} \operatorname{depth}_X(Y),$$

where the minimum is taken over all finite generating sets of H.

I will now provide some elementary properties of exponential growth rates needed for the following.

LEMMA 2.1. *Let G be a group with a given finite generating set X. Then the following assertions are true.*

(1) *Suppose R is a finitely generated subgroup of G; then $\omega(G, X) \geq (\omega(R))^{1/\operatorname{depth}_X(R)}$.*
(2) *Suppose R is a normal subgroup of G; then $\omega(G/R) \leq \omega(G)$.*
(3) *Suppose R is a subgroup of finite index in G; then $\omega(R) \leq \omega(G)^{(2[G:R]-1)}$.*

Proof. The proofs of claims (1) and (2) are straightforward and left as an exercise to the reader. The claim (3) is also quite trivial and follows from Proposition 3.3 of [**28**]. □

Let us introduce some notation. Suppose G is a group generated by a finite set X. Then we put $W_1(X) = X$ and

$$W_i(X) = \{[u, v] : u \in W(i_1), v \in W(i_2), i_1, i_2 \in \mathbb{N}, i_1 + i_2 = i\}$$

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for any $i > 1$. Also, consider the function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(1) = 1 \quad \text{and} \quad f(n+1) = 2f(n) + 2 \tag{3}$$

for any $n \in \mathbb{N}$. It can easily be checked that $f(n) = 3 \cdot 2^{n-1} - 2$. The proof of the following lemma is quite trivial and left to the reader.

LEMMA 2.2. *Let f be the function given by* (3). *Then for any $i, j \in \mathbb{N}$, one has* $2(f(i) + f(j)) \leq f(i+j)$.

LEMMA 2.3. *For any group G with a given finite generating set X, one has*

$$\operatorname{depth}_X(W_n(X)) \leq f(n). \tag{4}$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial. Next, for $n > 1$, we observe that if $u \in W_{i_1}(X)$, $v \in W_{i_2}(X)$ and $i_1 + i_2 = n$, then

$$\begin{aligned} \|[u, v]\|_X &\leq 2(\|u\|_X + \|v\|_X) \leq 2(\operatorname{depth}_X(W_{i_1}(X)) + \operatorname{depth}_X(W_{i_2}(X))) \\ &\leq 2(f(i_1) + f(i_2)) \leq f(n) \end{aligned}$$

by the inductive hypothesis and Lemma 2.2. □

As an exercise, one can show that if G is a non-Abelian free group and X is a basis in G, then $\operatorname{depth}_X(W_n(X)) = f(n)$.

LEMMA 2.4. *Suppose G is a group generated by a finite set X and for $i = 1, \dots, t$, the sets $W_i(X) = \{w_{i1}, \dots, w_{iq_i}\}$ are defined as before. Consider two collections of integer constants $\{\lambda_{ij} : i = 1, \dots, t, j = 1, \dots, q_i\}$ and $\{\mu_{ij} : i = 1, \dots, t, j = 1, \dots, q_i\}$. Assume that $\lambda_{ij} = 0$ whenever $i < r_1$ and $\mu_{ij} = 0$ whenever $i < r_2$ for some $r_1, r_2 \in \mathbb{N}$. Then we have*

$$\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\lambda_{ij}}\Big)\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\mu_{ij}}\Big) = \Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\lambda_{ij}+\mu_{ij}}\Big)g, \tag{5}$$

where $g \in \gamma_{r_1+r_2}G$.

Proof. First let us prove that

$$\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\lambda_{ij}}\Big)w_{rs} = \Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\nu_{ij}}\Big)h, \tag{6}$$

where $h \in \gamma_{r+r_1}G$ and

$$\nu_{ij} = \begin{cases} \lambda_{ij}, & \text{if } (i,j) \neq (r,s), \\ \lambda_{ij} + 1, & \text{if } (i,j) = (r,s). \end{cases}$$

Indeed, suppose

$$u = \Big(\prod_{i=1}^{r-1}\prod_{j=1}^{q_i} w_{ij}^{\lambda_{ij}}\Big)\prod_{j=1}^{s-1} w_{rj}^{\lambda_{rj}}, \quad v = \prod_{j=s+1}^{q_r} w_{rj}^{\lambda_{rj}}\Big(\prod_{i=r+1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\lambda_{ij}}\Big).$$

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Then we have

$$uw_{rs}^{\lambda_{rs}} v w_{rs} = uw_{rs}^{\lambda_{rs}+1} v[v, w_{rs}].$$

Clearly, $[v, w_{rs}] \in \gamma_{r+r_1} G$. Thus, (6) is proved. It can easily be checked that (5) follows from (6) by induction. □

PROPOSITION 2.5. *Suppose $G \in \mathfrak{AN}_t$ is generated by a finite set X and f is given by* (3). *If any* 2*-generated Abelian-by-cyclic subgroup $H \leq G$ of depth at most $f(t+1)$ with respect to X is polycyclic, then G is polycyclic.*

Proof. Denote by B the normal closure of $W_{t+1}(X)$ in G. Note that $B = \gamma_{t+1} G$ (see [**18**, §5.3]). Since $G \in \mathfrak{AN}_t$, B is an Abelian subgroup of G. Suppose $W_i(X) = \{w_{i1}, \ldots, w_{iq_i}\}$ for $i = 1, \ldots, t+1$. Taking into account (4) and the assumptions of our proposition, we obtain that $\langle u, v \rangle$ is polycyclic for any $u \in W_{t+1}(X)$, $v \in W_i(X)$, $i = 1, \ldots, t$. Hence the Abelian subgroup $\langle v^{-l} u v^l : l \in \mathbb{Z} \rangle$ is finitely generated. This implies that for any $u \in W_{t+1}(X)$, $v \in W_i(X)$, $i = 1, \ldots, t$, there exists $L(u, v) \in \mathbb{N}$ such that $\langle v^{-l} u v^l : l \in \mathbb{Z} \rangle$ is generated by $\{v^{-l} u v^l : l \in \mathbb{Z}, |l| \leq L(u, v)\}$. Let us put

$$L = \max\left\{ L(u, v) : u \in W_{t+1}(X), v \in \bigcup_{i=1}^{t} W_i(X) \right\}.$$

Consider the set

$$Z = \left\{ \left(\prod_{i=1}^{t} \prod_{j=1}^{q_i} w_{ij}^{\alpha_{ij}} \right)^{-1} w_{t+1,k} \left(\prod_{i=1}^{t} \prod_{j=1}^{q_i} w_{ij}^{\alpha_{ij}} \right) : k = 1, \ldots, q_{t+1}, |\alpha_{ij}| \leq L \right\}.$$

Let us show that $B = \langle Z \rangle$, i.e. B is generated by Z as a subgroup. To prove this it is sufficient to check that $Z^G \subseteq \langle Z \rangle$, where, as usual, Z^G denotes the set $\{g^{-1} z g : g \in G, z \in Z\}$. We will show that

$$Z^{\gamma_r G} \subseteq \langle Z \rangle \tag{7}$$

by induction on r. The case $r = t+1$ is trivial since $\gamma_{t+1} G \leq B$ and hence $Z^{\gamma_{t+1} G} = Z$. Further, suppose $1 \leq r < t+1$. Note that $\gamma_r G$ is generated by $\left(\bigcup_{i=1}^{r} W_i(X) \right) \cup B$. Therefore in order to prove (7), it is sufficient to check that $w^{-1} z w \in \langle Z \rangle$ and $w z w^{-1} \in \langle Z \rangle$ for any $z \in Z$, $w \in W_r(X)$. Assume

$$z = \left(\prod_{i=1}^{t} \prod_{j=1}^{q_i} w_{ij}^{\alpha_{ij}} \right)^{-1} w_{t+1,k} \left(\prod_{i=1}^{t} \prod_{j=1}^{q_i} w_{ij}^{\alpha_{ij}} \right), \quad w = w_{rs}.$$

Let us consider two cases.

(1) $\alpha_{rs} < L$. Using Lemma 2.4, we obtain

$$w^{-1} z w = g^{-1} \left(\prod_{i=1}^{t} \prod_{j=1}^{q_i} w_{ij}^{\beta_{ij}} \right)^{-1} w_{t+1,k} \left(\prod_{i=1}^{t} \prod_{j=1}^{q_i} w_{ij}^{\beta_{ij}} \right) g = g^{-1} z' g,$$

where

$$\beta_{ij} = \begin{cases} \alpha_{ij}, & \text{if } (i, j) \neq (r, s), \\ \alpha_{ij} + 1, & \text{if } (i, j) = (r, s), \end{cases}$$

$g \in \gamma_{r+1} G$, and $z' \in Z$. It follows from the inductive assumption that $w^{-1} z w \in \langle Z \rangle$.

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(2) $\alpha_{rs} = L$. By definition of L, we have

$$w_{rs}^{-L-1} w_{t+1,k} w_{rs}^{L+1} = (w_{rs}^{-L} w_{t+1,k} w_{rs}^{L})^{\xi_{-L}} \cdots (w_{rs}^{-L} w_{t+1,k} w_{rs}^{L})^{\xi_L}$$

for some $\xi_{-L}, \dots, \xi_L \in \mathbb{Z}$. Using Lemma 2.4, we get

$$\begin{aligned} w^{-1}zw &= h^{-1}\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\delta_{ij}}\Big)^{-1} w_{rs}^{-L-1} w_{t+1,k} w_{rs}^{L+1}\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\delta_{ij}}\Big)h \\ &= h^{-1}\prod_{m=-L}^{L}\Big(\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\delta_{ij}}\Big)^{-1} w_{rs}^{-m} w_{t+1,k}^{\xi_m} w_{rs}^{m}\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\delta_{ij}}\Big)\Big)h \\ &= h^{-1}\prod_{m=-L}^{L}\Big(f_m^{-1}\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\epsilon_{ijm}}\Big)^{-1} w_{t+1,k}^{\xi_m}\Big(\prod_{i=1}^{t}\prod_{j=1}^{q_i} w_{ij}^{\epsilon_{ijm}}\Big) f_m\Big)h, \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} \alpha_{ij}, & \text{if } (i,j) \neq (r,s), \\ 0, & \text{if } (i,j) = (r,s), \end{cases} \qquad \epsilon_{ijm} = \begin{cases} \delta_{ij}, & \text{if } (i,j) \neq (r,s) \\ m, & \text{if } (i,j) = (r,s), \end{cases}$$

and $h, f_{-L}, \dots, f_L \in \gamma_{r+1}G$. Taking into account the fact that $|\epsilon_{ijm}| \leq L$ for all admissible i, j, m, we obtain $w^{-1}zw = h^{-1}z'h$, where $z' \in \langle Z \rangle$. The inductive hypothesis implies $\langle Z \rangle^{\gamma_{r+1}G} \subseteq \langle Z \rangle$ and so $w^{-1}zw \in \langle Z \rangle$.

In the same way we obtain $wzw^{-1} \in \langle Z \rangle$ for any $w \in W_r(X)$. Thus (7) is proved. Since Z is finite, B is finitely generated and hence G is polycyclic. □

Proving the following lemma, we use some basic properties of nilpotent groups [**15**].

LEMMA 2.6. *Let N be a nilpotent group generated by a finite set $X = \{x_1, \dots, x_k\}$. Then the subgroup $M = \langle x_1^n, \dots, x_k^n \rangle$ is of finite index in N for any $n \in \mathbb{N}$. Moreover, $\gamma_i M$ is of finite index in $\gamma_i N$ for all i.*

Proof. Let $\gamma_s N$ be the last non-trivial term of the lower central series of N. Denote by $w = w(x_{i_1}, \dots, x_{i_k})$ some element of $W_s(X)$ and consider $w(x_{i_1}^n, \dots, x_{i_k}^n)$, that is the element obtained from w by substitution $x_{i_j}^n$ for x_{i_j}, $j = 1, \dots, k$. Clearly,

$$w(x_{i_1}^n, \dots, x_{is}^n) = w(x_{i_1}, \dots, x_{is})^{n^s}.$$

Therefore $w^{n^s} \in \gamma_s M$ for any $w \in W_s(X)$. Since $\gamma_s N$ is generated by $W_s(X)$, $\gamma_s M$ is of finite index in $\gamma_s N$. By inductive arguments, we can assume that $\overline{M} = (M \cdot \gamma_s N)/\gamma_s N$ is of finite index in $\overline{N} = N/\gamma_s N$ and, moreover, $\gamma_i \overline{M}$ has a finite index in $\gamma_i \overline{N}$ for all i. Hence $\gamma_i M$ is of finite index in $\gamma_i N$ for all i. In particular, M is of finite index in N. □

LEMMA 2.7. *Suppose H is a group generated by a finite set $\{h_1, \dots, h_k\}$, D is a finitely generated free Abelian normal subgroup of H, and H/D is virtually nilpotent. The action of H on D by conjugation induces a representation $\sigma : H \to GL(D \otimes_{\mathbb{Z}} \mathbb{C})$. If there exists a constant $c \in \mathbb{N}$ such that $\sigma(h_i^c)$ is unipotent for any $i = 1, \dots, k$, then H is virtually nilpotent.*

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Proof. Since H is virtually solvable, it follows from the Kolchin–Mal'tsev Theorem that there is a subgroup H_1 of finite index in H such that $\sigma(H_1)$ is in triangular form. Without loss of generality we can assume that $D \leq H_1$. Let us put $n = [H : H_1]$ and consider the subgroup $H_0 = \langle h_1^{cn}, \ldots, h_k^{cn}, D\rangle$. Clearly, $H_0 \leq H_1$. Since H/D is nilpotent, H_0/D has a finite index in H/D by Lemma 2.6. Therefore H_0 is of finite index in H. Moreover, $\sigma(H_0)$ is generated by unipotent elements. Since the set of all upper triangular unipotent matrices forms a subgroup in $GL(n, \mathbb{C})$, it follows that $\sigma(h)$ is unipotent for any $h \in H_0$. Further, there exists a sequence

$$1 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d = V$$

of subspaces in $V = D \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\dim(V_{i+1}/V_i) = 1$ and V_{i+1}/V_i is H_0-invariant for all $i = 0, \ldots, d-1$. Let us put $C_i = V_i \cap D$. Let

$$1 = \zeta_0 H_0 \leq \zeta_1 H_0 \leq \cdots \leq \zeta_k H_0 \leq \cdots$$

be the upper central series of H_0. Now we are going to prove that

$$C_i \leq \zeta_i H_0, \quad i = 0, \ldots, d \tag{8}$$

by induction on i. The case $i = 0$ is trivial. Next, suppose $i > 0$. Since V_i/V_{i-1} is centralized by H_0, we obtain $C_i/C_{i-1} \leq Z(H_0/C_{i-1})$ and (8) follows immediately. To conclude the proof it remains to note that $C_d = D$ and thus H_0 is nilpotent. □

LEMMA 2.8. *Let $G \in \mathfrak{AN}_t$ be a polycyclic group with a given finite generating set X and let $A = \gamma_{t+1}(G)$. Then there exists an Abelian subgroup U of A with the following properties.*

(1) $\mathrm{depth}_X(U) \leq 2\chi(G) + f(t+1)$, *where $\chi(G)$ is the Hirsch number of G.*

(2) *U has finite index in A.*

Proof. Let us recall that $A = \langle W_{t+1}(X)\rangle^G$ [**18**, §5.3]. We put

$$U_i = \langle g^{-1} w g : w \in W_{t+1}(X), \|g\|_X \leq i\rangle$$

for all $i \in \mathbb{N}$. Since any polycyclic group satisfies the ascending chain condition for subgroups, we obtain a finite filtration

$$U_1 \leq U_2 \leq \cdots \leq U_r = A.$$

Consider a minimal $n \in \mathbb{N}$ such that $[U_{n+1} : U_n] < \infty$. This means that there is an $N \in \mathbb{N}$ such that $g^{-1} w^N g \in U_n$ whenever $\|g\|_X \leq n+1$. Let us take an element $h \in G$ of length $\|h\|_X \leq n+2$. Then $h = gx$, where $\|g\|_X \leq n+1$, $x \in X \cup X^{-1}$, and we have $h^{-1} w^N h \in x^{-1} U_n x \leq U_{n+1}$. Therefore U_{n+1} is of finite index in U_{n+2} and so on. Proceeding by induction, we establish that U_n is a subgroup of finite index in A. Finally, let us note that $\chi(G) \geq \chi(A) \geq n$ and thus $U = U_{\chi(G)}$ is of finite index in A. The inequality $\mathrm{depth}_X(U) \leq 2\chi(G) + f(t+1)$ follows immediately from the definition of U_i. □

3. *The entropy of polycyclic groups*

During this section, P will denote a split extension of type $A \rtimes_\theta \mathbb{Z}$, where $A \cong \mathbb{Z}^d$, $\theta \in GL(d, \mathbb{Z})$ and $\det(\theta) = \pm 1$. Obviously such a group is polycyclic. Our next goal is

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to obtain an estimate for the entropy of P. Let us introduce some notation. We denote by $\mathrm{Spec}(\theta)$ the set of all (complex) eigenvalues of θ and put

$$\Lambda(\theta) = \max_{\lambda \in \mathrm{Spec}(\theta)} |\lambda|.$$

In fact, the group P is virtually nilpotent if and only if $\Lambda(\theta) = 1$, i.e. any eigenvalue of θ is of absolute value one [**33**] (see also [**30, 31**]).

LEMMA 3.1. *In the previous notation, let us suppose P is not virtually nilpotent. Then we have*

$$\omega(P) \geq 2^{\log \Lambda(\theta)/(\log 2 + 5 \log \Lambda(\theta))}. \tag{9}$$

Proof. Let us note that the previous bound of the entropy was essentially proved by Tits in [**30**] as well as in [**31**] although it was not formulated in an explicit form. Also, this bound was obtained by Bucher and de la Harpe [**3**] up to some notation. We repeat this proof (up to some small changes) for the convenience of the reader.

Denote the characteristic polynomial of θ by p. First we assume that p is irreducible over $\mathbb{Q}$. This means precisely that θ is irreducible over $\mathbb{Q}$. Let S be a finite generating set of P. If $\pi : P \to \mathbb{Z}$ denotes the canonical projection, there exists an $s \in S$ such that $\pi(s) = m \neq 0$; upon changing s to s^{-1}, we can assume that $m \geq 1$. As P is not Abelian, there exists a $t \in S$ such that $u = sts^{-1}t^{-1} \neq 1$. Since $u \in \mathbb{Z}^d$, we can identify sus^{-1} with $\theta^m(u)$. Let $L : A \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}$ be a non-trivial linear form such that $L \circ \theta = \lambda L$, where $|\lambda| = \Lambda(\theta) > 1$. Since θ is irreducible over $\mathbb{Q}$, $L(a) \neq 0$ whenever a is a non-trivial element of A. Set $N = [\log_{\Lambda(\theta)} 2] + 1$, where $[x]$ denotes the integer part of x. For $k \geq 1$ and $\epsilon\ 0, \ldots, \epsilon\ k \in \{0, 1\}$, consider the elements

$$p_{\epsilon_0,\ldots,\epsilon_k} = u^{\epsilon_0} \prod_{j=1}^{k} (s^{-N} u^{\epsilon_j}) = \prod_{j=0}^{k} (\theta^{jmN}(u^{\epsilon_j})) s^{-kN}.$$

We have

$$L(p_{\epsilon_0,\ldots,\epsilon_k} s^{kN}) = \left(\sum_{j=0}^{k} \epsilon_j \lambda^{jmN} \right) L(u).$$

Since $|\lambda^{mN}| \geq 2$, the numbers $\sum_{j=0}^{k-1} \epsilon_j (\lambda^{mN})^j$ are pairwise distinct. This implies $p_{\epsilon_0,\ldots,\epsilon_k} \neq p_{\delta_0,\ldots,\delta_l}$ whenever $(\epsilon_0, \ldots, \epsilon_k) \neq (\delta_0, \ldots, \delta_l)$. Now we have

$$\|p_{\epsilon_0,\ldots,\epsilon_k}\|_S \leq kN + (k+1)\|u\|_S \leq k(N+4) + 4.$$

Thus the growth function of P satisfies $\gamma_P^S(k(N+4)+4) \geq 2^{k+1}$, and we have finally that $\omega(P, S) \geq 2^{1/(N+4)}$. This implies (9).

Now suppose $p = p_1 p_2$, where p_1, p_2 are polynomials over $\mathbb{Q}$ of degree at least 1. Without loss of generality we can assume that p_1 is irreducible over $\mathbb{Q}$ and has a root λ of absolute value $|\lambda| = \Lambda$. Consider the set $A_0 = \{a \in A \mid p_2(\theta)(a) = 0\}$. It is not hard to check that A_0 is a normal subgroup in P. So we can take the quotient group P/A_0 that is a group of type $\mathbb{Z}^{d'} \rtimes_\psi \mathbb{Z}$, where the characteristic polynomial of ψ is p_1. To conclude the proof it remains to refer to assertion (2) of Lemma 2.1. □

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Proving the following two lemmas, we use some well-known facts about field extensions [**21**].

LEMMA 3.2. *Let $S \subseteq GL_n(\mathbb{Z})$ be a finitely generated solvable group. Then the set*

$$\mathrm{Spec}(S) = \bigcup_{\sigma \in S} \mathrm{Spec}(\sigma)$$

lies in the ring of algebraic integers of a finite algebraic extension of $\mathbb{Q}$.

Proof. Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$. By the Kolchin–Mal'tsev Theorem, there exists a $\rho \in GL_n(\overline{\mathbb{Q}})$ such that $\rho^{-1}S\rho$ contains a subgroup S_0 of finite index consisting of upper-triangular matrices. Note that $\rho^{-1}S\rho \subseteq GL_n(K)$, where K is a finite extension of $\mathbb{Q}$. Consider a finite generating set $\sigma_1, \ldots, \sigma k$ of S_0 and denote by L the finite extension of K obtained by adjoining of all eigenvalues of σ_i, $i = 1, \ldots, k$. Clearly, $\mathrm{Spec}(S_0)$ is a subset of the ring of algebraic integers over of L. Now let $n = [S : S_0]$. Denote by M the field obtained from L by joining of the set $\{x : x^n \in \mathrm{Spec}(S_0)\}$. Observe that if $\sigma \in S$ then $\sigma^n \in S_0$. This implies that $\mathrm{Spec}(S)$ is a subset of the ring of algebraic integers over of M. □

Let us recall some facts from algebraic number theory. Let K be a finite algebraic extension of $\mathbb{Q}$, K_0 be the ring of algebraic integers of K and K_0^* be the group of units of K_0. Denote real isomorphisms $k \to \mathbb{C}$ by $\sigma_1, \ldots, \sigma_s$ and complex ones by $\sigma_{s+1}, \bar{\sigma}_{s+1}, \ldots, \sigma_{s+t}, \bar{\sigma}_{s+t}$. Let us define $\tau : K_0^* \to \mathbb{R}^{s+t}$ by putting

$$\tau(x) = (\ln|\sigma_1(x)|, \ldots, \ln|\sigma_{s+t}(x)|). \tag{10}$$

Then as is well known, the set $\tau(K_0^*)$ is discrete in $\mathbb{R}^{s+t}$. The kernel of τ consists of the roots of the identity. I refer the reader to [**19**, Appendix] or [**17**, III.28] for details.

LEMMA 3.3. *Let G be an arbitrary polycyclic group, A an Abelian normal subgroup of G and $\rho : G \to GL(A \otimes_{\mathbb{Z}} \mathbb{C})$ the corresponding representation. There exists a constant $\epsilon = \epsilon(G, A) > 0$ such that if*

$$\Lambda(\rho(g)) < 1 + \epsilon \tag{11}$$

then $\rho(g^k)$ is unipotent for some $k \in \mathbb{N}$.

Proof. Let K be a finite extension of $\mathbb{Q}$ such that $\mathrm{Spec}(\rho(G)) \subseteq K_0$. Note that $\det(\rho(g)) = \pm 1$ for any $g \in G$. Therefore $\mathrm{Spec}(\rho(G)) \subseteq K_0^*$. Let us define a norm on $\mathbb{R}^{s+t}$ by putting

$$\|(x_1, \ldots, x_{s+t})\| = \max_{i=\overline{1,s+t}} |x_i|$$

and denote by ϵ a positive constant such that there are no elements of $\tau(K_0^*)$ in the ball $B_\epsilon(0) \subset R^{s+t}$ except for zero. Let g be an element of G satisfying (11). Given an embedding $\sigma : K \to \mathbb{C}$ such that $\sigma(\mathbb{Q}) = \mathbb{Q}$, $\lambda \in \mathrm{Spec}(\rho(g))$ implies $\sigma(\lambda) \in \mathrm{Spec}(\rho(g))$. Therefore (11) implies $\|\tau(\lambda)\| < \ln(1+\epsilon) < \epsilon$ and hence $\tau(\lambda) = 0$. Since $\mathrm{Ker}(\tau)$ consists of roots of the identity, there is a constant $k \in \mathbb{N}$ such that $\rho(g^k)$ is unipotent. □

PROPOSITION 3.4. *Let G be a polycyclic group of zero entropy. Then G is virtually nilpotent.*

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Proof. Note that any polycyclic group contains a subgroup of finite index that is nilpotent-by-Abelian [**26**]. Thus without loss of generality we can assume that $G \in \mathfrak{AN}_t$. Let $A = \gamma_{t+1}G$ and U be a subgroup of G satisfying conditions (1) and (2) of Lemma 2.8. We denote by $\rho : G \to GL(A \otimes_{\mathbb{Z}} \mathbb{C})$ the corresponding representation and by ϵ the constant $\epsilon(G, A)$ from Lemma 3.3.

Now let X be a finite generating set of G such that

$$\omega(G, X) < 2^{\log(1+\epsilon)/(2\chi(G)+f(t+1))(\log 2+5\log(1+\epsilon))}. \tag{12}$$

First we assume that there exists an $x \in X$ such that $\Lambda(\rho(x)) \geq 1 + \epsilon$. Consider the subgroup $P = \langle x, U\rangle$. Note that

$$\mathrm{depth}_X(P) \leq 2\chi(G) + f(t + 1). \tag{13}$$

Let us put $P_0 = P \cap A$ and denote by $\rho' : P \to GL(P_0 \otimes_{\mathbb{Z}} \mathbb{C})$ the corresponding representation. Since U is of finite index in A, P_0 is of finite index in A also and hence

$$\Lambda(\rho'(x)) = \Lambda(\rho(x)) \geq 1 + \epsilon. \tag{14}$$

Using (13), (14), Lemma 2.1 and Lemma 3.1, we get a contradiction to (12).

Therefore for any $x \in X$ we have $\Lambda(\rho(x)) < 1 + \epsilon$. By Lemma 3.3, there exists a $k \in \mathbb{N}$ such that $\rho(x^k)$ is unipotent for all $x \in X$. Applying Lemma 2.7, we establish that G is virtually nilpotent. □

4. *The general case*

LEMMA 4.1. *Suppose H is a 2-generated Abelian-by-cyclic group. If, in addition, $\omega(H) < \sqrt[5]{2}$, then H is polycyclic.*

Proof. Proving this lemma, we essentially follow Milnor's idea in [**25**]. Consider an exact sequence

$$1 \longrightarrow A \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where A is an Abelian group. Denote by x, y some generators of H and by $\pi : H \to \mathbb{Z}$ the canonical homomorphism. Without loss of generality we can assume that $\pi(x) = m \neq 0$. Let us put $a = [x, y]$, $T = \langle a, x\rangle$, and consider the elements

$$t_{\alpha_1,\ldots,\alpha_r} = xa^{\alpha_1}xa^{\alpha_2}\cdots xa^{\alpha_r},$$

where $\alpha_i \in \{0, 1\}$ for all i.

Suppose that for any generating set $\{x, y\}$ of H we have $t_{\alpha_1,\ldots,\alpha_{r_1}} \neq t_{\beta_1,\ldots,\beta_{r_2}}$ whenever $(\alpha_1, \ldots, \alpha_{r_1}) \neq (\beta_1, \ldots, \beta_{r_2})$. Then

$$\omega(H) \geq \sqrt[5]{2}. \tag{15}$$

Indeed, we have $\|t_{\alpha_1,\ldots,\alpha_r}\|_{\{x,y\}} \leq 5r$. Therefore

$$\gamma_H^{\{x,y\}}(5r) \geq 2^n. \tag{16}$$

Since (16) is true for any generating set $\{x, y\}$ of H, we obtain (15) which contradicts the assumption of the lemma. Thus there exists a generating set $\{x, y\}$ of H such that we have

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$t_{\alpha_1,\ldots,\alpha_{r_1}} = t_{\beta_1,\ldots,\beta_{r_2}}$ for some $(\alpha_1,\ldots,\alpha_{r_1}) \neq (\beta_1,\ldots,\beta_{r_2})$. Clearly, $r_1 = r_2$ and we obtain

$$a_1^{\alpha_1-\beta_1}\cdots a_r^{\alpha_r-\beta_r} = 1,$$

where $r = r_1 = r_2$ and $a_i = x^i a x^{-i}$. Without loss of generality we can assume that $\alpha_1 - \beta_1 \neq 0$ and $\alpha_r - \beta_r \neq 0$. Thus $a_r \in \langle a_1,\ldots,a_{r-1}\rangle$. Conjugating by x, we obtain $a_{r+q} \in \langle a_1,\ldots,a_{r-1}\rangle$ for all $q \geq 0$. Similarly, we can show that $a_{1-q} \in \langle a_2,\ldots,a_r\rangle$ for all $q \geq 0$. Therefore $\langle a_1,\ldots,a_r\rangle = \langle a\rangle^T$ and thus T is polycyclic. Let us denote by z an element of H such that zA generates H/A. The reader will have no difficulty in showing that the set $\{z^{-j}a_i z^j : i = 1,\ldots,r,\ j = 1,\ldots,m-1\}$ generates $\langle a\rangle^H$. Hence H is polycyclic. □

Proof of Theorem 1.1. Let S be a finitely generated solvable group and let

$$1 = S_k \lhd S_{k-1} \lhd \cdots \lhd S_0 = S$$

be the derived series of S. Assume that $h(S) = 0$. Consider a maximal $r \in \mathbb{N}$ such that S/S_r is virtually nilpotent. If $r = k$ then there is nothing to prove. Now suppose $r < k$ and denote by N a nilpotent subgroup of finite index in S/S_r. Consider the natural homomorphism $\alpha : S/S_{r+1} \to S/S_r$ and denote by G the preimage of N in S/S_{r+1}. Evidently G is of finite index in S/S_{r+1}. By Lemma 2.1, $h(G) = 0$. Now let t denote the degree of nilpotency of N. Then $G \in \mathfrak{AN}_t$.

Let us fix a generating set X of G such that

$$\omega(G, X) < (\sqrt[5]{2})^{1/f(t+1)}.$$

Using Lemma 2.1 again, we obtain that any 2-generated Abelian-by-cyclic subgroup of depth at most $f(t+1)$ with respect to X has an entropy less than $1/5$. By Lemma 4.1, this implies that any such a subgroup is polycyclic. Therefore G is polycyclic by Proposition 2.5 and we can apply Proposition 3.4. Thus G is virtually nilpotent as well as S/S_{r+1} that contradicts to the choice of r. The theorem is proved. □

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